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Nonlinear resonant absorption of fast magnetoacoustic waves in strongly anisotropic and dispersive plasmas

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The nonlinear theory of driven magnetohydrodynamics (MHD) waves in strongly anisotropic and dispersive plasmas, developed for slow resonance by Clack & Ballai [Phys. Plasmas **15**(8), 2310 (2008)] and Alfvén resonance by Clack *et al.* [A&A **494**, 317 (2009)], is used to study the weakly nonlinear interaction of fast magnetoacoustic (FMA) waves in a one-dimensional planar plasma. The magnetic configuration consists of an inhomogeneous magnetic slab sandwiched between two regions of semi-infinite homogeneous magnetic plasmas. Laterally driven FMA waves penetrate the inhomogeneous slab interacting with the localized slow or Alfvén dissipative layer and are partly reflected, dissipated and transmitted by this region. The nonlinearity parameter defined by Clack & Ballai (2008) is assumed to be small and a regular perturbation method is used to obtain analytical solutions in the slow dissipative layer. The effect of dispersion in the slow dissipative layer is to further decrease the coefficient of energy absorption, compared to its standard weakly nonlinear counterpart, and the generation of higher harmonics in the outgoing wave in addition to the fundamental one. The absorption of external drivers at the Alfvén resonance is described within the linear MHD with great accuracy.

I. INTRODUCTION

The problem of interacting fast magnetoacoustic (FMA) waves with different magnetic structures is not only important in the context of astrophysics and solar physics, but also in laboratory plasma devices. Space and laboratory plasmas are highly non-uniform and dynamical systems and as a consequence they are a natural medium for magnetohydrodynamic (MHD) waves. When the magnetic plasma configuration is inhomogeneous in the transversal direction relative to the ambient magnetic field a phenomenon, known as resonant absorption, occurs (see, e.g., Appert *et al.* [1] and Ionson [2]). Some of the wave energy can be converted into heat in a thin layer which embraces the ideal resonant magnetic surface when dissipative processes are taken into account.

In the context of solar physics, the resonant coupling of waves was first suggested by Ionson(author?) [3] as a possible mechanism for heating coronal loops. Shortly after, several studies on the efficiency of resonant absorption in the complicated process of coronal heating were published by, e.g., Ionson(author?) [2], Kuperus *et al.*(author?) [4], Davila(author?) [5] and Hollweg(author?) [6]. The same principle was used to explain the observed loss of power of acoustic oscillations in the vicinity of sunspots by, e.g., Hollweg(author?) [7], Lou(author?) [8], Sakurai *et al.*(author?) [9], Goossens and Poedts(author?) [10], Goossens and Hollweg(author?) [11] and Stenuit *et al.*(author?) [12]. All these studies dealt with the Alfvén resonant position. Although happening at lower frequencies, slow resonance is also important as shown in a study by Keppens(author?) [13] where he investigated the interaction of sound waves with hot evacuated magnetic fibrils.

Most of the analytical studies of resonant absorption were based on the linear theory due to its relative simplicity.

A new approach to the problem of resonant absorption in the context of high Reynolds number plasmas was given by Ruderman *et al.*(author?) [14] who developed a nonlinear theory of resonant absorption for slow waves in isotropic plasmas. They pointed out that nonlinearity has to be taken into account under typical solar conditions near resonance. The theory of nonlinear resonant slow waves was extended to strongly anisotropic plasmas in Ballai *et al.*(author?) [15] to describe conditions typical for the solar chromosphere and corona. Over the next few years there was an enormous amount of effort put into studying resonant absorption, including the investigation of the effect of equilibria flows at the slow resonance (see, e.g., Ballai and Erdélyi [16]), the absorption of sound waves at the slow dissipative layers in isotropic and anisotropic plasmas (see, e.g., Ruderman *et al.* [17] and Ballai *et al.* [18]) and the effect of an equilibrium flow on the absorption of sound and FMA waves due to the coupling in the slow continua (see, e.g., Erdélyi and Ballai [19] and Erdélyi *et al.* [20]). In a recent paper, Clack and Ballai(author?) [21] showed that in strongly anisotropic and dispersive plasmas the dispersion, dissipation and nonlinearity are all of the same order inside the dissipative layer.

A study by Clack *et al.*(author?) [22] on the nonlinear effects at the Alfvén dissipative layer found that nonlinearity and dispersion are always negligible in comparison to the linear terms describing dissipation. This implies that the linear theory is always applicable for resonant absorption at the Alfvén resonance if the dimensionless amplitude of perturbations inside the dissipative layer are less than unity. Moreover, Clack *et al.*(author?) [22]

showed that the largest nonlinear and dispersive terms cancel out - leaving only small corrections to linear theory.

Many studies of resonant absorption considered only the sound (or slow) and Alfvén waves as excellent candidates for coronal heating. Alfvén waves can only carry energy along the magnetic field lines and slow waves are only able to carry $1 - 2\%$ of energy under coronal (low plasma-β) conditions. However, FMA waves might have an important role in explaining the coronal temperatures, as has been shown by, e.g., Čadež et al.(author?) [23] and Csík et al.(author?) [24].

The aim of the present paper is to study the nonlinear (linear) resonant interaction of externally driven FMA waves with the slow (Alfvén) dissipative layer in strongly anisotropic and dispersive static plasmas. The governing equations and jump conditions derived earlier by Clack and Ballai(author?) [21] and Clack et al.(author?) [22] will be used to study the efficiency of absorption at the slow and Alfvén resonance. The paper is organized as follows. In the next section we introduce the governing equations, the equilibrium state and the fundamental assumptions which allow analytical progress. In Sec. III we find the solutions describing the waves outside the dissipative layers. Section IV is devoted to the nonlinear solution inside the slow dissipative layer. In Sec. V we derive the solution inside the Alfvén dissipative layer. In Sec. VI we will calculate the absorption coefficient in the case of slow/Alfvén resonance. Finally, in Sec. VII we summarize our results and draw our conclusions.

II. GOVERNING EQUATIONS AND ASSUMPTIONS

The dynamics and absorption of the waves will be studied in a Cartesian coordinate system. The equilibrium state is shown in Figure 1. The configuration consists of an inhomogeneous magnetized plasma $0 < x < x_0$ (Region II) sandwiched between two semi-infinite homogeneous magnetized plasmas $x < 0$ and $x > x_0$ (Regions I and III, respectively). We have chosen this model to obtain analytical results. Our intention is to have a model which gives us the trend in the absorption of an incident wave on a magnetic structure. It is obvious that real magnetic structures are more complicated (and far from being fully understood), however, the magnetic field has been simplified to be unidirectional in order to make the model more transparent, such that the role of the dispersion at the resonance and the change in the absorption can be investigated more fully, and compared to previous studies. We took inspiration for this model from seminal studies such as Ruderman et al.(author?) [17], Ballai et al.(author?) [18], Erdélyi(author?) [20], Roberts(author?) [25], Edwin and Roberts(author?) [26] and Ruderman(author?) [27].

