Stokes Equations and Elliptic Systems With Non Standard Boundary Conditions

Chérif Amrouche, Nour El Houda Seloula

Stokes Equations and Elliptic Systems With Non Standard Boundary Conditions

Chérif Amrouche [a], Nour El Houda Seloula [a,b]

[a] *Laboratoire de Mathématiques Appliquées, CNRS UMR 5142, Université de Pau et des Pays de l'Adour, IPRA, Avenue de l'Université- 64000 Pau (France)*

[b] *EPI Concha, LMA UMR CNRS 5142, INRIA Bordeaux-Sud-Ouest, 64000 Pau (France)*

Received *****; accepted after revision +++++

Presented by

Abstract

In a three dimensional bounded possibly multiply-connected domain of class $\mathcal{C}^{1,1}$, we consider the stationary Stokes equations with nonstandard boundary conditions of the form $\boldsymbol{u}\cdot\boldsymbol{n} = g$ and $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ or $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ and $\pi = \pi_0$ on the boundary Γ. We prove the existence and uniqueness of weak, strong and very weak solutions corresponding to each boundary condition in L^p theory. Our proofs are based on obtaining $Inf-Sup$ conditions that play a fundamental role. And finally, we give two Helmholtz decompositions that consist of two kinds of boundary conditions such as $\boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u}\times\boldsymbol{n}$ on Γ.

To cite this article: C. Amrouche, N. Seloula, C. R. Acad. Sci. Paris, Ser. I 340 (2010).

Résumé

Équations de Stokes et systèmes elliptiques avec des conditions aux limites non standard. Dans un ouvert borné tridimensionnel, éventuellement multiplement connexe de classe $\mathcal{C}^{1,1}$, nous considérons les équations stationnaires de Stokes avec des conditions aux limites de la forme $\boldsymbol{u}\cdot\boldsymbol{n} = g$ et $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ ou $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ et $\pi = \pi_0$ sur le bord Γ. Nous prouvons l'existence et l'unicité des solutions faibles, fortes et très faibles en théorie L^p. Nos preuves sont basées sur l'obtention de conditions $Inf-Sup$ qui jouent un rôle fondamental. Finalement, on donne deux décompositions d'Helmholtz qui tiennent compte des deux types de conditions aux limites $\boldsymbol{u}\cdot\boldsymbol{n}$ et $\boldsymbol{u}\times\boldsymbol{n}$ sur Γ.

Pour citer cet article : C. Amrouche, N. Seloula, C. R. Acad. Sci. Paris, Ser. I 340 (2010).

Email addresses: `cherif.amrouche@univ-pau.fr` (Chérif Amrouche), `nourelhouda.seloula@etud.univ-pau.fr` (Nour El Houda Seloula).

Version française abrégée

L'objet de cette Note consiste essentiellement à étudier en théorie L^p avec $1 < p < \infty$, l'existence et l'unicité de solutions faibles, fortes et très faibles pour les équations stationnaires de Stokes $(\mathcal{S}_T)$ dans le cas des conditions aux limites : $\boldsymbol{u}\cdot\boldsymbol{n} = g$ et $\mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n}$ sur Γ et $(\mathcal{S}_N)$ dans le cas des conditions aux limites : $\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ et $\pi = \pi_0$ sur Γ. Les résultats concernant l'existence de solutions faibles et fortes pour $(\mathcal{S}_T)$ sont donnés dans le Théorème 2.1 ; et en ce qui a trait à $(\mathcal{S}_N)$, les résultats sont donnés dans le Théorème 3.2. Pour la preuve de solutions très faibles pour $(\mathcal{S}_T)$ et $(\mathcal{S}_N)$, l'une des difficultées consiste à donner un sens aux traces sur le bord.

1. Introduction

Let Ω a bounded open connected set of $\mathbb{R}^3$ of class $\mathrm{C}^{1,1}$ whith boundary Γ. Let Γ_i, $0 \le i \le I$, denote the connected components of the boundary Γ, Γ_0 being the exterior boundary of Ω. We do not assume that Ω is simply-connected but we suppose that there exist J connected open surfaces Σ_j, $1 \le j \le J$, called 'cuts', contained in Ω, such that each surface Σ_j is an open subset of a smooth manifold, the boundary of Σ_j is contained in Γ. The intersection $\overline{\Sigma_i}\cap\overline{\Sigma_j}$ is empty for $i \ne j$, and finally the open set $\Omega^\circ = \Omega\setminus\cup_{j=1}^{J}\Sigma_j$ is simply-connected and pseudo-Lipschitz (see [1]). We are interested in some questions concerning the stationary Stokes equations with non standard boundary conditions, that generally can be written as:

$$(\mathcal{S}_T)\ \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\cdot\boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} & \text{on}\,\Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0, \quad 1 \le j \le J, \end{cases} \quad (\mathcal{S}_N)\ \begin{cases} -\Delta\,\boldsymbol{u} + \nabla\,\pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div}\boldsymbol{u} = 0 & \text{in}\,\Omega,\\ \boldsymbol{u}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\,\Gamma,\\ \langle \boldsymbol{u}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0, \quad 1 \le i \le I, \end{cases}$$

