Exhaustive Family of Energies Minimizable Exactly by a Graph Cut

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In a nutshell

- There exist many multi-label **non-submodular energies** that can be minimized exactly by a graph cut
- Characterization of all energies minimizable exactly (**exhaustive family**), in particular full characterization of a generating subfamily

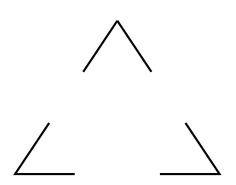
Motivation

Image segmentation is often based on:

- **local descriptions**, such as **texture** classification/comparison
- **edge detection**
- a functional to **optimize**, expressing segmentation as a pixel labeling problem

However:

- local **texture descriptors** are most often **not precise nor reliable**
- **edge detectors** are usually **precise and more reliable**
- standard graph cuts require **submodularity** of interaction matrices between labels of neighboring pixels:
  - $\hookrightarrow$ can favor spatial **homogeneity but not label variations**
  - $\Longrightarrow$ **most information is lost** (edges)



Questions:

- Submodularity really required? What about other graph constructions?
- Exhaustive list of energies minimizable by a graph cut?

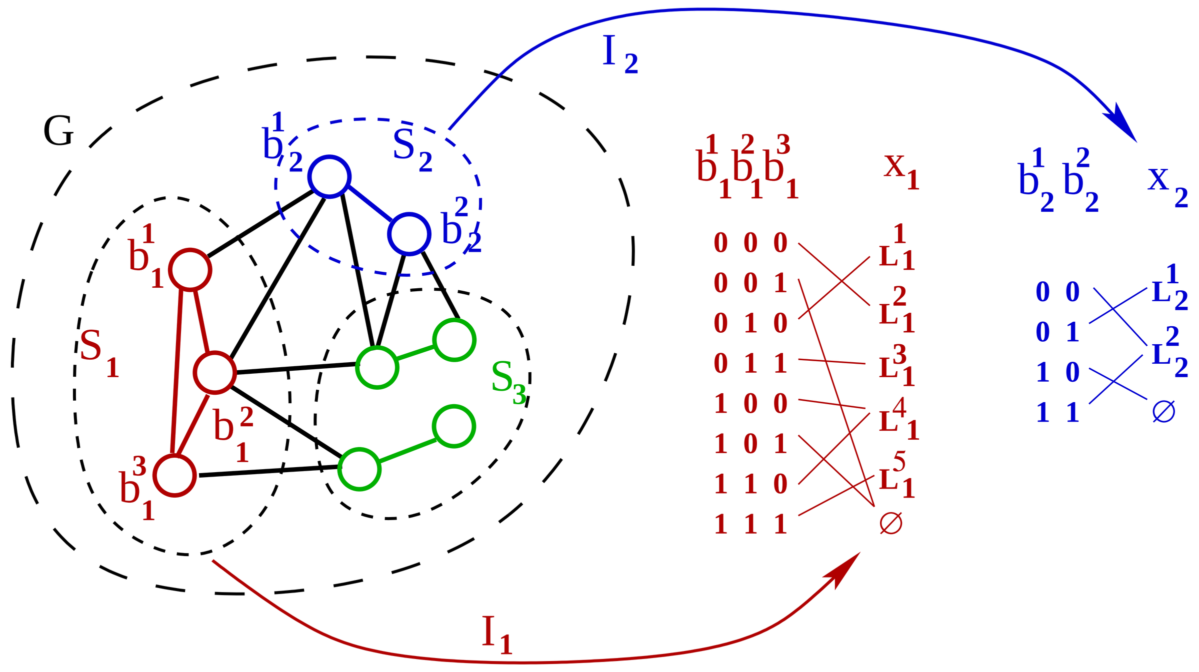
Summary of contributions

- **Def: Subfamily** $\mathcal{F}_0$ of energies minimizable when bijection *graph states* $\leftrightarrow$ *states of energy variables*
- **Prop 1: Exhaustive family** $\mathcal{F}$ **of energies minimizable by a graph cut** is generated by partial infima of energies in $\mathcal{F}_0$
- **Prop 2: Canonical form of Markov random fields** (of clique size 2) by removing useless constants (local means and interaction means)
- **Prop 3: Characterization of** $\mathcal{F}_0$, for fixed label ordering, as product of vector half-spaces, with explicit bases
- **Prop 4: Checking** $\mathcal{F}_0$ **membership and finding orderings** in linear complexity a.s.! (as a function of the data size)
- **Corr: Uniqueness of representation of an energy in** $\mathcal{F}_0$: only one ordering possible a.s. (up to an uninteresting permutation group)
- **Prop 5: Building graph** of an energy in $\mathcal{F}_0$ in quasilinear time a.s.

