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3 **ON THE STRONG PARITY CHROMATIC NUMBER**

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12 **Abstract**

13 A vertex colouring of a 2-connected plane graph  $G$  is a *strong parity*  
14 *vertex colouring* if for every face  $f$  and each colour  $c$ , the number of  
15 vertices incident with  $f$  coloured by  $c$  is either zero or odd.

16 Czap *et al.* in [9] proved that every 2-connected plane graph has a  
17 proper strong parity vertex colouring with at most 118 colours.

18 In this paper we improve this upper bound for some classes of plane  
19 graphs.

20 **Keywords:** plane graph,  $k$ -planar graph, vertex colouring, strong par-  
21 ity vertex colouring.

22 **2010 Mathematics Subject Classification:** 05C15.

23 1. INTRODUCTION

24 We adapt the convention that a graph (as a combinatorial object) is *k-planar*  
25 if it can be drawn in the plane (on the sphere) so that each its edge is crossed  
26 by at most  $k$  other edges; such a drawing is then called a *k-plane graph* (a  
27 geometrical object). Specially, for  $k = 0$  we have planar or plane graphs.

28 If a plane graph  $G$  is drawn in the plane  $\mathcal{M}$ , then the maximal connected  
29 regions of  $\mathcal{M} \setminus G$  are called the *faces* of  $G$ . The *facial walk* of a face  $f$  of

30 a connected plane graph  $G$  is the shortest closed walk traversing all edges  
 31 incident with  $f$ . The *size* of a face  $f$  is the length of its facial walk. Let a  
 32  $d$ -*face* be a face of size  $d$ . A 3-face is called a *triangle* and a face of size at  
 33 least 4 is called a *non-triangle* face.

34 A *triangulation* is a simple plane graph which contains only 3-faces.  
 35 A *near-triangulation* is a simple plane graph which contains at most one  
 36 non-triangle face.

37 The *degree* of a vertex  $v$  of a graph  $G$  is the number of edges incident  
 38 with  $v$ .

39 Let the set of vertices, edges, and faces of a connected plane graph  $G$   
 40 be denoted by  $V(G)$ ,  $E(G)$ , and  $F(G)$ , respectively, or by  $V$ ,  $E$ , and  $F$  if  $G$   
 41 is known from the context.

42 A  $k$ -*colouring* of the graph  $G$  is a mapping  $\varphi : V(G) \rightarrow \{1, \dots, k\}$ . A  
 43 colouring of a graph in which no two adjacent vertices have the same colour  
 44 is a *proper colouring*. A graph which has a proper  $k$ -colouring is called  
 45  $k$ -*colourable*.

46 Let  $\varphi$  be a vertex colouring of a connected plane graph  $G$ . We say that  
 47 a face  $f$  of  $G$  uses a colour  $c$  under the colouring  $\varphi$   $k$  times if this colour  
 48 appears  $k$  times along the facial walk of  $f$ . (The first and the last vertex of  
 49 the facial walk is counted as one appearance only.)

50 The problems of graph colouring, in particular the well-known Four  
 51 Colour Problem [14], have motivated the development of different graph  
 52 colourings, which brought many problems and questions in this area. Colour-  
 53 ings of graphs embedded on surfaces with face constraints have recently  
 54 drawn a substantial amount of attention, see e.g. [4, 5, 10, 11, 12, 16]. Two  
 55 problems of this kind are the following.

56 **Problem 1.** A vertex colouring  $\varphi$  is a *weak parity vertex colouring* of a  
 57 connected plane graph  $G$  if each face of  $G$  uses at least one colour an odd  
 58 number of times. The problem is to determine the minimum number  $\chi_w(G)$   
 59 of colours used in such a colouring of  $G$ . The number  $\chi_w(G)$  is called the  
 60 *weak parity chromatic number* of  $G$ .

61 **Problem 2.** A vertex colouring  $\varphi$  is a *strong parity vertex colouring* of a  
 62 2-connected plane graph  $G$  if for each face  $f$  and each colour  $c$ , either no  
 63 vertex or an odd number of vertices incident with  $f$  is coloured by  $c$ . The  
 64 problem is to find the minimum number  $\chi_s(G)$  of colours used in such a  
 65 colouring of a given graph  $G$ . The number  $\chi_s(G)$  is called the *strong parity*  
 66 *chromatic number* of  $G$ .

67 Our research has been motivated by the paper [6] which deals with parity  
68 edge colourings in graphs. Recall that a parity edge colouring is such a  
69 colouring in which each walk uses some colour an odd number of times. The  
70 parity edge chromatic number  $p(G)$  is the minimum number of colours in  
71 a parity edge colouring of  $G$ . Computing  $p(G)$  is NP-hard even when  $G$  is  
72 a tree, but the problem of recognizing parity edge colourings of graphs is  
73 solvable in polynomial time. The vertex version of this problem is introduced  
74 in [5]. This article deals with parity vertex colourings of plane graphs focused  
75 on facial walks.

