Filtering Solid Gabor Noise
Supplemental Material

Ares Lagae
George Drettakis

Report CW 605, April 2011

Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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Filtering Solid Gabor Noise
Supplemental Material

Ares Lagae[1,2] George Drettakis[2]
[1]Katholieke Universiteit Leuven [2]REVES/INRIA Sophia-Antipolis

Abstract

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1 Filtering Perlin and Wavelet Noise using Frequency Clamping

In this section, we give more details on how we filter Perlin and wavelet noise using frequency clamping.

Whether or not aliasing will occur is determined by how the frequency content of the noise $[f_{min}, f_{max})$, i.e. the range of frequencies in the noise, is positioned with request to the Nyquist frequency f_c, i.e., the maximum frequency that can be represented. Frequencies in the noise smaller than f_c correspond to detail, while frequencies larger than f_c will cause aliasing.

We filter the noise by multiplying it with a filtering weight w, depending on how $[f_{min}, f_{max})$ is positioned w.r.t. f_c: (i) if $f_{max} <= f_c$, then all frequencies correspond to detail, and $w = 1$; (ii) if $f_{min} > f_c$, then all frequencies will cause aliasing, and $w = 0$; (iii) if $f_{min} \leq f_c < f_{max}$, then one part of the frequencies correspond to detail $[f_{min}, f_c)$ and another part will cause aliasing $[f_c, f_{max})$, and we multiply with $w = \text{smoothstep}\,(f_{min}, f_{max}, f_c)$.

We approximate the Nyquist frequency in texture space f_c by $1/2\Delta$, where Δ, the sampling interval in texture space, is approximated by

$$\Delta \approx \max\left(\left| \frac{\partial \mathbf{x_{tex}}}{\partial x_{scr,x}} \right|, \left| \frac{\partial \mathbf{x_{tex}}}{\partial x_{scr,y}} \right| \right). \quad (1)$$

For wavelet noise, we use $f_{min} = 1/4$ and $f_{max} = 1/2$, motivated by the theoretical framework of wavelet noise, and we optionally slightly adjust both parameters for optimal results. For Perlin noise, we use $f_{min} = 0.25$ and $f_{max} = 0.95$, which we determined experimentally.

This derivation shows that frequency clamping is inherently subject to an aliasing vs. detail loss trade-off. This is because of case (iii) above, where a part of the noise corresponds to detail and another part causes aliasing: lowering w results in less aliasing but also less detail, while increasing w results in more detail but also more aliasing. This is why the weight w is chosen proportionally to the ratio of the parts corresponding to detail and aliasing. This problem improves as the noise becomes more narrowly band-limited: as the interval $[f_{min}, f_{max})$ becomes smaller, case (iii) becomes less frequent.

[1,2]e-mail: ares.lagae@cs.kuleuven.be, george.drettakis@inria.fr

2 Slicing Random-Phase Gabor Noise

In this section, we show that random-phase Gabor noise is closed under slicing, i.e., that slicing a n-dimensional random-phase Gabor noise results in a $(n-1)$-dimensional random-phase Gabor noise.

We slice an n-dimensional random-phase Gabor noise n with a hyperplane Π, which results in

$$\begin{aligned} \mathcal{S}_\Pi \left[n\left(\mathbf{x}; a, \omega\right)\right]\left(\mathbf{x}'\right) &= \sum_i w_i^s g\left(\mathbf{x}' - \mathbf{x_i}'; a, \omega^{\mathbf{s}}, \phi_i^s\right) \\ &= \sum_i w_i^s \delta\left(\mathbf{x}' - \mathbf{x_i}'\right) * g\left(\mathbf{x}'; a, \omega^{\mathbf{s}}, \phi_i^s\right), \end{aligned} \quad (2)$$

where $w_i^s = e^{-\pi a^2 d_i^2}$, $d_i = \mathbf{n} \cdot \mathbf{x_i} + d$ and $\mathbf{x_i}' = \text{proj}_\Pi \mathbf{x_i}$. This is because $\mathcal{S}$ is a linear operator and because the random-phase Gabor kernel is closed under slicing.