Graphs and Interpretations

- Min-cut algorithms: find a partition (*source* vs. *sink*) with minimal cost, of graphs with edge weights $\geqslant 0$, hence **graph nodes** seen **as binary variables**
- Min-cut as **energy optimizer**: minimizes the cost of the *s*-*t* cut, over node states $\Longrightarrow$ energies writable as an *s*-*t* cut cost?

Simple case: bijection ***graph states*** $\leftrightarrow$ ***states of energy variables***



- Necessarily, finite number of variables and finite number of labels
- **Def: Subfamily** $\mathcal{F}_0$**:** when bijection btw. states: each variable $\leftrightarrow$ a subgraph

$$\Longleftrightarrow \quad E(x) = \sum_i D_i(x_i) + \sum_{i,j} V_{i,j}(x_i, x_j)$$

with

$$\begin{cases} D_i(x_i) = \displaystyle\sum_{k=1}^{q_i} \left(w_i^{\text{source}\to k}\, b_i^k + w_i^{k\to\text{sink}}\, \overline{b_i^k} \right) + \sum_{k=1}^{q_i} \sum_{k'} w_i^{k\to k'}\, \overline{b_i^k}\, b_i^{k'} \\ V_{i,j}(x_i, x_j) = \displaystyle\sum_{k=1}^{q_i} \sum_{l=1}^{q_j} w_{i,j}^{k\to l}\, \overline{b_i^k}\, b_j^l + w_{i,j}^{k\leftarrow l}\, b_i^k\, \overline{b_j^l} \end{cases}$$

$b_i^k = k$-th digit of interpretation $I_i^{-1}(x_i)$, and $w_i^{k\to k'}$, $w_{i,j}^{k\to l}$ = weights of edges

General case: any interpretation ***graph states*** $\leftrightarrow$ ***states of variables of E***

- **Prop 1: Exhaustive family** $\mathcal{F}$ **of energies minimizable by a graph cut** is generated by partial infima of energies in $\mathcal{F}_0$
- Ex: $f(x_1, x_2) = \min_{x_3} g(x_1, x_2, x_3)$ with $g \in \mathcal{F}_0$
- or $f(x_1, x_2, x_3) = \min_{y_3 \in \mathcal{L}(x_3)} g(x_1, x_2, y_3)$ with $g \in \mathcal{F}_0$

Canonical Form of MRF

The following Markov random fields are equivalent:

$$E = D_1(x_1) + V_{12}(x_1, x_2) + D_2(x_2) + V_{23}(x_2, x_3) + D_3(x_3)$$

$$C_1 \quad \begin{pmatrix}2\\3\end{pmatrix} \quad \begin{pmatrix}4 & 7\\5 & 0\end{pmatrix} \quad \begin{pmatrix}-4\\1\end{pmatrix} \quad \begin{pmatrix}-1 & 0\\2 & 2\end{pmatrix} \quad \begin{pmatrix}0\\0\end{pmatrix}$$

$$C_2 \quad \begin{pmatrix}6\\0\end{pmatrix} \quad \begin{pmatrix}0 & 0\\8 & 0\end{pmatrix} \quad \begin{pmatrix}-7\\3\end{pmatrix} \quad \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \quad \begin{pmatrix}2\\3\end{pmatrix}$$

$\Longrightarrow$ need for a unique representation!

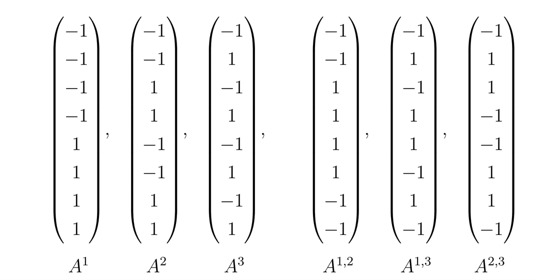
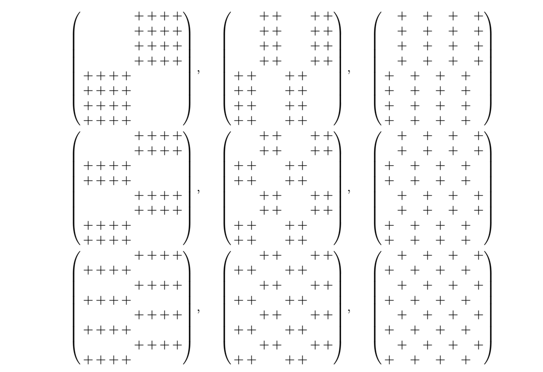
Prop 2: Canonization: means of vectors, matrix rows & columns = 0