76 The first problem has been investigated in [7]. The authors have found  
77 a general upper bound for this parameter.

78 **Theorem 1** (Czap and Jendroř [7]). *Let  $G$  be a 2-connected plane graph.*  
79 *Then there is a proper weak parity vertex 4-colouring of  $G$ , such that each*  
80 *face of  $G$  uses some colour exactly once.*

81 Czap and Jendroř [7] conjecture that  $\chi_w(G) \leq 3$  for all simple plane graphs  $G$   
82 and they have proved that this conjecture is true for the class of 2-connected  
83 simple cubic plane graphs. This conjecture is still open in general.

84 In this paper, we focus on the second problem.

## 85 2. STRONG PARITY VERTEX COLOURING

86 In the paper [7] there is posed a conjecture that for any 2-connected plane  
87 graph  $G$  the strong parity chromatic number can be bounded from above  
88 by a constant. The conjecture was proved by Czap *et al.* in the following  
89 form.

90 **Theorem 2** (Czap, Jendroř, and Voigt [9]). *Let  $G$  be a 2-connected plane*  
91 *graph. Then  $G$  has a proper strong parity vertex colouring with at most 118*  
92 *colours.*

93 The constant 118 was recently improved to 97 by Kaiser *et al.* [13]. In this  
94 section, we improve this upper bound for 3-connected simple plane graphs  
95 having property that the faces of a certain size are in a sense far from each  
96 other.

97 The following lemma is fundamental. Remind that a cycle can be con-  
98 sidered as a connected 2-regular plane graph.

99 **Lemma 3.** *Let  $C = v_1, \dots, v_k$  be a cycle on  $k$  vertices. Then there is a*  
 100 *proper strong parity vertex colouring  $\varphi$  of  $C$  using the colours  $a, b, c, d, e$ ,*  
 101 *where the colours  $a, b, c$  are used at most once.*

102 **Proof.** We define the colouring  $\varphi$  of  $C$  in the following way:

- 103 •  $k = 4t$ , then  $\varphi(v_1) = a$ ,  $\varphi(v_2) = b$ ,  $\varphi(v_i) = d$  for  $i \equiv 1 \pmod{2}$ ,  $i > 1$ ,  
 104 and  $\varphi(v_i) = e$  for  $i \equiv 0 \pmod{2}$ ,  $i > 2$ .
- 105 • If  $k = 4t + 1$ , then  $\varphi(v_1) = a$ ,  $\varphi(v_2) = b$ ,  $\varphi(v_3) = c$ ,  $\varphi(v_i) = d$  for  $i \equiv 1$   
 106  $\pmod{2}$ ,  $i > 3$ , and  $\varphi(v_i) = e$  for  $i \equiv 0 \pmod{2}$ ,  $i > 2$ .
- 107 • If  $k = 4t + 2$ , then  $\varphi(v_i) = d$  for  $i \equiv 1 \pmod{2}$  and  $\varphi(v_i) = e$  for  $i \equiv 0$   
 108  $\pmod{2}$ .
- 109 • If  $k = 4t + 3$ , then  $\varphi(v_1) = a$ ,  $\varphi(v_i) = d$  for  $i \equiv 1 \pmod{2}$ ,  $i > 1$ , and  
 110  $\varphi(v_i) = e$  for  $i \equiv 0 \pmod{2}$ .

111 Clearly, this colouring satisfies our requirements in each case. ■

112 **Lemma 4.** *Let  $G$  be a 3-connected near-triangulation. Then there is a*  
 113 *proper strong parity vertex colouring of  $G$  which uses at most 6 colours.*  
 114 *Moreover, this bound is best possible.*

115 **Proof.** If  $G$  is a triangulation, then by the Four Colour Theorem [1] we  
 116 can colour the vertices of  $G$  with at most 4 colours in such a way that the  
 117 vertices incident with the same face receive different colours. Clearly, this  
 118 colouring is a strong parity vertex one.

119 Now we suppose that  $G$  contains a  $d$ -face  $f$ ,  $d \geq 4$ . Let  $v_1, \dots, v_d$  be  
 120 the vertices incident with  $f$  in this order. Next we insert the diagonals  $v_1v_i$ ,  
 121  $i \in \{3, \dots, d-1\}$  and we get a new graph  $T$ . The graph  $T$  has a proper  
 122 colouring which uses at most four colours, since it is a plane triangulation.  
 123 This colouring induces the colouring  $\varphi$  of  $G$  in the natural way.

124 We can assume that  $\varphi(v_1) = 1$ ,  $\varphi(v_2) = 2$ , and  $\varphi(v_3) = 3$ . Next we use  
 125 Lemma 3 and we recolour the vertices incident with the face  $f$ . We use the  
 126 following colours:  $a = 1$ ,  $b = 2$ ,  $c = 3$ ,  $d = 5$ , and  $e = 6$ .

