Clausal Presentation of Theories in Deduction Modulo

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Abstract. Resolution modulo is an extension of first-order resolution where axioms are replaced by rewrite rules, used to rewrite, or more generally narrow, clauses during the search. In the first version of this method, clauses were rewritten to arbitrary propositions, that needed to be dynamically transformed into clauses. This unpleasant feature can be eliminated when the rewrite system is clausal, i.e. when it transforms clauses to clauses. We show in this paper that how to transform any rewrite system into a clausal one, preserving the existence of cut free proof of any sequent.

Keywords: Resolution; Deduction Modulo; Cut Free Proof; Clause;

1 Motivations

1.1 Resolution modulo

Resolution Modulo [7] is an extension of first-order resolution [1, 2, 6] where axioms are replaced by rewrite rules used to narrow clauses during the search. The paper [7] defined a sequent calculus modulo which is an extension of pure sequent calculus, see Figure 1, and gave a complete proof search method called *Resolution Modulo*. According to this method, for the confluent rewrite systems $\mathcal{R}$, the sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ if and only if the empty clause can be derived from the clauses of $Cl(\Gamma, \neg\Delta)$ with two rules: the usual resolution rule and the narrowing rule that permits to rewrite, or more generally narrow, a clause.

1.2 Main Problem

In Resolution Modulo, rules rewrite clauses to arbitrary propositions, that need to be dynamically transformed into clauses. For instance, the rule $P \longrightarrow (Q \Rightarrow R)$ rewrites the clause P to a non clausal proposition $Q \Rightarrow R$. In the process of

$$\frac{}{A \vdash B} \text{ axiom if } A \longrightarrow_-^* P, B \longrightarrow_+^* P \text{ and } P \text{ atomic}$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma \vdash C, \Delta}{\Gamma \vdash \Delta} \text{ cut if } A \longrightarrow_-^* B, A \longrightarrow_+^* C$$

$$\frac{\Gamma, B, C \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr-left if } A \longrightarrow_-^* B, A \longrightarrow_-^* C$$

$$\frac{\Gamma \vdash B, C, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-right if } A \longrightarrow_+^* B, A \longrightarrow_+^* C$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ weak-left}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ weak-right}$$

$$\frac{}{\Gamma \vdash A, \Delta} \top\text{-right if } A \longrightarrow_+^* \top$$

$$\frac{}{\Gamma, A \vdash \Delta} \bot\text{-left if } A \longrightarrow_-^* \bot$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma, A \vdash \Delta} \neg\text{-left if } A \longrightarrow_-^* \neg B$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma \vdash A, \Delta} \neg\text{-right if } A \longrightarrow_+^* \neg B$$

$$\frac{\Gamma, B, C \vdash \Delta}{\Gamma, A \vdash \Delta} \wedge\text{-left if } A \longrightarrow_-^* (B \wedge C)$$

$$\frac{\Gamma \vdash B, \Delta \quad \Gamma \vdash C, \Delta}{\Gamma \vdash A, \Delta} \wedge\text{-right if } A \longrightarrow_+^* (B \wedge C)$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \vee\text{-left if } A \longrightarrow_-^* (B \vee C)$$

$$\frac{\Gamma \vdash B, C, \Delta}{\Gamma \vdash A, \Delta} \vee\text{-right if } A \longrightarrow_+^* (B \vee C)$$

$$\frac{\Gamma \vdash B, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \Rightarrow\text{-left if } A \longrightarrow_-^* (B \Rightarrow C)$$

$$\frac{\Gamma, B \vdash C, \Delta}{\Gamma \vdash A, \Delta} \Rightarrow\text{-right if } A \longrightarrow_+^* (B \Rightarrow C)$$

$$\frac{\Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \langle x, B, t\rangle \ \forall\text{-left if } A \longrightarrow_-^* \forall x\ B,\ (t/x)B \longrightarrow_-^* C$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta} \langle x, B\rangle \ \forall\text{-right if } A \longrightarrow_+^* \forall x\ B,\ x \notin FV(\Gamma\Delta)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta} \langle x, B\rangle \ \exists\text{-left if } A \longrightarrow_-^* \exists x\ B,\ x \notin FV(\Gamma\Delta)$$

$$\frac{\Gamma \vdash C, \Delta}{\Gamma \vdash A, \Delta} \langle x, B, t\rangle \ \exists\text{-right if } A \longrightarrow_+^* \exists x\ B,\ (t/x)B \longrightarrow_+^* C$$

