State dependent sampling: an LMI based mapping approach

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State Dependent Sampling: an LMI Based Mapping Approach *,**

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Abstract: The uprising use of embedded systems and Networked Control Systems (NCS) requires reductions of the use of processor and network loads. In this work, we present a state dependent sampling control that maximizes the sampling intervals of state feedback control. We consider linear time invariant systems and guarantee the exponential stability of the system origin for a chosen decay rate α. The proof of the α-stability is based on a quadratic Lyapunov function which is computed, thanks to LMIs, so as to optimize some performance criterion on the sampling intervals. A mapping of the state space is then designed offline: it computes for each state of the state space the maximum allowable sampling interval, which makes it possible to reduce the number of actuations during the real-time control of the system.

Keywords: networked/embedded control systems, state dependent sampling, self-triggered control, linear matrix inequality, convex polytope

1. INTRODUCTION

In the last decades, a large attention has been given to embedded and Networked Control Systems (Zhang et al. [2001]). Such systems present several advantages such as: reduced system wiring, plug and play devices, increased system agility, and ease of system maintenance. However, from the control theory point of view, they bring up new challenges: these systems are often required to share a limited number of computational and transmission resources. In practice, this often leads to fluctuations of the sampling interval, which may have a destabilizing effect if it is not properly taken into account.

Several studies about robust stability with respect to sampling period variations have been made (see Fridman [2010], Seuret [2009], Fujioka [2009], Fridman et al. [2004], and Cloosterman et al. [2010]). Also, intensive research has been conducted to adapt dynamically the sampling in order to ensure the desired control performances. Two main approaches exist in the literature:
Event-triggered control (Tabuada [2007], Heemels et al. [2008], Lunze and Lehmann [2010]), in which sensors are equiped with special intelligence so that information is sent to the controller only when special events occur (i.e. crossing a frontier of the state space, or a level of a Lyapunov function). However, the main drawback of this approach is that it generally requires dedicated hardware to continuously monitor the plant state and check the defined stability conditions.
Self-triggered control (Velasco et al. [2003], Mazo-Jr. et al. [2010], Anta and Tabuada [2010], Wang and Lemmon [2010]), in which at each sampling instant one computes a lower bound estimation of the next largest admissible sampling interval, so as to emulate event-triggered control without resorting to extra hardware. However, in these works, the computations for the next sampling times are made online. Moreover, most of the works found in the literature require the use of a Lyapunov function, but there is no method to compute such a Lyapunov function guaranteeing stability while optimizing a performance criterion on the sampling intervals.

In this paper, we design offline a state dependent sampling function maximizing the sampling intervals under some Lyapunov exponential stability conditions. The approach is based on a mapping of the state space defining the sampling intervals. Moreover, we provide a formal method based on LMIs to compute the adequate Lyapunov function, in order to maximize the lower bound of the sampling function.

The paper is organized as follows. First, we introduce the studied problem in Section 2. Then, Sections 3 and 4 describe the proposed method and the guaranteed performances. Finally, some simulation results are shown in Section 5 before concluding in Section 6. The proof of the first theorem is given in Appendix 7 along with two lemmas.

Notations: Throughout the paper, the superscript 'T' stands for matrix transposition. $\mathcal{M}_n(\mathbb{R})$ is the set for all

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$n \times n$ matrices, and the notation $P \succ 0$ (resp. $P \succeq 0$) for a symmetric matrix $P \in \mathcal{M}_n(\mathbb{R})$ means that P is definite positive (resp. semi-definite positive). The set of eigenvalues of a matrix $M \in \mathcal{M}_n(\mathbb{R})$ will be written $\mathrm{eig}(M)$. We also denote by $\lfloor x \rfloor$ the floor of x (i.e. the largest integer n not greater than x: $x - 1 < n \leq x$). Eventually, the notations $\|.\|$ and $\|.\|_\infty$ will stand for the Euclidean and the infinity norm respectively. We recall that for a bounded function $f : \mathbb{R}^p \to \mathbb{R}^q$, $\|f\|_\infty = \mathrm{Sup}_{x\in\mathbb{R}^p}\|f(x)\|$.

2. PROBLEM STATEMENT

Consider the linear time invariant (LTI) system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \forall t \in \mathbb{R}_+ \\ x(t) &= x_0, \forall t \leq 0, \end{aligned} \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$ and $u : \mathbb{R} \to \mathbb{R}^m$ represent the system state and the control function, and the matrices A and B are constant and of appropriate dimensions. The control function is defined as a piecewise constant state feedback

$$u(t) = -Kx(t_k),\ \forall t \in [t_k; t_{k+1}), \tag{2}$$

where $0 = t_0 < t_1 < \cdots < t_k < \cdots$ are the sampling instants satisfying $\lim_{k\to\infty} t_k = \infty$ and defined by

$$t_{k+1} = t_k + \tau(x(t_k)),\ \forall k \in \mathbb{N}, \tag{3}$$

with a state dependent sampling function $\tau : \mathbb{R}^n \to \mathbb{R}_+$.