127 Observe, that each triangle face of  $G$  uses three different colours and  
 128 from Lemma 3 it follows that the face  $f$  uses each colour which appears on  
 129 its boundary an odd number of times.

130 To see that the bound 6 is best possible it suffices to consider the graph  
 131 of a wheel  $W_5$  depicted in Figure 1. ■

132 We write  $v \in f$  if a vertex  $v$  is incident with a face  $f$ . Two distinct faces  
 133  $f$  and  $g$  touch each other, if there is a vertex  $v$  such that  $v \in f$  and  $v \in g$ .

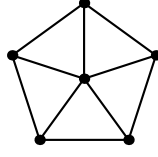


Figure 1: An example of a graph with no proper strong parity vertex colouring using less than 6 colours.

134 Two distinct faces  $f$  and  $g$  influence each other, if they touch, or there is a  
 135 face  $h$  such that  $h$  touches both  $f$  and  $g$ .

136 **Theorem 5.** *Let  $G$  be a 3-connected plane graph in which no two non-*  
 137 *triangle faces influence each other. Then there is a proper strong parity*  
 138 *vertex colouring of  $G$  which uses at most six colours  $1, \dots, 6$  such that each*  
 139 *vertex which is not incident with any non-triangle face has a colour from*  
 140 *the set  $\{1, 2, 3, 4\}$ . Moreover, this bound is sharp.*

141 **Proof.** We apply induction on the number of non-triangle faces. If  $G$  con-  
 142 tains one non-triangle face then the claim follows from Lemma 4.

143 Assume that  $G$  contains  $j$  non-triangle faces,  $j \geq 2$ . Let  $f = v_1, \dots, v_m$   
 144 be one of them. We insert the diagonals  $v_1v_i$ ,  $i \in \{3, \dots, m-1\}$  to the  
 145 face  $f$  and we get a new graph  $H$ . The graph  $H$  has  $(j-1)$  non-triangle  
 146 faces, hence, by induction, it has a proper strong parity vertex colouring  
 147 which uses at most six colours  $1, \dots, 6$ . Moreover, each vertex which is not  
 148 incident with any non-triangle face has a colour from the set  $\{1, 2, 3, 4\}$ .  
 149 This colouring of  $H$  induces the colouring  $\varphi$  of  $G$ .

150 Observe, that the vertices incident with  $f$  or the faces which touch  $f$   
 151 have colours from the set  $\{1, 2, 3, 4\}$  (else  $G$  contains two non-triangle faces  
 152 that influence each other). We use the colouring from Lemma 3 with the  
 153 colours  $a = \varphi(v_1)$ ,  $b = \varphi(v_2)$ ,  $c = \varphi(v_3)$ ,  $d = 5$ , and  $e = 6$  to recolour the  
 154 vertices incident with  $f$  so that we obtain a required colouring of  $G$ .

155 To see that the bound 6 is best possible it suffices to consider a trian-  
 156 gulation  $T$  such that it contains  $\ell$  triangle faces  $f_1, f_2, \dots, f_\ell$  which do not  
 157 influence each other, and insert a wheel-like configuration into each of them,  
 158 see Figure 2 for illustration. ■

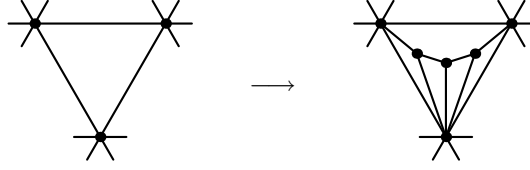


Figure 2: Inserting a path on three vertices into a triangle face yields a configuration without a required colouring using less than 6 colours.

159 2.1. STRONG PARITY COLOURING VERSUS CYCLIC COLOURING

160 A *cyclic colouring* of a plane graph is a vertex colouring in which, for each  
 161 face  $f$ , all the vertices on the boundary of  $f$  have distinct colours. The *cyclic*  
 162 *chromatic number*  $\chi_c(G)$  of a plane graph  $G$  is the minimum number of  
 163 colours in a cyclic colouring. Clearly, every cyclic colouring of a 2-connected  
 164 plane graph is also a strong parity vertex colouring, hence,  $\chi_s(G) \leq \chi_c(G)$ .  
 165 Therefore, every upper bound on  $\chi_c(G)$  also applies for  $\chi_s(G)$ . There are  
 166 several known bounds on  $\chi_c(G)$  depending on  $\Delta^*(G)$ , the maximum face  
 167 size of a plane graph  $G$ . The results of [16], [11], [12], and [10] immediately  
 168 give the following statements.