The equilibrium density and pressure are denoted by ρ and p. The equilibrium magnetic field, **B**, is unidi-
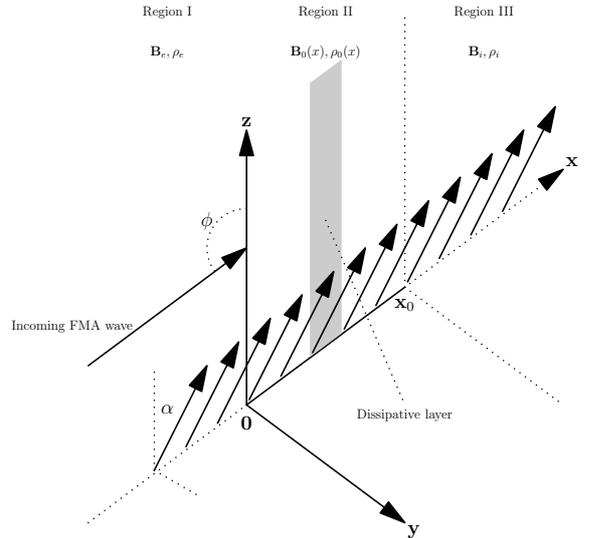
FIG. 1: Illustration of the equilibrium state. Regions I ($x < 0$) and III ($x > 0$) contain a homogeneous magnetized plasma and Region II ($0 < x < x_0$) an inhomogeneous magnetized plasma. The shaded strip shows the dissipative layer embracing the ideal resonant position x_c.

rectional and lies in the yz-plane. In what follows the subscripts "e", "0" and "i" denote the equilibrium quantities in the three regions (Regions I, II, III, respectively). It is convenient to introduce the angle, α, between the z-axis and the direction of the equilibrium magnetic field, so that the components of the equilibrium magnetic field are; $B_y = B \sin \alpha$ and $B_z = B \cos \alpha$. All equilibrium quantities are continuous at the boundaries of Region II, so they satisfy the equation of total pressure balance. It follows from the equation of total pressure that the density ratio between Regions I and III satisfy the relation

$$\frac{\rho_i}{\rho_e} = \frac{2c_{Se}^2 + \gamma v_{Ae}^2}{2c_{Si}^2 + \gamma v_{Ai}^2}, \tag{1}$$

where the squares of the Alfvén and sound speed are $v_A^2 = B_0^2/\mu_0 \rho_0$ and $c_S^2 = \gamma p_0/\rho_0$. Where μ_0 is the magnetic permeability of free space and γ is the adiabatic constant. Replace the subscript "0" with "e" for Region I and "i" for Region III. We consider a hot magnetized plasma such that $c_{Si}^2 > c_{Se}^2$, and $v_{Ai}^2 > v_{Ae}^2$.

The objective of the present paper is to study (i) the combined effect of nonlinearity and dispersion on the interaction of incoming *fast waves* with *slow dissipative layers* and (ii) the interaction of incoming *fast waves* with *Alfvén dissipative layers*. We, therefore, have two different criteria. For interaction of FMA waves with the slow dissipative layer we assume that the frequency of the incoming fast wave is within the slow continuum of the inhomogeneous plasma, so that there is a slow resonant position at $x = x_c$ in Region II. Interactions with the Alfvén dissipative layer leads to the assumption that the frequency of the incoming fast wave is within the Alfvén continuum of the inhomogeneous plasma, so that there

is an Alfvén resonant point at $x = x_a$ in Region II. This leads to the inequality, $c_{Te} < \omega/k < c_{Ti}$, at the slow resonance. Where the square of the cusp speed, c_T^2, is defined by $c_T^2 = c_S^2 v_A^2/(c_S^2 + v_A^2)$. We also have the inequality, $v_{Ae} < \omega/k < v_{Ai}$, for Alfvén resonance. Here ω is the frequency of the incoming fast wave and $k = (k_x^2 + k_z^2)^{1/2}$ is the wave number. Even though, in principle, when a slow resonance occurs in this manner an Alfvén resonance is also present we ignore the Alfvén resonance that occurs alongside the slow resonance as this would complicate the analysis and obscure the results associated with the slow resonance. We study the Alfvén resonance separately to the slow resonance. We note that the Alfvén resonance would, in simple terms, act to *restrict the energy* available at the slow resonance. We intend to address the issue of coupled resonances in our next paper, where we will show that the governing equations derived here remain the same (meaning the work here is valid), however, the interaction of the waves between the resonant positions changes the absorption of wave energy.

In an attempt to remove other effects from the analysis we consider the incoming fast wave to be entirely in the xz-plane, i.e. $k_y = 0$. Ruderman *et al.*(author?) [17] suggests aligning the equilibrium magnetic field with the z-axis, to remove the Alfvén resonance (if we consider planar waves) from the analysis for slow resonance, however, this is not possible nor necessary here. The dispersion is dependent on the angle between the equilibrium magnetic field and the z-axis (α), hence if $\alpha = 0$ the dispersion effects disappear, and we recover the governing equation studied by Ballai *et al.*(author?) [18].

The inequalities above guarantee that the slow and Alfvén resonances appears in Region II when studying in the upper chromosphere and the solar corona, respectively. The resonant positions, therefore, are defined mathematically as: $\omega_c = k c_T(x_c) \cos \alpha$ and $\omega_a = k v_A(x_a) \cos \alpha$. The position of the resonant points also provides us with some information about the plasma condition. First, in conjunction with Eq. (1) we obtain that

$$\frac{\rho_i}{\rho_e} = \frac{2c_{Se}^2 + \gamma v_{Ae}^2}{2c_{Si}^2 + \gamma v_{Ai}^2} < 1. \tag{2}$$

Hence, the plasma in region III is more rarefied than in Region I. Secondly, it follows that $c_{Te} < c_{Ti}$ and the plasma in Region III is hotter than the plasma in Region I.

The dispersion relation for the impinging propagating fast waves takes the form

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left\{ (v_A^2 + c_S^2) + \left[(v_A^2 + c_S^2)^2 - 4v_A^2 c_S^2 \cos^2 \phi \right]^{1/2} \right\}, \tag{3}$$

where ϕ is the angle between the direction of propagation and the background magnetic field within the xz-plane and $\mathbf{k} = k_x \mathbf{e}_x + k_z \mathbf{e}_z$. For the sake of simplicity, we denote κ_e as the ratio k_x/k_z. Since the equilibrium magnetic field in the xz-plane is aligned with the z-axis, the

dispersion relation (3) becomes

$$\frac{\omega^2}{k^2} = \frac{1}{2}\left\{ (v_A^2 + c_S^2) + \left[(v_A^2 + c_S^2)^2 - 4\frac{v_A^2 c_S^2}{1 + \kappa_e^2} \right]^{1/2} \right\}, \tag{4}$$

where $1 + \kappa_e^2 = 1/\cos^2 \phi$.

We assume the plasma is *strongly* magnetized in the three regions, such that the conditions $\omega_{i(e)}\tau_{i(e)} \gg 1$ are satisfied, here $\omega_{i(e)}$ is the ion (electron) gyrofrequency and $\tau_{i(e)}$ is the ion (electron) collision time. Due to the strong magnetic field, transport processes are derived from Braginskii's stress tensor (see, e.g., Braginskii[28]; Ruderman *et al.*[29]). As we deal with two separate waves (slow and Alfvén), we will need to choose the particular dissipative process which is most efficient for these waves. For slow waves, it is a good approximation to retain only the first term of Braginksii's expression for viscosity, namely compressional viscosity(author?) [30]. In addition, in the solar upper atmosphere slow waves are sensitive to thermal conduction. In a strongly magnetized plasma, the thermal conductivity parallel to the magnetic field lines dwarfs the perpendicular component, hence the heat flux can be approximated by the parallel component only(author?) [31]. On the other hand, since Alfvén waves are transversal and incompressible they are affected by the second and third components of Braginskii's stress tensor, called shear viscosity(author?) [22]. Finally, Alfvén waves are efficiently damped by finite electrical conductivity, which becomes anisotropic under coronal conditions. The parallel and perpendicular components, however, only differ by a factor of 2, so we will only consider one of them without loss of generality. All other transport mechanisms can be neglected. For further details, please refer to, for example, Clack *et al.*(author?) [22], Braginskii(author?) [28], Ruderman *et al.*(author?) [29], Hollweg(author?) [30], Priest(author?) [31] and Porter *et al.*(author?) [32].

The dynamics of nonlinear resonant MHD waves in anisotropic and dispersive plasmas was studied by Clack and Ballai(author?) [21] and Clack *et al.*(author?) [22]. They derived the governing equations and connection formulae necessary to study resonant absorption in slow/Alfvén dissipative layers. We recall the key steps and necessary results found by Clack and Ballai(author?) [21] and Clack *et al.*(author?) [22].

