Optimal integrated maintenance/production policy for randomly failing systems with variable failure rate

Hajej Zied, Sofiene Dellagi, Nidhal Rezg

Optimal integrated maintenance/production policy for randomly failing systems with variable failure rate

Journal:	*International Journal of Production Research*
Manuscript ID:	TPRS-2010-IJPR-0856
Manuscript Type:	Original Manuscript
Date Submitted by the Author:	27-Aug-2010
Complete List of Authors:	Zied, Hajej; LGIPM DELLAGI, Sofiene; Université Paul Verlaine, Metz REZG, Nidhal; Universite Paul Verlaine, Metz
Keywords:	PREVENTIVE MAINTENANCE, MAINTENANCE PLANNING, PRODUCTION PLANNING, MANUFACTURING STRATEGY, OPTIMIZATION
Keywords (user):	PREVENTIVE MAINTENANCE, PRODUCTION PLANNING

Optimal integrated maintenance/production policy for randomly failing systems with variable failure rate

Hajej Zied, Dellagi Sofiene, Rezg Nidhal

LGIPM-INRIA, Université Paul Verlaine, Metz, France

{hajej, dellagi, rezg}@univ-metz.fr

Abstract

This paper deals with combined production and maintenance plans for a manufacturing system satisfying a random demand. We first establish an optimal production plan which minimizes the average total inventory and production cost. Secondly, using this optimal production plan, and taking into account the deterioration of the machine according to its production rate, we derive an optimal maintenance schedule which minimizes the maintenance cost. A numerical example illustrates the proposed approach, this analytical approach , based on a stochastic optimization model and using the operational age concept, reveals the significant influence of the production rate on the deterioration of the manufacturing system and consequently on the integrated production/maintenance policy.

Keywords

Failure rate, Maintenance strategies, Operational age, Linear quadratic model, Maintenance scheduling, Production plan.

I. Introduction

Recently, maintenance and production scheduling using stochastic optimal control techniques has drawn much attention among researchers. Due to the complexity of the manufacturing systems, decisions pertaining to marketing, production and maintenance have traditionally been treated separately. Clearly, however, analyzing these decisions simultaneously is more realistic and useful from a practical point of view. Accordingly, this study seeks to find the joint optimal production and maintenance strategy for a randomly failing manufacturing system which must satisfy a random product demand over future periods. This is indeed a complex task due to the various uncertainties caused by exogenous and endogenous factors. While exogenous factors are typically due to demand randomness, an example of an endogenous factor would be the availability of the production system. As a direct effect of these random elements, the inventory variable cannot be computed precisely, giving rise to the need to adopt a stochastic optimal control approach. Moreover, it is interesting to develop an intelligent optimal maintenance strategy considering the deterioration of the manufacturing system as a function of the production rate. Little research has been conducted in this area. Akella and Kumar (1986) formulated a one-machine one-part-type production problem as a stochastic optimal control problem, in which the part demand is assumed to be constant, the state of the machine is assumed to be a two-state continuous-time Markov chain, and the objective function is a discounted inventory/shortage cost over an infinite time horizon.

(Silva and Wagner, 2004) deal with a chance-constrained stochastic production-planning problem under hypotheses of imperfect information of inventory variables. The optimal production plan is obtained by the minimizing of the expected cost. Barták et al. (2009) describe a constraint programming approach solving scheduling problems with earliness and tardiness costs. In the same vein, Kelle et al. (1994) considered a single-product with random demand along with a single-machine with setups in the process industry. They formulated a model that incorporates mean and standard deviation of demand in each period. Though only one product was being made, start-ups after periods of idleness required significant setups.

In the situation of interest here, the stochastic nature of the system is due to machines which are subject to breakdowns and repairs or maintenance actions. The traditional maintenance strategies proposed in the literature are mainly policies involving the critical age of a machine or a set of machines. These policies are based on models describing the equipment failure law. The basic assumptions related to repair efficiency are known as minimal repair or as bad as old (ABAO) and perfect repair or as good as new (AGAN). In the ABAO case, each repair restores the system to the operating state to leaves it with the same failure rate level, he had before failure. In the AGAN case, each repair is perfect and restores the system was new. Obviously, reality lies somewhere between these two extreme cases: standard maintenance reduces the failure rate but does not return the system to the as good as new condition. This is sometimes known as imperfect or better-than-minimal repair. Along these lines, Brown and Proschan (1983) considered a model in which a perfect repair occurs with probability p whereas a minimal repair occurs with probability $(1-p)$. Another class of models of interest is the one of virtual age models proposed by Kijima (1988). Usually, these models are defined by the conditional distributions of successive inter-failure times.

The cost/time of maintenance/repair is supposed to be known and consequently the impact of a maintenance/failure can be analyzed. Under these conditions, it can be shown that the optimal policy is of the critical age type which consists in carrying out a preventive maintenance action at its critical age. In this context, Boukas and Yang (1996) assumed the simultaneous planning of production and maintenance in a flexible manufacturing system. The system is composed of a single machine subject to random failures which produces a given commodity. The probability of machine failure is supposed to be an increasing function of its age. The objective is to minimize the discounted inventory and maintenance cost subject to meeting the demand.

Moreover, under production control policies such as just-in-time, which requires the availability of machines at the right time, an integrated approach of maintenance and production control becomes essential. In this context, Rezg et. al. (2004) proposed a method for the joint optimization of preventive maintenance and stock control in a production line made up of N machines. Rezg et al. (2008) similarly presented a mathematical model and a numerical procedure for determining simultaneously an optimal inventory control policy and an age-based preventive maintenance policy for a randomly failing production system. Boukas and Haurie (1990) considered a system which has two machines with age-dependent failure rates and where a preventive maintenance decision must be made. They used a numerical method to evaluate the optimal control policy and showed that in their context the optimal hedging surfaces can be defined to represent the optimal production policies. Van der Dyun Schouten and Vanneste (1995) proposed an age-based preventive maintenance policy considering the capacity of a buffer stock between two machines. Moreover, maintenance/production strategies taking into account the context of a subcontractor are studied by Dellagi, et al. (2007), while Cheung

and Hausmann (1997) considered the simultaneous optimization of the strategic stock and the maintenance policy of the critical age type.

