A triangular canonical form for a class of 0-flat nonlinear systems

Driss Boutat, Soray Bououden, Driss Boutat, Gang Zheng, Jean-Pierre Barbot, Frédéric Kratz

▶ To cite this version:

HAL Id: hal-00627655
https://hal.archives-ouvertes.fr/hal-00627655

Submitted on 29 Sep 2011

A triangular canonical form for a class of 0-flat nonlinear systems

S. Bououden[a], D. Boutat[a], G. Zheng[b]*, J.-P. Barbot[bc] and F. Kratz[a]

[a]*Loire Valley University, ENSI de Bourges, Institut PRISME, 88 Boulevard Lahitolle,
18020 Bourges Cedex, France;* [b]*INRIA, 40 Avenue Halley, Villeneuve d'Ascq 59650,
France;* [c]*ECS/ENSEA, 6 Avenue du Ponceau, Cergy-Pontoise 95014, France*

This article proposes a triangular canonical form for a class of 0-flat nonlinear systems. Necessary and sufficient geometrical conditions are given in order to guarantee the existence of a local diffeomorphism to transform the studied nonlinear systems into the proposed 0-flat canonical form, which enables us to compute the flat output as well.

Keywords: flatness; canonical form; geometrical condition

1. Introduction

Differential flatness is a well-known concept in control theory, proposed by Fliess, Lévine, Martin and Rouchon (Fliess, Lévine, Martin, and Rouchon 1992, 1993, 1995, 1997, 1998, 1999a,b; Sira-Ramirez and Agrawal 2004; Lévine 2009) and nowadays is widely used in industrial applications, such as trajectory planning problem (Rotella and Zambettakis 2007). Other problems, such as isolation of the faults, left invertibility for nonlinear systems, synchronisation of chaotic systems, sometimes can be seen as a problem whether the system is flat (Pomet 1995a; Tilbury, Murray, and Sastry 1995; van Nieuwstadt, Rathinam, and Murray 1998; Hagenmeyer and Delaleau 2003; Lévine 2004; Schlacher and Schöberl 2007).

In fact, flat systems are a generalisation of linear dynamical systems in the sense that all linear controllable dynamical systems are flat and static feedback linearisable (in Brunovsky's form) (Jakubczyk and Respondek 1980; Singh 1981; Shadwick 1990; Sluis 1993; Shadwick and Sluis 1994; Rouchon 1995; Rudolph 1995; Sluis and Tilbury 1996). In contrast to the feedback linearisation techniques, the property of flatness does not need to convert nonlinear systems into linear ones to design different kinds of feedback laws (Hagenmeyer and Delaleau 2003; Lévine 2004; Schlacher and Schöberl 2007). Therefore, when a system is flat, one can use its structure to design control for motion, trajectory generation and stabilisation.

In order to apply the concept of flatness, one key problem is to determine the flat outputs for nonlinear systems. For this, many approaches have been proposed in the literature. One approach to deal with flatness is exterior differential systems where a control dynamical system is regarded as a Pfaffian system on an appropriate jet space (Anderson and Ibragimov 1979; Descusse and Moog 1985; Pomet 1995a; Delaleau and da Silva 1998; van Nieuwstadt et al. 1998; da Silva 2000a), and flatness is related to absolute equivalence introduced by Cartan (1914). Another geometrical approach by means of Lie–Bäcklund equivalence was addressed in Fliess et al. (1993, 1997, 1998) and Martin, Murray, and Rouchon (1997). In addition, since feedback linearisable dynamical systems are flat, some results in this direction are stated, such that controllable codimension 1 affine dynamical systems. In da Silva (2000) authors gave a characterisation of the so-called k-flatness with the Cartan–Kähler approach. There are also many interesting results on flatness of driftless dynamical systems. We can cite the work of Martin and Rouchon (1995a,b) where the authors proved that any (controllable) driftless system with m inputs and $m+2$ states is flat. Another interesting result on 1-flatness can be found in Pomet (1995b) for four-dimensional dynamical systems.

However, for general nonlinear systems, it is still an open problem to construct flat outputs. As an extension of Bououden, Boutat, Barbot, and Kratz (2009), this article studies this problem from normal form point of view, by proposing a triangular 0-flat canonical form for a class of nonlinear systems, whose outputs are flat. Some results on triangular flat forms are reported by using states transformation and static states feedback. For instance, for five states and two controls, every 0-flat system can be transformed into a

*Corresponding author. Email: gang.zheng@inria.fr

triangular form by state transformation and regular static-state feedback (Silveira and da Silva 2007). Other results about the relationship between triangular forms and flatness can be found in Silveira (2009). It has been proved that a system can be k-flat but not $(k-1)$-flat and an example, which is not 0-flat but 1-flat, is given in da Silva (2000b). It should be noted that we treat the linearisation problem only through state transformations without static states feedback in this article. Necessary and sufficient geometrical conditions are given in order to guarantee the existence of a local diffeomorphism to transform the studied nonlinear systems into the proposed 0-flat canonical form.

This article is organised as follows. Notations and definitions are given in Section 2. In Section 3, a class of triangular 0-flat canonical form is proposed and its associated geometrical interpretation is discussed. Section 4 presents the necessary and sufficient geometrical conditions to locally transform nonlinear systems into the proposed 0-flat canonical form.

2. Notations and definitions

Let us first recall the notion of flatness of the nonlinear system of the following form:

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$ and f is assumed to be a smooth vector field on $\mathcal{X} \times \mathcal{U}$.

Definition 2.1: Dynamical system (1) is flat if there exist m functions $y = (y_1, \ldots, y_m)^T$, called flat outputs, where m is the number of inputs, such that

(1) the flat output is a unique function of the state x, the input u and its derivatives $u^{(i)}$, i.e. $y(x, u, \dot{u}, \ldots, u^{(r_1)})$.
(2) the state can be written as a function of the flat output and its derivatives, i.e. $x = \varphi(y, \dot{y}, \ldots, y^{(r_2)})$.
(3) the input can be written as a function of the flat output and its derivatives, i.e. $u = \gamma(y, \dot{y}, \ldots, y^{(r_2+1)})$.

Without loss of generalities, in this article we are interested in affine dynamical systems in the following form:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i. \tag{2}$$

Remark 1: A system of the form (1) can be brought into the form (2) by adding an integrator to each input.