The efficiency of dissipation, when studying slow dissipative layers, in an anisotropic plasma is given by the (compressional) viscous Reynolds number ($R_{e(c)}$) and the Pechlet number (P_e), combining to define the total Reynolds number: $R_c^{-1} = R_{e(c)}^{-1} + P_e^{-1}$, where $R_{e(c)}$ and P_e are defined by $R_{e(c)} = v_{ch} l_{ch} \rho_{0c}/\bar{\eta}_0$ and $P_e = v_{ch} l_{ch} \rho_{0c} \widetilde{R}/\bar{\kappa}_\parallel$. Here v_{ch} is the characteristic velocity (e.g. the slow magnetoacoustic velocity at $x = x_c$), l_{ch} is the characteristic length, $\rho_{0c} = \rho_0(x_c)$, $\widetilde{R}$ denotes the gas constant and $\kappa_\parallel$ is the coefficient of thermal conductivity parallel to the equilibrium magnetic field lines.

The efficiency of dissipation, when studying Alfvén dissipative layers, in an anisotropic plasma is measured in a slightly different way. Now dissipative processes are described by the (shear) viscous Reynolds number (R_e) and the magnetic Reynolds number (R_m), combining to define the total Reynolds number: $R_a^{-1} = R_e^{-1} + R_m^{-1}$, where $R_e = v_{ch}l_{ch}\rho_{0_a}/\bar{\eta}_1$ and $R_m = v_{ch}l_{ch}/\bar{\eta}$. Here v_{ch} is the characteristic velocity (e.g. the Alfvén velocity at $x = x_a$), l_{ch} is the characteristic length, η_1 is the coefficient of shear viscosity and η is the coefficient of finite electrical resistivity. Originally, these total Reynolds numbers were introduced based on intuition, simplicity and linear theory (see, e.g., Sakurai *et al.* [9], Goossens *et al.* [34] and Goossens and Ruderman [35]). However, it turned out that using these definitions the *strength* of dissipation is the same order of magnitude as the inverse of the total Reynolds numbers. Under chromospheric and coronal conditions $R \gg 1$ which means that dissipation is only important inside the dissipative layer. Far away from the dissipative layer amplitudes are small, therefore we can use the linear ideal MHD equations to describe the plasma motions far from the resonant position. These equations can be reduced to a system of coupled first order PDE's for the total pressure perturbation, P, and the normal component of the velocity, u,

$$\frac{\partial u}{\partial x} = \frac{V}{F}\frac{\partial P}{\partial \theta}, \quad \frac{\partial P}{\partial x} = \frac{\rho_0 A}{V}\frac{\partial u}{\partial \theta}. \tag{5}$$

Here

$$
\begin{aligned}
F &= \frac{\rho_0 A C}{V^4 - V^2\left(v_A^2 + c_S^2\right) + v_A^2 c_S^2 \cos^2\alpha}, \\
C &= \left(v_A^2 + c_S^2\right)\left(V^2 - c_T^2\cos^2\alpha\right), \\
A &= V^2 - v_A^2\cos^2\alpha.
\end{aligned} \tag{6}
$$

The system (5) describes the wave motion far from the ideal resonant position. The singularities in the coefficients A and C give the conditions of Alfvén and slow resonance. All perturbations depend on the combination $\theta = z - Vt$, where $V = \omega/k$ is the phase speed.

Inside the thin dissipative layers (where the dynamics is described by the nonlinear and dissipative MHD equations) embracing the ideal resonant surfaces ($x = x_c$, $x = x_a$) we must use the governing equations derived by Clack and Ballai(author?) [21] and Clack *et al.*(author?) [22]. The characteristic thickness of the slow dissipative layer, δ_c, is

$$\delta_c = \frac{V^3 k\lambda}{|\Delta_c|\left(c_{S_c}^2 + v_{A_c}^2\right)}. \tag{7}$$

Here $k = 2\pi/L$ with L the wavelength, the subscript "c" indicates that the quantity has been calculated at the slow resonant position. The quantity λ is defined by

$$\lambda = \frac{\eta_0\left(2v_{A_c}^2 + 3c_{S_c}^2\right)^2}{3\rho_{0_c}v_{A_c}^2 c_{S_c}^2} + \frac{(\gamma-1)^2\kappa_\parallel\left(v_{A_c}^2 + c_{S_c}^2\right)}{\gamma\rho_{0_c}\widetilde{R}c_{S_c}^2}, \tag{8}$$

and Δ_c is simply the gradient of the cusp speed given by $\Delta_c = -\left(dc_T^2/dx\right)_c\cos^2\alpha$. Clack and Ballai(author?) [21] showed that nonlinearity and dispersion are important in the slow dissipative layer if the nonlinearity parameter is greater than unity, $N^2 = \epsilon R_c^2 \gtrsim 1$, where ϵ is the dimensionless wave amplitude far from the dissipative layer. The concept of nonlinear parameters was introduced by Ruderman *et al.*(author?) [14] for slow waves and Clack *et al.*(author?) [22] for Alfvén waves. The two parameters are different not only in their form but also in the values the Reynolds numbers take. In the case of slow waves (damped by compressional viscosity, i.e. the first term in the Braginskii's viscosity tensor) the Reynolds number that corresponds to a characteristic length of 200Mm, a speed of 200kms^{-1}, a density of 10^{-13}kgm^{-3} and a compressional viscosity coefficient of 5×10^{-2}kgm^{-1}s^{-1} is about 80. Alfvén waves are efficiently damped by shear viscosity which is given by the second and third coefficients of the Braginskii's tensor (here denoted cumulatively as η_1). Since $\eta_1 = \eta_0/(\omega_i\tau_i)^2$ and under coronal conditions $\omega_i\tau_i$ is of the order of 10^5, we obtain that the coefficient of shear viscosity is about 10 orders of magnitude smaller than the coefficient of compressional viscosity. Now, using the characteristic speed of 1000kms^{-1}, the Reynolds number used in calculating the nonlinear parameter in the case of Alfvén nonlinearity is 4×10^{12}. The nonlinearity parameter for resonant Alfvén waves is $\epsilon R^{2/3} \ll 1$. However, it was shown by Clack *et al.*(author?) [22] that the waves in this situation remain linear anyway. In the present paper, therefore, we do not need the nonlinearity parameter for resonant Alfvén waves. The characteristic thickness of the Alfvén dissipative layer, δ_a, is

$$\delta_a = \left[\frac{V}{k|\Delta_a|}\left(\bar{\eta} + \frac{\bar{\eta}_1}{\rho_{0a}}\right)\right]^{1/3}, \tag{9}$$

with Δ_a being the gradient of the Alfvén speed given by $\Delta_a = -\left(dv_A^2/dx\right)_a\cos^2\alpha$.

The governing equation inside the slow dissipative layer is(author?) [21]

$$
\begin{aligned}
\sigma_c\frac{\partial q_c}{\partial \theta} + \Lambda q_c\frac{\partial q_c}{\partial \theta} - k^{-1}\frac{\partial^2 q_c}{\partial \theta^2} \\
- \Psi\frac{\partial q_c}{\partial \sigma}\frac{\partial q_c}{\partial \theta} = -\frac{kV^4}{\rho_{0_c}v_{A_c}^2|\Delta_c|}\frac{d\widetilde{P}}{d\theta},
\end{aligned} \tag{10}
$$

where

$$\sigma_c = \frac{x - x_c}{\delta_c}, \tag{11}$$

$$\Lambda = R^2\frac{v_{A_c}^4|\Delta|\left[(\gamma+1)v_{A_c}^2 + 3c_{S_c}^2\right]}{kV^8}, \tag{12}$$

$$\Psi = R^2\frac{\chi|\Delta|^2 c_{S_c}^2 v_{A_c}^2\left(v_{A_c}^2 + c_{S_c}^2\right)\sin\alpha}{kV^{13}}, \tag{13}$$

$$\chi = \eta\omega_e\tau_e. \tag{14}$$

Here the first term of the governing equation appears due to the inhomogeneity in the cusp speed, the second

term describes the nonlinearity of waves, the third term stands for the dissipative effects while the last term on the left-hand side describes the nonlinear dispersive effects generated after taking into account Hall currents by Clack and Ballai**(author?)** [21]. The term on the right-hand side can be considered as a driver. We also note that $q_c(\sigma_c, \theta)$ is the dimensionless component of velocity parallel to the equilibrium magnetic field and $\chi = \eta \omega_e \tau_e$ is the coefficient of Hall conduction**(author?)** [21].