**Proposition 6.** *Let  $G$  be a 2-connected plane graph with maximum face size  $\Delta^*$ . Then*

$$\chi_s(G) \leq \left\lceil \frac{5\Delta^*}{3} \right\rceil.$$

Moreover, if  $G$  is 3-connected, then

$$\chi_s(G) \leq \begin{cases} \Delta^* + 1 & \text{for } \Delta^* \geq 60, \\ \Delta^* + 2 & \text{for } \Delta^* \geq 18, \\ \Delta^* + 5 & \text{for } \Delta^* \geq 3. \end{cases}$$

169 Borodin *et al.* proved the following.

170 **Theorem 7** (Borodin [2]). *Let  $G$  be a 2-connected plane graph with maxi-*  
 171 *mum face size  $\Delta^* \leq 4$ . Then  $\chi_c(G) \leq 6$ .*

172 **Theorem 8** (Borodin, Sanders and Zhao [4]). *Let  $G$  be a 2-connected plane*  
 173 *graph with maximum face size  $\Delta^* \leq 5$ . Then  $\chi_c(G) \leq 8$ .*

174 We use these theorems to improve the general upper bound on  $\chi_s(G)$  for  
 175 several graph classes with arbitrary large faces.

176 **Theorem 9.** *Let  $G$  be a 3-connected plane graph in which the faces of size*  
 177 *at least 5 do not influence each other. Then there is a proper strong parity*  
 178 *vertex colouring of  $G$  which uses at most 8 colours.*

179 **Proof.** Let  $B = \{f_1, \dots, f_\ell\}$  be the set of faces of size at least 5 and let  $d_i$   
 180 denote the size of the face  $f_i$ . Let the face  $f_i$  be incident with the vertices  
 181  $v_{i,1}, \dots, v_{i,d_i}$ ,  $i \in \{1, \dots, \ell\}$ . Next we insert the diagonals  $v_{i,1}v_{i,m}$ ,  $m \in$   
 182  $\{3, \dots, d_i - 1\}$ , to the face  $f_i$  for  $i \in \{1, \dots, \ell\}$ , and we get a new graph  $H$ .  
 183 Observe, that  $H$  contains only 3-faces and 4-faces.

184 From Theorem 7 it follows that  $H$  has a cyclic colouring with at most  
 185 six colours. This colouring defines the colouring  $\varphi$  of  $G$ . Clearly, each face  
 186 of  $G$  of size  $j$ ,  $j \in \{3, 4\}$ , uses  $j$  different colours. Finally, we recolour the  
 187 vertices incident with the faces from  $B$  in such a way that we get a proper  
 188 strong parity vertex colouring of  $G$ . For the face  $f_i$  we use the same colouring  
 189 as in Lemma 3 with  $a = \varphi(v_{i,1})$ ,  $b = \varphi(v_{i,2})$ ,  $c = \varphi(v_{i,3})$ ,  $d = 7$ ,  $e = 8$ .

190 It is easy to check that this colouring of  $G$  satisfies our requirements. ■

191

192 **Theorem 10.** *Let  $G$  be a 3-connected plane graph such that the faces of*  
 193 *size at least 6 do not influence each other. Then there is a proper strong*  
 194 *parity vertex colouring of  $G$  which uses at most 10 colours.*

195 **Proof.** We create a graph  $H$  from  $G$  analogously as in the proof of Theorem  
 196 9. Using Theorem 8 we colour the vertices of  $H$  cyclically with at most 8  
 197 colours. By this colouring we get the colouring  $\varphi$  of  $G$ .

198 At this time each face of  $G$  of size  $j$ ,  $j \in \{3, 4, 5\}$ , uses  $j$  different colours.  
 199 We recolour the vertices incident with  $f_i$  by the colouring defined in Lemma  
 200 3. We use the following colours:  $a = \varphi(v_{i,1})$ ,  $b = \varphi(v_{i,2})$ ,  $c = \varphi(v_{i,3})$ ,  $d = 9$ ,  
 201  $e = 10$ . ■

## 202 2.2. STRONG PARITY COLOURING VERSUS $k$ -PLANARITY

203 Recall that a graph is  $k$ -planar if it can be drawn in the plane so that each  
 204 its edge is crossed by at most  $k$  other edges. In this section we investigate  
 205 the structure of  $k$ -planar graphs. We will use only one operation, namely  
 206 the *contraction*. The contraction of an edge  $e = uv$  in the graph  $G$ , denoted  
 207 by  $G \circ e$ , is defined as follows: identify the vertices  $u$  and  $v$ , delete the loop  
 208  $uv$  and replace all multiple edges arisen by single edges.