The objective of this article is to characterise a class of dynamical systems for which the flat outputs are

only functions of states x. This implies that in (1) of Definition 2.1 we have only $y(x)$. In the literature, these systems are called 0-flat (da Silva 2000a).

Let us denote for $i = 1 : m$ by $ad_f g_i$ the Lie bracket of f with g_i, i.e.

$$ad_f g_i = [f, g_i]$$

and by induction we have $ad_f^k g_i = [ad_f^{k-1} g_i, g_i]$ for $0 \leq k$ with conventional notation $ad_f^0 g_i = g_i$. It is well-known that, if (2) can be linearised by a diffeomorphism $z = \phi(x)$ and a static states feedback $u = \alpha(x) + \beta(x)v$ into the following form:

$$\begin{aligned} \dot{z}_{i,j} &= z_{i+1,j}, \quad \text{for } 1 \leq j \leq m \text{ and } 1 \leq i \leq \nu_j - 1, \\ \dot{z}_{\nu_j,j} &= v_j \end{aligned} \tag{3}$$

with $\sum_{j=1}^{m} \nu_j = n$, then $y_j = z_{1,j}$ for $1 \leq j \leq m$ are the 0-flat outputs of (2). To transform a dynamical system into the form (3) is called the static states feedback linearisation problem and we have the following famous result (see e.g. Jakubczyk and Respondek 1980; Hunt, Su, and Meyer 1983).

Theorem 2.2: *The linearisation problem for* (2) *can be solved if and only if*

(1) *the dimension of $\mathcal{G}_i$ is constant on $\mathcal{X}$ for $1 \leq i \leq n-1$,*
(2) *the dimension of $\mathcal{G}_{n-1}$ is of n,*
(3) *the distribution $\mathcal{G}_k$ is involutive for each $1 \leq k \leq n-2$,*

where

$$\mathcal{G}_0 = \text{span}\{g_1, \ldots, g_m\},$$
$$\mathcal{G}_1 = \text{span}\{g_1, \ldots, g_m, ad_f g_1, \ldots, ad_f g_m\},$$
$$\vdots$$
$$\mathcal{G}_k = \text{span}\{ad_f^j g_i, \text{ for } 0 \leq j \leq k, 1 \leq i \leq m\}.$$

Another famous class of 0-flat dynamical systems is controllable affine systems with n states and $n-1$ inputs (Martin et al. 1997; Rotella and Zambettakis 2007). In what follows, we will propose a more general triangular canonical form than (3) and prove that it is 0-flat. Then we will give sufficient and necessary geometrical conditions to guarantee the existence of a local diffeomorphism which transforms (2) into the proposed canonical form.

3. A class of 0-flat dynamical system

This section first gives a cascade 0-flat normal form which represents a class of affine flat dynamical systems, then gives a geometrical characterisation of

such a normal form. It should be noted that the proposed conditions are only sufficient and the considered dynamical systems are only a small class. Therefore there exists many dynamical systems, in particular flat driftless systems do not check the proposed conditions, and examples can be found in Martin and Rouchon (1995a,b) and Pomet (1995b).

3.1 Triangular 0-flat canonical form

In this section, we give a class of 0-flat dynamical systems, represented by its normal form, which generalises the well-known controllable affine systems with n states and $n-1$ inputs. For this, it is assumed that there exists a list of integers:

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_m$$

such that $\sum_{j=1}^{m} \nu_j = n$.

Now, for all $1 \leq j \leq m$, denote $z_j = (z_{1,j}, \ldots, z_{\nu_j,j})^T$ and define the following system:

$$\dot{z}_{i,j} = z_{i+1,j} + \sum_{l=1}^{m} \alpha_{i,j}^l(z)u_l \quad \text{for } 1 \leq j \leq m, \ 1 \leq i \leq \nu_j - 1,$$

$$(4)$$

$$\dot{z}_{\nu_j,j} = a_j(z) + \sum_{l=1}^{m} \alpha_{\nu_j,j}^l(z)u_l, \quad (5)$$

where $\alpha_{i,j}^l(z)$ and $a_j(z)$ satisfy the following property.

Property 3.1:

(1) Functions $\alpha_{i,j}^l$ for $1 \leq l \leq m$, $1 \leq j \leq m$ and $1 \leq i \leq \nu_j$ are as follows:

- $\alpha_{\nu_j,j}^j \neq 0$ on $\mathcal{X}$;
- if $\nu_l > i$, then $\alpha_{i,j}^l = 0$;
- if $\nu_l \leq i$, then $\alpha_{i,j}^l$ depend only on $z_{s,k}$ for $1 \leq k \leq m$ and $1 \leq s \leq \min\{i+1, \nu_k\}$.

(2) Functions a_j for $1 \leq j \leq m$ depend only on the following variables:

- $z_{i,k}$ for $k \in \{k \in [1,m] \mid \nu_k > \nu_j\}$ and $1 \leq i \leq \nu_j + 1$;
- $z_{i,k}$ for $k \in \{k \in [1,m] \mid \nu_k \leq \nu_j\}$ and $1 \leq i \leq \nu_k$.

Remark 2: According to the form (4), (5), for $1 \leq j \leq m$, the dynamics $\dot{z}_{1,j}$ depends only on u_l for $l \in \{l \in [1,m] \mid \nu_l = 1\}$, $z_{1,k}$ for $1 \leq k \leq m$ and $z_{2,k}$ for $k \in \{k \in [1,m] \mid \nu_k \geq 2\}$. For $1 \leq j \leq m$, the dynamics $\dot{z}_{2,j}$ depends only on u_l for $l \in \{l \in [1,m] \mid \nu_l \leq 2\}$, $z_{1,k}$ for $1 \leq k \leq m$, $z_{2,k}$ for $1 \leq k \leq m$ and $z_{3,k}$ for $k \in \{k \in [1,m] \mid \nu_k \geq 3\}$.

By induction, it can be seen that the dynamics $\dot{z}_{i,j}$ for $1 \leq j \leq m$ and $1 \leq i \leq \nu_j$ depend on the following variables:

- u_l for $l \in \{l \in [1,m] \mid \nu_l \leq i\}$;
- $z_{s,k}$ for $1 \leq k \leq m$ and $1 \leq s \leq \min\{i+1, \nu_k\}$,

which implicitly yield a triangular structure allowing to calculate $z_{i+1,j}$ for $1 \leq j \leq m$ and $1 \leq i \leq \nu_j - 1$ and u_l for $1 \leq l \leq m$ step by step.