The governing equation inside the Alfvén dissipative layer is**(author?)** [22]

$$\sigma_a \frac{\partial q_a}{\partial \theta} - k \frac{\partial^2 q_a}{\partial \sigma_a^2} = \frac{k \sin \alpha}{\rho_{0_a} |\Delta_a|} \frac{d\widetilde{P}}{d\theta}, \qquad (15)$$

with $\sigma_a = (x - x_a)/\delta_a$. Here $q_a(\sigma_a, \theta)$ is the dimensionless component of velocity perpendicular to the equilibrium magnetic field. We should point out here that although nonlinearity and dispersion have been considered when deriving the dynamics of the Alfvén resonance, the governing equation remains linear regardless of the degree of nonlinearity (for details see Clack *et al.* [22]).

When studying resonant MHD waves, we are generally not interested in the solution inside the dissipative layer and can consider the dissipative layer as a surface of discontinuity. Instead, we solve the system (5) and match the solutions at the boundaries of the discontinuity using connection formulae. These connection formulae determine the jumps in u and P across the dissipative layer. In the context of solar plasmas, they were first introduced by Sakurai *et al.***(author?)** [9]. It was shown by Clack and Ballai**(author?)** [21] (in complete agreement with Ruderman *et al.***(author?)** [14] and Ballai *et al.***(author?)** [15]) that the first connection formula is $[P] = 0$, where the square brackets denote the jump across the dissipative layer. It can also be shown, in a similar manner, that the same jump condition exists for the Alfvén resonance. The second connection formula for slow resonance can only be written in implicit form, i.e.

$$[u_c] = -\frac{V}{k \cos^2 \alpha} \mathscr{P} \int_{-\infty}^{\infty} \frac{\partial q_c}{\partial \theta} \, d\sigma, \qquad (16)$$

where we use the Cauchy principal value of the integral because the integral is divergent at infinity. As a result we must solve Eqs. (5) and (10) along with the boundary conditions, $[P] = 0$ and Eq. (16). In an attempt to follow the same procedure utilized for finding solutions at the slow resonance we can write the jump in the normal component of velocity for the Alfvén resonance in an implicit form. For the sake of brevity, we do not show the derivation here, but it follows the procedure to find the jump in the normal component of velocity completed by Clack and Ballai**(author?)** [21]. This jump is given by

$$[u_a] = \frac{V \sin \alpha}{k} \mathscr{P} \int_{-\infty}^{\infty} \frac{\partial q_a}{\partial \theta} \, d\sigma. \qquad (17)$$

Finally, we should note some critical assumption we make to allow analytical progress. From the very beginning we must assume that the nonlinearity parameter is small so that regular perturbation theory can be applied at the slow resonance. We also assume that the inhomogeneous region is thin in comparison with the wavelength of the impinging wave, i.e. $kx_0 \ll 1$. Ruderman**(author?)** [27] investigated the absorption of sound waves at the slow dissipative layer in the limit of strong nonlinearity. In his analysis nonlinearity dominated dissipation in the *resonant layer* which embraces the dissipative layer. He concluded that nonlinearity decreases absorption in the long wavelength approximation, but increases it at intermediate values of kx_0, however, the increase is never more than 20%. To the best of our knowledge, at present, we cannot solve the governing equation (10) in the limit of strong nonlinearity due to the nonlinear dispersive term, therefore we restrict our analysis to the weak nonlinear limit. We mention that no such assumptions are needed for studying the Alfvén dissipative layer since the governing equation (15) is linear.

III. SOLUTIONS OUTSIDE THE DISSIPATIVE LAYERS

In what follows we derive a solution for the system (5) in Regions I, II and III. In Region II we only find the solution outside the dissipative layers. Section IV is devoted to finding a solution to Eq. (10) inside the slow dissipative layer and Section V is used to find a solution to Eq. (15) inside the Alfvén dissipative layer. Outside the dissipative layers, the solutions take identical forms.

A. Region I

The solution of Eq. (5) in Region I is given in the form of an incoming and outgoing fast wave of the form

$$P = \epsilon \left\{ p_e \cos\left[k\left(\theta + \kappa_e x\right)\right] + A \cos\left[k\left(\theta - \kappa_e x\right)\right] \right\}, \quad (18)$$

$$u = \epsilon \frac{\kappa_e V \left\{ p_e \cos\left[k\left(\theta + \kappa_e x\right)\right] - A \cos\left[k\left(\theta - \kappa_e x\right)\right] \right\}}{\rho_e \left(V^2 - v_{Ae}^2 \cos^2 \alpha\right)}, \tag{19}$$

where $\epsilon \ll 1$ is the dimensionless amplitude of perturbation far from the dissipative layer. The frequency of the incoming wave is given by Eq. (4) and must lie within the slow or Alfvén continuum depending on which dissipative layer we are studying. The first term in Eqs. (18) and (19) describes the incoming wave, while the second term describes the outgoing wave which will be obtained in Section IV for slow dissipative layers and in Section V for Alfvén dissipative layers.

B. Region II

In Region II, the equation for the total pressure, P, is obtained by eliminating u from the system (5),

$$F \frac{\partial}{\partial x} \left[\frac{1}{\rho_0 \left(V^2 - v_A^2 \cos^2 \alpha \right)} \frac{\partial P}{\partial x} \right] = \frac{\partial^2 P}{\partial \theta^2}. \qquad (20)$$

Since we have assumed $kx_0 \ll 1$, the ratio of the right-hand side and the left-hand side is of the order of $k^2 x_0^2$. It follows that

$$\frac{\partial P}{\partial x} = \rho_0 \left(V^2 - v_A^2 \cos^2 \alpha \right) f(\theta) + \mathcal{O}(k^2 x_0^2), \qquad (21)$$

where the function $f(\theta)$ is determined by the second equation of (5) and the boundary conditions at $x = 0$. Equation (21) yields

$$P = \widetilde{P}(\theta) + f(\theta) \int_0^x \rho_0 \left[V^2 - v_A^2 \cos^2 \alpha \right] dx + \mathcal{O}(k^2 x_0^2). \qquad (22)$$

The function $\widetilde{P}(\theta)$ has to be determined by the boundary conditions at $x = 0$. It can be shown that, because $[P] = 0$, the functions $f(\theta)$ and $\widetilde{P}(\theta)$ take the same values throughout Region II. Noting that the second term in Eq. (22) is of the order of kx_0 we can express P in a simplified form

$$P = \widetilde{P}(\theta) + (kx_0) P'(x, \theta) + \mathcal{O}(k^2 x_0^2). \qquad (23)$$

C. Region III

To derive the governing equation for Region III we eliminate the normal component of the velocity from the system (5) to arrive at

$$\frac{\partial^2 P}{\partial x^2} + \kappa_i^2 \frac{\partial^2 P}{\partial \theta^2} = 0, \qquad (24)$$

where κ_i^2 is defined as

$$\kappa_i^2 = -\frac{V^4 - V^2 \left(c_{Si}^2 + v_{Ai}^2 \right) + c_{Si}^2 v_{Ai}^2 \cos^2 \alpha}{\left(c_{Si}^2 + v_{Ai}^2 \right) \left(V^2 - c_{Ti}^2 \cos^2 \alpha \right)}. \qquad (25)$$

Since, for slow dissipative layers, $V < c_{Ti} \cos \alpha$, it follows that $\kappa_i^2 > 0$. It also follows that for Alfvén dissipative layers $\kappa_i^2 > 0$ because $V > c_{Ti} \cos \alpha > v_{Ae} \cos \alpha$. Therefore, Eq. (24) is an elliptical differential equation and the wave motion is evanescent in Region III. In reality, there could be wave leakage. The existence of wave leakage depends on the profile of the slow and Alfvén speeds in the inhomogeneous region (Region II). For simplicity, we have assumed that the slow and Alfvén resonances take place at a single location (obviously different for the two resonances), which means the profiles of the slow and Alfvén speeds are monotonically increasing inside Region II. Should we have a more complex model, the possibility of wave leakage would need to be taken into account.