209 **Lemma 11.** *Let  $G$  be a drawing of a  $k$ -planar graph, and let  $e$  be an edge*  
 210 *which is not crossed by any other edge. Then  $G \circ e$  is a  $k$ -planar graph.*

211 **Proof.** While contracting the edge  $e$ , the number of crossings of any edge  
 212 does not increase, therefore, the graph remains  $k$ -planar. ■

213 **Lemma 12.** *Let  $G$  be a drawing of a  $k$ -planar graph, and let  $C = v_1, \dots, v_t$*   
 214 *be a cycle in  $G$  such that the edges  $v_i v_{i+1}$ ,  $i \in \{1, \dots, t\}$ ,  $v_{t+1} = v_1$ , are*  
 215 *not crossed by any other edge and the inner part of  $C$  does not contain any*  
 216 *vertex. Let  $H$  be a graph obtained from  $G$  by collapsing  $C$  into a single*  
 217 *vertex (and replacing multiple edges by single edges). Then the graph  $H$  is*  
 218 *a  $k$ -planar graph.*

219 **Proof.** We successively contract the edges  $v_1 v_2, \dots, v_{t-1} v_t$ . After the con-  
 220 traction of  $v_1 v_2$  we obtain a  $k$ -planar graph (see Lemma 11). Clearly, there  
 221 exists a plane drawing of  $G$  such that the edges on the cycle corresponding  
 222 to  $C$  are not crossed by any other edge and the cycle has an empty inner  
 223 part. When we contract the last edge  $v_{t-1} v_t$  we get the graph  $H$ . ■

224 We say that a face  $f$  of size  $i$  is *isolated* if there is no face  $g$  of size at least  
 225  $i$  touching  $f$ .

226 **Lemma 13.** *Let  $j$  be a fixed integer from the set  $\{3, 4, 5\}$ . Let  $G$  be a 2-*  
 227 *connected plane graph such that any face of size at least  $j + 1$  is isolated.*  
 228 *Let  $H$  be a graph obtained from  $G$  in the following way: for each face in  $G$*   
 229 *of size at least  $j + 1$  insert a vertex to  $H$ , join two vertices of  $H$  by an edge*  
 230 *if the corresponding faces influence each other in  $G$ . Then*

231 1 *If  $j = 3$  then  $H$  is a planar graph.*

232 2 *If  $j = 4$  then  $H$  is a 1-planar graph.*

233 3 *If  $j = 5$  then  $H$  is a 2-planar graph.*

234 **Proof.** Let  $B = \{f_1, \dots, f_\ell\}$  be a set of faces which have sizes at least  
 235  $j + 1$ . Let  $V(f_i)$  denote the set of vertices of  $G$  incident with the face  $f_i$ ,  
 236  $i \in \{1, \dots, \ell\}$ . Clearly,  $V(f_i) \cap V(f_j) = \emptyset$ , for  $i \neq j$ , because  $f_i$  and  $f_j$  do  
 237 not touch each other.

238 Given the sets  $V(f_i)$ , we colour the vertices of  $G$  in the following way:  
 239 Vertices contained in  $V(f_i)$  receive the colour  $i$ ; vertices not contained in  
 240 any  $V(f_i)$  receive the colour 0.

241 To each face  $g$  with a facial walk  $u_1, \dots, u_p$ ,  $4 \leq p \leq j$  we insert the  
 242 diagonal  $u_n u_m$ ,  $n, m \in \{1, \dots, p\}$ , if the vertices  $u_n$  and  $u_m$  have distinct

243 colours and these colours are different from 0. So we get the graph  $G_1$ . Let  
 244  $G_2$  be a graph induced on the vertices of  $G_1$  which have colours different  
 245 from 0 and let  $G_3$  be a graph obtained from  $G_2$  by collapsing the vertices  
 246 from  $V(f_i)$  to the vertex  $v_i$ ,  $i \in \{1, \dots, \ell\}$ .

247 Observe that,

248 1. If  $j = 3$  then  $G = G_1$ , hence  $G_2$  is a plane graph. From Lemma 12 it  
 249 follows that  $G_3$  is a plane graph.

250 2. If  $j = 4$  then to each face of size 4 we add at most 2 diagonals, hence,  
 251  $G_1$  is a 1-plane graph.  $G_2$  is a subgraph of  $G_1$  therefore it is 1-plane too.  
 252 Lemma 12 ensures that  $G_3$  is 1-plane.

253 3. If  $j = 5$  then  $G_1$  and  $G_2$  are 2-plane graphs because the complete  
 254 graph on 5 vertices is 2-planar. From Lemma 12 it follows that  $G_3$  is 2-plane.

255 Observe, the vertices  $v_s, v_t$  of  $G_3$ ,  $s, t \in \{1, \dots, \ell\}$ , are joined by an edge  
 256 if and only if the corresponding faces  $f_s, f_t$  of  $G$  influence each other. Hence,  
 257 the graph  $G_3$  is the plane drawing of  $H$ . ■

258 In the rest of the paper let  $B_i(G)$  (or  $B_i$  if  $G$  is known from the context)  
 259 denote the set of faces of  $G$  of size at least  $i$ ,  $i \in \{4, 5, 6\}$ , and let  $\ell_i$  denote  
 260 the number of faces in  $B_i(G)$ . Let  $H_i$  be a graph obtained from  $G$  in the  
 261 following way: for each face  $f \in B_i \subseteq F(G)$  insert a vertex to  $H_i$ , join two  
 262 vertices of  $H_i$  if the corresponding faces influence each other in  $G$ .