Fig. 1. Polarized sequent calculus modulo

searching for a proof of the sequent $P, Q, \neg R \vdash$, we attempt to derive the empty clause from the clauses $\{P\}$, $\{Q\}$, $\{\neg R\}$. In this case, we first derive $\{Q \Rightarrow R\}$ from $\{P\}$ and then we need to transform $\{Q \Rightarrow R\}$ into a clause $\{\neg Q, R\}$, see Figure 2. In another example, attempting to derive the empty clause from $\{P\}$, $\{\neg Q(x)\}$ with the rewrite rule $P \longrightarrow \exists x Q(x)$, we first derive $\{\exists x Q(x)\}$ from $\{P\}$ and then $\{\exists x Q(x)\}$ needs to be transformed into a clause $\{Q(c)\}$ with a new Skolem symbol c. The problem we adreess in this paper is to avoid this dynamic transformation.

$$\dfrac{\dfrac{\dfrac{}{R, Q \vdash R}\text{axiom}}{R, Q, \neg R \vdash}\neg\text{-left} \qquad \dfrac{}{Q, \neg R \vdash Q}\text{axiom}}{P, Q, \neg R \vdash}\Rightarrow\text{-left}$$

$$\begin{tikzcd}
\{\neg Q \vee R\}, \{Q\}, \{\neg R\} \arrow[r, rightsquigarrow] & \{\neg Q, R\}, \{Q\}, \{\neg R\} \arrow[d, rightsquigarrow] \\
\{Q \Rightarrow R\}, \{Q\}, \{\neg R\} \arrow[u, rightsquigarrow] & \{R\}, \{\neg R\} \arrow[d, rightsquigarrow] \\
\{P\}, \{Q\}, \{\neg R\} \arrow[u, rightsquigarrow] & \square
\end{tikzcd}$$

Fig. 2. Example of Resolution Modulo

1.3 Solution

This unpleasant dynamical transformation can be eliminated when the rewrite system is clausal, i.e. when it transforms clauses to clauses. This is the idea of Polarized Resolution Modulo [5]. See Figure 3 for a presentation of Polarized Resolution Modulo where unification problems are kept as constraints. The sequent $\Gamma \vdash \Delta$ has a cut free proof if and only if a clause $\square[\mathcal{C}]$ with a $\mathcal{E}$-unifiable constraints can be derived from $Cl(\Gamma, \neg\Delta)$. See Figure 4 for an example of Polarized Resolution Module applying on the clausal rewrite system with one rule $P \longrightarrow_{-} \neg Q \vee R$. See [3] for an efficient implementation of Polarized Resolution Modulo. Thus the problem can be reformulated as that of translating a rewrite system into a clausal one.

$$\dfrac{(U, P_1, \ldots, P_n)[\mathcal{C}_1] \quad (V, \neg Q_1, \ldots, \neg Q_p)[\mathcal{C}_2]}{(U \cup V)[\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{P_1 = \ldots = P_n = Q_1 = \ldots = Q_p\}]}\ \textbf{Resolution}$$

$$\dfrac{(U, P)[\mathcal{C}]}{(U \cup |V|)[\mathcal{C} \cup \{P = Q\}]}\ \text{if } Q \longrightarrow_{-} V \text{ is one rule of } \mathcal{R}\ \textbf{Extended Narrowing}$$

$$\dfrac{(U, \neg P)[\mathcal{C}]}{(U \cup |V|)[\mathcal{C} \cup \{P = Q\}]}\ \text{if } Q \longrightarrow_{+} \neg V \text{ is one rule of } \mathcal{R}\ \textbf{Extended Narrowing}$$