where $\boldsymbol{u}$ denotes the velocity field and π the pressure, both being unknown, and $\boldsymbol{f}$, g, $\boldsymbol{h}$, $\boldsymbol{g}$ and π_0 are given.
To prove the existence of solutions of problems $(\mathcal{S}_T)$ and $(\mathcal{S}_N)$ (see the sketch of the proofs of Theorem 2.1 for $(\mathcal{S}_T)$ and Theorem 3.2 for $(\mathcal{S}_N)$) we begin by solving pressure π as a solution of a Neumann problem or Dirichlet problem. Then, we are reduced to solve the following elliptic problems:

$$(E_T)\ \ -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \ \text{ and }\ \operatorname{div}\boldsymbol{\xi} = 0 \ \text{ in }\Omega,\ \ \boldsymbol{\xi}\cdot\boldsymbol{n} = g \ \text{ and }\ \mathbf{curl}\,\boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{h}\times\boldsymbol{n} \ \text{ on }\Gamma,\ \ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j} = 0,$$

$$(E_N)\quad -\Delta\,\boldsymbol{\xi} = \boldsymbol{f} \ \text{ and }\ \operatorname{div}\boldsymbol{\xi} = 0 \ \text{ in }\Omega, \quad \boldsymbol{\xi}\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n} \ \text{ on }\Gamma,\ \ \langle\boldsymbol{\xi}\cdot\boldsymbol{n},\,1\rangle_{\Gamma_i} = 0,\ \ 1 \le i \le I.$$

We denote by $[\cdot]_j$ the jump of a function over Σ_j, for $1 \le j \le J$ and $\langle\cdot,\cdot\rangle_{X,X'}$ denotes the duality product between a space X and X'. For any $1 < p < \infty$, we then define the spaces:

$$\boldsymbol{H}^{\,p}(\mathbf{curl},\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \mathbf{curl}\ \boldsymbol{v}\in\boldsymbol{L}^p(\Omega)\}\,,\ \ \boldsymbol{H}^{\,p}(\operatorname{div},\Omega) = \{\boldsymbol{v}\in\boldsymbol{L}^p(\Omega);\ \operatorname{div}\ \boldsymbol{v}\in L^p(\Omega)\}$$

$$\boldsymbol{X}^{\,p}(\Omega) = \boldsymbol{H}^{\,p}(\mathbf{curl},\Omega)\cap\boldsymbol{H}^{\,p}(\operatorname{div},\Omega),$$

which are equipped with the graph norm, and their subspaces:

$$\boldsymbol{H}^{p}_{0}(\mathbf{curl},\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^{\,p}(\mathbf{curl},\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}\text{ on }\Gamma\}\,,\ \boldsymbol{H}^{p}_{0}(\operatorname{div},\Omega) = \{\boldsymbol{v}\in\boldsymbol{H}^{\,p}(\operatorname{div},\Omega);\ \boldsymbol{v}\cdot\boldsymbol{n} = 0\text{ on }\Gamma\}\,,$$

$$\boldsymbol{X}^{\,p}_{N}(\Omega) = \{\boldsymbol{v}\in\boldsymbol{X}^{\,p}(\Omega);\ \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}\text{ on }\Gamma\}\,,\quad \boldsymbol{X}^{\,p}_{T}(\Omega) = \{\boldsymbol{v}\in\boldsymbol{X}^{\,p}(\Omega);\boldsymbol{v}\cdot\boldsymbol{n} = 0\text{ on }\Gamma\}$$

and $\boldsymbol{X}^{\,p}_{0}(\Omega) = \boldsymbol{X}^{\,p}_{N}(\Omega)\cap\boldsymbol{X}^{\,p}_{T}(\Omega)$. We also define the space $\boldsymbol{W}^{1,p}_{\sigma}(\Omega) = \{\boldsymbol{v}\in\boldsymbol{W}^{\,1,p}(\Omega),\ \operatorname{div}\boldsymbol{v} = 0\text{ in }\Omega\}$. For any function q in $W^{\,1,p}(\Omega^\circ)$, $\mathbf{grad}\,q$ can be extented to $\boldsymbol{L}^p(\Omega)$. We denote this extension by $\widetilde{\mathbf{grad}}\,q$. We finally define the spaces:

$$\boldsymbol{K}^{p}_{T}(\Omega)=\{\boldsymbol{v}\in \boldsymbol{X}^{p}_{T}(\Omega),\ \mathbf{curl}\,\boldsymbol{v}=\mathbf{0},\ \mathrm{div}\,\boldsymbol{v}=0\ \mathrm{in}\,\Omega\},$$

$$\boldsymbol{K}^{p}_{N}(\Omega)=\{\boldsymbol{v}\in \boldsymbol{X}^{p}_{N}(\Omega),\ \mathbf{curl}\,\boldsymbol{v}=\mathbf{0},\ \mathrm{div}\,\boldsymbol{v}=0\ \mathrm{in}\,\Omega\}.$$

We know due to [4] (see also [1] for the case $p=2$) that the space $\boldsymbol{K}^{p}_{T}(\Omega)$ is of dimension J and that it is spanned by functions $\widetilde{\mathbf{grad}}\,q^T_j$, $1\le j\le J$, where each $q^T_j\in W^{1,p}(\Omega^\circ)$. Similarly, the dimension of the space $\boldsymbol{K}^{p}_{N}(\Omega)$ is I and that it is spanned by the functions $\mathbf{grad}\,q^N_i$, $1\le i\le I$, where each $q^N_i\in W^{1,p}(\Omega)$. In what follows, the letter C denotes a constant that does not necessarily have the same value. The detailled proofs of the results announced in this Note are given in [4].