263 The previous theorems give upper bounds for the strong parity chromatic  
 264 number for graphs in which any two faces of size at least 4, 5 or 6 do  
 265 not influence each other. In the next part of this article we provide another  
 266 upper bound in the case when the faces of size at least six do not touch but  
 267 they can influence one another.

268 **Theorem 14.** *Let  $G$  be a 3-connected plane graph such that any face of size*  
 269 *at least 4 is isolated. Then there is a proper strong parity vertex colouring*  
 270 *of  $G$  which uses at most 12 colours.*

271 **Proof.** If  $G$  does not contain any two non-triangle faces influencing each  
 272 other then from Theorem 5 it follows that  $G$  has a required colouring.

273 Assume that  $G$  contains at least two non-triangle faces which influence  
 274 each other. Let the face  $f_i \in B_4$  be incident with the vertices  $v_{i,1}, \dots, v_{i,d_i}$ ,  
 275  $i \in \{1, \dots, \ell_4\}$ , where  $d_i$  is the size of  $f_i$ . We insert the diagonals  $v_{i,1}v_{i,m}$ ,  
 276  $m \in \{3, \dots, d_i - 1\}$ , to the face  $f_i$  for  $i \in \{1, \dots, \ell_4\}$ , and we get a triangulation  
 277  $T$ . Using the Four Colour Theorem we colour the vertices of  $T$  with at  
 278 most four colours such that adjacent vertices receive distinct colours. This  
 279 colouring induces the colouring  $\varphi$  of  $G$ .

280 From Lemma 13 it follows that  $H_4$  is a planar graph, hence, we can as-  
 281 sign to each vertex of  $H_4$  one pair of colours from  $\{(5, 6), (7, 8), (9, 10), (11, 12)\}$   
 282 in such a way that two adjacent vertices receive distinct pairs. It means that  
 283 we can assign a pair of colours to each face of  $G$  of size at least four in such  
 284 a way that two faces which influence each other receive distinct pairs.

285 Assume that we assigned the pair  $(x_i, y_i)$  to the face  $f_i$ . Now we recolour  
 286 the vertices incident with  $f_i$ ,  $i \in \{1, \dots, \ell_4\}$ . We use the same colouring  
 287 as in Lemma 3 with colours  $a = \varphi(v_{i,1})$ ,  $b = \varphi(v_{i,2})$ ,  $c = \varphi(v_{i,3})$ ,  $d = x_i$ ,  
 288 and  $e = y_i$ . If we perform this recolouring of vertices on all faces of size at  
 289 least 4 we obtain such a colouring that if a colour appears on a face  $f_i \in B_4$ ,  
 290  $i \in \{1, \dots, \ell_4\}$ , then it appears an odd number of times. Moreover, if we  
 291 recolour at least two vertices on a triangle face of  $G$  then we recolour them  
 292 with distinct colours, because the corresponding faces influence each other.  
 293 ■

294 There is a lot of papers about plane graphs and their colourings but little is  
 295 known about  $k$ -planar graphs,  $k \geq 1$ . We use the following result of Borodin  
 296 to find an upper bound on  $\chi_s(G)$  for the class of 3-connected plane graphs  
 297 for which the faces of size at least five are in a sense far from each other.

298 **Theorem 15** (Borodin [3]). *If a graph is 1-planar, then it is vertex 6-*  
 299 *colourable.*

300 **Theorem 16.** *Let  $G$  be a 3-connected plane graph such that any face of size*  
 301 *at least 5 is isolated. Then there is a proper strong parity vertex colouring*  
 302 *of  $G$  which uses at most 18 colours.*

303 **Proof.** Assume that  $G$  contains at least two faces of size at least 5 which  
 304 influence each other. Let the face  $f_i \in B_5$  be incident with the vertices  
 305  $v_{i,1}, \dots, v_{i,d_i}$ ,  $i \in \{1, \dots, \ell_5\}$ , where  $d_i$  is the size of  $f_i$ . Next we insert the  
 306 diagonals  $v_{i,1}v_{i,m}$ ,  $m \in \{3, \dots, d_i - 1\}$ , to the face  $f_i$  for  $i \in \{1, \dots, \ell_5\}$ ,  
 307 and we get a graph  $W$ . Observe, that each face of  $W$  has size at most 4.  
 308 Applying Theorem 7 we colour the vertices of  $W$  with at most 6 colours  
 309 cyclically. This colouring defines the colouring  $\varphi$  of  $G$ .