In reality, the failure rate increases with time and according to the utilization of the equipment, a situation rarely studied in the literature. Many maintenance models assume that the system is maintained under fixed operational and environmental conditions. For example, fixed operational conditions assume that the manufacturing system operates at the maximal production rate (hence ignoring the production rate variation). Schutz et al. (2009) proposed model periodic and sequential preventive maintenance policies for a system that performs various missions over a finite planning horizon. Each mission can have different characteristics that depend on operational and environmental conditions. To account for variable environmental conditions, Özekici (1995) proposes to take an intrinsic age of the system instead of the actual age, while Martorell et al. (1999) use models of accelerated life.

Motivated by the lack of consideration of the systems failure rate variation according to the production rate change, we propose a new approach to model an integrated maintenance/production policy taking into account this fact.

The paper is organized as follows. Section II describes the production/maintenance problem at hand together with the assumptions and a general stochastic control model. In Section III we develop the analytical models for evaluating maintenance and production strategies based on the operational age approach considering the influence of the production plan on the deterioration of the manufacturing system. In Section IV we present a simple numerical example in order to illustrate the analytical results and to compare solutions obtained, on the one hand, by maintenance schedule combined with an optimal production plan and, on the other hand, by a maintenance schedule combined with a nominal production plan. Finally, the conclusion of the paper is given in comprises Section V.

II. Problem description

We develop a model for jointly planning the production and maintenance activities of a single machine M producing one part-type through a single operation in order to satisfy a random demand. The latter is characterized by a normal distribution whose mean and standard deviation are respectively denoted by $\hat{d}$ and σ_d. The problem is illustrated in Figure.1.

Fig 1. Problem description

During the horizon H, machine M is subject to random failure. The probability density function of time to failure is $f(t)$, while the failure rate $\lambda(t)$ is increasing in both time and production rate $u(t)$. Failures of machine M can be reduced through preventive maintenance activities. Preventive maintenance (PM), usually scheduled periodically at certain time intervals, is a policy aimed at improving the overall reliability and availability of a system. Ideally, one would like to define a PM policy such that the overall cost of system failure, maintenance, and replacement during its production horizon H is minimized.

Under the constraint that the total time needed to perform both maintenance activities (preventive and corrective) is not greater than the finite horizon H, the Cox model (Cox, 1972), given below for a period k during an horizon

H, provides an estimate of the treatment effect on survival after adjustment for other explanatory variables. Thus he establishes a parametric relationship between risk factors (related to the operational and environmental conditions of each period) and the hazard rate. The model relies mainly on the assumption of proportional hazards, which implies that each factor affects the life steadily over time.

Let

$\lambda_0(t)$ the hazard rate for nominal conditions

$g(u_k)$ the risk function of period *k*

$\lambda(t,u_k)$ the hazard rate representing the instantaneous failure risk at time *t* under condition u_k

For a period *k*, the Cox model is given by:

$$\lambda(t,u_k) = \lambda_0(t) \cdot g(u_k)$$

Our first objective is to establish an economical production plan satisfying the random demand. Secondly, using this optimal production plan, we establish the optimal preventive maintenance plan. The use of the optimal production plan as an input to the maintenance study is justified by the influence which the production rate at each period exerts on the failure rate of the machine. Since the Cox model is used to define the failure law, each period has its distinct failure rate. Meantime, the operational and environmental conditions will impact the optimal scheduling of maintenance actions through the minimization of the average number of failures. The cost and duration of a PM activity are respectively assumed to be strictly lower than the cost and duration of a corrective maintenance action.

II.1. Notation

The main decision variables, cost coefficients and parameters associated with the stochastic problem at hand are listed below:

H : finite production horizon

Δt: period length of production

$s(k)$: inventory level at the end of the period k ($k=1,\ldots\ldots,H/\Delta t$)

$u(k)$: production level at period k ($k=1,\ldots\ldots,H/\Delta t$)

$d(k)$: demand level at period k ($k=1,\ldots\ldots,H/\Delta t$)

C_{pr} : unit production cost

C_s: holding cost of a product unit during the period k

$f(t)$: probability density function of time to failure for the machine

$R(t)$: reliability function

Cp: preventive maintenance action cost

C_c: corrective maintenance action cost

mu: monetary unit

U_{max}: maximal production rate

Z: total expected cost of production and inventory over the finite horizon H

C: total expected maintenance cost per time unit

α: probabilistic index (related to customer satisfaction)

II.2. Problem formulation

It is assumed that the horizon H is partitioned equally into N periods of length $H/\Delta t$. Let $\{f_k, k = 1,....., N\}$ represent holding and production costs (they will be formulated in the next subsection), and $E\{\}$ denotes the mathematical expectation operator. The following aggregate sequential stochastic linear programming problem provides an optimal production plan over the planning horizon:

$$\min_{u(k)} E\left\{\sum_{k=0}^{N-1} f_k\left(s(k), u(k)\right) + f_N\left(s(N)\right)\right\}$$

Subject to:

$$s(k+1) = s(k) + u(k) - d(k) \qquad k = 0,1,..., N-1 \quad (1)$$

$$\text{Prob}\ \left[s(k+1) \geq 0\right] \geq \alpha \qquad k = 0,1,..., N-1 \quad (2)$$

$$0 \leq u(k) \leq U_{\max} \qquad k = 0,1,..., N-1 \quad (3)$$

Constraint (1) defines the inventory balance equation for each time period. The constraint (2) imposes the service level requirement for each period as well as a lower bound on inventory variables so as to prevent stockouts. Note that the non-negative lower limit in (2) represents a safety stock. Finally, the last constraint defines an upper bound on the production level during each period k.

II.3. The stochastic production policy

The purpose of this subsection is to develop and optimize the expected production and holding costs $E\{f(.)\}$ over the finite time horizon H. As mentioned above, the demand d is a random variable with mean $\hat{d}(k)$ and standard–deviation $\sigma_d(k)$, which are known for each period k. The randomness of demand turns the inventory balance equation (1) into a stochastic process that also has a probability distribution. Since demand must be satisfied at the end of each period, the problem can be formulated as a linear-stochastic optimal control problem under a threshold inventory level constraint, as follows:

$$u^* = \min_u \left(Z(u)\right)$$

with:

$$u = \left(u(1), u(2),....u(k)....u(N)\right)$$

The model is described by a hybrid state with continuous component, namely the inventory level as given by equation (1) above, with $s(0) = s_0$, where s_0 is the given initial inventory.

The expected production and holding costs for period k are given by:

$$f_k(s(k),u(k)) = C_s \cdot E\{(s(k)^2)\} + C_{pr} \cdot (u(k)^2) \quad (4)$$

Remark:

The use of quadratic costs allows penalizing both excess and shortage of inventory.