2. The Stokes equations with the tangential boundary conditions

We can prove that by assuming appropriate conditions on $\boldsymbol{f}$ and $\boldsymbol{h}$, the pressure in the problem (S_T) may be constant, and we are reduced to solve the elliptic system (E_T):

Proposition 2.1. *Let* $\boldsymbol{f}$ *belongs to* $\boldsymbol{L}^p(\Omega)$ *with* $\mathrm{div}\,\boldsymbol{f}=0$ *in* Ω , $g\in W^{1-\frac{1}{p},p}(\Gamma)$ *and* $\boldsymbol{h}\in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)$ *verify the following compatibility conditions:*

$$\boldsymbol{f}\cdot\boldsymbol{n}+\mathrm{div}_\Gamma\,(\boldsymbol{h}\times\boldsymbol{n})=0\quad\text{on }\Gamma, \tag{1}$$

$$\forall \boldsymbol{v}\in \boldsymbol{K}^{p'}_{T}(\Omega),\quad \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}+\langle \boldsymbol{h}\times\boldsymbol{n},\,\boldsymbol{v}\rangle_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)\times \boldsymbol{W}^{\frac{1}{p},p'}(\Gamma)}=0\quad\text{and}\quad \int_\Gamma g\,\mathrm{d}\boldsymbol{\sigma}=0, \tag{2}$$

where div_Γ *is the surface divergence on* Γ. *Then, the problem* (E_T) *has a unique solution* $\boldsymbol{u}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *satisfying the estimate:*

$$\|\,\boldsymbol{u}\,\|_{\boldsymbol{W}^{1,p}(\Omega)}\le C\Big(\|\,\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}+\|\,g\,\|_{W^{1-1/p,p}(\Gamma)}+\|\,\boldsymbol{h}\times\boldsymbol{n}\,\|_{\boldsymbol{W}^{-1/p,p}(\Gamma)}\Big).$$

Moreover, if $g\in W^{2-1/p,p}(\Gamma)$ *and* $\boldsymbol{h}\in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, *then the solution* $\boldsymbol{u}$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega)$ *and satisfies the corresponding estimate.*

Sketch of the proof. For the proof of weak solutions, we reduce (E_T) to a problem having homogeneous normal boundary condition on Γ, where it is easy to solve it by using the $Inf-Sup$ condition: (see [4])

$$\inf_{\boldsymbol{\varphi}\in \boldsymbol{V}^{p'}_{T}(\Omega)}\ \sup_{\boldsymbol{u}\in \boldsymbol{V}^{p}_{T}(\Omega)}\frac{\int_\Omega \mathbf{curl}\,\boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}}{\|\boldsymbol{u}\,\|_{\boldsymbol{X}^p_T(\Omega)}\|\boldsymbol{\varphi}\,\|_{\boldsymbol{X}^{p'}_T(\Omega)}}>0, \tag{3}$$

with $\boldsymbol{V}^{p}_{T}(\Omega)=\{\boldsymbol{v}\in \boldsymbol{X}^{p}_{T}(\Omega),\ \mathrm{div}\,\boldsymbol{v}=0\text{ in }\Omega,\ \langle \boldsymbol{v}\cdot\boldsymbol{n},\,1\rangle_{\Sigma_j}=0\}$. For the regularity, we set $\boldsymbol{z}=\mathbf{curl}\,\boldsymbol{u}$. Since $\boldsymbol{z}\times\boldsymbol{n}\in \boldsymbol{W}^{1-1/p,p}(\Gamma)$, we deduce from [4] that $\boldsymbol{z}\in \boldsymbol{W}^{1,p}(\Omega)$. Therefore, since $\boldsymbol{u}\cdot\boldsymbol{n}\in W^{2-1/p,p}(\Gamma)$, then $\boldsymbol{u}\in \boldsymbol{W}^{2,p}(\Omega)$. □

Theorem 2.1. **(Weak and Strong solutions for** $(\mathcal{S}_\mathrm{T})$**)** *Let* $\boldsymbol{f}$, g, $\boldsymbol{h}$ *with:*

$$\boldsymbol{f}\in(\boldsymbol{H}^{p'}_0(\mathrm{div},\,\Omega))',\quad g\in W^{1-\frac{1}{p},p}(\Gamma),\quad \boldsymbol{h}\in \boldsymbol{W}^{-\frac{1}{p},p}(\Gamma), \tag{4}$$

and verify the compatibility conditions (2). *Then, the Stokes problem* $(\mathcal{S}_T)$ *has a unique solution* $(\boldsymbol{u},\,\pi)\in \boldsymbol{W}^{1,p}(\Omega)\times L^p(\Omega)/\mathbb{R}$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)}+\|\pi\|_{L^p(\Omega)}\le C\big(\|\,\boldsymbol{f}\|_{(\boldsymbol{H}^{p'}_0(\mathrm{div},\,\Omega))'}+\|g\|_{W^{1-\frac{1}{p},p}(\Gamma)}+\|\boldsymbol{h}\times\boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