The feedback control matrix K is supposed to be fixed such that the continuous state feedback $u(t) = -Kx(t)$ stabilizes the ideal control loop system ($A - BK$ is Hurwitz).

The objective of this work is to study how to update the given control law as few times as possible, while insuring the exponential stability of the system origin for a chosen decay rate α. At each sampling instant, we want to determine for how long it is possible to let the control input unchanged, without additional sampling. A first, literal problem formulation can be expressed as:

General Problem: Given the LTI system (1) and the linear state feedback control (2), find the state dependent sampling function τ that maximizes the sampling intervals while insuring the exponential stability of the system origin for a chosen decay rate α.

Before computing the state dependent sampling function, we need to decide how to check the system exponential stability. In this respect, the Lyapunov stability theory has proved to be a very useful tool. In this paper, in order to keep things simple and easy to read, we will work exclusively with quadratic Lyapunov functions. We will use the following well known property:

Proposition 1. Let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a quadratic Lyapunov candidate function satisfying $V(x) = x^T P x, \forall x \in \mathbb{R}^n$, with $P = P^T \succ 0$. If the condition

$$\dot{V}(x) + 2\alpha V(x) \leq 0 \tag{4}$$

is satisfied for all trajectories of (1), for a given scalar $\alpha > 0$, then the system origin is globally α-stable (i.e. there exists a scalar β such that the trajectories satisfy $\|x(t)\| \leq \beta e^{-\alpha t}\|x_0\|$ for any initial condition x_0).

Throughout this work, we will focus on solving two main problems. The first one concerns the design of the sampling function and is formulated as:

Problem 1: Given system (1), the state feedback control (2), and a Lyapunov function $V(x(t))$, find the state dependent sampling function τ satisfying the stability condition (4) from Proposition 1 that maximizes the sampling time $\tau(x)$ for all $x \in \mathbb{R}^n$.

One can see in that formulation that the Lyapunov function is supposed to be given, which makes us wonder if there is a clever way to choose it. Since the objective is to sample as few times as possible, one will also want to make sure the minimal sampling interval is as big as possible by solving the following problem:

Problem 2: Given system (1) and the state feedback control (2), find a Lyapunov function $V(x(t))$ such that there exists a sampling function τ satisfying the stability condition (4) from Proposition 1 and maximizing the minimal sampling time $\underline{\tau} = \inf_{x\in\mathbb{R}^n} \tau(x)$.

3. A GENERIC STABILITY PROPERTY

The goal of this section is to provide checkable stability conditions from Proposition 1. For that purpose, we prove the following Lemma that will be used as a stability condition basis throughout the whole here reported work:

Lemma 2. Given a scalar $\alpha > 0$, if there exist a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ and a bounded function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$ and $\sigma \in [0; \tau(x)]$:

$$x^T \Phi_{P,\alpha}(\sigma) x \leq 0, \tag{5}$$

with

$$\Phi_{P,\alpha}(\sigma) = \begin{pmatrix} \Lambda(\sigma) \\ I \end{pmatrix}^T \begin{pmatrix} A^TP + PA + 2\alpha P & -PBK \\ -K^TB^TP & 0 \end{pmatrix} \begin{pmatrix} \Lambda(\sigma) \\ I \end{pmatrix} \tag{6}$$

and

$$\Lambda(\sigma) = I + \int_0^\sigma e^{sA} ds (A - BK), \tag{7}$$

then the origin of system (1) (2) is globally α-stable for any time varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ that defines sampling instant sequences by the sampling law $t_{k+1} = t_k + \tilde{\tau}(t_k, x(t_k)), k \in \mathbb{N}$ and which satisfies $0 < \tilde{\tau}(t, x) \leq \tau(x)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$.

Proof: Let $\alpha > 0$ be given. Let $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ and $V(x) = x^T P x$ the associated quadratic function. For the studied system, the stability condition (4) from Proposition 1 can be written as: for all $k \in \mathbb{N}$, for all $t \in [t_k; t_{k+1})$,

$$\begin{pmatrix} x(t) \\ x(t_k) \end{pmatrix}^T \begin{pmatrix} A^TP + PA + 2\alpha P & -PBK \\ -K^TB^TP & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x(t_k) \end{pmatrix} \leq 0.$$