Remark 3: By using a linear change of coordinates, we can always assume that a_j for $1 \leq j \leq m$ contains only the terms with orders greater than 1, i.e. $O^2(z)$, and $\alpha_{i,j}^l$ for $1 \leq l \leq m$, $1 \leq j \leq m$ and $1 \leq i < \nu_j$ is of $O^1(z)$.

Remark 4: By setting $\alpha_{\nu_j,j}^l = 1$, $a_j = 0$ and $\alpha_{\nu_j,j}^l = 0$ for $i \neq \nu_j$, we see that the form (3) is a special case of the form (4), (5).

3.2 Geometrical interpretation

Before giving a geometrical interpretation of the above conditions, let us rewrite dynamical system (4), (5) into the following compact form:

$$\dot{z} = \bar{f} + \sum_{k=1}^{m} \bar{g}_k u_k, \quad (6)$$

with

$$\bar{f} = \text{col}(\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_{m-1}, \bar{f}_m)$$

and

$$\bar{f}_j = \text{col}(z_{2,j}, z_{3,j}, \ldots, z_{\nu_j,j}, a_j)$$

for $1 \leq j \leq m$, and

$$\bar{g}_l = \text{col}(\bar{g}_1^l, \bar{g}_2^l, \ldots, \bar{g}_m^l)$$

and

$$\bar{g}_j^l = \text{col}(\alpha_{1,j}^l, \alpha_{2,j}^l, \ldots, \alpha_{\nu_j,j}^l)$$

for $1 \leq l \leq m$ and $1 \leq j \leq m$.

Since $\alpha_{i,j}^l$ for $1 \leq l \leq m$, $1 \leq j \leq m$ and $1 \leq i \leq \nu_j$ satisfies Property 3.1, for $1 \leq l \leq m$ we have

$$\bar{g}_j^l = 0 \quad \text{if } \nu_j < \nu_l,$$

which gives the geometrical characteristic of (4), (5) in the compact form (6) as follows.

Proposition 3.2: *The canonical form* (4), (5) *in the compact form* (6) *satisfies the following conditions:*

(1) *The distribution*

$$\Delta = \text{span}\{ad_f^k g_j \text{ for } 1 \leq j \leq m, 0 \leq k \leq \nu_j - 1\}$$

is of dimension n. In this case we say that (2) *is controllable and* v_j *for* $1 \leq j \leq m$ *are the controllability indices.*[1]

(2) *The distribution* $\bar{\Delta}$ *defined as follows*:

$$\bar{\Delta} = \text{span}\{ad_f^k \bar{g}_i, \text{ for } j \in \{j \in [1, m] \mid v_j \geq 2\},$$

$$0 \leq k \leq v_j - 2\}$$

is involutive.

(3) *The item* (2) *of Property* 3.1 *is equivalent to the following condition*:

$$[\bar{g}_l, ad_f^{v_k - v_l - 1 - s} \bar{g}_k] \in \text{span}\{ad_f^j \bar{g}_j \text{ for } 0 \leq j \leq v_i - v_l - s$$

$$and \ j \in \{j \in [1, m] \mid v_j \geq v_l + s\}\},$$

where $k \in \{k \in [1, m] \mid v_k \geq 2\}$, $l \in \{l \in [1, m] \mid v_l < v_k\}$ *and* $0 \leq s \leq v_k - v_l - 1$.

Proof: According to the triangular structure of (4), (5), condition (1) of Proposition 3.2 is easy to be proved. Moreover, the structure of (4), (5) allows to calculate $\bar{g}_l$ for $1 \leq l \leq m$, and by some straightforward calculations one can check that (3) of Proposition 3.2 is evident, and thus we only prove here (2) of Proposition 3.2.

Denote by $\bar{\Delta}^\perp$ the annihilator of $\bar{\Delta}$. According to the form (6) with Property 3.1, it can be proved that

$$\bar{\Delta}^\perp = \text{span}\{dz_{1,j}, \text{ for } 1 \leq j \leq m\},$$

which is spanned by the exact differential forms. Thus according to Frobenius Theorem, the distribution $\bar{\Delta}$ is involutive.   □

Then we have the following preliminary result.

Proposition 3.3: *Dynamical system* (4), (5) *is locally 0-flat and the variables* $z_{1,j}$ *for* $1 \leq j \leq m$ *are the flat outputs.*

Proof: As mentioned in Remark 2, we can obtain a triangular structure to calculate the states and inputs step by step. For this, denote $\mathfrak{S}_s$ for $1 \leq s \leq \max\{v_j, 1 \leq j \leq m\}$ the following set of dynamics:

$$\mathfrak{S}_s = \{\dot{z}_{i,j}, 1 \leq j \leq m \text{ and } 1 \leq i \leq \min\{s, v_j\}\}, \quad (7)$$

which in fact depends only on the following variables:

- u_l for $l \in \{l \in [1, m] \mid v_l \leq i\}$;
- $z_{s,k}$ for $1 \leq k \leq m$ and $1 \leq s \leq \min\{i+1, v_k\}$;

and specially we denote $\mathfrak{S}_0 = \{z_{1,j}, 1 \leq j \leq m\}$ as the set of the flat outputs, thus we have $\mathfrak{S}_1 = \{\dot{z}_{1,j}, 1 \leq j \leq m\} = \{\xi, \xi \in \dot{\mathfrak{S}}_0\}$.

Now we will show that $y_j = z_{1,j} \in \mathfrak{S}_0$ for $1 \leq j \leq m$ are the 0-flat outputs. First let us compute the following variables:

- $z_{2,j}$ for $v_j \geq 2$,
- u_l for $v_l = 1$

from the flat outputs and their derivatives. According to the form (4), (5),we have

$$\dot{z}_{1,j} - z_{2,j} - \sum_{l=1}^m \alpha_{i,j}^l u_l = 0 \quad \text{for } j \in \{j \in [1, m] \mid v_j \geq 2\},$$

$$\dot{z}_{1,j} - a_j - \sum_{l=1}^m \alpha_{i,j}^l u_l = 0 \quad \text{for } j \in \{j \in [1, m] \mid v_j = 1\},$$

$$(8)$$

where a_j and $\alpha_{1,j}^l$ satisfy Property 3.1, and thus depend only on the known variable $z_{1,k}$ for $1 \leq k \leq m$ and the unknown variables u_k for $k \in \{k \in [1, m] \mid v_k = 1\}$ and $z_{2,k}$ for $k \in \{k \in [1, m] \mid v_k \geq 2\}$.