Moreover, if $\boldsymbol{f}\in \boldsymbol{L}^p(\Omega)$, $g\in W^{2-\frac{1}{p},p}(\Gamma)$, $\boldsymbol{h}\in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, *the solution* $(\boldsymbol{u},\,\pi)$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega)\times W^{1,p}(\Omega)$ *and satisfies the corresponding estimate.*

Sketch of the proof. We reduce $(\mathcal{S}_T)$ to a problem with the homogeneous normal boundary condition on Γ. We use again the $Inf-Sup$ condition (3) in order to prove the existence of a unique $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ solution of $(\mathcal{S}_T)$ and by using De Rham's Theorem, we prove the existence of a unique $\pi \in L^p(\Omega)$. For the regularity of the solution, we observe that π satisfies: $\operatorname{div}(\nabla \pi - \boldsymbol{f}) = 0$ in Ω and $(\nabla \pi - \boldsymbol{f}) \cdot \boldsymbol{n} = -\operatorname{div}_\Gamma(\boldsymbol{h} \times \boldsymbol{n})$ on Γ which implies that π belongs to $W^{1,p}(\Omega)$. We deduce the regularity of $\boldsymbol{u}$ from Proposition 2.1 since $\boldsymbol{u}$ is a solution of a problem (E_T) with the right hand side $\boldsymbol{F} = \boldsymbol{f} - \nabla \pi$. □

Remark 2.2. We can also treat the case when the divergence operator does not vanish. So we consider the following Stokes problem

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} \text{ and } \operatorname{div} \boldsymbol{u} = \chi & \text{in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = g \quad \text{and} \quad \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{h} \times \boldsymbol{n} & \text{on } \Gamma, \qquad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Sigma_j} = 0, \quad 1 \le j \le J. \end{cases} \tag{5}$$

If we suppose that χ belongs to $L^p(\Omega)$, $\boldsymbol{f}$, g, $\boldsymbol{h}$ as in (4) satisfying the first compatibility condition in (2) and such that

$$\int_\Omega \chi \,\mathrm{d}\boldsymbol{x} = \int_\Gamma g \,\mathrm{d}\boldsymbol{\sigma}, \tag{6}$$

then, we can prove that the Stokes problem (5) has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times L^p(\Omega)$ satisfying the estimate:

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p}(\Omega)} + \|\pi\|_{L^p(\Omega)} \le C\big(\|\boldsymbol{f}\|_{(\boldsymbol{H}_0^{p'}(\mathrm{div},\,\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{1-\frac{1}{p},p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-\frac{1}{p},p}(\Gamma)}\big).$$

Moreover, if we suppose that $\chi \in W^{1,p}(\Omega)$ with $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $g \in W^{2-\frac{1}{p},p}(\Gamma)$, $\boldsymbol{h} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma)$, then the solution $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satifies the corresponding estimate.

We define now the spaces: $\boldsymbol{T}^p(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{H}_0^p(\mathrm{div},\,\Omega);\ \operatorname{div} \boldsymbol{\varphi} \in W_0^{1,p}(\Omega)\}$, $\boldsymbol{Y}_T^p(\Omega) = \{\boldsymbol{\varphi} \in \boldsymbol{W}^{2,p}(\Omega);\ \boldsymbol{\varphi} \cdot \boldsymbol{n} = 0,\ \operatorname{div} \boldsymbol{\varphi} = 0,\ \mathbf{curl}\, \boldsymbol{\varphi} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}$ and $\boldsymbol{H}_p(\Delta;\,\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \Delta \boldsymbol{v} \in (\boldsymbol{T}^{p'}(\Omega))'\}$, endowed with the corresponding graph norms. Note that $\boldsymbol{\mathcal{D}}(\Omega)$ is dense in $\boldsymbol{T}^p(\Omega)$ and then $[\boldsymbol{T}^p(\Omega)]'$ is a subspace of $\boldsymbol{\mathcal{D}}'(\Omega)$.