Fig. 3. Polarized resolution modulo

$$\dfrac{\dfrac{\dfrac{}{R, Q \vdash R}\ \text{axiom}}{R, Q, \neg R \vdash}\ \neg\text{-left} \qquad \dfrac{}{Q, \neg R \vdash Q}\ \text{axiom}}{P, Q, \neg R \vdash}\ \Rightarrow\text{-left}$$

$$\begin{tikzcd}
\{\neg Q \vee R\}, \{Q\}, \{\neg R\} \arrow[r, squiggly] & \{\neg Q, R\}, \{Q\}, \{\neg R\} \arrow[d, squiggly] \\
 & \{R\}, \{\neg R\} \arrow[d, squiggly] \\
\{P\}, \{Q\}, \{\neg R\} \arrow[uu, squiggly] & \Box
\end{tikzcd}$$

Fig. 4. Example of Polarized Resolution Modulo

A proposition is a *literal* if it is either atomic or the negation of an atomic proposition. A *clause* is a set of literals. A proposition is *clausal* if it is $\bot$ or of the form $L_1 \vee \ldots \vee L_n$ where $L_1, \ldots, L_n$ are literals. If $A = L_1 \vee \ldots \vee L_n$ is a clausal proposition, we write $|A|$ for the clause $\{L_1, \ldots, L_n\}$.

A *polarized rewrite system* is a triple $\mathcal{R} = \langle \mathcal{E}, \mathcal{R}_-, \mathcal{R}_+ \rangle$ where $\mathcal{E}$ is a set of equations between terms, $\mathcal{R}_-$ and $\mathcal{R}_+$ are sets of rewrite rules whose left hand sides are atomic propositions and right hand sides are arbitrary propositions. The rules of $\mathcal{R}_-$ are called *negative* rules and those of $\mathcal{R}_+$ are called *positive* rules. A rewrite system is *clausal* if negative rules rewrite atomic propositions to clausal propositions and positive rules atomic propositions to negations of clausal propositions.

Let $\mathcal{R} = \langle \mathcal{E}, \mathcal{R}_-, \mathcal{R}_+ \rangle$ be a polarized rewrite system. We define the equivalence relation $=_\mathcal{E}$ as the congruence on terms generated by the equations of $\mathcal{E}$. We then define the one step negative and positive rewriting relations $\longrightarrow_-$ and $\longrightarrow_+$ as follows.

- If $t_i =_\mathcal{E} u$ then both $P(t_1, \ldots, t_i, \ldots, t_n) \longrightarrow_- P(t_1, \ldots, u, \ldots, t_n)$ and $P(t_1, \ldots, t_i, \ldots, t_n) \longrightarrow_+ P(t_1, \ldots, u, \ldots, t_n)$.
- If $P \longrightarrow A$ is a rule of $\mathcal{R}_s$ and σ is a substitution then $\sigma P \longrightarrow_s \sigma A$, where s is either $-$ or $+$.
- If $A \longrightarrow_{\bar{s}} A'$ then $\neg A \longrightarrow_s \neg A'$, where $\bar{\cdot}$ swaps $-$ and $+$.
- If ($A \longrightarrow_s A'$ and $B = B'$) or ($A = A'$ and $B \longrightarrow_s B'$), then $A \wedge B \longrightarrow_s A' \wedge B'$ and $A \vee B \longrightarrow_s A' \vee B'$.
- If ($A \longrightarrow_{\bar{s}} A'$ and $B = B'$) or ($A = A'$ and $B \longrightarrow_s B'$), then $A \Rightarrow B \longrightarrow_s A' \Rightarrow B'$.
- If $A \longrightarrow_s A'$ then $\forall x\ A \longrightarrow_s \forall x\ A'$ and $\exists x\ A \longrightarrow_s \exists x\ A'$.

We define the sequent one step term rewriting relation $\longrightarrow$ as follows.

- If $A \longrightarrow_- A'$ then $(\Gamma, A \vdash \Delta) \longrightarrow (\Gamma, A' \vdash \Delta)$.
- If $A \longrightarrow_+ A'$ then $(\Gamma \vdash A, \Delta) \longrightarrow (\Gamma \vdash A', \Delta)$.