So (8) is formed with m equations and contains m unknown variables. According to Property 3.1 and Remark 3, partial differentiation of the left-hand side of (8) with respect to $\frac{\partial}{\partial(z_{2,j}, u_k)}$ is equal to $I + O^1(z, u)$. Therefore, thanks to the implicit function theorem one can find m functions: φ_j for $v_j \geq 2$ and γ_l for $v_l = 1$ such that

$$z_{2,j} = \varphi_j(z_{1,k}, \dot{z}_{1,k}) \quad \text{for } 1 \leq k \leq m,$$
$$u_l = \gamma_l(z_{1,k}, \dot{z}_{1,k}) \quad \text{for } 1 \leq k \leq m. \quad (9)$$

Thus it can be stated that

$$z_{2,j} \in \{\xi, \xi \in \mathfrak{S}_0 \cup \mathfrak{S}_1\} \quad \text{for } j \in \{j \in [1, m] \mid v_j \geq 2\},$$
$$u_l \in \{\xi, \xi \in \mathfrak{S}_0 \cup \mathfrak{S}_1\}, \quad \text{for } l \in \{l \in [1, m] \mid v_l = 1\}$$

and $\mathfrak{S}_2 = \{\xi, \xi \in \dot{\mathfrak{S}}_0 \cup \ddot{\mathfrak{S}}_0\}$.

Now, similar to the first step, for the following variables:

- $z_{3,j}$ for $v_j \geq 3$,
- u_l for $v_l = 2$,

one can obtain

$$z_{3,j} = \varphi_j(z_{1,k}, \dot{z}_{1,k}, z_{2,k}, \dot{z}_{2,k}),$$
$$u_l = \gamma_l(z_{1,k}, \dot{z}_{1,k}, z_{2,k}, \dot{z}_{2,k}), \quad (10)$$

which implies that

$$z_{3,j} \in \{\xi, \xi \in \mathfrak{S}_0 \cup \dot{\mathfrak{S}}_0 \cup \ddot{\mathfrak{S}}_0\} \quad \text{for } j \in \{j \in [1, m] \mid v_j \geq 3\},$$
$$u_j \in \{\xi, \xi \in \mathfrak{S}_0 \cup \dot{\mathfrak{S}}_0 \cup \ddot{\mathfrak{S}}_0\} \quad \text{for } j \in \{j \in [1, m] \mid v_j = 2\}$$

and $\mathfrak{S}_3 = \{\xi, \xi \in \cup_{i=1}^3 \mathfrak{S}_0^{(i)}\}$.

By induction, assume that for $1 \leq s < \max\{v_j, 1 \leq j \leq m\}$ we have computed $\mathfrak{S}_s = \{\xi, \xi \in \sup_{i=1}^s \mathfrak{S}_0^{(i)}\}$, and one has

$$z_{s+1,j} \in \{\xi, \xi \in \cup_{i=1}^s \mathfrak{S}_0^{(i)}\} \quad \text{for } j \in \{j \in [1, m] \mid v_j \geq s\},$$
$$u_l \in \{\xi, \xi \in \cup_{k=1}^s \mathfrak{S}_0^{(k)}\} \quad \text{for } l \in \{l \in [1, m] \mid v_l = s - 1\}.$$

Finally, we show that all states and inputs can be written as functions of $y_{1,j}$ for $1 \leq j \leq m$ and its derivatives, and thus prove Proposition 3.3.   □

Let us give an example which is already in the 0-flat canonical form (4), (5), in order to show the procedure of computation of the state variables and the inputs proposed in the above proof.

Example 3.4: Consider the following dynamical system:

$$\begin{cases} \dot{z}_{1,1} = z_{2,1}, \\ \dot{z}_{2,1} = z_{3,1} + \dfrac{z_{2,1}}{z_{2,2} - 1} u_2, \\ \dot{z}_{3,1} = (z_{2,2} - 1)u_1 + \dfrac{z_{3,1}}{z_{2,2} - 1} u_2, \\ \dot{z}_{1,2} = z_{2,2}, \\ \dot{z}_{2,2} = u_2, \end{cases} \quad (11)$$

and suppose that $y_1 = z_{1,1}$ and $y_2 = z_{1,2}$, we will give the procedure to compute all state variables and inputs from the outputs. For this, let us consider the following dynamics:

$$\begin{cases} \dot{z}_{1,1} - z_{2,1} = 0, \\ \dot{z}_{1,2} - z_{2,2} = 0, \end{cases}$$

we obtain $z_{2,1} = \dot{y}_1$ and $z_{2,2} = \dot{y}_2$.

Let us again consider the derivatives of these obtained variables

$$\begin{cases} \dot{z}_{2,1} = z_{3,1} + \dfrac{z_{2,1}}{z_{2,2} - 1} u_2, \\ \dot{z}_{2,2} = u_2, \end{cases}$$

which gives $u_2 = \ddot{y}_2$ and $z_{3,1} = \ddot{y}_1 - \dfrac{\dot{y}_1}{\dot{y}_2 - 1} \ddot{y}_2$.

Finally, from the third equation of (11) we obtain

$$\dot{z}_{3,1} - (z_{2,2} - 1)u_1 - \dfrac{z_{3,1}}{z_{2,2} - 1} u_2 = 0,$$

which gives

$$u_1 = \frac{1}{\dot{y}_2 - 1}\left(y_1^{(3)} - \frac{\partial\left(\frac{\dot{y}_1}{\dot{y}_2 - 1}\ddot{y}_2\right)}{\partial t} - \frac{\ddot{y}_1 - \frac{\dot{y}_1}{\dot{y}_2 - 1}\ddot{y}_2}{\dot{y}_2 - 1}\ddot{y}_2 \right).$$

4. Geometrical conditions to transform nonlinear systems into the proposed canonical form

In this section, we will show that conditions (1)–(3) of Proposition 3.2 are sufficient for the existence of a local diffeomorphism which transforms an affine dynamical system of the form (2) into (4), (5).