Theorem 2.3. **(Very weak solutions for $(\mathcal{S}_\mathrm{T})$)** *Let* $\boldsymbol{f}$, χ, g, *and* $\boldsymbol{h}$ *with*

$$\boldsymbol{f} \in (\boldsymbol{T}^{p'}(\Omega))',\ \chi \in L^p(\Omega),\ g \in W^{-1/p,p}(\Gamma),\ \boldsymbol{h} \in \boldsymbol{W}^{-1-1/p,p}(\Gamma),$$

and satisfying the first compatibility condition in (2) *and* (6). *Then, the Stokes problem* (5) *has exactly one solution* $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta;\,\Omega)$ *and* $\pi \in W^{-1,p}(\Omega)/\mathbb{R}$ *satisfying the estimate:*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}_p(\Delta;\,\Omega)} + \|\pi\|_{W^{-1,p}(\Omega)/\mathbb{R}} \le C\Big(\|\boldsymbol{f}\|_{(\boldsymbol{T}^{p'}(\Omega))'} + \|\chi\|_{L^p(\Omega)} + \|g\|_{W^{-1/p,p}(\Gamma)} + \|\boldsymbol{h} \times \boldsymbol{n}\|_{\boldsymbol{W}^{-1-1/p,p}(\Gamma)}\Big).$$

Sketch of the proof. First, we prove the density of the space $\mathcal{D}(\overline{\Omega})$ in $\boldsymbol{H}_p(\Delta;\,\Omega)$. Second, we prove that the mapping $\gamma:\ \boldsymbol{u} \mapsto \mathbf{curl}\, \boldsymbol{u}|_\Gamma \times \boldsymbol{n}$ on the space $\boldsymbol{\mathcal{D}}(\overline{\Omega})$ can be extended by continuity to a linear and continuous mapping still denoted by γ, from $\boldsymbol{H}_p(\Delta;\,\Omega)$ into $\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma)$ and we have the following Green formula: for any $\boldsymbol{u} \in \boldsymbol{H}_p(\Delta;\,\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{Y}_T^{p'}(\Omega)$,

$$\langle \Delta \boldsymbol{u},\, \boldsymbol{\varphi} \rangle_{(\boldsymbol{T}^{p'}(\Omega))' \times \boldsymbol{T}^{p'}(\Omega)} = \int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{\varphi} \,\mathrm{d}\boldsymbol{x} + \langle \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n},\, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-1-\frac{1}{p},p}(\Gamma) \times \boldsymbol{W}^{1+1/p,p'}(\Gamma)}, \tag{7}$$

Finally, using the formula (7), we can write an equivalent variational formulation of the problem (5) and we are able to conclude by using the regularity result for its dual problem presented in Corollary **??**. □

3. The Stokes equations with the normal boundary conditions

In this section, we focus on the study of the Stokes problem $(\mathcal{S}_N)$. Observe that the pressure p can be obtained independently of the velocity as a solution of a Dirichlet problem. So, the velocity $\boldsymbol{u}$ is a solution of an elliptic system of type (E_N).

Proposition 3.1. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega))'$ *with* $\operatorname{div} \boldsymbol{f} = 0$ *in* Ω *and* $\boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma)$ *satisfying the compatibility condition:*

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} = 0 \quad \text{for } 0 \leq i \leq I. \tag{8}$$

Then, the problem (E_N) *has a unique solution* $\boldsymbol{u}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *satisfying the estimate:*

$$\|\, \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\big(\| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'} + \| \boldsymbol{g} \times \boldsymbol{n} \|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)}\big).$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$ *and* $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$*, then the solution* $\boldsymbol{u}$ *is in* $\boldsymbol{W}^{2,p}(\Omega)$ *and satisfies the corresponding estimate.*

Sketch of the proof. First, we lift the boundary condition and we write an equivalent variational formulation for the homogeneous problem as follows: find $\boldsymbol{u} \in \boldsymbol{V}_N^p(\Omega)$ such that

$$\forall \boldsymbol{\varphi} \in \boldsymbol{V}_N^{p'}(\Omega), \; \int_\Omega \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} = \langle \boldsymbol{f},\, \boldsymbol{\varphi} \rangle_\Omega, \tag{9}$$

where $\boldsymbol{V}_N^p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{X}_N^p(\Omega);\, \operatorname{div} \boldsymbol{w} = 0 \text{ in}\,\Omega \text{ and } \langle \boldsymbol{w} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0,\ 1 \leq i \leq I\}$. Next, using a result concerning normal vector potential [4], we establish a similar Inf-Sup condition to (3), where the spaces $\boldsymbol{X}_T^p(\Omega)$ and $\boldsymbol{V}_T^p(\Omega)$ are replaced by the spaces $\boldsymbol{X}_N^p(\Omega)$ and $\boldsymbol{V}_N^p(\Omega)$ respectively. This conclude the proof of weak solution. For the regularity of the velocity, we need some additional properties. We prove the following trace formula for any $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega)$:

$$\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} = \big(\sum_{j=1}^{2} \frac{\partial\, \boldsymbol{u}}{\partial\, s_j} \times \boldsymbol{\tau}_j\big) \cdot \boldsymbol{n} \quad \text{on}\,\Gamma, \quad \text{in the sense of } \boldsymbol{W}^{-1/\mathrm{p},\mathrm{p}}(\Gamma). \tag{10}$$

As a consequence, if we suppose that $\boldsymbol{u} \times \boldsymbol{n} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, then $\mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n} \in W^{1-1/p,p}(\Gamma)$. This implies that $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ and thereafter from [4], we have $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$. □