The total expected cost of production and inventory over the finite horizon H can then be expressed as:

$$F(u) = \sum_{k=0}^{N} f_k(u(k), s(k)) = C_s \times E\{(s(N)^2)\} + \sum_{k=0}^{N-1} \left[C_s \times E\{s(k)^2\} + C_{pr} \times u(k)^2\right] \quad (5)$$

Remark:
$(u(N))^2$ is not included in the cost formulation because we don't consider the production order at the end of the horizon H.

Thus the problem becomes:

$$(\text{P1}): \min_u \left[C_s \times E\{(s(N)^2)\} + \sum_{k=0}^{N-1} \left(C_s \times E\{s(k)^2\} + C_{pr} \times u(k)^2\right)\right]$$

Subject to:

$$s(k+1) = s(k) + u(k) - d(k) \qquad k = 0,1,\ldots, N-1$$

$$\text{Prob}[s(k+1) \geq 0] \geq \alpha \qquad k = 0,1,\ldots,N-1$$

$$0 \leq u(k) \leq U_{\max} \qquad k = 0,1,\ldots, N-1$$

The following figure describes dynamic system evolution in discrete time:

Fig 2. Discrete time

II.4. Maintenance policy

The maintenance strategy under consideration is the well known preventive maintenance policy with minimal repair at failure (Faulkner, 2005). Perfect preventive maintenance is performed periodically at times $k.T$, $k=0,1,\ldots,N$, following which the unit is as good as new. Whenever a failure occurs between preventive maintenance actions, the system undergoes a minimal repair to allow it to continue operating during the current period and hence the failure rate is undisturbed. It is assumed that the repair and replacement times are negligible. It has been proved in the literature that the average maintenance total cost per time unit is expressed as follows:

$$C_T = \frac{C_p + C_c \times \int_0^T \lambda(t)dt}{T} \quad (6)$$

$\lambda(t)$ being the machine failure rate function

The existence of an optimal preventive maintenance period T^* has been proved in the case of an increasing failure rate.

We next seek to determine the optimal interval k^* at which the preventive maintenance actions must be carried out considering the production plan previously established (in the above subsection) for the N periods of the planning horizon. For the case where k^* exceeds $N.\Delta T$, no preventive maintenance is done. In order to calculate the average total maintenance cost per time unit, the analytical model is developed.

For each period k we use the production rate $u(k)$ earlier established by the optimal production plan. The machine failure rate in each interval will vary according to the interval's production rate. We determine the failure law according to the prognosis approach, of which Byington et al (2003) proposed three categories. In particular, the first approach is based on a physical model, which assumes that a mathematical formulation of the deterioration mechanism is available. The second approach is based on some indicators of deterioration whose forecast is determined by statistical means. The last approach, experience-based, is used when it is too difficult to develop a physical model for monitoring the state of deterioration, as in the present case. Following the Cox approach, we define the machine rate as follows

$$\lambda_k\left(t,u\left(k\right)\right)=\lambda_0\left(t\right)\cdot g\left(u\left(k\right)\right) \qquad (7)$$

$\lambda_k\left(t,u\left(k\right)\right)$ representing the instantaneous failure rate function at period k according to the production rate $u(k)$

$\lambda_0\left(t\right)$; Failure rate for nominal conditions which is equivalent to the Failure rate with maximal production over the period H.

$g\left(u\left(k\right)\right)=\dfrac{u\left(k\right)}{U_{\max}}$ The production function represents the operational condition for each period k.

III. Analytical determination of the joint production-maintenance policy

III.1.Production Policy

This section focuses on determining the optimal production plan characterized by the best combination of production rates and inventory levels so as to minimize the total costs over the planning horizon H. In practice, the model provides a linear decision rule for inventory and production bearing in mind the requirement of satisfying the random demand.

Recall that our problem formulated in subsection *II.3* is:

$$\min_u F\left(u\right)=\min_u\left[C_s\times E\left\{\left(s\left(N\right)^2\right)\right\}+\sum_{k=0}^{N-1}\left(C_s\times E\left\{s\left(k\right)^2\right\}+C_{pr}\times u\left(k\right)^2\right)\right] \qquad (8)$$

Subject to:

$$s(k+1) = s(k) + u(k) - d(k) \qquad k = 0,1,\ldots, N-1$$

$$\text{Prob}\left[s(k+1) \geq 0\right] \geq \alpha \qquad k = 0,1,\ldots, N-1$$

$$0 \leq u(k) \leq U_{\max} \qquad k = 0,1,\ldots, N-1$$

Solving such a sequential stochastic linear programming problem under constraints is generally difficult. Let us proceed by transforming the stochastic problem into an equivalent deterministic problem which will then be easier to solve.

<u>Transformation to an equivalent deterministic problem</u>

- The objective function :

We can simplify the expected value of the production/inventory costs of eq. (8) as follows:

Lemma1:

$$F(u) = C_s \times \left(\hat{s}(N)^2\right) + \sum_{k=0}^{N-1}\left[C_s \cdot \hat{s}(k)^2 \times + C_{pr} \times u(k)^2\right] + C_s \times (\sigma_d)^2 \times \frac{N(N+1)}{2} \qquad (9)$$

Where $\hat{S}(k)$ represents mean stock level at the end of period k

- The inventory balance equation:

Letting $d_k = \hat{d}_k$, the inventory balance equation (1) can be converted to:

$$\hat{s}(k+1) = \hat{s}(k) + u(k) - \hat{d}(k)$$

Since $u(k)$ is constant for each interval Δt, we have $\hat{u}(k) = u(k)$ and $Var(u(k) = 0)$

Proof of equation (9):