IV. WEAK NONLINEAR SOLUTION INSIDE THE SLOW DISSIPATIVE LAYER

Since we are not able to solve the governing equation (10) inside the slow dissipative layer analytically, we consider the limit of weak nonlinearity ($N^2 \ll 1$). In accordance with this assumption we rewrite the governing equation (10) and the jump condition (16) as

$$\sigma \frac{\partial \overline{q_c}}{\partial \theta} + \epsilon^{-1} \zeta \left(\frac{\Lambda}{\Psi} \right) \overline{q_c} \frac{\partial \overline{q_c}}{\partial \theta} - \epsilon^{-1} \zeta \frac{\partial \overline{q_c}}{\partial \sigma} \frac{\partial \overline{q_c}}{\partial \theta}$$
$$- k^{-1} \frac{\partial^2 \overline{q_c}}{\partial \theta^2} = -\frac{V^4}{\rho_{0c} v_{Ac}^4 |\Delta_c| x_0} \frac{dP_c}{d\theta}, \qquad (26)$$

$$[u_c] = -\frac{V x_0}{\cos^2 \alpha} \mathscr{P} \int_{-\infty}^{\infty} \frac{\partial \overline{q_c}}{\partial \theta} \, d\sigma, \qquad (27)$$

where

$$\overline{q_c} = \frac{q_c}{kx_0}, \quad \zeta = \frac{kx_0 D_d^2 \Psi}{R^4}, \quad D_d^2 = \epsilon R^4 = R^2 N^2. \quad (28)$$

Note that ζ is of the order of ϵR^2, the ratio (Λ/Ψ) is of the order of unity and $\overline{q_c}$ is of the order of ϵ. In what follows we drop the bar notation and for the rest of this section we drop the subscript "c" on the dimensionless variable q.

We proceed by using a regular perturbation method and look for solutions in the form

$$f = \epsilon \sum_{n=1}^{\infty} \zeta^{n-1} f_n, \qquad (29)$$

where f represents any of the quantities P, u and q.

A. First order approximation

In the first order approximation, from Eq. (26), we obtain

$$\sigma \frac{\partial q_1}{\partial \theta} - k^{-1} \frac{\partial^2 q_1}{\partial \theta^2} = -\frac{V^4}{\rho_{0c} v_{Ac}^4 |\Delta| x_0} \frac{dP_{1c}}{d\theta}. \qquad (30)$$

Since the total pressure, P, is continuous throughout the dissipative layer *and* is periodical with respect to θ, we look for a solution in the form $g_1 = \Re(\hat{g}_1 e^{ik\theta})$, where g_1 represents P_1, u_1 and q_1 and $\Re$ indicates the real part of a quantity.

In Region I the solutions for the pressure and velocity exactly recover the results found in linear theory, i.e.

$$\hat{P}_1 = p_e e^{ik\kappa_e x} + A_1 e^{-ik\kappa_e x}, \qquad (31)$$

$$\hat{u}_1 = \frac{\kappa_e V \left(p_e e^{ik\kappa_e x} - A_1 e^{-ik\kappa_e x} \right)}{\rho_e \left(V^2 - v_{Ae}^2 \cos^2 \alpha \right)}, \qquad (32)$$

where A_1 (and subsequent values of A_i) is the amplitude of the outgoing wave. The first terms of the right-hand

side of $\hat{P}_1$ and $\hat{u}_1$ represent the incoming wave, while the second terms are the outgoing (reflected) wave. The continuity of the total pressure perturbation at $x = 0$ and $x = x_0$ in combination with Eq. (23) yields $\hat{P}_1$, in Region II, as

$$\hat{P}_1 = p_e + A_1 + (kx_0)\hat{h}_1, \tag{33}$$

where $\hat{h}_n = \hat{h}_n(x) = \hat{P}'_n(x) - \hat{P}'_n(0), \quad n \geq 1$. The solution in Region III is obtained by using Eqs. (5), (24) and (33) with the continuity conditions at $x = x_0$. The solution takes the form

$$\hat{P}_1 = \left\{ p_e + A_1 + (kx_0)\hat{h}_1 \right\} e^{-k\kappa_i(x-x_0)}, \tag{34}$$

$$\hat{u}_1 = \frac{i\kappa_i V \left\{ p_e + A_1 + (kx_0)\hat{h}_1 \right\}}{\rho_i \left(V^2 - v_{Ai}^2 \cos^2 \alpha \right)} e^{-k\kappa_i(x-x_0)}. \tag{35}$$

Utilizing the fact that $\hat{u}_1$ is continuous at $x = 0$ and $x = x_0$, and employing Eqs. (5) and (33) we find that the jump in the normal component of velocity across the dissipative layer is

$$[\hat{u}_1] = \frac{i\kappa_i V (p_e + A_1)}{\rho_i \left(V^2 - v_{Ai}^2 \cos^2 \alpha \right)} - \frac{\kappa_e V (p_e - A_1)}{\rho_e \left(V^2 - v_{Ae}^2 \cos^2 \alpha \right)}$$
$$- ikV (p_e + A_1) \mathscr{P} \int_0^{x_0} F^{-1}(x) \, dx$$
$$- ikV(kx_0) \mathscr{P} \int_0^{x_0} \frac{\hat{h}_1(x)}{F(x)} \, dx, \tag{36}$$

where the expression of $F(x)$ is given by Eq. (6).

Solving Eq. (30) reveals $\hat{q}_1$ to be

$$\hat{q}_1 = -\frac{V^4 (p_e + A_1) \left\{ 1 + \mathscr{O}(kx_0) \right\}}{\rho_{0c} v_{Ac}^2 |\Delta| x_0 (\sigma - i)}. \tag{37}$$

Substitution of this result into Eq. (27) leads to another definition of the jump in the normal component of velocity across the dissipative layer, namely,

$$[\hat{u}_1] = \frac{-\pi k V^5 (p_e + A_1) \left\{ 1 + \mathscr{O}(kx_0) \right\}}{\rho_{0c} v_{Ac}^4 |\Delta| \cos^2 \alpha}. \tag{38}$$

Comparing Eqs. (36) and (38) we obtain that

$$A_1 = -p_e \frac{\tau - \mu + i\upsilon}{\tau + \mu + i\upsilon} + \mathscr{O}(k^2 x_0^2), \tag{39}$$

where

$$\tau = \frac{\pi k V^5}{\rho_{0c} v_{Ac}^4 |\Delta| \cos^2 \alpha}, \quad \mu = \frac{\kappa_e V}{\rho_e \left(V^2 - v_{Ae}^2 \cos^2 \alpha \right)}$$
$$\upsilon = \frac{\kappa_i V}{\rho_i \left(V^2 - v_{Ai}^2 \cos^2 \alpha \right)} - kV \mathscr{P} \int_0^{x_0} F^{-1}(x) \, dx. \tag{40}$$

When deriving Eq. (39) we have employed the estimate that $k \mathscr{P} \int_0^{x_0} \hat{h}_n(x)/F(x) \, dx = \mathscr{O}(kx_0)$. The quantity A_1 is a complex value. This means that the outgoing (reflected) wave has a phase alteration compared with the

incoming wave. The *true amplitude* of the outgoing wave is given by $\widetilde{A_1} = (A_{1(r)}^2 + A_{1(im)}^2)^{1/2}$ (where the subscripts "r" and "im" mean the real and imaginary parts, respectively). The Fourier analysis allows A_1 to be complex. In general, a complex value of A_n means the true amplitude of the outgoing harmonic is defined as above and a phase of the outgoing wave is shifted by $\tan^{-1}(A_{n(im)}^2/A_{n(r)}^2)$. This definition of A_n applies to all subsequent orders of approximation.