Let us then take a bounded function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ and a sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ that defines sampling instant sequences from the sampling law $t_{k+1} = t_k + \tilde{\tau}(t_k, x(t_k)), k \in \mathbb{N}$ and which satisfies $0 < \tilde{\tau}(t, x) \leq \tau(x)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$. For a given trajectory of (1), $k \in \mathbb{N}$, and for $t \in [t_k; t_{k+1})$, when using the notations $x = x(t_k)$ and $\sigma = t - t_k \leq \tilde{\tau}(t_k, x(t_k)) \leq \tau(x(t_k))$, one can write $x(t) = \Lambda(\sigma)x$, with $\Lambda(\sigma)$ defined in (7), so that the stability conditions can be rewritten as: for all $x \in \mathbb{R}^n$, for all $\sigma \in [0; \tau(x)]$, $x^T \Phi_{P,\alpha}(\sigma) x \leq 0$, with $\Phi_{P,\alpha}(\sigma)$ defined in (6). This ends the proof. □

Remark 1: Since the LTI system (1) is supposed to be asymptotically stable with $u(t) = -Kx(t)$, one can show that there always exist a couple of parameters α, P satisfying $\Phi_{P,\alpha}(0) = (A-BK)^T P + P(A-BK) + 2\alpha P \prec 0$ and that, for such parameters, we can find sampling functions τ satisfying the stability conditions of Lemma 2 that are lower-bounded by a strictly positive scalar, hence avoiding any Zeno phenomenon issue.

Remark 2: This lemma also says that if a state dependent function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ satisfies the conditions from Lemma 2, with given P and α, then any time varying sampling law $\tau_k = t_{k+1} - t_k$ satisfying at each sampling instant $0 < \tau_k \leq \tau(x(t_k))$ will make the system origin globally α-stable. In particular, if τ is a constant function, $\tau(x) = \tau^*, \forall x \in \mathbb{R}^n$, the system origin will be globally α-stable for any varying sampling interval bounded by τ^*.

Remark 3: For any given state $x \neq 0$, the stability conditions from Lemma 2 are the same for any state $y = \lambda x, \lambda \in \mathbb{R}^*$. Therefore it is sufficient to work with homogeneous state dependent sampling functions of degree 0 (i.e. satisfying $\tau(\lambda x) = \tau(x)$ for all $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^*$) and to check Lemma 2 stability conditions on the n-dimensional sphere when attempting to solve Problem 1.

Lemma 2 gives some preliminary stability conditions for a state feedback control system with a state dependent sampling. However, one can see that there is an infinite number of inequalities to check because of both temporal and spatial dependencies in the stability conditions.

4. A NUMERICAL METHOD TO DERIVE A FINITE NUMBER OF STABILITY CONDITIONS

Here a two step tractable methodology is proposed to derive a finite number of stability conditions from Lemma 2:
Convex embedding according to time: The matrix function $\Phi_{P,\alpha}$ is replaced by a finite number of constant matrices whose convex hull embed this matrix: for this, a Taylor expansion of $\Phi_{P,\alpha}$ is used.
Space discretization: Finally, the state space is divided into conic regions in order to design a state dependent sampling period such that Lemma 2 holds.

A finite number of LMI stability conditions will then be derived so to compute the Lyapunov function $V(x) = x^T P x$ solving Problem 2 and build offline (once for all) the associated state dependent sampling function τ solving Problem 1.

4.1 Convex embedding - Technical result

In this part, the objective is to obtain a finite number of sufficient conditions to satisfy $x^T \Phi_{P,\alpha}(\sigma) x \leq 0, \forall \sigma \in [0; \underline{\sigma}]$ for given state x, scalar $\underline{\sigma}$ and parameters P, α and τ in order to get rid of the time dependency of the stability conditions in Lemma 2.

The idea behind the convex embedding is to use the knowledge we have of the system to predict the evolution of the state in order to design a convex polytope around the function $\Phi_{P,\alpha}(.)$ and derive a finite number of stability conditions on the vertices. The method is proposed as follows:
Theorem 3. Let scalars $\alpha > 0$, $\bar{\sigma} > 0$ and integers $N \geq 0$, $l \geq 1$ be given.
If there exist a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ and a bounded function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $\|\tau\|_\infty \leq \bar{\sigma}$ and such that for all x in $\mathbb{R}^n$, for all $i \in \{0; \cdots ; N\}$ and for all $j \in \{0; \cdots ; \lfloor \frac{\tau(x) l}{\bar{\sigma}} \rfloor\}$, the conditions $x^T \Phi_{i,j} x \leq 0$ are satisfied, with

$$\Phi_{i,j} = \hat{\Phi}_{i,j} + \nu I, \tag{8}$$

$$\begin{cases} \hat{\Phi}_{i,j} = \left(\sum_{k=0}^{i} L_{k,j} \left(\frac{\bar{\sigma}}{l} \right)^k \right) \text{ if } j < \lfloor \frac{\tau(x) l}{\bar{\sigma}} \rfloor, \\ \hat{\Phi}_{i,j} = \left(\sum_{k=0}^{i} L_{k,j} \left(\tau(x) - \frac{j\bar{\sigma}}{l} \right)^k \right) \text{ otherwise}, \end{cases} \tag{9}$$