In Polarized Resolution Modulo, for the clausal rewrite system $\mathcal{R}$, the sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ if and only if the empty clause with an $\mathcal{E}$-unifiable constraint. can be derived from the clauses of $Cl(\Gamma, \neg\Delta)$. In Polarized Resolution Modulo, rewrite systems distinguish rules as positive and negative, with negative rules rewriting atomic propositions to clausal propositions and positive rules rewriting atomic propositions to negation of clausal propositions. This is needed because the extended narrowing rule with $P \longrightarrow \neg Q \vee R$ for example transforms the clause $\{P\}$ to the clause $\{\neg Q, R\}$. But when we have for example, the clauses $\{\neg P\}$ and $\{Q\}$, we can not use the same rewrite rule, that would transform $\{\neg P\}$ into $\{\neg(\neg Q \vee R)\}$ which is not a clause. Instead we want to use the positive rule $P \longrightarrow \neg\neg Q$. Using this rewrite rule, the extended narrowing of Figure 3 transforms the clause $\{\neg P\}$ into the clause $\{\neg Q\}$ and we can conclude with resolution rule.

In this paper, we show how to transform any rewrite system into a clausal one, preserving the existence of cut free proof of any sequent. So that Polarized Resolution Modulo can be applied to the system directly.

2 Translator

In this section we will show how to translate a polarized rewrite system into a clausal one. We translate negative rules into rules rewriting atomic propositions to clausal propositions. And translate positive rules into rules rewriting atomic propositions to negations of clausal propositions.

We first add a symbol $\bot'$ into the language which is just another $\bot$ with a mark. This symbol is used to prove the termination of the translator only.

Definition 1. Step 1 *Translate the rewrite rule $P \longrightarrow_- A$, $P \longrightarrow_+ A$ into $P \longrightarrow_- A \vee \bot'$ and $P \longrightarrow_+ \neg(\neg A \vee \bot')$ respectively.*

Step 2 *Translate any source rule in Table 1 into its target rule. Keep on recurring in Step 2 until termination.*

Step 3 *Translate the rewrite rule $P \longrightarrow_- \bot' \vee B$ and $P \longrightarrow_+ \neg(\bot' \vee B)$ into $P \longrightarrow_- B$, $P \longrightarrow_+ \neg B$ respectively.*

If the rewrite rule r is translated into r′, we denoted this by $r \rhd r'$. The polarized rewrite system $\mathcal{R}$ is translated into $\mathcal{R}'$ in one step (denoted by $\mathcal{R} \blacktriangleright \mathcal{R}'$) if $\mathcal{R}'$ is obtained from $\mathcal{R}$ by translating one rule of $\mathcal{R}$.

Proposition 1. *For any Polarized rewrite system $\mathcal{R}_0$, the translator will finally stop at $\mathcal{R}_f$ and $\mathcal{R}_f$ is clausal. We said $\mathcal{R}_f$ is the* final polarized rewrite system *of $\mathcal{R}_0$.*

$$\mathcal{R}_0 \blacktriangleright \mathcal{R}_1 \cdots \blacktriangleright \mathcal{R}_f$$

For example, given the polarized rewrite system containing only two rules $P \longrightarrow_- (Q \Rightarrow R)$ and $P \longrightarrow_+ (Q \Rightarrow R)$. Here we start Step 2 with $P \longrightarrow_- (Q \Rightarrow R) \vee \bot'$ and $P \longrightarrow_+ (Q \Rightarrow R) \vee \bot'$, see Figure 5. Finally, we get the rules $P \longrightarrow_- \neg Q \vee R$, $P \longrightarrow_+ \neg Q$, $P \longrightarrow_+ \neg(\neg R)$.

Notice that when applying the translator to HOL we get the system $HOL^{\pm}$ of [4].