310 From Lemma 13 it follows that  $H_5$  is a 1-planar graph. By Theorem 15 we  
 311 can assign to each vertex of  $H_5$  one pair of colours from  $\{(7, 8), \dots, (17, 18)\}$   
 312 so that two adjacent vertices receive distinct pairs. Ergo, we assign distinct  
 313 pairs of colours to faces of  $G$  of size at least 5 which influence each other.

314 Assume that the face  $f_i$  receives the pair  $(x_i, y_i)$ . Now we recolour the  
 315 vertices incident with  $f_i$  by the colouring defined in Lemma 3. We use the  
 316 following colours:  $a = \varphi(v_{i,1})$ ,  $b = \varphi(v_{i,2})$ ,  $c = \varphi(v_{i,3})$ ,  $d = x_i$ , and  $e = y_i$ .

317 If we perform this recolouring of vertices on all faces of size at least 5  
 318 we obtain a required colouring of  $G$ . ■

319 The class of 2-planar graphs has not been sufficiently investigated. Pach and  
 320 Tóth tried to answer the following question: What is the maximum number  
 321 of edges that a simple graph of  $n$  vertices can have if it can be drawn in  
 322 the plane so that every edge crosses at most  $k$  others? They proved the  
 323 following.

324 **Theorem 17** (Pach and Tóth [15]). *Let  $G$  be a simple graph drawn in the*  
 325 *plane so that every edge is crossed by at most  $k$  others. If  $0 \leq k \leq 4$ , then*  
 326 *we have  $|E(G)| \leq (k + 3) \cdot (|V(G)| - 2)$ .*

327 Using this result we can prove that every 2-planar graph has a vertex of  
 328 degree at most 9, therefore 2-planar graphs are 10-colourable. In the next  
 329 lemma let  $\delta(G)$  denote the minimum vertex degree of a graph  $G$ .

330 **Lemma 18.** *Let  $G$  be a 2-planar graph. Then  $\delta(G) \leq 9$ .*

**Proof.** From Theorem 17 it follows that  $|E(G)| \leq 5 \cdot |V(G)| - 10$ . For  
 every graph it holds  $2 \cdot |E(G)| = \sum_{v \in V(G)} \deg(v) \geq \delta(G) \cdot |V(G)|$ . Hence, we  
 get

$$10 \cdot |V(G)| - 20 \geq |V(G)| \cdot \delta(G)$$

$$\delta(G) \leq \frac{10 \cdot |V(G)| - 20}{|V(G)|} < 10.$$

331 ■

332 **Corollary 19.** *If a graph is 2-planar, then it is vertex 10-colourable.*

333 This information about 2-planar graphs helps us to prove the following the-  
 334 orem.

335 **Theorem 20.** *Let  $G$  be a 3-connected plane graph such that any face of size*  
 336 *at least 6 is isolated. Then there is a proper strong parity vertex colouring*  
 337 *of  $G$  which uses at most 28 colours.*

338 **Proof.** The proof follows the scheme of the proof of Theorem 16. We omit  
 339 the details. ■

## 3. APPLICATIONS

340

341 Two edges of a plane graph are *face-adjacent* if they are consecutive edges  
 342 of a facial walk of some face. The *facial parity edge colouring* of a connected  
 343 bridgeless plane graph is an edge colouring such that no two face-adjacent  
 344 edges receive the same colour, and for each face  $f$  and each colour  $c$ , either  
 345 no edge or an odd number of edges incident with  $f$  is coloured by  $c$ . The  
 346 minimum number of colours  $\chi'_{fp}(G)$  used in such a colouring is called the  
 347 *facial parity chromatic index* of  $G$ . In [8] it is proved that  $\chi'_{fp}(G) \leq 92$  for  
 348 an arbitrary connected bridgeless plane graph  $G$ .

349 The *medial graph*  $M(G)$  of a plane graph  $G$  is obtained as follows. For  
 350 each edge  $e$  of  $G$  insert a vertex  $m(e)$  in  $M(G)$ . Join two vertices of  $M(G)$   
 351 if the corresponding edges are face-adjacent (see [14], pp. 47).

352 **Lemma 21.** *Let  $G$  be a 3-connected plane graph. Then the graph  $M(G)$  is*  
 353 *3-connected too.*

354 **Proof.** By contradiction, suppose that  $m(e_1)$  and  $m(e_2)$  form a 2-vertex-  
 355 cut in  $M(G)$ . Let  $M_1, M_2$  be the components of  $M(G) \setminus \{m(e_1), m(e_2)\}$ ; let  
 356  $E_1$  and  $E_2$  be the corresponding decomposition of  $E(G) \setminus \{e_1, e_2\}$ . Let the  
 357 edges from  $E_1$  (resp.  $E_2$ ) be white (resp. black); let  $e_1$  and  $e_2$  be red.