For this, assume that for (2) there exist m integers $v_1 \geq v_2 \geq \cdots \geq v_m$ such that

(1) $\sum_{i=1}^{m} v_i = n$,
(2) $\Delta = \text{span}\{ad_f^k g_j \text{ for } 1 \leq j \leq m \text{ and } 1 \leq k \leq v_k - 1\}$ is of constant dimension n on $\mathcal{X}$.

Let us also define the following distribution:

$$\bar{\Delta} = \text{span}\{ad_f^k g_j \text{ for } i \in \{i \in [1, m] \mid v_j \geq 2\}, \, 0 \leq k \leq v_j - 2\}, \quad (12)$$

then we are ready to give our main result.

Theorem 4.1: *There exists a local diffeomorphism which transforms dynamical system* (2) *into the* (4), (5) *form if and only if*

(1) $\bar{\Delta}$ *defined in* (12) *is involutive;*
(2)

$$[g_l, ad_f^{v_k - v_l - 1 - s} g_k] \in \text{span}\{ad_f^i g_j \text{ for } 0 \leq i \leq v_j - v_l - s$$
$$\text{and } j \in \{j \in [1, m] \mid v_j \geq v_l + s\}\},$$

where $k \in \{k \in [1, m] \mid v_k \geq 2\}$, $l \in \{l \in [1, m] \mid v_l < v_k\}$ *and* $0 \leq s \leq v_k - v_l - 1$.

Proof: According to Proposition 3.2, the necessity of Theorem 4.1 is obvious. Thus we prove only the sufficiency. For this, without loss of, generality, assume that there exists $1 \leq r \leq m$ such that

$$v_j \geq 2 \quad \text{for } 1 \leq j \leq r$$

and

$$v_j = 1 \quad \text{for } r + 1 \leq j \leq m.$$

Thus, according to the definition of $\bar{\Delta}$ in (12), since $v_{r+1} + \cdots + v_m = m - r$, one has

$$\begin{aligned} \dim \bar{\Delta} &= v_1 + \cdots + v_r - r \\ &= n - (v_{r+1} + \cdots + v_m) - r \\ &= n - m, \end{aligned}$$

which means that the codistribution $\bar{\Delta}^\perp$ has

$$\dim \bar{\Delta}^\perp = m.$$

If $\bar{\Delta}$ is involutive, then by Frobenius Theorem, there exists a set of functions $\{h_1, \ldots, h_m\}$ locally defined around some $x_0 \in \mathcal{X}$, such that the set $\{dh_1, \ldots, dh_m\}$ is locally independent around x_0 and

$$dh_j(\bar{\Delta}) = 0 \quad \text{for } 1 \leq j \leq m,$$
$$dh_j(ad_f^{v_k - 1} g_j) \neq 0 \quad \text{on } \mathcal{X} \quad \text{for } 1 \leq j \leq m, 1 \leq k \leq m.$$

After having h_j for $1 \leq j \leq m$, define the following new variables:

$$z_{i,j} = L_f^{i-1} h_j \quad \text{for } 1 \leq i \leq v_j.$$

Set $z_j = \text{col}(z_{i,j})$ for $1 \leq j \leq m$ and $1 \leq i \leq v_j$, and $z = \text{col}(z_j)$ for $1 \leq j \leq m$ which then defines a local diffeomorphism, noted as $z = \phi(x) = \text{col}(\phi_{i,j})$ for $1 \leq j \leq m$ and $1 \leq i \leq v_j$ with $\phi_{i,j} = z_{i,j} = L_f^{i-1} h_j$.

Let us now analyse the expression of the system in the new coordinates. Denote $\bar{f} = \phi_*(f)$ and $\bar{g}_s = \phi_*(g_l)$

for $1 \le l \le m$ with $\bar{g}_l = \mathrm{col}(\alpha_j^l)$ for $1 \le j \le m$ where $\alpha_j^l = \mathrm{col}(\alpha_{i,j}^l)$ for $1 \le i \le v_l$.

First, according to the definition of the above diffeomorphism $z = \phi(x)$, it is easy to see that for $1 \le j \le m$ and $1 \le i < v_j$ one has

$$\phi_{i,j*} f = dL_f^{i-1} h_j f = L_f^i h_j = \phi_{i+1,j} = z_{i+1,j}.$$

Moreover, by the involutivity condition, functions a_j for $1 \le j \le m$ fulfil (2) of Property 3.1, thus $\phi_* f$ is in the form $\bar{f}$ of (4), (5).

By the definition of the diffeomorphism, for $1 \le j \le m$, $1 \le i \le v_j$ and $1 \le l \le m$ we have

$$\alpha_{i,j}^l = [\phi_*(g_l)]_{i,j} = dL_f^{i-1} h_j g_l.$$

Since $dh_j(\bar{\Delta}) = 0$ for $1 \le j \le m$, then for $l \in \{l \in [1,m] \mid v_l \ge 2\}$, $k \le v_l - 2$ and $1 \le j \le m$ one has

$$dh_j ad_f^k g_l = dL_f^k h_j g_l = \alpha_{k+1,j}^l = 0,$$

which means that for $1 \le j \le m$, $1 \le i \le v_j$ and $1 \le l \le m$, one obtains

$$\alpha_{i,j}^l = 0 \quad \text{if } v_l > i.$$

Moreover, because ϕ is a diffeomorphism, thus for $1 \le l \le m$ and $k \ge 0$ one has the following equality:

$$\phi_*(ad_f^k g_l) = \phi_*[ad_f^{k-1} g_l, g_l] = [\phi_* ad_f^{k-1} g_l, \phi_* g_l]$$
$$= [ad_f^{k-1} \bar{g}_l, \bar{g}_l] = ad_f^k \bar{g}_l.$$

Thus, by applying the diffeomorphism ϕ, the second condition of Theorem 4.1 yields

$$[\bar{g}_l, ad_f^{v_k - v_l - 1 - s} \bar{g}_k] \in \mathrm{span}\{ad_f^i \bar{g}_j \text{ for } 0 \le i \le v_j - v_l - s$$
$$\text{and } j \in \{j \in [1,m] \mid v_j \ge v_l + s\}\},$$

where $k \in \{k \in [1,m] \mid v_k \ge 2\}$, $l \in \{l \in [1,m] \mid v_l < v_k\}$ and $0 \le s \le v_k - v_l - 1$. According to Proposition 3.2, this means that, for $1 \le l \le m$, $\phi_*(g_l)$ is transformed into $\bar{g}_l$ of the form defined in (4), (5). $\qquad \square$