Number	Source rule	Target rule
Case 1	$P \longrightarrow_{-} \bot \vee R$	$P \longrightarrow_{-} R$
Case 2	$P \longrightarrow_{-} Q \vee R$ (Q is atomic)	$P \longrightarrow_{-} R \vee Q$
Case 3	$P \longrightarrow_{-} (Q_1 \wedge Q_2) \vee R$	$P \longrightarrow_{-} (Q_1 \vee R)$ $P \longrightarrow_{-} (Q_2 \vee R)$
Case 4	$P \longrightarrow_{-} (Q_1 \vee Q_2) \vee R$	$P \longrightarrow_{-} Q_1 \vee (Q_2 \vee R)$
Case 5	$P \longrightarrow_{-} (Q_1 \Rightarrow Q_2) \vee R$	$P \longrightarrow_{-} (\neg Q_1 \vee (Q_2 \vee R)$
Case 6	$P \longrightarrow_{-} \forall x Q \vee R$	$P \longrightarrow_{-} (Q \vee R)$
Case 7	$P \longrightarrow_{-} \exists x Q \vee R$	$P \longrightarrow_{-} ((f(l)/x)Q \vee R)$ l free variables of $P, \exists x Q, R$
Case 8	$P \longrightarrow_{-} \neg\bot \vee R$	drop this rule
Case 9	$P \longrightarrow_{-} \neg Q \vee R$ (Q is atomic)	$P \longrightarrow_{-} R \vee \neg Q$
Case 10	$P \longrightarrow_{-} \neg(\neg Q) \vee R$	$P \longrightarrow_{-} Q \vee R$
Case 11	$P \longrightarrow_{-} \neg(Q_1 \wedge Q_2) \vee R$	$P \longrightarrow_{-} \neg Q_1 \vee (\neg Q_2 \vee R)$
Case 12	$P \longrightarrow_{-} \neg(Q_1 \vee Q_2) \vee R$	$P \longrightarrow_{-} \neg Q_1 \vee R$ $P \longrightarrow_{-} \neg Q_2 \vee R$
Case 13	$P \longrightarrow_{-} \neg(Q_1 \Rightarrow Q_2) \vee R$	$P \longrightarrow_{-} Q_1 \vee R$ $P \longrightarrow_{-} \neg Q_2 \vee R$
Case 14	$P \longrightarrow_{-} \neg(\forall x Q) \vee R$	$P \longrightarrow_{-} ((f(l)/x)\neg Q) \vee R$ l free variables of $P, \forall x Q, R$
Case 15	$P \longrightarrow_{-} \neg(\exists x Q) \vee R$	$P \longrightarrow_{-} (\neg Q) \vee R$
Case 16	$P \longrightarrow_{+} \neg(\bot \vee R)$	$P \longrightarrow_{+} \neg R$
Case 17	$P \longrightarrow_{+} \neg(Q \vee R)$ (Q is atomic)	$P \longrightarrow_{+} \neg(R \vee Q)$
Case 18	$P \longrightarrow_{+} \neg((Q_1 \wedge Q_2) \vee R)$	$P \longrightarrow_{+} \neg(Q_1 \vee R)$ $P \longrightarrow_{+} \neg(Q_2 \vee R)$
Case 19	$P \longrightarrow_{+} \neg((Q_1 \vee Q_2) \vee R)$	$P \longrightarrow_{+} \neg(Q_1 \vee (Q_2 \vee R))$
Case 20	$P \longrightarrow_{+} \neg((Q_1 \Rightarrow Q_2) \vee R)$	$P \longrightarrow_{+} \neg(\neg Q_1 \vee (Q_2 \vee R))$
Case 21	$P \longrightarrow_{+} \neg(\forall x Q \vee R)$	$P \longrightarrow_{+} \neg(Q \vee R)$
Case 22	$P \longrightarrow_{+} \neg(\exists x Q \vee R)$	$P \longrightarrow_{+} \neg((f(l)/x)Q \vee R)$ l free variables of $P, \exists x Q, R$
Case 23	$P \longrightarrow_{+} \neg(\neg\bot \vee R)$	drop this rule
Case 24	$P \longrightarrow_{+} \neg((\neg Q) \vee R)$ (Q is atomic)	$P \longrightarrow_{+} \neg(R \vee (\neg Q))$
Case 25	$P \longrightarrow_{+} \neg(\neg(\neg Q) \vee R)$	$P \longrightarrow_{+} \neg(Q \vee R)$
Case 26	$P \longrightarrow_{+} \neg(\neg(Q_1 \wedge Q_2) \vee R)$	$P \longrightarrow_{+} \neg(\neg Q_1 \vee (\neg Q_2 \vee R))$
Case 27	$P \longrightarrow_{+} \neg(\neg(Q_1 \vee Q_2) \vee R)$	$P \longrightarrow_{+} \neg(\neg Q_1 \vee R)$ $P \longrightarrow_{+} \neg(\neg Q_2 \vee R)$
Case 28	$P \longrightarrow_{+} \neg(\neg(Q_1 \Rightarrow Q_2) \vee R)$	$P \longrightarrow_{+} \neg(Q_1 \vee R)$ $P \longrightarrow_{+} \neg(\neg Q_2 \vee R)$
Case 29	$P \longrightarrow_{+} \neg(\neg(\forall x Q) \vee R)$	$P \longrightarrow_{+} \neg(((f(l)/x)\neg Q) \vee R)$ l free variables of $P, \forall x Q, R$
Case 30	$P \longrightarrow_{+} \neg(\neg(\exists x Q) \vee R)$	$P \longrightarrow_{+} \neg(\neg Q \vee R)$