In Ruderman *et al.* [17] and Ballai *et al.* [18] a similar procedure was carried out. Our results are similar with theirs if we consider $B_e = 0$ and $\alpha = 0$. This conclusion is not surprising because the first order approximation with respect to the nonlinearity parameter coincides with linear theory. In addition, dispersion due to the Hall effect at the slow resonance does not alter linear theory either since dispersion effects appear as a nonlinear term in the governing equation.

B. Second order approximation

Nonlinear effects start to be important from the second order approximation onwards, but they are always due to the nonlinear combination of lower order harmonics. In this order of approximation Eq. (26) is reduced to

$$\sigma \frac{\partial q_2}{\partial \theta} - k^{-1} \frac{\partial^2 q_2}{\partial \theta^2} = -\frac{V^4}{\rho_{0c} v_{Ac}^4 |\Delta| x_0} \frac{dP_{2c}}{d\theta}$$
$$- q_1 \frac{\partial q_1}{\partial \theta} + \frac{\partial q_1}{\partial \sigma} \frac{\partial q_1}{\partial \theta}. \tag{41}$$

Taking advantage of the form of the first order approximation terms enables us to rewrite the second term on the right-hand side of this equation as

$$q_1 \frac{\partial q_1}{\partial \theta} = \Re \left(\frac{ik}{2} \hat{q}_1^2 e^{2ik\theta} \right). \tag{42}$$

Since the nonlinear terms are proportional to $\Re \left(e^{2ik\theta} \right)$ it is appropriate to seek a solution of the form $g_2 = \Re \left(\hat{g}_2 e^{2ik\theta} \right)$, where g_2 represents P_2, u_2 and q_2.

Using the same techniques as in the first order approximation, it is straightforward to find the jump in the normal component of velocity in Region II

$$[\hat{u}_2] = \frac{i\kappa_i V A_2}{\rho_i \left(V^2 - v_{Ai}^2 \cos^2 \alpha \right)} + \frac{\kappa_e V A_2}{\rho_e \left(V^2 - v_{Ae}^2 \cos^2 \alpha \right)}$$
$$- 2ikV A_2 \mathscr{P} \int_0^{x_0} F^{-1}(x) \, dx$$
$$- 2ikV(kx_0) \mathscr{P} \int_0^{x_0} \frac{\hat{h}_2(x)}{F(x)} \, dx. \tag{43}$$

Using Eqs. (37) and (42) we can solve Eq. (41) to ob-

tain

$$\hat{q}_2 = -\frac{1}{\sigma - 2i}\left[\frac{V^4 A_2}{\rho_{0c} v_{Ac}^4 |\Delta| x_0} + \frac{V^8 (p_e + A_1)^2 (1 + 4\Omega_2)}{4\rho_{0c}^2 v_{Ac}^8 |\Delta|^2 x_0^2 (\sigma - i)^2}\right], \quad (44)$$

where $\Omega_2 = 1/(\sigma - i)$ is the additional factor due to the nonlinear dispersion (as are all subsequent values of Ω_i, $i > 2$). We substitute the expression for $\hat{q}_2$ into Eq. (27) to find

$$[\hat{u}_2] = -\frac{2\pi k V^5 A_2}{\rho_{0c} v_{Ac}^4 |\Delta| \cos^2 \alpha}, \quad (45)$$

where the terms of the order of $k^2 x_0^2$ are not indicated. To calculate A_2 we compare the jump in the normal component of velocity across the dissipative layer defined by Eqs. (43) and (45). This leads to $A_2 = \mathscr{O}(k^2 x_0^2)$. This result implies that all quantities in the second order approximation are zero outside the dissipative layer up to an accuracy of $\mathscr{O}(kx_0)$. With this restriction the outgoing wave remains monochromatic in the second order approximation. This result coincides with the results of Ruderman *et al.*(author?) [17], Ballai *et al.*(author?) [18], Erdélyi *et al.*(author?) [20] and Ruderman(author?) [27] (this is especially surprising because in this paper nonlinearity is strong).

C. Third order approximation

The third order approximation with respect to ζ is governed by

$$\sigma\frac{\partial q_3}{\partial \theta} - k^{-1}\frac{\partial^2 q_3}{\partial \theta^2} = -\frac{V^4}{\rho_{0c} v_{Ac}^4 |\Delta| x_0}\frac{dP_{3c}}{d\theta} - \frac{\partial (q_1 q_2)}{\partial \theta} + \frac{\partial q_1}{\partial \sigma}\frac{\partial q_2}{\partial \theta} + \frac{\partial q_2}{\partial \sigma}\frac{\partial q_1}{\partial \theta}. \quad (46)$$

Taking into account the form of the solutions in the previous two orders of approximation we can rewrite the second term on the right-hand side of Eq. (46) as

$$\frac{\partial (q_1 q_2)}{\partial \theta} = \frac{k}{2}\Re\left(3i\hat{q}_1\hat{q}_2 e^{3ik\theta} + i\hat{q}_1^*\hat{q}_2 e^{ik\theta}\right), \quad (47)$$

where $q_n = \Re\left(\hat{q}_n e^{ink\theta} + \hat{q}_n^* e^{-ink\theta}\right)$ and the asterisk denotes a complex conjugate. This result inspires us to seek solutions in the third order approximation in the form $g_3 = \Re\left(\hat{g}_{31} e^{ik\theta} + \hat{g}_{33} e^{3ik\theta}\right)$, where g_3 represents P_3, u_3 and q_3. Considering the length of this paper we only calculate the $\hat{g}_{31}$ quantities, as it can be shown that $A_{33} = \mathscr{O}(k^2 x_0^2)$.

In a similar manner as the first and second order approximations, we find that the jump in the normal com-

ponent of velocity across the slow dissipative layer to be

$$[\hat{u}_{31}] = \frac{i\kappa_i V A_{31}}{\rho_i (V^2 - v_{Ai}^2 \cos^2 \alpha)} + \frac{\kappa_e V A_{31}}{\rho_e (V^2 - v_{Ae}^2 \cos^2 \alpha)}$$
$$- ikV A_{31}\mathscr{P}\int_0^{x_0} F^{-1}(x)\,dx$$
$$- ikV(kx_0)\mathscr{P}\int_0^{x_0}\frac{\hat{h}_{31}(x)}{F(x)}\,dx, \quad (48)$$

To find $\hat{q}_{31}$ we must exploit Eqs. (37), (44) and (47) to solve Eq. (46). The calculation is analogous to the first and second order approximation calculations and we arrive at the solution

$$\hat{q}_{31} = -\frac{V^4 A_{31}}{\rho_{0c} v_{Ac}^4 |\Delta| x_0 (\sigma - i)}$$
$$- \frac{V^{12}(p_e + A_1)|p_e + A_1|^2 (1 + 2\Omega_{31})}{8\rho_{0c}^3 v_{Ac}^{12}|\Delta|^3 x_0^3 (\sigma - i)^2 (\sigma - 2i)(\sigma^2 + 1)}, \quad (49)$$

where Ω_{31} and is given by

$$\Omega_{31} = \Omega_2\frac{\sigma^3 - (8 + 7i)\sigma^2 - (11 + 12i)\sigma - (44 - 5i)}{(\sigma - 2i)(\sigma^2 + 1)},$$

We substitute this expression for $\hat{q}_{31}$ to find a second definition for the jump in the normal component of velocity across the slow dissipative layer (up to an accuracy of kx_0)

$$[\hat{u}_{31}] = -\frac{\pi k V^5 A_{31}}{\rho_{0c} v_{Ac}^4 |\Delta| \cos^2 \alpha}$$
$$+ \frac{\pi k V^{13}(p_e + A_1)|p_e + A_1|^2 (27 - 8i)}{96\rho_{0c}^3 v_{Ac}^{12}|\Delta|^3 x_0^2 \cos^2 \alpha}, \quad (50)$$

Similar to the first two orders of approximation, we can compare Eqs. (48) with (50) to find the coefficients A_{31}

$$A_{31} = \frac{p_e^3 \tau^3 \mu^3 (27 - 8i)\cos^4 \alpha}{12\pi^2 V^2 k^2 x_0^2 (\mu + iv)^2 (\mu^2 + v^2)}, \quad (51)$$

When calculating A_{31} we have used the estimates $\tau = \mathscr{O}(kx_0)$ and $kV\mathscr{P}\int_0^{x_0} F^{-1}(x)\,dx = \mathscr{O}(kx_0)$, and retain only the terms of lowest order with respect to kx_0, as we have assumed that $kx_0 \ll 1$. Equation (51) illustrates that with an accuracy of up to $\mathscr{O}(kx_0)$ the outgoing (reflected) wave remains monochromatic in the third order approximation. Nevertheless, there is a slight alteration to the amplitude of the fundamental harmonic of the outgoing wave from A_1 to $A_1 + \zeta^2 A_{31}$. These results coincide, qualitatively, with the findings by Ruderman *et al.*(author?) [17], Ballai *et al.*(author?) [18] and Erdélyi *et al.*(author?) [20], however, A_{31} is quantitatively larger than that of previous studies and has an imaginary component. This implies that the amplitude of the wave is greater and the phase of the correction is changed when compared with those studies. The expression for A_{31}, Eq. (51), is different to the ones they obtained because of the inclusion of dispersion through the Hall current.