Corollary 4.2: *Suppose that indices v_j and v_k for $1 \le j \le m$, $1 \le k \le m$ and $i \ne j$ satisfy $|v_j - v_k| \le 1$. The dynamical system (2) can be transformed into (4), (5) if and only if the distribution $\bar{\Delta}$ defined in (12) is involutive.*

Moreover, for any $1 \le j \le m$ and $1 \le k \le m$, if $v_j = v_k$, then the 0-flat canonical form (4), (5) can be written in the following simpler form:

$$\begin{cases} \dot{z}_{i,j} = z_{i+1,j} & \text{for } 1 \le i \le v_j - 2, \\ \dot{z}_{v_j - 1, j} = z_{v_j, j} + \sum_{l=1}^{m} \alpha_{v_j - 1, j}^l u_l, \\ \dot{z}_{v_j, j} = a_j + \sum_{l=1}^{m} \alpha_{v_j, j}^l u_l, \end{cases}$$

where a_j and $\alpha_{i,j}^l$ satisfy Property 3.1.

In particular, a codimension 1 dynamical system, i.e. $m = n - 1$, is flat (the well-known result in Charlet and Lévine (1989)).

Remark 5: If $v_j \le 2$ for all $1 \le j \le m$, then there exists a local diffeomorphism which transforms dynamical system (2) into (4), (5) if and only if the distribution

$$\bar{\Delta} = \{g_j, \text{ for } j \in \{j \in [1,m] \mid v_j = 2\}$$

is involutive, since the second condition of Theorem 4.1 is always fulfilled. In fact, it is a special case of Corollary 4.2.

Remark 6: For single input systems, i.e. $m = 1$, we only need the first condition of Theorem 4.1 and this condition is equivalent to the condition for the linearisation problem by means of a diffeomorphism and a static feedback.

Remark 7: For the case of codimension 2, i.e. $m = n - 2$, by reordering g_j for $1 \le j \le m$ if necessary, we have the following two cases:

(1) $v_1 = 2$ and $v_2 = 2$;
(2) $v_1 = 3$.

The first case is similar to Remark 5, thus we need to check the involutivity of the distribution $\bar{\Delta} = \mathrm{span}\{g_1, g_2\}$.

For the second case, we have to check the following two conditions:

- the distribution $\bar{\Delta} = \{g_1, ad_f g_1\}$ is involutive;
- for all $2 \le k \le m$, $[g_k, g_1] \in \mathrm{span}\{g_1, ad_f g_1\}$.

Example 4.3: Consider the following academic example (da Silva 2000a) modified for a regularity question:

$$\begin{cases} \dot{x}_1 = x_2 + x_4 x_3, \\ \dot{x}_2 = x_4, \\ \dot{x}_3 = x_5, \\ \dot{x}_4 = u_1, \\ \dot{x}_5 = u_2. \end{cases}$$

A simple calculation shows that distribution Δ is spanned by the following vector fields:

$$g_1 = \frac{\partial}{\partial x_4},$$
$$ad_f g_1 = -\frac{\partial}{\partial x_2} - x_3 \frac{\partial}{\partial x_1},$$
$$ad_f^2 g_1 = (1 - x_5) \frac{\partial}{\partial x_1},$$
$$g_2 = \frac{\partial}{\partial x_5},$$
$$ad_f g_2 = -\frac{\partial}{\partial x_3},$$

with dim $\Delta = 5$ on an open set of 0 such that $x_5 \neq 1$ and $v_1 = 3$ and $v_2 = 2$.

Then it is easy to check that the distribution

$$\bar{\Delta} = \text{span}\{g_1, ad_f g_1, g_2\}$$

is involutive. Thus, the first condition of Theorem 4.1 is fulfilled. The second condition of Theorem 4.1 is obviously satisfied, because g_2 commutes with g_1 and $ad_f g_1$, i.e. $[g_2, g_1] = [g_2, ad_f g_1] = 0$. Hence Theorem 4.1 is satisfied and the studied system can be locally transformed into the proposed 0-flat canonical form by a local diffeomorphism.

In order to deduce the local diffeomorphism, let us calculate the codistribution of $\bar{\Delta}$. It is easy to see that

$$\bar{\Delta}^\perp = \text{span}\{dh_1, dh_2\},$$

where $h_1 = x_3 x_2 - x_1$ and $h_2 = x_3$, which gives the diffeomorphism

$$
\begin{aligned}
z_{1,1} &= h_1, \\
z_{2,1} &= L_f h_1 = (x_5 - 1)x_2, \\
z_{3,1} &= L_f^2 h_1 = (x_5 - 1)x_4, \\
z_{1,2} &= h_2, \\
z_{2,2} &= L_f h_2 = x_5
\end{aligned}
$$

by which the studied system is locally transformed into the following 0-flat form:

$$
\begin{cases}
\dot{z}_{1,1} = z_{2,1}, \\
\dot{z}_{2,1} = z_{3,1} + \dfrac{z_{2,1}}{z_{2,2} - 1} u_2, \\
\dot{z}_{3,1} = (z_{2,2} - 1)u_1 + \dfrac{z_{3,1}}{z_{2,2} - 1} u_2, \\
\dot{z}_{1,2} = z_{2,2}, \\
\dot{z}_{2,2} = u_2.
\end{cases}
$$

Remark 8:

- The specialists of linearisation by dynamic states feedback will argue that the above discussed system is linearisable, but we should remark that here we do not need the derivatives of u.
- If, instead of the first dynamic $\dot{x}_1 = x_2 + x_4 x_3$ we take the same dynamic $\dot{x}_1 = x_4 x_3$ as in da Silva (2000a), then, Δ is of rank 5 on an open dense of 0. In this case, the same flat outputs work well except that $dy_1(ad_f^3 g_1) \neq 0$ and $dy_2(ad_f^2 g_2) \neq 0$ on an open dense subset.
- We think that we can generalise Theorem 4.1 by assuming that distribution Δ is of dimension n in a dense subset of $\mathcal{X}$ and $\bar{\Delta}$ is regular on $\mathcal{X}$.