Table 1. Translator for the negative and positive rules

$$\boxed{\begin{array}{l} P \longrightarrow_- (Q \Rightarrow R) \vee \bot' \\ P \longrightarrow_+ \neg(\neg(Q \Rightarrow R) \vee \bot') \end{array}} \blacktriangleright^* \boxed{\begin{array}{l} P \longrightarrow_- (\neg Q \vee R) \vee \bot' \\ P \longrightarrow_+ \neg(\neg(\neg Q \vee R) \vee \bot') \end{array}} \blacktriangleright^*$$

$$\boxed{\begin{array}{l} P \longrightarrow_- \neg Q \vee (R \vee \bot') \\ P \longrightarrow_+ \neg((\neg\neg Q \wedge \neg R) \vee \bot') \end{array}} \blacktriangleright^* \boxed{\begin{array}{l} P \longrightarrow_- (R \vee \bot') \vee \neg Q \\ P \longrightarrow_+ \neg((\neg\neg Q) \vee \bot') \\ P \longrightarrow_+ \neg(\neg R \vee \bot') \end{array}} \blacktriangleright^*$$

$$\boxed{\begin{array}{l} P \longrightarrow_- \bot' \vee \neg Q \vee R \\ P \longrightarrow_+ \neg(Q \vee \bot') \\ P \longrightarrow_+ \neg(\bot' \vee \neg R) \end{array}} \blacktriangleright^* \boxed{\begin{array}{l} P \longrightarrow_- \bot' \vee \neg Q \vee R \\ P \longrightarrow_+ \neg(\bot' \vee Q) \\ P \longrightarrow_+ \neg(\bot' \vee \neg R) \end{array}}$$

Fig. 5. Translation Example

3 Equivalence

In this section we prove the two sides of the translator are equivalent. Our final goal is to prove that when Dedution modulo the rewite system $\mathcal{R}_0$ has the cut elimination property, $\mathcal{R}_0 \blacktriangleright \mathcal{R}_1 \cdots \blacktriangleright \mathcal{R}_f$ and $\Gamma \vdash \Delta$ is a sequent in the language $\mathcal{L}$ of of the theory $\mathcal{R}_0$,

$$\Gamma \vdash_{\mathcal{R}_0} \Delta \iff Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f} \Box$$

It has been proved in [5] following the lines of [7,8] that $\Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta$ if and only if $Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f} \Box$. So it is sufficient to prove $(\Gamma \vdash_{\mathcal{R}_0} \Delta) \iff (\Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta)$.