We show that the r.h.s. of Eqn. 2 is a $(n-1)$-dimensional random-phase Gabor noise. This follows from two elements. (i) The r.h.s. of the convolution in Eqn. 2 is an $(n-1)$-dimensional random-phase Gabor kernel. The phases $\{\phi_i^s\}$ with $\phi_i^s = \phi_i - 2\pi d_i \mathbf{n} \cdot \omega$ are uniformly distributed over $[0, \pi)$, i.e., the phase-shifted random phases are still random. This is because ϕ_i is uniformly distributed over $[0, \pi)$, ϕ_i and d_i are independent, and the sum is modulo 2π [Scheinok 1965, Eqn. 3.3 with $g_Y(y) = 1/a$]. (ii) The l.h.s. of the convolution in Eqn. 2 is an $(n-1)$-dimensional weighted Poisson process on Π. The random positions $\{\mathbf{x_i}'\}$ with $\mathbf{x_i}' = \text{proj}_\Pi \mathbf{x_i}$ are the perpendicular projections of $\{\mathbf{x_i}\}$ onto Π, the random weights $\{w_i^s\}$ with $w_i^s = e^{-\pi a^2 d_i^2}$ and $d_i = \mathbf{n} \cdot \mathbf{x_i} + d$ are the Gaussian-weighted distances of $\{\mathbf{x_i}\}$ to Π, and the random variables corresponding to $\mathbf{x_i}'$ and w_i^s are independent.

Eqn. 2 and Eqn. 7 in the paper have the same form, except for the extra weight w_i^s, i.e., the Poisson process in Eqn. 2 is weighted while the Poisson process in Eqn. 7 is not. This results in an extra factor w^s in the expressions for the variance and the power spectrum of sliced random-phase Gabor noise, where

$$w^s = \int_{-\infty}^{\infty} \left(e^{-\pi a^2 x^2}\right)^2 dx = \frac{1}{\sqrt{2}a}, \quad (3)$$

which follows from the derivation of the shot noise equations for this weighted Poisson process [van Etten 2005, Ch. 8].

We conclude that slicing an n-dimensional random-phase Gabor noise n with a hyperplane Π results in

$$\mathcal{S}_\Pi \left[n\left(\mathbf{x}; a, \omega\right)\right]\left(\mathbf{x}'\right) = \sqrt{w^s} n\left(\mathbf{x}'; a, \omega^{\mathbf{s}}\right), \quad (4)$$

an $(n-1)$-dimensional random-phase Gabor noise, i.e., random-phase Gabor noise is closed under slicing.

This derivation shows that the random phases are essential because they ensure invariance w.r.t. slicing: since the phases of the Gabor kernels are random, the phase-shifted phases of the sliced kernels are random as well.

3 Analytical Expressions for Random-Phase Gabor Noise

In this section, we give the analytical expressions for random-phase Gabor noise. For the definitions of all symbols, please see the paper.

3.1 The Phase-Augmented Gabor Kernel

The phase-augmented Gabor kernel is

$$g(\mathbf{x}; a, \omega, \phi) = e^{-\pi a^2 |\mathbf{x}|^2} \cos(2\pi \mathbf{x} \cdot \omega + \phi). \quad (5)$$

The Fourier transform of the phase-augmented Gabor kernel is

$$G(\xi; a, \omega, \phi) = \frac{1}{2a^n} \left(e^{-\frac{\pi}{a^2}|\xi - \omega|^2} e^{i\phi} + e^{-\frac{\pi}{a^2}|\xi + \omega|^2} e^{-i\phi} \right). \quad (6)$$

The scale factor equals $1/2a$, $1/2a^2$ and $1/2a^3$ for $n = 1$, $n = 2$ and $n = 3$ respectively.