Let us give another example to highlight the second condition of Theorem 4.1.

Example 4.4: Consider the following dynamical system:

$$
\begin{cases}
\dot{x}_1 = x_2 + \beta u_2 + ((1 + x_3)\beta + x_5)u_3, \\
\dot{x}_2 = x_3 + x_4 u_2 + x_3 u_3, \\
\dot{x}_3 = u_1, \\
\dot{x}_4 = x_5, \\
\dot{x}_5 = u_2 + x_3 u_3, \\
\dot{x}_6 = x_3 x_5 e^{x_4} + e^{x_4} u_1 + u_3,
\end{cases}
$$

where $\beta = x_6 - x_3 e^{x_4}$.

The straightforward calculation shows that the distribution Δ is generated by the following vector fields:

$$g_1 = \frac{\partial}{\partial x_3} + e^{x_4} \frac{\partial}{\partial x_6},$$

$$ad_f g_1 = -\frac{\partial}{\partial x_2},$$

$$ad_f^2 g_1 = \frac{\partial}{\partial x_1},$$

$$g_2 = \frac{\partial}{\partial x_5} + \beta \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_2},$$

$$ad_f g_2 = -\frac{\partial}{\partial x_4} - x_4 \frac{\partial}{\partial x_1} - x_3 e^{x_4} \frac{\partial}{\partial x_6} + x_5 \frac{\partial}{\partial x_2},$$

$$g_3 = \frac{\partial}{\partial x_6} + x_3 \frac{\partial}{\partial x_5} + x_3 \frac{\partial}{\partial x_2} + (x_5 + (1 + x_3)\beta) \frac{\partial}{\partial x_1}.$$

Thus $v_1 = 3$, $v_2 = 2$ and $v_3 = 1$.

It is easy to see that

$$[g_2, g_1] = 0 \in \text{span}\{g_1, ad_f g_1\},$$

$$[g_3, g_1] = -\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_5} - \beta \frac{\partial}{\partial x_1} \in \text{span}\{g_1, ad_f g_1, g_2\}.$$

Thus, the second condition of Theorem 4.1 is fulfilled.

In fact, the first condition of Theorem 4.1 is also fulfilled, since the distribution

$$\bar{\Delta} = \text{span}\{g_1, ad_f g_1, g_2\}$$

is involutive, which gives the following annihilator:

$$\bar{\Delta}^\perp = \text{span}\{dh_1, dh_2, dh_3\},$$

where $h_1 = x_1 - x_5(x_6 - x_3 e^{x_4})$, $h_2 = x_4$ and $h_3 = x_6 - x_3 e^{x_4}$.

By setting $z_{1,1} = h_1$, $z_{1,2} = h_2$ and $z_{1,3} = h_3$, we obtain the following diffeomorphism:

$$
\begin{aligned}
z_{1,1} &= x_1 - x_5(x_6 - x_3 e^{x_4}), \\
z_{2,1} &= L_f h_1 = x_2 \quad \text{and} \quad z_{3,1} = L_f^2 h_1 = x_3, \\
z_{1,2} &= h_2 = x_4 \quad \text{and} \quad z_{2,2} = L_f h_2 = x_5, \\
z_{1,3} &= h_3 = x_6 - x_3 e^{x_4},
\end{aligned}
$$

which locally transforms the dynamic into the following 0-flat form (4), (5):

$$\begin{cases} \dot{z}_{1,1} = z_{2,1} + z_{1,3}u_3, \\ \dot{z}_{2,1} = z_{3,1} + z_{1,2}u_2 + z_{3,1}u_3, \\ \dot{z}_{3,1} = u_1, \\ \dot{z}_{1,2} = z_{2.2}, \\ \dot{z}_{2,2} = u_2 + z_{3.1}u_3, \\ \dot{z}_{1,3} = u_3. \end{cases}$$

5. Conclusion

A triangular 0-flat canonical form was proposed in this article, which characterises a class of nonlinear dynamical systems. Sufficient and necessary geometrical conditions were given in order to deduce a local diffeomorphism with which the studied systems can be locally transformed into the proposed triangular 0-flat canonical form. Those conditions appear to be a natural generalisation of that of codimension 1 dynamical systems.

Acknowledgements

Note

1. Other definitions of observability indices can be found in Charlet, Lévine, and Marino (1989).

References

Anderson, R., and Ibragimov, N (1979), *Lie–Bäcklund Transformations in Applications*, Philadelphia: SIAM.

Bououden, S., Boutat, D., Barbot, J.P., and Kratz, F. (2009), 'A Geometrical Characterisation of a Class of 0-flat Affine Dynamical Systems', in *Proceedings of IEEE ACC*, St. Louis, MO, pp. 796–801.

Cartan, E. (1914), 'Sur l'Équivalence Absolue De Certains Systèmess d'Équations Différentielles et sur Certaines Familles de Courbes', *Bulletin de Société Mathématique de France*, 42, 12–48.

Charlet, B., Lévine, J., and Marino, R. (1989), 'On Dynamic Feedback Linearisation', *Systems & Control Letters*, 13, 143–151.

da Silva, P.S.P. (2000a), 'Flatness of Nonlinear Control Systems: A Cartan–Kähler Approach', in *Proceedings of Mathematical Theory of Networks and Systems*, pp. 1–11.

da Silva, P.S.P. (2000b), 'Flatness of Nonlinear Control Systems and Exterior Differential Systems', in *Nonlinear Control in the Year 2000*, eds. A. Isidori, F. Lamnabhi-Lagarrigue and W. Respondek, Berlin: Springer-Verlag, pp. 205–227.

Delaleau, E., and da Silva, P.S.P. (1998), 'Filtrations in Feedback Synthesis: Part I – Systems and Feedbacks', *Forum Mathematicum*, 10, 147–174.