$$\begin{cases} L_{0,j} = \Pi_{3,j}^T \Pi_1 \Pi_{3,j} - \Pi_{3,j}^T \Pi_2 - \Pi_2^T \Pi_{3,j}, \\ L_{1,j} = \Pi_{4,j}^T (\Pi_1 \Pi_{3,j} - \Pi_2) + (\Pi_{3,j}^T \Pi_1^T - \Pi_2^T) \Pi_{4,j}, \\ L_{k \geq 2,j} = \Pi_{4,j}^T \frac{(A^{k-1})^T}{k!} (\Pi_1 \Pi_{3,j} - \Pi_2) \\ \quad + (\Pi_{3,j}^T \Pi_1^T - \Pi_2^T) \frac{A^{k-1}}{k!} \Pi_{4,j} \\ \quad + \Pi_{4,j}^T \left(\sum_{i=1}^{k-1} \frac{(A^{i-1})^T}{i!} \Pi_1 \frac{A^{k-i-1}}{(k-i)!} \right) \Pi_{4,j}, \end{cases} \tag{10}$$

$$\begin{cases} \Pi_1 = A^T P + PA + 2\alpha P, \ \Pi_2 = PBK, \\ \Pi_{3,j} = I + M_j (A - BK), \ \Pi_{4,j} = N_j (A - BK), \end{cases} \tag{11}$$

$$M_j = \int_0^{j\frac{\bar{\sigma}}{l}} \mathrm{e}^{As} ds, \ N_j = A M_j + I, \tag{12}$$

and

$$\nu \geq \max_{\substack{\sigma' \in [0; \frac{\bar{\sigma}}{l}] \\ r \in \{0; \cdots ; l-1\}}} \left(\max_{\lambda \in \mathrm{eig}\left(\Phi_{P,\alpha}\left(\sigma' + r\frac{\bar{\sigma}}{l}\right) - \hat{\Phi}_{P,\alpha,N,r}(\sigma')\right)} \lambda \right), \tag{13}$$

with the function $\hat{\Phi}_{P,\alpha,N,r}$ defined on $[0; \frac{\bar{\sigma}}{l}]$ as

$$\hat{\Phi}_{P,\alpha,N,r}(\sigma') = \sum_{k=0}^{N} L_{k,r} \sigma'^k, \tag{14}$$

then the origin of system (1) is globally exponentially stable with a decay rate α regarding the control (2) for any time varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ that defines sampling instant sequences by the sampling law $t_{k+1} = t_k + \tilde{\tau}(t_k, x(t_k)), k \in \mathbb{N}$ and which satisfies for all t in $\mathbb{R}_+$ and for all x in $\mathbb{R}^n$, $0 < \tilde{\tau}(t, x) \leq \tau(x)$.

The proof, which describes the principle in details, can be found in the Appendix. Compared to Lemma 2, Theorem 3 reduces the number of α-stability conditions. They depend on a finite number of matrices $\hat{\Phi}_{i,j}$ representing the vertices of a convex polytope built around a polynomial approximation of the function $\Phi_{P,\alpha}(.)$, and on a scalar ν which bounds the approximation error. It is possible to compute an approximation of such a bound ν by using a gridding. In this theorem, N represents the order of the polynomial approximation, while l is the number of polytope subdivisions, as described in the first step of the proof as well as in Figure 4, in the Appendix.

The number of α-stability conditions to check has been reduced, but there is still an infinite number of conditions regarding the state x.

4.2 Main result

Remember that when trying to solve Problem 1, it is sufficient to work with homogeneous state dependent sampling functions of degree 0 (see Remark 3). Therefore, in order to derive a finite number of conditions, one will want to divide the state space into a finite number of subspaces $\mathcal{R}_s$ defined by conics centered on the origin and try to find for each subspace its maximum allowable sampling time τ_s, as shown for a 2 dimensional system in Figure 1. For higher dimensions, one can use the generalized spherical coordinates in $\mathbb{R}^n$, each region $\mathcal{R}_s$ being associated to some range of the $(n-1)$ angular coordinates θ_i: $\theta_i \in [\theta_{i,s}^-, \theta_{i,s}^+]$.

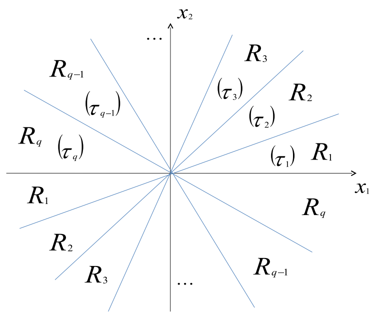

Fig. 1. Dividing the space into conic subspaces

The following theorem gives a finite number of conditions to solve this problem.