The inventory variable $s(k)$ is statistically described by its mean $E\{s(k)\} = \hat{s}(k)$ and variance $Var(s(k))$

$$E\left\{\left(s(k) - \hat{s}(k)\right)^2\right\} = Var\left(s(k)\right).$$

The balance equation (1) can be converted into an equivalent inventory balance equation, as follows

$$(1) \Rightarrow E\{s(k+1)\} = E\{s(k) + u(k) - d(k)\}$$

$$\Rightarrow \hat{s}(k+1) = \hat{s}(k) + u(k) - \hat{d}(k) \qquad (10)$$

Equation (10) represents the mean variation of inventory at each period k, $k \in \{1,2,.....,N-1\}$. Furthermore, *u(k)* is deterministic, since it does not depend on the random variables *d(k)* and *s(k)*. That is, $E\{u\} = u(k)$ with $V(u(k)) = 0 \quad \forall k$. Taking the difference between (1) and (10):

$$s(k+1) - \hat{s}(k+1) = s(k) - \hat{s}(k) - \left(d(k) - \hat{d}(k)\right)$$

$$\Rightarrow \left(s(k+1) - \hat{s}(k+1)\right)^2 = \left(\left(s(k) - \hat{s}(k)\right) - \left(d(k) - \hat{d}(k)\right)\right)^2$$

$$\Rightarrow E\left(\left(s(k+1) - \hat{s}(k+1)\right)^2\right) = E\left(\left(\left(s(k) - \hat{s}(k)\right) - \left(d(k) - \hat{d}(k)\right)\right)^2\right)$$

$$\Rightarrow E\left(\left(s(k+1) - \hat{s}(k+1)\right)^2\right) = E\left(\left(s(k) - \hat{s}(k)\right)^2 + \left(d(k) - \hat{d}(k)\right)^2 - 2\cdot\left(s(k) - \hat{s}(k)\right)\cdot\left(d(k) - \hat{d}(k)\right)\right)$$

$$\Rightarrow E\left(\left(s(k+1) - \hat{s}(k+1)\right)^2\right) = E\left(\left(s(k) - \hat{s}(k)\right)^2\right) + E\left(\left(d(k) - \hat{d}(k)\right)^2\right) - 2\cdot E\left(\left(s(k) - \hat{s}(k)\right)\cdot\left(d(k) - \hat{d}(k)\right)\right)$$

Since *s(k)* and *d(k)* are independent random variables we can deduce that:

$$E\left(\left(s(k) - \hat{s}(k)\right)\cdot\left(d(k) - \hat{d}(k)\right)\right) = E\left(\left(s(k) - \hat{s}(k)\right)\right)\cdot E\left(\left(d(k) - \hat{d}(k)\right)\right)$$

Also, it is easy to see that:

$$E\left(\left(s(k) - \hat{s}(k)\right)\right) = E\left(s(k)\right) - E\left(\hat{s}(k)\right) = 0$$

$$E\left(\left(d(k) - \hat{d}(k)\right)\right) = E\left(d(k)\right) - E\left(\hat{d}(k)\right) = 0$$

Consequently,

$$E\left(\left(s(k+1) - \hat{s}(k+1)\right)^2\right) = E\left(\left(s(k) - \hat{s}(k)\right)^2\right) + E\left(\left(d(k) - \hat{d}(k)\right)^2\right)$$

$$V_s(k+1) = V_s(k) + V_d(k) = V_s(k) + \sigma_{d_k}^2$$

If we assume that $V_s(k=0) = 0$ and $\sigma_d(k)$ is constant and equal to σ_d for all k's, we can deduce that:

$$V_s(k) = k.\sigma_{d_k}^2$$

$$\Rightarrow E\left(\left(s(k)-\hat{s}(k)\right)^2\right) = E\left(s(k)^2\right) - \hat{s}(k)^2$$

$$\Rightarrow E\left(s(k)^2\right) - \hat{s}(k)^2 = V_s(k) = k.\sigma_d^2$$

Thus

$$E\left(s(k)^2\right) = k \cdot (\sigma_d)^2 + \hat{s}(k)^2 \qquad (11)$$

Substituting (11) in the expected cost (8):

$$F(u) = C_s \times \left(\hat{s}(N)^2\right) + \sum_{k=0}^{N-1}\left[C_s \cdot \hat{s}(k)^2 \times + C_{pr} \times u(k)^2\right] + C_s \times (\sigma_d)^2 \times \sum_{k=0}^{N} k$$

$$\Rightarrow F(u) = C_s \times \left(\hat{s}(N)^2\right) + \sum_{k=0}^{N-1}\left[C_s \cdot \hat{s}(k)^2 \times + C_{pr} \times u(k)^2\right] + C_s \times (\sigma_d)^2 \times \frac{N(N+1)}{2}$$

- The service level constraint (2):

Another step toward transforming the problem into a deterministic equivalent is to cast the service level constraint in a deterministic form by specifying certain minimum cumulative production quantities that depend on the service level requirements. It is necessary first to determine the change of the variance of inventory over the planning horizon.

Lemma 2:

$$\Pr ob\left(s(k+1) \geq 0\right) \geq \alpha \quad \Rightarrow \quad \left(u(k) \geq U_\alpha\left(s(k), \alpha\right)\right) \quad k = 0,1,....,N-1$$

where

$U_\alpha(\ )$: Minimum cumulative production quantity

$$U_\alpha\left(s(k), \alpha\right) = V_{d,k} \cdot \varphi_{d,k}^{-1}(\alpha) + \hat{d}(k) - s(k) \qquad k = 0,1,....,N-1$$

$V_{d,k}$: Variance of demand d at period k

$\varphi_{d,k}$: Cumulative Gaussian distribution function with mean $\hat{d}_k$ and finite variance $Var(d_k) = V_{d,k} \geq 0$

$\varphi_{d,k}^{-1}$: Inverse distribution function

Proof of lemma1:

$$s(k+1) = s(k) + u(k) - d(k)$$

$$\Rightarrow \text{Prob}\left(s(k+1)\geq 0\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(s(k)+u(k)-d(k)\geq 0\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(s(k)+u(k)\geq d(k)\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(s(k)+u(k)-\hat{d}(k)\geq d(k)-\hat{d}(k)\right)\geq \alpha$$

$$\Rightarrow \text{Prob}\left(\frac{s(k)+u(k)-\hat{d}(k)}{V_{d,k}}\geq \frac{d(k)-\hat{d}(k)}{V_{d,k}}\right)\geq \alpha \quad (12)$$

Note that $\left(\frac{d(k)-\hat{d}(k)}{V_{d,k}}\right)$ is a Gaussian random variable with an identical distribution as *d(k)*.

It is possible from (12) to determine a lower bound for the control variable, assuming that φ is a probability distribution function and *f* a probability density function. Hence,

$$(12)\Rightarrow \varphi_{d,k}\left(\frac{s(k)+u(k)-\hat{d}(k)}{V_{d,k}}\right)\geq \alpha \qquad (13)$$

Since $\lim_{-\infty}\varphi_{d,k}\to 0$ and $\lim_{+\infty}\varphi_{d,k}\to 1$ we conclude that $\varphi_{d,k}$ is strictly increasing. We note that $\varphi_{d,k}$ is indefinitely differentiable, so we conclude that $\varphi_{d,k}$ is invertible.

$$\text{Thus (13)}\Rightarrow \frac{s(k)+u(k)-\hat{d}(k)}{V_{d,k}}\geq \varphi_{d,k}^{-1}(\alpha)$$

$$\Leftrightarrow s(k)+u(k)-\hat{d}(k)\geq V_{d,k}\cdot\varphi_{d,k}^{-1}(\alpha)$$

$$\Leftrightarrow u(k)\geq V_{d,k}\cdot\varphi_{d,k}^{-1}(\alpha)+\hat{d}(k)-s(k)$$

Thus

$$\text{Prob}\left(s(k+1)\geq 0\right)\geq \alpha \quad \Rightarrow \quad \left(u(k)\geq V_{d,k}\cdot\varphi_{d,k}^{-1}(\alpha)+\hat{d}(k)-s(k)\right)$$

This completes the proof.