We can also treat the case of the following elliptic system, which is similar to (E_N) but where we have replaced the condition $\operatorname{div} \boldsymbol{u} = 0$ in Ω by $\operatorname{div} \boldsymbol{u} = 0$ on Γ.

$$(E'_N) \quad -\Delta\, \boldsymbol{u} = \boldsymbol{f} \text{ in } \Omega, \quad \operatorname{div} \boldsymbol{u} = 0 \text{ on } \Gamma, \quad \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma, \quad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, \boldsymbol{1} \rangle_{\Gamma_\mathrm{i}} = 0 \text{ for any } 1 \leq \mathrm{i} \leq \mathrm{I}.$$

Theorem 3.1. *Let* $\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega))'$ *satisfying the compatibility condition* (8)*. Then, the problem* (E'_N) *has a unique solution* $\boldsymbol{u}$ *in* $\boldsymbol{W}^{1,p}(\Omega)$ *satisfying the estimate:*

$$\|\, \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C \| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]'}. \tag{11}$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$*, then the solution* $\boldsymbol{u}$ *is in* $\boldsymbol{W}^{2,p}(\Omega)$ *and satisfies the corresponding estimate.*

Theorem 3.2. **(Weak and Strong solutions for** $(\mathcal{S}_\mathrm{N})$**)** *Let* $\boldsymbol{f}$, $\boldsymbol{g}$, π_0 *such that*

$$\boldsymbol{f} \in (\boldsymbol{H}_0^{p'}(\mathbf{curl}, \Omega))', \;\; \boldsymbol{g} \in \boldsymbol{W}^{1-1/p,p}(\Gamma), \;\; \pi_0 \in W^{1-1/p,p}(\Gamma), \tag{12}$$

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)]' \times \boldsymbol{H}_0^{p'}(\mathbf{curl},\Omega)} - \int_\Gamma \pi_0\, \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{\sigma} = 0, \tag{13}$$

then, the Stokes problem $(\mathcal{S}_N)$ *has a unique solution* $(\boldsymbol{u},\, \pi) \in \boldsymbol{W}^{1,p}(\Omega) \times W^{1,p}(\Omega)$ *satisfying the estimate*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)} \leq C\big(\| \boldsymbol{f} \|_{(\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega))'} + \| \boldsymbol{g} \times \boldsymbol{n} \|_{\boldsymbol{W}^{1-1/p,p}(\Gamma)} + \| \pi_0 \|_{W^{1-1/p,p}(\Gamma)} \Big). \qquad (14)$$

Moreover, if $\boldsymbol{f} \in \boldsymbol{L}^p(\Omega)$, $\boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma)$, $\pi_0 \in W^{1-1/p,p}(\Gamma)$, *then the solution* $(\boldsymbol{u},\, \pi)$ *belongs to* $\boldsymbol{W}^{2,p}(\Omega) \times W^{1,p}(\Omega)$*and satisfies the corresponding estimate. .*

Sketch of the proof. We note that the pressure is a solution of the following Dirichlet problem: $-\Delta\, \pi = \operatorname{div} \boldsymbol{f}$ in Ω and $\pi = \pi_0$ on Γ. Since $\pi_0 \in W^{1-1/p,p}(\Gamma)$, then $\pi \in W^{1,p}(\Omega)$. The velocity is a solution of the problem (E_N) and it suffices to apply Proposition 3.1 to obtain weak and strong solutions. □

Remark 3.3. Let $\boldsymbol{u}$ is a solution of (E_N). We set $\boldsymbol{u} = \nabla\, v$. Then the function v satisfies: $\Delta\, v = 0$ in Ω and $(\nabla\, v)_t = \boldsymbol{g}$ on Γ, where $(\nabla\, v)_t$ is the tangential component of $\nabla\, v$.
And more generally, we can also solve the Stokes problem $(\mathcal{S}_N)$ when the divergence operator does not vanish. More precisely, we consider the following Stokes pobelem:

$$\begin{cases} -\Delta\, \boldsymbol{u} + \nabla\, \pi = \boldsymbol{f} \ \text{ and } \ \operatorname{div} \boldsymbol{u} = \chi & \text{in}\, \Omega, \\ \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{g} \times \boldsymbol{n} \quad \text{and} \quad \pi = \pi_0 & \text{on}\, \Gamma, \qquad \langle \boldsymbol{u} \cdot \boldsymbol{n},\, 1 \rangle_{\Gamma_i} = 0, \quad 1 \leq i \leq I. \end{cases} \qquad (15)$$

Notice that in the corresponding theorem for the problem $(\mathcal{S}_T)$, we took $\chi \in L^p(\Omega)$. In the case of the problem $(\mathcal{S}_N)$, we can not suppose the same, because we need to solve (15), the fact that $\nabla\, \chi \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]'$, which is not checked because χ is only $L^p(\Omega)$.