Theorem 4. Let a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$, scalars $\alpha > 0$ and $\bar{\sigma} > 0$ and integers $N \geq 0$ and $l \geq 1$ be given. Let us divide the state space into a partition of q conic subspaces $\mathcal{R}_s, s \in \{1; \cdots ; q\}$, defined for all $s \in \{1; \cdots ; q\}$ as $\mathcal{R}_s = \{x \in \mathbb{R}^n, x^T Q_s x \geq 0\}$, with $Q_s = Q_s^T \in \mathcal{M}_n(\mathbb{R})$, and let us define sampling times for these subspaces, $\tau_1, \cdots, \tau_q$, satisfying $\tau_s \leq \bar{\sigma}$ for all $s \in \{1; \cdots ; q\}$, and a bounded function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $\tau(x) = \tau_s$ for all $x \in \mathcal{R}_s$, $s \in \{1; \cdots ; q\}$.
If there exist scalars $\varepsilon_{s,i,j} \geq 0$, for $s \in \{1; \cdots ; q\}$, $i \in \{0; \cdots ; N\}$ and $j \in \{0; \cdots ; \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor\}$, such that the LMI conditions $\Phi_{i,j,s} + \varepsilon_{s,i,j} Q_s \preceq 0$ are satisfied for all $i \in \{0; \cdots ; N\}$, $s \in \{1; \cdots ; q\}$ and $j \in \{0; \cdots ; \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor\}$, with

$$\Phi_{i,j,s} = \hat{\Phi}_{i,j,s} + \nu I, \tag{15}$$

$$\begin{cases} \hat{\Phi}_{i,j,s} = \left(\sum_{k=0}^{i} L_{k,j} \left(\frac{\bar{\sigma}}{l} \right)^k \right) \text{ if } j < \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor, \\ \hat{\Phi}_{i,j,s} = \left(\sum_{k=0}^{i} L_{k,j} \left(\tau_s - \frac{j\bar{\sigma}}{l} \right)^k \right) \text{ otherwise,} \end{cases} \tag{16}$$

with the $L_{k,j}$ and ν defined by the equations (10) to (14), then the origin of system (1) is globally α-stable regarding control (2) for any time varying sampling function $\tilde{\tau} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ that defines sampling instant sequences by the sampling law $t_{k+1} = t_k + \tilde{\tau}(t_k, x(t_k)), k \in \mathbb{N}$ and which satisfies $0 < \tilde{\tau}(t,x) \leq \tau(x)$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$.

Proof: Let x be in $\mathbb{R}^n$. There exists a subspace $\mathcal{R}_s = \{x \in \mathbb{R}^n, x^T Q_s x \geq 0\}$, $s \in \{1; \cdots ; q\}$, $Q_s = Q_s^T$, such that $x \in \mathcal{R}_s$ and $\tau(x) = \tau_s$. Using the lossless version of the S-procedure, one can see that for any $i \in \{0; \cdots ; N\}$ and any $j \in \{0; \cdots ; \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor\}$ the condition $x^T \Phi_{i,j,s} x \leq 0, x \in \mathcal{R}_s$ is satisfied if and only if there exists a scalar $\varepsilon_{s,i,j} \geq 0$ such that $\Phi_{i,j,s} + \varepsilon_{s,i,j} Q_s \preceq 0$. As a consequence, if the condition $\Phi_{i,j,s} + \varepsilon_{s,i,j} Q_s \preceq 0$ is satisfied for all $i \in \{0; \cdots ; N\}$, for all $s \in \{1; \cdots ; q\}$ and for all $j \in \{0; \cdots ; \lfloor \frac{\tau_s l}{\bar{\sigma}} \rfloor\}$, then the stability conditions from Theorem 3 are satisfied for any $x \in \mathbb{R}^n$, which ends the proof. □

Corollary 5. Let scalars $\alpha > 0$ and $\bar{\sigma} > 0$ and integers $N \geq 0$ and $l \geq 1$ be given. Let us define a sampling time $\tau^* \leq \bar{\sigma}$ for the whole state space: the sampling function $\tau : \mathbb{R}^n \to \mathbb{R}_+$ satisfies $\tau(x) = \tau^*$ for all $x \in \mathbb{R}^n$.
If there exist a matrix $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ such that the LMI conditions $\Phi_{i,j} \preceq 0$ are satisfied for all $i \in \{0; \cdots ; N\}$ and $j \in \{0; \cdots ; \lfloor \frac{\tau^* l}{\bar{\sigma}} \rfloor\}$, with $\Phi_{i,j}$ defined by the equations (8) to (14), then the origin of system (1) with control (2) is globally α-stable for any time varying sampling bounded by τ^*.