Using Lemma 1 and Lemma 2, the equivalent deterministic model can now be formulated as follows:

$$\underset{u}{Min}\ C_s \times \left(\hat{S}(N)^2\right) + \sum_{k=0}^{N-1}\left[C_s \cdot \hat{S}(k)^2 \times + C_{pr} \times u(k)^2\right] + C_s \times (\sigma_d)^2 \times \frac{N(N+1)}{2}$$

Subject to:

$$\hat{s}(k+1) = \hat{s}(k) + u(k) - \hat{d}(k) \qquad k = 0,1,\ldots,N-1$$

$$u(k) \geq V_{d,k} \cdot \varphi_{d,k}^{-1}(\alpha) + \hat{d}(k) - S(k) \qquad k = 0,1,\ldots.,N-1$$

$$0 \leq u(k) \leq U_{\max} \qquad k = 0,1,\ldots,\ N-1$$

III.2. Optimal maintenance plan considering the influence of the production plan on the deterioration of the manufacturing system

For the maintenance policy, we seek to find the cost associated with a given schedule of future preventive maintenance and replacement activities. The joint optimization strategy considers these costs based on optimal production rates previously found by the production policy in order to optimize the maintenance strategy characterized by the optimal time interval between successive preventive maintenance or replacement activities, $k^*\Delta t$.

The analytic expression of the average cost per unit time of maintenance actions is defined by:

$$C(k) = \frac{C_p + C_c \times A_k}{k \cdot \Delta t}$$

Where A_k corresponds to the expected number of failure, i.e. the average number of failures that can occur during the horizon *H,* considering the production rate variation for each production period Δt. We recall that the manufacturing system considered in this study is composed of a machine *M* which produces a single product at the rate *u(k)* during each Δt period with the reliability function $R_k(t,u(k))$ $(k=0,1\ldots N-1\ ;\ N.\Delta t=H)$.

Since $u(k)$ varies in each production period Δt, it is complex to formulate directly the analytical expression of A_k , which is why we do so by employing the operational age method. Using the maximal production rate and the failure rate, i.e. the nominal failure rate, we determine the expected failure number as follows:

$$A_k = \sum_{i=1}^{k} \int_{\Gamma_i}^{\Gamma_i+\Delta t} \lambda_i(t,u(i))dt$$

where

$$\lambda_i\left(t,u(i)\right) = \frac{u(i)}{U_{\max}} \cdot \lambda_0(t)\ ,$$

Remark:

$\lambda_0(t)$: the nominal failure rate

We assume that the nominal failure rate is the failure rate where the production level is maximal.

$\boldsymbol{\Gamma_i}$: Time at which the reliability at the end of period *i-1* is identical to that at the beginning of the next period *i*

We now determine an analytical expression for $\boldsymbol{\Gamma_i}$.

Lemma 3:

$$\Gamma_j = R_j^{-1}\left(R_{j-1}\left(\Gamma_{j-1} + \Delta t\right)\right) \quad \text{For} \quad j \geq 3$$

$$\begin{cases} \Gamma_2 = R_2^{-1}\left(R_1\left(\Delta t\right)\right) \\ \Gamma_1 = 0 \end{cases}$$

$R_{\max}^{-1}$: inverse of the reliability with the nominal (maximum) production.

R_{i+1} *:* reliability at the production rate *u(i+1).*

Γ_{i+1} : time at which the reliability at the end of period *i* is identical to that at the beginning of period *i+1*.

Γ_j : time at which the reliability at the end of period *j-1* is identical to that at the beginning of the next period *j*.

Proof:

The operational age model considers that the reliability associated with the beginning of the period *i+1* is equal to the reliability at the end of the previous period *i*.

$$R_{u_i}(t_{u_i}) = R_{u_{i+1}}(t_{u_i})$$

With

t_{u_i} : the system age at the end of the period where the rate of production equal to *u(i)*

R_{ui} *:* reliability at the production rate *u(i).*

R_{ui+1} *:* reliability at the production rate *u(i+1).*

To verify that equation (knowing that each period characterized by a production rate). If a period characterized by a production rate *u (i)* for a period *i.Δt*, is equivalent to the period characterized by a production rate *u (i +1)*, but during a different duration d_{ui+1}. This new duration characterizes the operational age system and the previous relation becomes:

$$R_{u_i}(t_{u_i}) = R_{u_{i+1}}(\Gamma_{u_{i+1}})$$

Where

$\Gamma_{u_{j+1}} = \sum_{i=1}^{j} d_{u_j}$ is the operational age at the beginning of the next period *i* +1 characterized by the production rate *u (i +1)*.

As production rates may vary between periods, the expected number of failure number is given by:

$$A_k = \sum_{i=1}^{k} \int_{\Gamma_i}^{\Gamma_i+\Delta t} \lambda_i(t, u(i))\, dt$$

From equation (7):

$$\lambda_i\left(t, u(i)\right) = \frac{u(i)}{U_{\max}} \cdot \lambda_0(t)$$

$$A_k = \sum_{i=1}^{k} \int_{\Gamma_i}^{\Gamma_i+\Delta t} \frac{u(i)}{U_{\max}} \cdot \lambda_0(t) dt \qquad (14)$$

where:

Γ_i: time at which the reliability at the end of period *i-1* is identical to that at the beginning of period *i*.

$R_i(\Gamma_i)$: reliability function associated with period *i*.

$R_i^{-1}(R_i(\Gamma_i))$: inverse of $R_i(\Gamma_i)$.

Fig 3. Policy example

Assuming continuity, we must have,

$$R_i\left(\Gamma_i\right) = R_{i-1}\left(\Gamma_{i-1} + \Delta t\right) \qquad i \geq 3 \qquad (15)$$

and $R_2\left(\Gamma_2\right) = \left(R_1\left(\Delta t\right)\right)$, where R_i denotes the reliability at the production rate $u(i) \leq U_{max}$.