D. Higher order approximations

In the fourth order of approximation the outgoing (reflected) wave becomes non-monochromatic. This means the energy from this order of approximation no longer contribute to the fundamental harmonic, but to a higher one. For full details of the calculation please refer to the Appendix.

Continuing calculations to even higher order approximations it can be shown that the higher order harmonics (third, fourth, etc.) are generated in the outgoing (reflected) fast wave. The pressure perturbation of the outgoing wave can be written as

$$P' = \epsilon \Re \left\{ \sum_{n=1}^{\infty} \overline{A}_n e^{ink(\theta - \kappa_e x)} \right\}. \quad (52)$$

The second harmonic only appears in the outgoing wave in the fourth order approximation, whereas, higher harmonics appear in higher orders of approximation. This implies that the estimate $\overline{A}_n = \mathcal{O}(\zeta^3)$, $n \geq 2$ is valid.

V. SOLUTION INSIDE THE ALFVÉN DISSIPATIVE LAYER

We can find the jump in the normal component of velocity at the Alfvén resonance explicitly, however, in an attempt to follow the procedure in the last section (and to verify the theory), we proceed to use the implicit form of the jump conditions. As the governing equation (15) is linear we only need to calculate one order of approximation.

Although the Alfvén resonant position is at $x = x_a$, compared with $x = x_c$ for the slow resonant position, we can use some of the same formulae as in the previous section. First, we look for a solution in the form of $g_1 = \Re(\hat{g}_1 e^{ik\theta})$. In Region I, we use Eqs. (31) and (32) to represent the pressure and normal component of velocity perturbations, respectively. For Region II, due to the first connection formula, $[P] = 0$, we can write the pressure perturbation as Eq. (33). We also find that Eqs. (34) and (35) can be used to represent the pressure and normal component of velocity perturbations, respectively, in Region III. The fact we can employ the same equations (as in slow resonance) in the three regions leads to one of the definitions of the jump in the normal component of velocity over the Alfvén dissipative layer being defined as Eq. (36). It should come as no surprise that this definition of the jump across the Alfvén dissipative layer coincides with the jump across the slow dissipative layer in the first order approximation. We are using linear theory to obtain both expressions and are not looking inside the, respective, dissipative layers', so the forms should be identical.

To find $\hat{q}_a$, so that we find the other definition of the jump in u_a, requires a different approach to the one utilized in the section before. After Fourier analyzing Eq.

(15), we are left with

$$i\sigma \hat{q}_a - \frac{d^2 \hat{q}_a}{d\sigma_a^2} = \frac{ik \sin \alpha}{\rho_{0a} |\Delta_a|} P_a. \quad (53)$$

To solve Eq. (53) we introduce the Fourier transform with respect to σ:

$$\mathscr{F}[f(\sigma)] = \int_{-\infty}^{\infty} f(\sigma) e^{-i\sigma r} \, d\sigma. \quad (54)$$

Then from Eq. (53) we have

$$\frac{d\mathscr{F}[\hat{q}_a]}{dr} - r^2 \mathscr{F}[\hat{q}_a] = -\frac{2\pi ik \sin \alpha \, (p_e + A)}{\rho_{0a} |\Delta_a|} \delta(r), \quad (55)$$

where $\delta(r)$ is the delta-function. We find that the solution to Eq. (55) that is bounded for $|r| \to \infty$ is

$$\mathscr{F}[\hat{q}_a] = \frac{2i\pi k \sin \alpha (p_e + A)}{\rho_{0a} |\Delta_a|} H(-r) e^{r^3/3}. \quad (56)$$

Here $H(r)$ denotes the Heaviside function. It was shown by Ruderman and Goossens**(author?)** [17] that

$$\mathscr{P} \int_{-\infty}^{\infty} f(\sigma) \, d\sigma = \frac{1}{2} \left(\lim_{r \to +0} \mathscr{F}[f] + \lim_{r \to -0} \mathscr{F}[f] \right). \quad (57)$$

With the aid of Eqs. (17), (56) and (57) we find that

$$[\hat{u}_a] = -\frac{\pi k V (p_e + A) \sin^2 \alpha}{\rho_{0a} |\Delta_a|}. \quad (58)$$

Comparing Eqs. (36) and (58) we derive that

$$A = -p_e \frac{\tau_a - \mu + i\upsilon}{\tau_a + \mu + i\upsilon} + \mathcal{O}(k^2 x_0^2), \quad (59)$$

where $\tau_a = \pi k V \sin^2 \alpha / (\rho_{0a} |\Delta_a|)$, and μ and υ have their forms given by Eq. (40). However, their values are different for the two resonances.

VI. COEFFICIENT OF WAVE ABSORPTION

The coefficient of wave absorption is defined as $\Gamma = (\Pi_{in} - \Pi_{out})/\Pi_{in}$, where Π_{in} and Π_{out} are the normal components of the energy fluxes, averaged over a period, of the incoming and outgoing waves, respectively. It is straightforward to obtain that

$$\Gamma = 1 - \frac{1}{p_e^2} \sum_{n=1}^{\infty} |\overline{A}_n|^2 \approx \Gamma_L + \zeta^2 \Gamma_{ND}, \quad (60)$$

where Γ_L is the linear coefficient of wave absorption and Γ_{ND} is the nonlinear and dispersive correction. Note that Γ_{ND} is multiplied by the small factor ζ^2 which means that this term will provide small corrections to linear results.

Carrying out calculations we find at the slow resonance, in agreement with linear theory, that

$$\Gamma_{\rm L} = \frac{4\tau\mu}{\mu^2 + \upsilon^2} + \mathscr{O}(k^2 x_0^2). \qquad (61)$$

The coefficient $\Gamma_{\rm ND}$ is defined as $\Gamma_{\rm ND} = -(2/p_e^2)\Re\{A_1^* A_{31}\}$, which can be rewritten using Eqs. (39) and (51) as

$$\Gamma_{\rm ND} = -\frac{27 p_e^2 \tau^3 \mu^3 \cos^4\alpha}{6\pi^2 V^2 k^2 x_0^2 (\mu^2 + \upsilon^2)^2} + \mathscr{O}(k^2 x_0^2). \qquad (62)$$

Both $\Gamma_{\rm L}$ and $\Gamma_{\rm ND}$ are of the order of kx_0. This result is qualitatively the same as Ruderman et al.(author?) [17] and Ballai *et al.*(author?) [18] results, however, the nonlinear correction is different. In fact, it is 270% times larger due to the Hall current having a dominant effect around the resonance. Moreover, the dispersion in the slow dissipative layer causes a further reduction in the coefficient of energy absorption, in comparison to the nonlinear regime alone.

At the Alfvén resonance dynamics can be described within the linear framework. Hence, using Eqs. (59) and (60) we obtain that

$$\Gamma_a = \frac{4\tau_a\mu}{(\tau_a + \mu)^2 + \upsilon^2}. \qquad (63)$$

Numerical verification of these results requires much more work than would first appear, and as such our next paper is to concentrates on this and further numerical analysis.