Proof: This comes naturally from Theorem 4 when working with a single subspace: $\mathbb{R}^n$ itself. □

4.3 General algorithm

Theorem 4 and Corollary 5 are solutions to Problem 1 and 2 respectively. While Corollary 5 gives a way to compute the Lyapunov function parameter P maximizing the lower bound τ^* of the sampling function τ under the stability conditions of Proposition 1, Theorem 4 gives a way to maximize the sampling function on state subspaces for a given P. A method to apply the proposed technique is the following:

First, use Corollary 5 with $\nu = 0$ at first to compute (using a line search algorithm) an approximation $\tilde{\tau}^*$ of the maximal admissible τ^* as well as the Lyapunov function parameter P that enables such a bound.

Then, compute the variable ν corresponding to that Lyapunov function. It is possible, though not needed, to compute the real lower bound τ^* verifying the stability conditions of Corollary 5 using the calculated P and ν.

Finally, divide the state space into a partition of q conic subspaces $\mathcal{R}_s, s \in \{1; \cdots ; q\}$ and use the LMI conditions from Theorem 4 with the computed values of P and ν to compute the maximal admissible sampling intervals τ_s for each subspace $\mathcal{R}_s$ (again using a line search algorithm). The state dependent sampling function τ can then be defined as $\tau(x) = \tau_s$, for all $x \in \mathcal{R}_s$, $s \in \{1, \cdots, q\}$.

5. NUMERICAL EXAMPLE

Consider the standard double integrator:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix} K x(t_k),$$
$$K = (2 \;\; 3).$$

After setting the polynomial approximation degree term $N = 5$, and the number of polytopic subdivisions $l = 100$, we can obtain a mapping of the state space that gives the maximal allowable sampling time for each state for a given decay rate $\alpha > 0$ thanks to Corollary 5 and Theorem 4. State dependent sampling functions obtained offline and insuring the exponential stability of the system origin for different decay rates α are presented Figure 2, using the spherical coordinate angle of the state, θ. These state dependent sampling functions were obtained using the partition of the state space shown in Figure 1 with a number of $q = 1000$ conic subspaces (which explains this impression of having smooth curves).

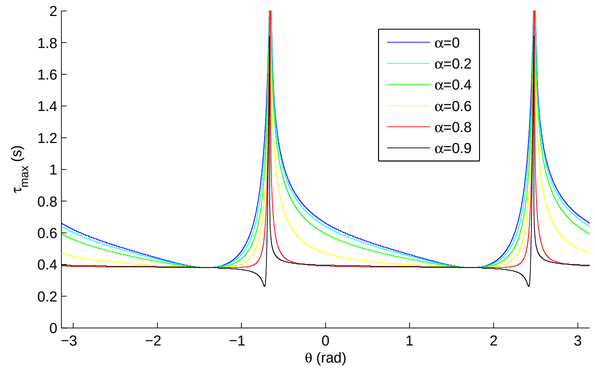

Fig. 2. Example 1: State-angle dependent sampling function τ for different decay rates α.

Note that for constant samplings greater than $T_{max} = 0.67s$ the discrete time model is not Schur anymore, so the system becomes unstable. However we can see that with the proposed technique, for some subspaces of the state space we can go beyond that limit T_{max}.

Figure 3 shows simulation results with $\alpha = 0$ and a random initial state. It first shows the sampling intervals (in blue), with the lower bound (in red) of the offline computed state dependent sampling function, and the "Schur limit" T_{max} (in magenta), before showing the decreasing Lyapunov function, and the control input.

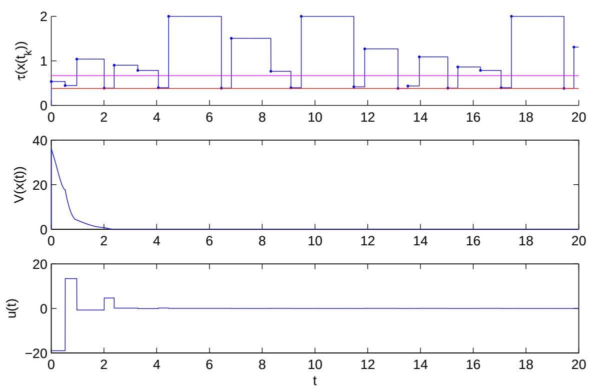

Fig. 3. Example 1: Inter-execution times $\tau(x(t_k))$, Lyapunov function $V(x) = x^T P x$ evolution, and control input $u(t)$ for a decay rate $\alpha = 0$.

6. CONCLUSION

We have introduced a method based on Lyapunov stability conditions to design a state-dependent sampling function τ that insures global exponential stability with a chosen decay rate α for linear state feedback control systems. It presents two main advantages.

The first advantage of the method is that it makes it possible to maximize the minimal sampling time τ^* of the state dependent sampling function τ and to compute the associated quadratic Lyapunov function.