Figure 3 illustrates variable Γ_i

Fig 4. Schematic representation of $\boldsymbol{\Gamma_i}$

Note that in Figure 3, we assume that $u_{i-1} \leq u_{i+1} \leq u_i$.

Since

$$g\left(u(i-1)\right) \leq g\left(u(i+1)\right) \leq g\left(u(i)\right)$$

and $\lambda_{i-1}\left(t, u(i-1)\right) \leq \lambda_{i+1}\left(t, u(i+1)\right) \leq \lambda_i\left(t, u(i)\right)$.

Γ_i can be determined from equation (15),

$$\Gamma_i = R_i^{-1}\left(R_{i-1}\left(\Gamma_{i-1} + \Delta t\right)\right).$$

This completes the proof.

Using Lemma 2 the maintenance cost can now be written as follows:

$$C(k) = \frac{C_p + C_c \times \sum_{i=1}^{k} \int_{\Gamma_i}^{\Gamma_i + \Delta t} \lambda_i(t, u(i))\, dt}{k \cdot \Delta t}$$

The minimum maintenance cost is obtained by solving the following equation which yields k^*.

$$\frac{\partial C(k)}{\partial k} = 0 \quad (16)$$

Lemma 4 proves the existence of a local minimum.

Lemma 4:

$$\exists\, k^* \;\; if \;\; \theta_{k-1} \le \frac{C_p}{C_c} \le \theta_k$$

With

$$\theta_k = k \cdot A_{k+1} - (k+1) \cdot A_k$$

Proof:

We recall that:

$$C(k) = \frac{C_p + C_c \times A_k}{k \cdot \Delta t}$$

with

$$A_k = \sum_{i=1}^{k} \int_{\Gamma_i}^{\Gamma_i + \Delta t} \frac{u(i)}{u_{\max}} \cdot \lambda_0(t) dt$$

Since we have:

$$\bullet C(k+1) - C(k) = \frac{C_p + C_c \times A_{k+1}}{(k+1)\cdot \Delta t} - \frac{C_p + C_c \times A_k}{k \cdot \Delta t}$$

$$= \frac{C_c \times \left(k \times A_{k+1} - (k+1) \times A_k\right) - C_p}{k(k+1)\cdot \Delta t}$$

$$C((k+1)) - C(k) \geq 0$$

$$\frac{C_c \times \left(k \times A_{k+1} - (k+1) \times A_k\right) - C_p}{k(k+1)\cdot \Delta t} \geq 0$$

$$C_c \times \left(k \times A_{k+1} - (k+1) \times A_k\right) - C_p \geq 0$$

$$\left(k \times A_{k+1} - (k+1) \times A_k\right) \geq \frac{C_p}{C_c} \qquad (17)$$

$$\bullet C(k) - C(k-1) = \frac{C_p + C_c \times A_k}{k \cdot \Delta t} - \frac{C_p + C_c \times A_{k-1}}{(k-1)\cdot \Delta t}$$

$$= \frac{C_c \times \left((k-1) \times A_k - k \times A_{k-1}\right) - C_p}{k(k+1)\cdot \Delta t}$$

$$C(k) - C((k-1)) \leq 0$$

$$\frac{C_c \times \left((k-1) \times A_k - k \times A_{k-1}\right) - C_p}{k(k+1)\cdot \Delta t} \leq 0$$

$$C_c \times \left((k-1) \times A_k - k \times A_{k-1}\right) - C_p \leq 0$$

$$\left((k-1) \times A_k - k \times A_{k-1}\right) \leq \frac{C_p}{C_c} \qquad (18)$$

When the failure time has a Weibull distribution, i.e., $A_k(t) = a \cdot t^{\gamma}$ (γ>1) and $a > 0$

$$(17) \Rightarrow k \times A_{k+1} - (k+1) \times A_k \geq \frac{C_p}{C_c}$$

$$\Rightarrow k \times a \times (k+1)^{\gamma} - (k+1) \times a \times k^{\gamma} \geq \frac{C_p}{C_c}$$

$$\Rightarrow k \times (k+1)^{\gamma} - (k+1) \times k^{\gamma} \geq \frac{1}{a} \times \frac{C_p}{C_c}$$

Since it is easily proved that $k \times (k+1)^{\gamma} + (k+1) \times k^{\gamma}$ is strictly increasing in $k \to \infty$ and $\gamma - 1 > 0$

$$(18) \Rightarrow (k-1)\times A_k - k\times A_{k-1} \le \frac{C_p}{C_c}$$

$$\Rightarrow (k-1)\times a\times k^{\gamma} - k\times a\times (k-1)^{\gamma} \le \frac{C_p}{C_c}$$

$$\Rightarrow (k-1)\times k^{\gamma} - k\times (k-1)^{\gamma} \le \frac{1}{a}\times\frac{C_p}{C_c}$$

The function $(k-1)\times k^{\gamma} - k\times(k-1)^{\gamma}$ is strictly decreasing in $k \to 1$ and $\gamma - 1 > 0$.

Thus:

$$k^* = \begin{cases} C(k+1) - C(k) \ge 0 \\ C(k) - C(k-1) \le 0 \\ C(k+1) - C(k) \text{ is increasing in in } k \to +\infty \\ C(k) - C(k-1) \text{ is decreasing in } k \to 1 \end{cases}$$

⇒Therefore, there exists as the production period where preventive maintenance should be performed

Since it is complex to solve equation (16) analytically, we next develop a numerical procedure for doing so which we illustrate via a numerical example.

IV. Optimal production and maintenance plans: A numerical example

In this section, the development of joint production-maintenance plans for a hypothetical company is introduced as an example. It is assumed that this company manufactures one product type whose demand fluctuates periodically. It is assumed that a production plan is generated for a planning horizon *H=18* months, and that the failure time of machine *M* is characterized by a Weibull distribution with increasing failure rate, implying the existence of an optimal maintenance schedule.

The main data of the problem are:

(i) the monthly mean demands $\hat{d}_k$ are given by the sequence :

Table 1. The mean demands

(ii) C_{pr} =3 *mu*, C_s =2 *mu*

(iii) U_{min}=2 *ut* and U_{max}=10 *ut*

(iv) $S(0)=10$ *ut*

(v) d_k, which is extracted from a historical sales report, is assumed Gaussian with $\sigma_d=1.42$

(vi) The degree of customer satisfaction, associated with the service level constraint (2), is equal to *90%* (α=0.9).