3.1 $\mathcal{R}_0 \Rightarrow \mathcal{R}_f$

We first prove $(\Gamma \vdash^{c.f.}_{\mathcal{R}_0} \Delta) \Rightarrow (\Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta)$. It is sufficient to prove that for each step $(\Gamma \vdash^{c.f.}_{\mathcal{R}_n} \Delta) \Rightarrow (\Gamma \vdash^{c.f.}_{\mathcal{R}_{n+1}} \Delta)$.

Proposition 2. *Let $\mathcal{R}_n$, $\mathcal{R}_{n+1}$ be a polarized rewrite system and $\mathcal{R}_n \blacktriangleright \mathcal{R}_{n+1}$. If the sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}_n$ then it has a cut free proof modulo $\mathcal{R}_{n+1}$.*

We prove the property for each case of Table 1. See the long version of the paper for the details.

3.2 $\mathcal{R}_f \Rightarrow \mathcal{R}_0$

In this subsection we will prove $(\Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta) \Rightarrow (\Gamma \vdash^{c.f.}_{\mathcal{R}_0} \Delta)$. The method to prove this is different from the first direction.

As $\mathcal{R}_0$ has the cut elimination property, it is sufficient to prove $(\Gamma \vdash_{\mathcal{R}_{n+1}} \Delta) \Rightarrow (\Gamma \vdash_{\mathcal{R}_n} \Delta)$. If a sequent $\Gamma \vdash \Delta$, has a cut free proof in $\mathcal{R}_f$ it has a proof in $\mathcal{R}_f$ and if we can prove $(\Gamma \vdash_{\mathcal{R}_{n+1}} \Delta) \Rightarrow (\Gamma \vdash_{\mathcal{R}_n} \Delta)$ then it has a proof in $\mathcal{R}_0$. Using the cut elimination theorem for $\mathcal{R}_0$, we get that it has a cut free proof in $\mathcal{R}_0$.

Before proving $(\Gamma \vdash_{\mathcal{R}_{n+1}} \Delta) \Rightarrow (\Gamma \vdash_{\mathcal{R}_n} \Delta)$, we need to transform the polarized rewrite systems to the theories in classical predicate logic with equality. The Definition 2 and Proposition 3 follow the lines of [5].

Definition 2. *Let $\mathcal{L}$ be a language containing an equality predicate in each sort. Let $\mathcal{R}$ be a polarized rewrite system in L. Let $\mathcal{U}_{\mathcal{R}}$ be the set of axioms containing*

- *the axioms of equality for $\mathcal{L}$.*
- *for each equational axiom $t = u$ of $\mathcal{E}$, the universal closure of the proposition $t = u$,*
- *for each rule $P \longrightarrow A$ of $\mathcal{R}_-$, the universal closure of the proposition $P \Rightarrow A$,*
- *for each rule $P \longrightarrow A$ of $\mathcal{R}_+$, the universal closure of the proposition $A \Rightarrow P$.*

Proposition 3. *Let $\mathcal{L}$ be a language and $\mathcal{R}$ be a polarized rewrite system in $\mathcal{L}$. Let L' be the language obtained by adding an equality symbol in each sort of $\mathcal{L}$. Then, a sequent $\Gamma \vdash \Delta$ of $\mathcal{L}$ is provable modulo $\mathcal{R}$ if and only if it is provable in $\mathcal{U}_{\mathcal{R}}$.*

Proposition 4. *Let $\mathcal{R}$, $\mathcal{R}'$ be polarized rewrite systems and $\mathcal{R} \blacktriangleright \mathcal{R}'$. If a sequent in the language $\mathcal{L}$, has a proof in $\mathcal{R}'$, then it has a proof in $\mathcal{R}$.*

Proof. Using Proposition 3, all we need to prove is that the theory $\mathcal{U}_{\mathcal{R}'}$ is a conservative extension of $\mathcal{U}_{\mathcal{R}}$. We can prove it case by case, using Skolem theorem for classical predicate logic with equality for the two cases of eliminating of existential quantifiers.