3.2 The Random-Phase Gabor Kernel

The integral of the random-phase Gabor kernel is

$$\frac{1}{2\pi} \int_{\phi=0}^{2\pi} \int_{\mathbb{R}^n} g(\mathbf{x}; a, \omega, \phi)\, d\mathbf{x} d\phi = 0. \quad (7)$$

The integral of the random-phase Gabor kernel squared is

$$\frac{1}{2\pi} \int_{\phi=0}^{2\pi} \int_{\mathbb{R}^n} g^2(\mathbf{x}; a, \omega, \phi)\, d\mathbf{x} d\phi = \frac{1}{2\left(\sqrt{2}a\right)^n}. \quad (8)$$

The scale factor equals $1/2\sqrt{2}a$, $1/4a^2$ and $1/4\sqrt{2}a^3$ for $n = 1$, $n = 2$ and $n = 3$ respectively.

The magnitude squared of the Fourier transform of the random-phase Gabor kernel is

$$\frac{1}{2\pi} \int_{\phi=0}^{2\pi} |G(\xi; a, \omega, \phi)|^2 d\phi = \frac{1}{4a^{2n}} \left(e^{-\frac{2\pi}{a^2}|\xi - \omega|^2} + e^{-\frac{2\pi}{a^2}|\xi + \omega|^2} \right). \quad (9)$$

The scale factor equals $1/4a^2$, $1/4a^4$ and $1/4a^6$ for $n = 1$, $n = 2$ and $n = 3$ respectively.

3.3 Random-Phase Gabor Noise

Random-phase Gabor noise is

$$n(\mathbf{x}; a, \omega) = \sum_i g(\mathbf{x} - \mathbf{x_i}; a, \omega, \phi_i). \quad (10)$$

The variance of random-phase Gabor noise is

$$\sigma_n^2 = \lambda \frac{1}{2\pi} \int_{\phi=0}^{2\pi} \int_{\mathbb{R}^n} g^2(\mathbf{x}; a, \omega, \phi)\, d\mathbf{x} d\phi. \quad (11)$$

The power spectrum of random-phase Gabor noise is

$$S_{nn}(\xi; a, \omega) = \lambda \frac{1}{2\pi} \int_{\phi=0}^{2\pi} |G(\xi; a, \omega, \phi)|^2 d\phi. \quad (12)$$

3.4 Sliced Random-Phase Gabor Noise

The slicing factor is

$$w_a^s = \int_{-\infty}^{\infty} \left(e^{-\pi a^2 x^2} \right)^2 dx = \frac{1}{\sqrt{2}a}. \quad (13)$$

The variance of sliced random-phase Gabor noise is

$$\sigma^2_{\mathcal{S}[n(\mathbf{x}; a, \omega)]} = w_a^s \sigma^2_{n(\mathbf{x'}; a, \omega^\mathbf{s})}. \quad (14)$$

The power spectrum of sliced random-phase Gabor noise is

$$S_{\mathcal{S}[n(\mathbf{x}; a, \omega)]\mathcal{S}[n(\mathbf{x}; a, \omega)]}(\xi'; a, \omega) = w_a^s S_{n(\mathbf{x'}; a, \omega^\mathbf{s}) n(\mathbf{x'}; a, \omega^\mathbf{s})}(\xi'; a, \omega). \quad (15)$$

4 Avoiding Matrix Inversions when Filtering

In this section, we give more details on how a matrix expression involving the inverse of the sum of a scaled identity matrix and a positive definite matrix can be simplified.

We use the expression

$$\left[(aI)^{-1} + \left(UU^T \right)^{-1} \right]^{-1} = aI - a^2 \left[aI + \left(UU^T \right) \right]^{-1}, \quad (16)$$

which follows from [Henderson and Searle 1981, Eqn. 1 with $A = aI$, $B = I$ and $V = U^T$]. This expression reduces the number of matrix inversions for an expression such as the one for Σ_{GF} (paper, Eqn. 10) from three to one.

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