In order to solve (P1) a numerical procedure consisting of dynamic programming is developed. Due to the additive structure of the functional production/inventory cost, the principle of optimality can be applied and, as a result, a sequence of sub problems can be defined and solved interactively during the horizon H. The problem (P1) becomes one of finding a sequence of control $\{u^*_k \in U_\alpha=[max(U_{min},u_\alpha(S(k),\alpha),U_{max}],k=0,1,\ldots,N-1\}$ where U_α is a sub-space that according to the observed state and the probability measure α at each period k.

The optimal production plan and the optimal maintenance period are exhibited respectively in Table 2 and Figure 5.

Table 2. Optimal production plan

For the maintenance policy, the scale and shape parameters of the Weibull distribution are respectively $\beta=16.79$ and $\delta=3$, while $C_c=3000$ mu, $C_p=500$ mu, and Δt=1.

We invoked Lemma 3 using the numerical data, which yielded the following:

$$\Gamma_j = \left(\frac{u(j-1)}{u(j)}\right)^{\frac{1}{\delta}} \cdot \left(\Gamma_{j-1} + \Delta t\right) \qquad j \geq 3$$

and

$$\begin{cases} \Gamma_2 = \left(\dfrac{u(1)}{u(2)}\right)^{\frac{1}{\delta}} \cdot \Delta t \\ \Gamma_1 = 0 \end{cases}$$

Fig 5. Curve of the average total maintenance cost as a function of k assuming optimal production rates

Figure 5 presents the curve of the average total maintenance cost per time unit, $C(k)$, as a function of k. We observe that the optimal preventive maintenance period is $k^*\Delta t=8\Delta t$ with a corresponding optimal cost of $C^*=111.114$ *mu*.

Fig 6. Curve of the average total maintenance cost as a function of k assuming the maximum production rate

Previous research assumed nominal (maximal) production rates in devising maintenance policies, corresponding to the result exhibited in Figure 6 for the numerical example at hand. By contrast, Figure 5 reveals the cost reduction engendered by using optimal instead of nominal production rates, in the order of 6% in this case.

v. Conclusion

A key purpose of this research was to show the effect of the production rate variation on the optimal maintenance strategy. A stochastic production planning and maintenance scheduling problem was investigated under the assumption of a single machine producing a single product. Firstly, given a random demand and a target customer service level, we formulated and solved a linear-quadratic stochastic programming problem which yielded an optimal production plan. Secondly, using this optimal production plan, we established an optimal maintenance schedule based on the operational age approach considering the influence of the production plan on the manufacturing system deterioration. A numerical example was finally developed which illustrates the cost benefit of our proposed approach.

References

Akella, R. & Kumar, P.R. (1986) .Optimal control of production rate in a failure-prone manufacturing system,*IEEE Transactions on Automatic Control*, AC-31. 116-126.

Brown, M., & Proschan. F. (1983). Imperfect repair. *Journal of Applied Probability* ,20 (4). 851-859.

Boukas, E.k,& Haurie, A. (1990). Manufacturing flow control and preventive maintenance:a stochastic control approach,*IEEE Transaction on Automatic control* ,35 1024-1031.

Boukas, E.k, & Yang, H.,(1996). Optimal control of manufacturing flow and preventive maintenance, *IEEE Transactions on Automatic Control* ,41 881-885.

Barták, R., Salido, M. A., & Rossi, F. (2009). Constraint satisfaction techniques in planning and scheduling. *Journal of Intelligent Manufacturing*. doi:10.1007/s10845-008-0203-4.

Cheung, K. L & Hausmann, W. H. (1997) . Joint optimization of preventive maintenance and safety stock in an unreliable production environment. *Naval Research Logistics,* 44: 257-272.

Cox D.R. (1972). Regression Models and Life-Tables. *Journal of the Royal Statistical Society*. 34 (2). 187-220.

Dellagi, S., N. Rezg, X. Xie,(2007) . Preventive Maintenance of Manufacturing Systems Under Environmental Constraints , *International Journal of Production Research*, vol. 45, Issue 5, p1233-1254.

Kijima, M., Morimura, H. & Suzuki Y. (1988). Periodical replacement problem without assuming minimal repair. *European Journal of Operational Research*, 37. 194-203.

Faulkner,L.L.(2005).*Maintenance,Replacement,andReliabilityTheoryand pplications*.Chapter 2 Component Replacement Decisions,Taylor & Francis Group,USA.

Martorell S., Ana Sanchez S., Serradell V. (1999) Age dependent reliability model considering effects of maintenance and working conditions. *Reliability Engineering & System Safety*, Volume 64, Issue 1: 19-31.

O.S.Silva, F. & Cezarino .W. (2004).An Optimal Production Policy applied to a Flow-shop Manufacturing System. *Brazilian Journal of Operations & Production Management*, Brazil.

Rezg, R.,Dellagi, S & Chelbi, A. (2008) Optimal Strategy of Inventory Control and Preventive Maintenance. *International Journal of Production Research* ,Vol. 46 No. 19: 5349–5365.

Rezg, N.,Xie,X.,& Mati,Y (2004) Joint optimization of preventive maintenance and inventory control in a production line using simulation. *International Journal of Production Research* 44:2029-2046.

Schutz, J. , Rezg, N.&J.-B. Léger.(2009) Periodic and sequential preventive maintenance policies over a finite planning horizon with a dynamic failure law. *Journal of Intelligent Manufacturing* ,doi:10.1007/s10845-009-0313-7.

Van der Duyn Schouen, F.A., and Vanneste, S.G.(1995) .Maintenance optimization of a production system with buffer capacity,*European Journal of Operational Research*, 82,323-338.

(Özekici) Özekici, S. (1995) Optimal maintenance policies in random environments. *European Journal of Operational Research*, Volume 82, Issue 2: 283-294.