Theorem 1. *Let $\mathcal{R}_0$ be a polarized rewrite system with cut elimination property and $\mathcal{R}_f$ be the final polarized rewrite system of $\mathcal{R}_0$. For a sequent $\Gamma \vdash \Delta$ containing no occurrence of Skolem symbols the following conditions are equivalent:*

1. $\Gamma \vdash^{c.f.}_{\mathcal{R}_0} \Delta$
2. $\Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta$
3. $\Gamma \vdash_{\mathcal{R}_f} \Delta$
4. $\Gamma \vdash_{\mathcal{R}_0} \Delta$

Proof. 1. ⇒ 2. is Proposition 2; 2. ⇒ 3. is trivial; 3. ⇒ 4. is Proposition 4; 4 ⇒ 1. is cut elimination property of $\mathcal{R}_0$.

Notice that as a side result we have a partial cut elimination property for $\mathcal{R}_f$. Indeed if $\Gamma \vdash \Delta$ is a sequent in the language $\mathcal{L}$ (i.e. it does not contain any Skolem symbol) $\Gamma \vdash_{\mathcal{R}_f} \Delta \Rightarrow \Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta$. This result does not extend to the full language. For instance, the sequent $Q(c) \vdash \forall x Q(x)$ has a proof in $\mathcal{R}_f$, see Figure 7, but it does not have a cut free proof. Fortunately, we do not need the cut elimination property of $\mathcal{R}_f$ to prove $\Gamma \vdash_{\mathcal{R}_0} \Delta$ if and only if $Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f} \Box$.

$$\begin{tikzcd}
\boxed{(\Gamma \vdash_{\mathcal{R}_0} \Delta)} \arrow[r, Leftrightarrow, "\text{cut}", "\text{elimination}"'] \arrow[d, Leftrightarrow, "\text{Proposition 3}"] & (\Gamma \vdash^{c.f.}_{\mathcal{R}_0} \Delta) \arrow[d, Rightarrow, "\text{Proposition 2}"] & \\
(\mathcal{U}_{\mathcal{R}_0}, \Gamma \vdash \Delta) \arrow[d, Leftrightarrow, "\text{Proposition 4}"] & (\Gamma \vdash^{c.f.}_{\mathcal{R}_f} \Delta) \arrow[r, Leftrightarrow, "\text{Polarized}", "\text{Rosolution Modulo}"'] \arrow[d, Rightarrow, "\text{trivial}"] & \boxed{(Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f} \Box)} \\
(\mathcal{U}_{\mathcal{R}_f}, \Gamma \vdash \Delta) \arrow[r, Leftrightarrow, "\text{Proposition 3}"] & (\Gamma \vdash_{\mathcal{R}_f} \Delta) &
\end{tikzcd}$$

Fig. 6. Equivalence

$$\boxed{P \longrightarrow \forall x Q(x)} \blacktriangleright \boxed{\begin{array}{c} P \longrightarrow_- \forall x Q(x) \\ P \longrightarrow_+ \forall x Q(x) \end{array}} \blacktriangleright \boxed{\begin{array}{c} P \longrightarrow_- Q(x) \\ P \longrightarrow_+ Q(c) \end{array}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q(x)}\ \text{Axiom}}{P \vdash \forall x Q(x)}\ \forall\text{-right}}{Q(c), P \vdash \forall x Q(x)}\ \text{weak-left} \qquad \dfrac{\dfrac{}{Q(c) \vdash P}\ \text{Axiom}}{Q(c) \vdash P, \forall x Q(x)}\ \text{weak-right}}{Q(c) \vdash \forall x Q(x)}\ \textit{cut}$$

Fig. 7. Proof with cut

In this paper, we translate any polarized rewrite system into a clausal one. We prove that the obtained clausal polarized rewrite system preserves the existence of cut free proof for any sequent. So that Polarized Resolution Modulo can be applied to the system directly. However the obtained clausal polarized rewrite system may lose cut elimination property. So there are two possibility for the future work. One is dropping the hypothesis that $\mathcal{R}_0$ has the cut elimination property. Another is fixing $\mathcal{R}_f$ such that $\mathcal{R}_f$ has the cut elimination property.

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