VII. CONCLUSIONS

In the present paper we have investigated (i) the effect of nonlinearity and dispersion on the interaction of fast magnetoacoustic (FMA) waves with a one-dimensional inhomogeneous magnetized plasma with strongly anisotropic transport processes in the slow dissipative layer (ii) the interaction of FMA waves with Alfvén dissipative layers. The study is based on the nonlinear theory of slow resonance in strongly anisotropic and dispersive plasmas developed by Clack and Ballai(author?) [21] and the theory of Alfvén resonance developed by Clack *et al.*(author?) [22].

We have assumed that (i) the thickness of the slab containing the inhomogeneous plasma (Region II) is small in comparison with the wavelength of the incoming fast wave (i.e. $kx_0 \ll 1$); and (ii) the nonlinearity in the dissipative layer is weak - the nonlinear term in the equation describing the plasma motion in the slow dissipative layer can be considered as a perturbation and nonlinearity gives only a correction to the linear results.

Applying a regular perturbation method, analytical solutions in the slow dissipative layer are obtained in the form of power expansions with respect to the nonlinearity parameter ζ. Our main results are the following: Nonlinearity in the dissipative layer generates higher harmonic contributions to the outgoing (reflected) wave in addition to the fundamental one. The dispersion does not alter this, however, the phase and amplitude of some of the higher harmonics are different from the standard nonlinear counterpart (see discussions before). Dispersion in the dissipative layer further decreases the coefficient of the wave energy absorption. The factor of alteration to the *nonlinear* correction of the coefficient of wave absorption due to dispersion is 270%. Remember, however, that the nonlinear correction is multiplied by the small parameter ζ^2, so the effect to the overall coefficient of wave energy absorption is still small.

Calculating the coefficient of wave absorption at the Alfvén resonance confirms the linear theory of the past and verifies the approach taken to be correct. As our physical set-up of the problem (for the Alfvén resonance) matches the typical conditions found in the solar corona, these results can be applied to it. The equilibrium state of the problem (for the slow resonance) can match conditions found in the upper chromosphere, where FMA waves may interact with slow dissipative layers, and if the reduction in the coefficient of wave energy absorption persists to the strong nonlinear case (as with the long wavelength approximation found by Ruderman [27]) dispersion may have further implications to the resonant absorption in the solar atmosphere.

In a forthcoming paper, we shall theoretically and numerically investigate coupled resonances, which builds from the work in the present paper to obtain a more realistic model for a solar physical description. In the same paper we will numerically analyze the absorption of fast waves at the Alfvén resonance as a possible scenario of the interaction of global fast waves (modelling EIT waves) and coronal loops.

APPENDIX: DETAILS FOR CALCULATION OF FOURTH ORDER APPROXIMATION

In the fourth order approximation Eq. (26) gives

$$\sigma \frac{\partial q_4}{\partial \theta} - k^{-1}\frac{\partial^2 q_4}{\partial \theta^2} = -\frac{V^4}{\rho_{0c}v_{Ac}^4|\Delta|x_0}\frac{dP_{4c}}{d\theta} + \frac{\partial q_2}{\partial \sigma}\frac{\partial q_2}{\partial \theta}$$
$$- \frac{\partial}{\partial \theta}\left(q_1 q_3 + \frac{1}{2}q_2^2\right) + \frac{\partial q_1}{\partial \sigma}\frac{\partial q_3}{\partial \theta} + \frac{\partial q_3}{\partial \sigma}\frac{\partial q_1}{\partial \theta}. \quad (A1)$$

We can rewrite the third term on the right-hand side of Eq. (A1) using our knowledge about the first three orders of approximation, so

$$\frac{\partial}{\partial \theta}\left(q_1 q_3 + \frac{1}{2}q_2^2\right) = k\Re\left\{i\left(\hat{q}_1\hat{q}_{31} + \hat{q}_1^*\hat{q}_{33}\right)e^{2ik\theta}\right.$$
$$\left. + i\left(2\hat{q}_1\hat{q}_{33} + \hat{q}_2^2\right)e^{4ik\theta}\right\}. \quad (A2)$$

This equation contains terms proportional to $e^{2ik\theta}$ and $e^{4ik\theta}$, so we can anticipate the solution to Eq. (A1) to be of the form $g_4 = \Re\left(\hat{g}_{42}e^{2ik\theta} + \hat{g}_{44}e^{4ik\theta}\right)$, where g_4 represents P_4, u_4 and q_4. We calculate the fourth order approximation to demonstrate that nonlinearity and dispersion in the dissipative layer generates overtones in the outgoing (reflected) fast wave. For brevity, we shall only derive the terms proportional to $e^{2ik\theta}$, but for completeness we note that it can be shown that terms proportional to $e^{4ik\theta}$ are only present in the solution inside the dissipative layer.

Using the continuity conditions at $x = 0$ and $x = x_0$ we find the jump in the normal component of velocity across the dissipative layer to be

$$[\hat{u}_{42}] = \frac{i\kappa_i V A_{42}}{\rho_i\left(V^2 - v_{Ai}^2\cos^2\alpha\right)} + \frac{\kappa_e V A_{42}}{\rho_e\left(V^2 - v_{Ae}^2\cos^2\alpha\right)}$$
$$- 2ikV A_2 \mathscr{P}\int_0^{x_0}F^{-1}(x)\,dx$$
$$- 2ikV(kx_0)\mathscr{P}\int_0^{x_0}\frac{\hat{h}_2(x)}{F(x)}\,dx. \quad (A4)$$

It is straightforward, but longwinded, to derive $\hat{q}_{42}$, so we skip all intermediate steps and give the result

$$\hat{q}_{42} = \frac{-1}{\sigma - 2i}\left\{\frac{V^4 A_{42}}{\rho_{0c}v_{Ac}^4|\Delta|x_0} + \frac{V^8\left(p_e + A_1\right)A_{31}\left(1 + \Omega_2\right)}{2\rho_{0c}v_{Ac}^8|\Delta|^2x_0^2(\sigma - i)^2}\right.$$
$$\left. + \frac{V^{16}\left(p_e + A_1\right)^2|p_e + A_1|^2\left(12 - \Omega_{42}\right)}{96\rho_{0c}^4v_{Ac}^{16}|\Delta|^4x_0^4(\sigma - i)^3(\sigma - 3i)(\sigma^2 + 1)}\right\}, \quad (A5)$$

with $\Omega_{42} = f(\sigma)$, where $f(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$, is the contribution due to the Hall effect. As it is not essential for forthcoming calculations, its exact form is not given here. The substitution of $\hat{q}_{42}$ into Eq. (27) yields

$$[\hat{u}_{42}] = -\frac{2\pi kV^5 A_{42}}{\rho_{0c}v_{Ac}^4|\Delta|\cos^2\alpha}$$
$$+ 0.082 \times \frac{\pi kV^{17}(p_e + A_1)^2|p_e + A_1|^2}{\rho_{0c}^4v_{Ac}^{16}|\Delta|^4x_0^3\cos^2\alpha}, \quad (A6)$$

Comparing Eqs. (A4) and (A6) we obtain that

$$A_{42} = 1.279 \times \frac{p_e^4\tau^4\mu^4\cos^6\alpha}{\pi^3V^3k^3x_0^3\left(\mu + iv\right)^3\left(\mu^2 + v^2\right)}. \quad (A7)$$

Here we have used the same estimations that were utilized for calculating A_{31} in the third order approximation and retain only the largest order terms with respect to kx_0. It is clear from this result that the outgoing wave becomes non-monochromatic in the fourth order approximation. We can also observe that the second harmonic appears in addition to the fundamental mode.

This result parallels the results obtained by Ruderman *et al.*(**author?**) [17] and Ballai *et al.*(**author?**) [18]. However, Eq. (A7) shows that the *phase* is inverted and the *amplitude* of the second harmonic is approximately 30 times greater than theirs due to the presence of the Hall effect. Remember, though, that this amplitude is multiplied by a very small term, ζ^3, which means the overall correction is very small.

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