Corollary 3.4. *For every* $\boldsymbol{f}$, χ, $\boldsymbol{g}$, π_0 *with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \ \ \chi \in W^{1,p}(\Omega), \ \ \boldsymbol{g} \in W^{1-1/p,p}(\Gamma), \ \ \pi_0 \in W^{1-1/p,p}(\Gamma) \qquad (16)$$

$$\forall \boldsymbol{v} \in \boldsymbol{K}_N^{p'}(\Omega), \qquad \langle \boldsymbol{f},\, \boldsymbol{v} \rangle_\Omega - \int_\Gamma (\pi_0 - \chi)\, \boldsymbol{v} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{\sigma} = 0. \qquad (17)$$

Then, the Stokes problem (15) *has exactly one solution* $\boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ *and* $\pi \in W^{1,p}(\Omega)/\mathbb{R}$. *Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{W}^{1,p}(\Omega)} + \| \pi \|_{W^{1,p}(\Omega)/\mathbb{R}} \leq C\Big(\| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \| \chi \|_{W^{1,p}(\Omega)} + \| \boldsymbol{g} \|_{W^{1-1/p,p}(\Gamma)} + \| \pi_0 \|_{W^{1-1/p,p}(\Gamma)} \Big).$$

Moreover, if we suppose that

$$\boldsymbol{f} \in \boldsymbol{L}^p(\Omega), \quad \boldsymbol{g} \in \boldsymbol{W}^{2-1/p,p}(\Gamma), \ \ \chi \in W^{1,p}(\Omega), \ \ \pi_0 \in W^{1-1/p,p}(\Gamma), \qquad (18)$$

then, the solution $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$ *and* $\pi \in W^{1,p}(\Omega)$ *satisfy the corresponding estimate.*

Theorem 3.5. **(Very weak solutions for** $(\mathcal{S}_\mathrm{N})$**)** *Let* $\boldsymbol{f}$, $\boldsymbol{g}$, *and* π_0 *with*

$$\boldsymbol{f} \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]', \ \ \boldsymbol{g} \in \boldsymbol{W}^{-1/p,p}(\Gamma), \ \ \pi_0 \in W^{-1/p,p}(\Gamma),$$

and satisfying the compatibility conditions (13). *Then, the Stokes problem* $(\mathcal{S}_N)$ *has exactly one solution* $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ *and* $\pi \in L^p(\Omega)/\mathbb{R}$. *Moreover, there exists a constant* $C > 0$ *depending only on* p *and* Ω *such that:*

$$\| \boldsymbol{u} \|_{\boldsymbol{L}^p(\Omega)} + \| \pi \|_{L^p(\Omega)/\mathbb{R}} \leq C\Big(\| \boldsymbol{f} \|_{[\boldsymbol{H}_0^{p'}(\mathbf{curl},\,\Omega)]'} + \| \boldsymbol{g} \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} + \| \pi_0 \|_{\boldsymbol{W}^{-1/p,p}(\Gamma)} \Big). \qquad (19)$$

Sketch of the proof. We use similar arguments presented for the case of problem $(\mathcal{S}_N)$ and the main difference between the two proofs is the fact that we prove a global Green formula. More precisely, we set the space

$$\boldsymbol{M}^p(\Omega) = \big\{ (\boldsymbol{u},\, \pi) \in \boldsymbol{Z}^p(\Omega) \times L^p(\Omega); \ \ -\Delta\, \boldsymbol{u} + \nabla\, \pi \in [\boldsymbol{H}_0^{p'}(\mathbf{curl},\, \Omega)]' \big\},$$

with $\boldsymbol{Z}^{\,p}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega \text{ and } \langle \boldsymbol{v}\cdot\boldsymbol{n},\, 1\rangle_{\Gamma_i},\ \ 1 \le i \le I\}$ and by establishing the density of $\boldsymbol{\mathcal{D}}_\sigma(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$ in $\boldsymbol{M}^{\,p}(\Omega)$, we prove that the trace of any $(\boldsymbol{u},\, \pi) \in \boldsymbol{M}^{\,p}(\Omega)$ belongs to $\boldsymbol{W}^{\,-1/p,p}(\Gamma) \times W^{\,-1/p,p}(\Gamma)$ with the following Green formula for any $\boldsymbol{\varphi} \in \boldsymbol{Y}_N^{\,p'}(\Omega)$:

$$\langle -\Delta\, \boldsymbol{u} + \nabla\, \pi,\ \boldsymbol{\varphi}\rangle_\Omega = -\int_\Omega \boldsymbol{u}\cdot\Delta\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{u}\times\boldsymbol{n},\, \mathbf{curl}\,\boldsymbol{\varphi}\rangle_\Gamma - \int_\Omega \pi\,\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle \pi,\, \boldsymbol{\varphi}\cdot\boldsymbol{n}\rangle_\Gamma, \tag{20}$$

where $\boldsymbol{Y}_N^{\,p'}(\Omega) = \big\{\boldsymbol{\varphi} \in \boldsymbol{W}^{\,2,p}(\Omega),\ \operatorname{div}\boldsymbol{\varphi} = 0 \text{ and } \boldsymbol{\varphi}\times\boldsymbol{n} = \boldsymbol{0} \text{ on}\,\Gamma\big\}$. In the first time, we prove the existence of a unique $\pi \in W^{\,-1,p}(\Omega)$, next we use [3] in order to prove that $\pi \in L^p(\Omega)$.□

4. Helmholtz decompositions

According to the two types $\boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{u}\times\boldsymbol{n}$ of boundary conditions on Γ, we give decompositions of vector fields $\boldsymbol{u}$ in $\boldsymbol{L}^p(\Omega)$. Our results may be regarded as an extension of the well-known De Rham-Hodge-Kodaira decomposition of $\mathcal{C}^\infty$-forms on compact Riemannian manifolds into $\boldsymbol{L}^p$-vector fields on Ω. We can find similar decompositions in [6], where the authors consider more regular domain with $\mathcal{C}^\infty$-boundary Γ. We can see also [8] for the case $p = 2$.