The second advantage is that the method makes it possible to compute offline a mapping of the state space with a maximum allowable sampling time for each subspace. The online number of computations are then reduced to the minimum since at each sampling instant t_k one only needs to compute the spherical coordinates of the state $x(t_k)$, check and memorize the offline computed associated sampling time $\tau(x(t_k))$ and compute the control input $u(t) = -Kx(t_k)$.

Extensions to perturbed, delayed, and nonlinear systems are currently being studied.

7. APPENDIX

Proof of Theorem 3: In order to prove Theorem 3, we first need to introduce two important lemmas.

Lemma 6. (From Hetel et al. [2007]) Consider the matrix polynomial function

$$L(\sigma) = L_0 + L_1\sigma + \cdots + L_N\sigma^N$$

such that the variable σ is positive bounded: $0 < \sigma < \bar{\sigma}$. Then we can find a convex polytope formed by $N + 1$ vertices which envelopes the matrix polynomial function $L(\sigma)$, i.e. there exists an indexed family $\mu_i(\sigma) > 0$, $i = 0..N$, verifying $\sum_{i=1}^{N} \mu_i(\sigma) = 1$, and such that

$$L(\sigma) = \sum_{i=1}^{N} \mu_i(\sigma) U_i$$

where the matrices U_i represent the vertices of the polytope and are given for all $i = 0..N$ by

$$U_i = \sum_{k=0}^{i} \bar{\sigma}^k L_k.$$

Lemma 7. Consider a state $x \in \mathbb{R}^n$, scalars $\bar{\sigma} > 0$, $0 < \underline{\sigma} \leq \bar{\sigma}$, integers $N \geq 0$, $l \geq 1$, and parameters $\alpha > 0$, $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$. If the conditions $x^T \Phi_{i,j} x \leq 0$ are satisfied for all $i \in \{0; \cdots; N\}$ and $j \in \{0; \cdots; \lfloor \frac{\underline{\sigma} l}{\bar{\sigma}} \rfloor\}$, with $\Phi_{i,j}$ defined by the equations (8) to (14) (with $\underline{\sigma}$ replacing $\tau(x)$ in the equations), then for all $\sigma \in [0; \underline{\sigma}]$, $x^T \Phi_{P,\alpha}(\sigma) x \leq 0$, with $\Phi_{P,\alpha}(\sigma)$ defined in (6).

Proof of Lemma 7: Let $x \in \mathbb{R}^n$, $\bar{\sigma} > 0$, $0 < \underline{\sigma} \leq \bar{\sigma}$, $N \geq 0$, $l \geq 1$, $P = P^T \succ 0 \in \mathcal{M}_n(\mathbb{R})$ and $\alpha > 0$. The proof of the Lemma is divided into 4 steps.

(1) First, we divide the time interval $[0; \bar{\sigma}]$ into l subdivisions and take a time $\sigma \leq \underline{\sigma}$ into one of these subdivisions. The aim of this step is to prepare the field to compute a precise estimation of $\Phi_{P,\alpha}(.)$ by building up to l small convex embeddings around it instead of building one big one (see Figure 4).
(2) Then, we compute a polynomial approximation of $\Phi_{P,\alpha}(.)$ for the chosen time interval subdivision.
(3) Afterwards, we bound the error term from this polynomial approximation with a constant term.
(4) Finally, we build a convex polytope around the polynomial approximation, using the method proposed in Hetel et al. [2007] (see Lemma 6), to obtain the desired sufficient finite number of conditions.

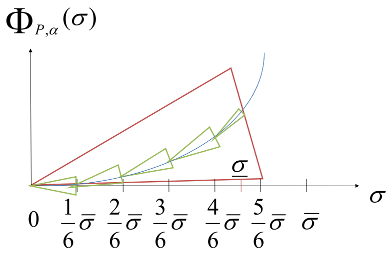

Fig. 4. Building polytope subdivisions (here with $l = 6$)

Step 1: Let us divide the time interval $[0;\bar{\sigma}]$ into l subdivisions $[j\frac{\bar{\sigma}}{l};(j+1)\frac{\bar{\sigma}}{l}]$, with $j \in \{0;\cdots;l-1\}$. Let $\sigma \in [0;\underline{\sigma}]$. There exists $j \in \{0;\cdots;\lfloor\frac{\underline{\sigma}l}{\bar{\sigma}}\rfloor\}$ such that $j\frac{\bar{\sigma}}{l} \leq \sigma \leq (j+1)\frac{\bar{\sigma}}{l}$. Let us then define $\sigma' = \sigma - j\frac{\bar{\sigma}}{l}$ ($\sigma' \in [0;\chi]$, with $\chi = \frac{\bar{\sigma}}{l}$ if $j < \lfloor\frac{\underline{\sigma}l}{\bar{\sigma}}\rfloor$, and $\chi = \underline{\sigma} - \frac{j\bar{\sigma}}{l}$ otherwise).