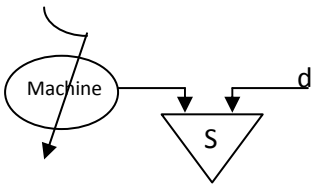


Fig 1. Problem description

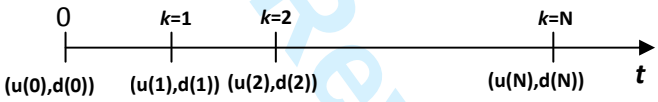


Fig 2. Discrete time

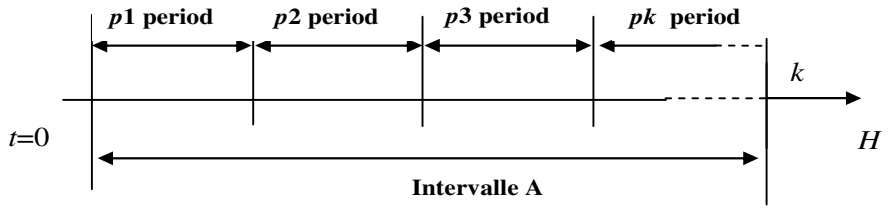
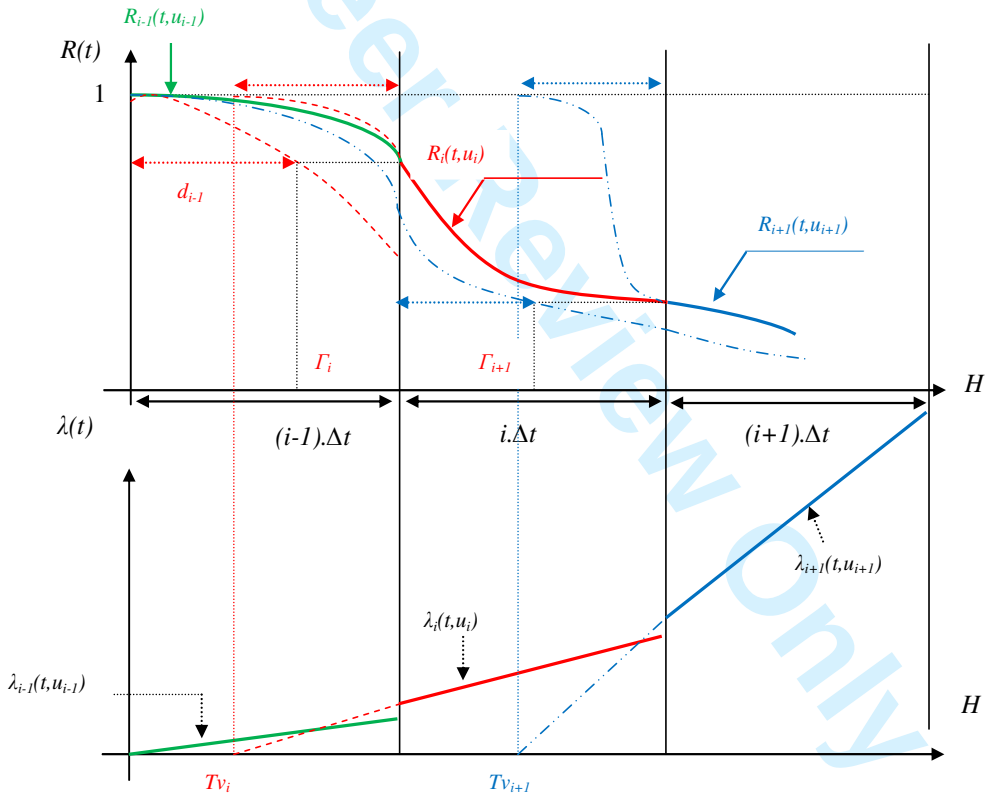


Fig 3. Policy example



- R_{i-1} : reliability with production rate u_{i-1}
- R_i : reliability with production rate u_i
- R_{i+1} : reliability with production rate u_{i+1}
- Γ_{i-1} : date for the reliability of period $i-1$
- Tv_i : virtual age for failure rate with production u_i

Fig 4. Schematic representation of Γ_i

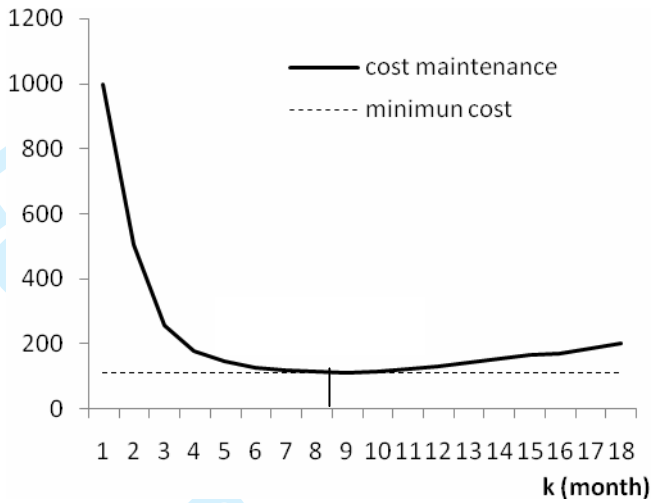


Fig 5. Curve of the average total maintenance cost as a function of *k* assuming optimal production rates

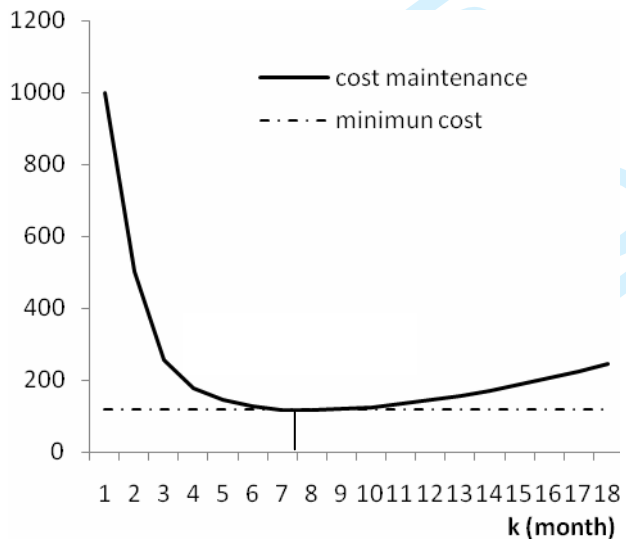


Fig 6. Curve of the average total maintenance cost as a function of *k* assuming the maximum production rate

d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
8	**8**	**9**	**8**	**8**	**8**	**7**	**6**	**4**
d_{10}	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	d_{16}	d_{17}	d_{18}
5	**7**	**8**	**10**	**8**	**9**	**5**	**6**	**6**

Table 1: the mean demands

k	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
d	8	8	9	8	8	8	7	6	4
$u^*(k)$	**10**	**10**	**10**	**9**	**8**	**8**	**5**	**4**	**2**
k	*10*	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*
d	5	7	8	10	8	9	5	6	6
$\boldsymbol{u^*(k)}$	**5**	**10**	**10**	**10**	**9**	**10**	**2**	**4**	**6**

Table 2. Optimal production plan