Descusse, J., and Moog, C. (1985), 'Decoupling with Dynamic Compensation for Strong Invertible Affine Nonlinear Systems', *International Journal of Control*, 42, 1387–1398.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1992), 'Sur les Systèmes Non-linénaires Différentiellement Plats', *Comptes Rendus Mathématique, Académie des Sciences, Paris, Séries I, Mathématiques*, 315, 619–624.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1993), 'Linéarisation par Bouclage Dynamique et Transformations de Lie–Bäcklund', *Comptes Rendus Mathématique, Académie des Sciences, Paris, Series I, Mathématiques*, 317, 981–986.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1995), 'Flatness and Defect of Non-linear Systems: Introductory Theory and Examples', *International Journal of Control*, 61, 1327–1361.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1997), 'Deux Applications de la géométrie Locale des Diffiétés', *Annales de l'Institut Henri Poincare, Physics Théorique*, 66, 275–292.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1998), 'Nonlinear Control and Diffieties, with an Application to Physics', in *Secondary Calculus and Cohomological Physics* (Vol. 219), eds. J. Krasilśhchik, M. Henneaux and A. Vinogradov, Providence, RI: American Mathematical Society, pp. 81–92.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1999a), 'A Lie–Bäcklund Approach to Equivalence and Flatness of Nonlinear Systems', *IEEE Transactions on Automatic Control*, 44, 922–937.

Fliess, M., Lévine, J., Martin, P., and Rouchon, P. (1999b), 'Some Open Question Related to Flat Nonlinear Systems', in *Open Problems in Mathematical Systems and Control Theory*, eds. V.D. Blondel, E. Sontag, M. Vidyasagar and J.C. Willems, London: Springer Verlag, pp. 99–103.

Hagenmeyer, V., and Delaleau, E. (2003), 'Robustness Analysis of Exact Feedforward Linearisation Based on Differential Fatness', *Automatica*, 39, 1941–1946.

Hunt, L.R., Su, R., and Meyer, G. (1983), 'Design for Multi-input Nonlinear Systems', in *Differential Geometric Methods in Nonlinear Control Theory*, eds. R. Brocket, R. Millmann and H.J. Sussmann, Boston, Ma: Birkhäuser, pp. 268–298.

Jakubczyk, B., and Respondek, W. (1980), 'On Linearisation of Control Systems', *Bulletin de l'Academie Polonaise des Sciences, Series des Sciences Mathematiques*, 28, 517–522.

Lévine, J. (2004), 'On Flatness Necessary and Sufficient Conditions', in *Proceedings of NOLCOS*, pp. 123–128.

Lévine, J. (2009), *Analysis and Control of Nonlinear Systems: A Flatness-based Approach*, Heidelberg: Springer-Verlag.

Martin, P., Murray, R., and Rouchon, P. (1997), 'Flat Systems', in *European Control Conference*, pp. 211–264.

Martin, P., and Rouchon, P. (1995a), 'Any (Controllable) Driftless System with 3 Inputs and 5 States is Flat', *Systems & Control Letters*, 25, 167–173.

Martin, P., and Rouchon, P. (1995b), 'Any (controllable) Driftless System with m Inputs and $m+2$ States is Flat', in *Proceedings of the IEEE Conference on Decision and Control*, pp. 2886–2891.

Pomet, J.B. (1995a), 'A Differential Geometric Setting for Dynamic Equivalence and Dynamic Linearization', in *Geometry in Nonlinear Control and Differential inclusions*, eds. B. Jackubczyk, W. Respondek and T. Rzezuchowski, Warsaw: Banach Center Publications, pp. 319–339.

Pomet, J.B. (1995b), 'On Dynamic Feedback Linearization of Four-dimensional Affine Control Systems with Two Inputs', Technical, Memorandum RR-2751, I.N.R.I.A., Books and Monographs, http://www.emath.fr/

Rotella, F. and Zambettakis, I. (2007), 'Commande des Systèmes par Platitude', Techniques de l'Ingènieurs, http://www.techniques-ingenieur.fr, 57450.

Rouchon, P. (1995), 'Necessary Condition and Genericity of Dynamic Feedback Linearisation', *Journal of Mathematical Systems, Estimation, and Control*, 5, 345–358.

Rudolph, J. (1995), 'Well-formed Dynamics Under Quasi-static State Feedback', in *Geometry in Nonlinear Control and Differential Inclusions*, eds. B. Jackubczyk, W. Respondek and T. Rzezuchowski, Warsaw: Banach Center Publications, pp. 349–360.

Schlacher, K., and Schöberl, M. (2007), 'Construction of Flat Outputs by Reduction and Elimination', in *Proceedings of NOLCOS*, pp. 666–671.

Shadwick, W. (1990), 'Absolute Equivalence and Dynamic Feedback Linearisation', *Systems and Control Letters*, 15, 35–39.

Shadwick, W., and Sluis, W. (1994), 'Dynamic Feedback for Classical Geometries', in *Differential Geometry and Mathematical Physics*, eds. J.K. Beem and K.L. Duggal, Providence, RI: American Mathematical Society, pp. 207–213.

Silveira, H. (2009), 'Formas Triangulares para Sistemas não Lineares com duas Entradas e Controle de Sistemas sem Arrasto em SU(n) com Aplicações em Mecânica Quântica', Ph.D. Thesis, Universidade de São Paulo, Brazil.

Silveira, H., and da Silva, P.S.P. (2007), 'A Triangular Form for Flat Nonlinear Systems with Two Controls and Five or Less States', in *Proceedings of the 3rd IFAC Symposium on System, Structure and Control*, pp. 127–132.

Singh, S. (1981), 'A Modified Algorithm for Invertibility in Nonlinear Systems', *IEEE Transactions on Automatic Control*, 26, 595–598.

Sira-Ramirez, H., and Agrawal, S. (2004), *Differentially Flat Systems*, New York: Marcel Dekker.

Sluis, W. (1993), 'A Necessary Condition for Dynamic Feedback Linearisation', *Systems and Control Letters*, 21, 277–283.

Sluis, W., and Tilbury, D. (1996), 'A Bound on the Number of Integrators Needed to Linearise a Control System', *Systems and Control Letters*, 29, 43–50.

Tilbury, D., Murray, R., and Sastry, S. (1995), 'Trajectory Generation for the n-trailer Problem using Goursat Normal Form', *IEEE Transactions on Automatic Control*, 40, 802–819.

van Nieuwstadt, M., Rathinam, M., and Murray, R. (1998), 'Differential Flatness and Absolute Equivalence of Nonlinear Control Systems', *SIAM Journal of Control Optimization*, 36, 1225–1239.