Step 2: In order to have lighter equations, let us define $\Pi_1 = A^TP + PA + 2\alpha P$ and $\Pi_2 = PBK$. From equations (6) and (7), we deduce that

$$\Phi_{P,\alpha}(\sigma) = \Lambda(\sigma)^T\Pi_1\Lambda(\sigma) - \Lambda^T(\sigma)\Pi_2 - \Pi_2^T\Lambda(\sigma). \quad (17)$$

In order to derive a useful expression of $\Lambda(\sigma)$ as a function of σ', we use the following equation obtained with some computations:

$$\int_0^{a+b} \mathrm{e}^{As}ds = \int_0^{a} \mathrm{e}^{As}ds + \int_0^{b} \mathrm{e}^{As}ds\left(A\int_0^{a} \mathrm{e}^{As}ds + I\right),$$

which is satisfied for any scalars a and b, in order to get

$$\begin{aligned}\Lambda(\sigma) &= I + \left(M_j + \int_0^{\sigma'} \mathrm{e}^{As}dsN_j\right)(A-BK)\\ &= \Pi_{3,j} + \int_0^{\sigma'} \mathrm{e}^{As}ds\Pi_{4,j},\end{aligned} \quad (18)$$

with $M_j = \int_0^{j\frac{\bar{\sigma}}{l}} \mathrm{e}^{As}ds$, $N_j = AM_j + I$, $\Pi_{3,j} = I + M_j(A-BK)$, and $\Pi_{4,j} = N_j(A-BK)$. Then, note that

$$\int_0^{\sigma'} \mathrm{e}^{As}ds = \sum_{i=1}^{\infty}\frac{A^{i-1}}{i!}\sigma'^i. \quad (19)$$

Combining equations (17), (18) and (19), one can compute

$$\Phi_{P,\alpha}(\sigma) = \sum_{k=0}^{\infty} L_{k,j}\sigma'^k, \quad (20)$$

with the $L_{k,j}$ defined in (10). It is then possible to express a polynomial approximation of order N of $\Phi_{P,\alpha}$ on the temporal interval subdivision $[j\frac{\bar{\sigma}}{l};(j+1)\frac{\bar{\sigma}}{l}]$ as

$$\hat{\Phi}_{P,\alpha,N,j}(\sigma') = \sum_{k=0}^{N} L_{k,j}\sigma'^k, \sigma' \in [0;\frac{\bar{\sigma}}{l}].$$

Step 3: Let us denote the approximation error term $R_{P,\alpha,N,j}(\sigma') = \Phi_{P,\alpha}(\sigma) - \hat{\Phi}_{P,\alpha,N,j}(\sigma')$. If we can compute a bound with a scalar ν independent of σ' such that $R_{P,\alpha,N,j}(\sigma') \preceq \nu I$ then the condition $x^T(\hat{\Phi}_{P,\alpha,N,j}(\sigma') + \nu I)x \leq 0$ will imply that $x^T\Phi_{P,\alpha}(\sigma)x \leq 0$. For a given σ', since $R_{P,\alpha,N,j}(\sigma') = \Phi_{P,\alpha}(\sigma) - \hat{\Phi}_{P,\alpha,N,j}(\sigma')$ is symmetric, then if we denote $\lambda_{\sigma'}$ the maximal eigenvalue of $R_{P,\alpha,N,j}(\sigma')$, we have $R_{P,\alpha,N,j}(\sigma') \preceq \lambda_{\sigma'}I$.
As a consequence, we can write $R_{P,\alpha,N,j}(\sigma') \preceq \nu I$ with ν a constant defined in (13).

Step 4: Since the function $\hat{\Phi}_{P,\alpha,N,j}(.) + \nu I : [0;\chi] \to \mathcal{M}_n(\mathbb{R})$ is polynomial, we can use the convex polytope given in Lemma 6 to prove that if $x^T\Phi_{i,j}x \leq 0$ for all $i \in \{1;\cdots;n\}$, with $\Phi_{i,j} = \left(\sum_{k=0}^{i} L_{k,j}\chi^k\right) + \nu I$, then $x^T(\hat{\Phi}_{P,\alpha,N,j}(\sigma') + \nu I)x \leq 0$, and therefore $x^T\Phi_{P,\alpha}(\sigma)x \leq 0$, which ends the proof of Lemma 7. □

The proof of Theorem 3 is obtained by implementing Lemma 7 conditions to satisfy $x^T\Phi_{P,\alpha}(\sigma)x \leq 0, \forall\sigma \in [0;\bar{\sigma}]$ in Lemma 2 with $\bar{\sigma} \geq \|\tau\|_\infty$ and $\underline{\sigma} = \tau(x)$, for all $x \in \mathbb{R}^n$. □

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