Recognition of Energies in $\mathcal{F}_0$

- **Prop 3: Characterization of** $\mathcal{F}_0$: MRF of clique size 2 s.t. $\exists$ interpretations s.t. in canonical form, $\begin{cases} D_i \in \text{Span}(A^k) + \text{Span}^+(A^{k,k'}) \\ V_{i,j} \in \text{Span}^+(W^{k,l}) \end{cases}$

  where vectors A^k and $A^{k,k'}$ and matrices $W^{k,l}$ are orthogonal families: $(A^k)_s = (-1)^{1+b^k(s)}$, $(A^{k,k'})_s = (-1)^{1+b^k(s)+b^{k'}(s)}$, and $W^{k,l} = -A^k \otimes A^l$

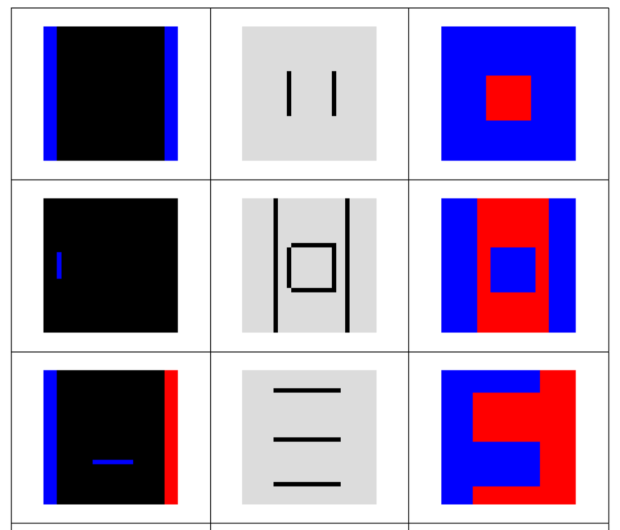
- **Ex:**

1 node	2 nodes	3 nodes
$A^1 = \begin{pmatrix}-1\\1\end{pmatrix}$	$\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\\-1\end{pmatrix}$ (A^1, A^2, $A^{1,2}$)	
1×1 node	**2×2 nodes**	**3×3 nodes**
$W^{1,1} = \begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}$	$\begin{pmatrix}-1&-1&1&1\\-1&-1&1&1\\1&1&-1&-1\\1&1&-1&-1\end{pmatrix}, \begin{pmatrix}-1&1&-1&1\\-1&1&-1&1\\1&-1&1&-1\\1&-1&1&-1\end{pmatrix}, \begin{pmatrix}-1&-1&1&1\\1&1&-1&-1\\-1&-1&1&1\\1&1&-1&-1\end{pmatrix}, \begin{pmatrix}-1&1&-1&1\\1&-1&1&-1\\-1&1&-1&1\\1&-1&1&-1\end{pmatrix}$	

- Includes $\begin{pmatrix}-1&0&0&1\\0&1&-1&0\\0&-1&1&0\\1&0&0&-1\end{pmatrix}$ e.g., hence **non-(permuted-)submodular energies**

- **Prop 4: Checking** $\mathcal{F}_0$ **membership and finding suitable orderings** in linear complexity a.s.! (instead of $m!^n$). **Proof sketch:** by construction:
  - $\hookrightarrow$ smallest coefficients in interaction matrices: (000...,000...) and (111...,111...)
  - $\hookrightarrow$ $M_{s_1,s_2} = \sum_{k,l} \alpha^{k,l} (-1)^{1+b^k(s_1)+b^l(s_2)} \Longrightarrow M_{000...,s} = -\sum_k (-1)^{b^k(s)} \beta_k$
  - $\hookrightarrow$ then: find the β_k from their sums $\pm$ by sorting frequencies of differences

Toy Examples of New Possibilities



Potentials	Edges	Results
Color: pref Black: no pref	Black: edge Grey: no edge	QPBO-like (?,Red,Blue,?)

Segmentation in $m+2$ **classes**

Vectors and matrices used for $m = 2$ (Red vs Blue):
$\begin{pmatrix}0\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}; \begin{pmatrix}-1&0&0&1\\0&-1&1&0\\0&1&-1&0\\1&0&0&-1\end{pmatrix}, \begin{pmatrix}-1&0&0&1\\0&1&-1&0\\0&-1&1&0\\1&0&0&-1\end{pmatrix}$

Most probable grey level locally	Idem after projection on $\mathcal{F}_0$	Edge detector	Result: digitalization in 6+2 classes

Discussion

Solved

- Subfamily $\mathcal{F}_0$ fully characterized
- Small complexity to recognize $\mathcal{F}_0$ & build graph
- Useful MRF canonical form

Not Solved Yet

- Family $\mathcal{F}$, not just $\mathcal{F}_0$: min operator?
- $\mathcal{F}_0$ charact.: not as simple as *submodularity*
- Algorithms and complexities: a.s.
- Approximation of any MRF with $\mathcal{F}$ or $\mathcal{F}_0$