Theorem 4.1.

i) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then, there exist $\chi \in W^{\,1,p}(\Omega)$, $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_N(\Omega)$, $\boldsymbol{z} \in \boldsymbol{K}^p_T(\Omega)$ such that: $\boldsymbol{u} = \boldsymbol{z} + \nabla\,\chi + \mathbf{curl}\,\boldsymbol{w}$ satisfiesthe estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{\,1,p}(\Omega)/\mathbb{R}} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_N(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

where $\boldsymbol{z}$ is unique, χ is unique up to an additive constant and $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_N(\Omega)$.

ii) Let $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$. Then, there exist $\chi \in W^{1,p}_0(\Omega)$, $\boldsymbol{w} \in \boldsymbol{W}^{1,p}_\sigma(\Omega) \cap \boldsymbol{X}^p_T(\Omega)$, $\boldsymbol{z} \in \boldsymbol{K}^p_N(\Omega)$ such that: $\boldsymbol{u} = \boldsymbol{z} + \nabla\,\chi + \mathbf{curl}\,\boldsymbol{w}$ satisfies the estimate:

$$\|\boldsymbol{z}\|_{\boldsymbol{L}^p(\Omega)} + \|\chi\|_{W^{\,1,p}(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p}(\Omega)/\boldsymbol{K}^p_T(\Omega)} \le C\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)},$$

where $\boldsymbol{z}$ and χ are unique and $\boldsymbol{w}$ is unique up to an additive element of $\boldsymbol{K}^p_T(\Omega)$.

Sketch of the proof. We give a short proof of the first point and the proof of the second one is similar. First, we introduce the solution χ in $W^{\,1,p}(\Omega)$, unique up to an additive constant, of the problem: $-\Delta\,\chi = \operatorname{div}\boldsymbol{u}$ in Ω and $(\mathbf{grad}\,\chi - \boldsymbol{u})\cdot\boldsymbol{n} = 0$ on Γ. Second, we solve the problem: $-\Delta\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u}$ in Ω and $\operatorname{div}\boldsymbol{w} = 0$ in Ω, $\boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, which has a solution $\boldsymbol{w} \in \boldsymbol{W}^{\,1,p}(\Omega)$, unique up to an additive element of $\boldsymbol{K}_N^{\,p}(\Omega)$. To finish, observe that the function $\boldsymbol{z} = \boldsymbol{u} - \nabla\,\chi - \mathbf{curl}\,\boldsymbol{w}$ belongs to $\boldsymbol{K}_T^{\,p}(\Omega)$. □

Remark 4.2. We can prove also similar decompositions for singular vector fields $\boldsymbol{u} \in (\boldsymbol{H}^p_0(\operatorname{div},\, \Omega))'$ and for $\boldsymbol{u} \in (\boldsymbol{H}^p_0(\mathbf{curl},\, \Omega))'$.

References

[1] C. Amrouche, C. Bernardi, M. Dauge, V. Girault, *Vector potentials in three-dimensional nonsmooth domains*, Math. Meth. Applied Sc., Vol. 21, pp. 823–864, 1998.

[2] C. Amrouche,V. Girault, *Decomposition of vector space and application to the Stokes problem in arbitrary dimension*, Czechosloval Math. J., Vol. 119, No. 44, pp. 109–140, 1994.

[3] C. Amrouche, M.A. Rodríguez-Bellido, *Stokes, Oseen and Navier-Stokes equations with singular data.*To appear in Archive. Rational. Mech. Anal.

[4] C. Amrouche, N. Seloula, *L^p-Inequalities for Vector Fields and Stokes Problems with Non Standard Boundary Conditions*, submitted for publication.

[5] Girault V. and Raviart, P.-A, *Finite Element Methods for the Navier-Stokes Equations, Theory and Algorithms*, Springer, Berling, 1986.

[6] H. Kosono, T. Yanagisawa, *L^r-variational Inequality for Vector Fielss and the Helmholtz-Weyl Decomposition in Bounded domains*, Indiana University Mathematics Journal, Vol. 58, No. 4 2009.

[7] W. Von. Wahl, *Estimating ∇u by* $\operatorname{div} u$*,* $\mathbf{curl}\, u$, Math. Meth. Appl. Sci., Vol. 15, pp. 123–143, 1992.

[8] Temam, R, *Theory and Numerical Analysis of the Navier-Stokes Equations*, North-Holland, Amsterdam, 1977.