358 Let  $V_i$  be the set of vertices incident only with edges from  $E_i \cup \{e_1, e_2\}$ ,  
 359  $i = 1, 2$ . Since the minimum degree of  $G$  is at least 3, we have  $V_1 \cap V_2 = \emptyset$ .  
 360 If  $V_1 \cup V_2 = V(G)$ , then there are no vertices incident both with white and  
 361 black edges. Hence,  $\{e_1, e_2\}$  is a 2-edge-cut in  $G$ , which is not possible since  
 362  $G$  is 3-connected.

363 Therefore,  $V_1 \cup V_2 \neq V(G)$ ; let  $v$  be a vertex incident both with a white and  
 364 a black edge. Since  $m(e_1), m(e_2)$  is a 2-vertex-cut in  $M(G)$  no white edge is  
 365 face-adjacent to any black edge in  $G$ . Hence  $v$  has to be incident to both red  
 366 edges  $e_1$  and  $e_2$ . Then  $V(G) \setminus (V_1 \cup V_2) = \{v\}$ , unless  $e_1$  and  $e_2$  are parallel  
 367 edges in  $G$ . Let  $e_1 = uv$ . We may assume  $u \in V_1$ , i.e. all edges incident  
 368 with  $u$  but  $e_1$  are white. Then on the boundary cycle of (at least) one of the  
 369 faces incident with  $e_1$  white and black edges meet, which is a contradiction.

370 ■

371 Observe that every proper strong parity vertex colouring of  $M(G)$  corre-  
 372 sponds to the facial parity edge colouring of a 3-connected plane graph  $G$ .  
 373 We can immediately derive the following upper bounds for the facial parity  
 374 chromatic index for some classes of plane graphs from Theorems 5, 9, 10,  
 375 14, 16, and 20.

- 376 **Corollary 22.** (a) *Let  $G$  be a 3-connected plane graph such that the non-*  
 377 *triangle faces of  $M(G)$  do not influence each other. Then  $\chi'_{fp}(G) \leq 6$ .*
- 378 (b) *Let  $G$  be a 3-connected plane graph such that the faces of  $M(G)$  of size*  
 379 *at least 5 do not influence each other. Then  $\chi'_{fp}(G) \leq 8$ .*
- 380 (c) *Let  $G$  be a 3-connected plane graph such that the faces of  $M(G)$  of size*  
 381 *at least 6 do not influence each other. Then  $\chi'_{fp}(G) \leq 10$ .*
- 382 (d) *Let  $G$  be a 3-connected plane graph such that any face of  $M(G)$  of size*  
 383 *at least 4 is isolated. Then  $\chi'_{fp}(G) \leq 12$ .*
- 384 (e) *Let  $G$  be a 3-connected plane graph such that any face of  $M(G)$  of size*  
 385 *at least 5 is isolated. Then  $\chi'_{fp}(G) \leq 18$ .*
- 386 (f) *Let  $G$  be a 3-connected plane graph such that any face of  $M(G)$  of size*  
 387 *at least 6 is isolated. Then  $\chi'_{fp}(G) \leq 28$ .*

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### REFERENCES

- 390
- 391 [1] K. Appel and W. Haken, *Every planar map is four colorable*, Bull. Amer.  
 392 Math. Soc. **82** (1976) 711–712.
- 393 [2] O.V. Borodin, *Solution of Ringel's problems on vertex-free coloring of plane*  
 394 *graphs and coloring of 1-planar graphs*, , Met. Diskret. Anal. **41** (1984) 12–26  
 395 (in Russian).
- 396 [3] O.V. Borodin, *A new proof of the 6 color theorem*, J. Graph Theory **19** (1995)  
 397 507–521.
- 398 [4] O.V. Borodin, D.P. Sanders, and Y. Zhao, *On cyclic coloring and their gener-*  
 399 *alizations*, Discrete Math. **203** (1999) 23–40.
- 400 [5] P. Borowiecki, K. Budajová, S. Jendroľ, and S. Krajčí, *Parity vertex colouring*  
 401 *of graphs*, Discuss. Math. Graph Theory **31** (2011) 183–195.
- 402 [6] D.P. Bunde, K. Milans, D.B. West, and H. Wu, *Parity and strong parity edge-*  
 403 *coloring of graphs*, Congressus Numerantium **187** (2007) 193–213.
- 404 [7] J. Czap and S. Jendroľ, *Colouring vertices of plane graphs under restrictions*  
 405 *given by faces*, Discuss. Math. Graph Theory **29** (2009) 521–543.
- 406 [8] J. Czap, S. Jendroľ, and F. Kardoš, *Facial parity edge colouring*, Ars Math.  
 407 Contemporanea **4** (2011) 255–269.

- 408 [9] J. Czap, S. JendroŮ, and M. Voigt, *Parity vertex colouring of plane graphs*,  
409 Discrete Math. **311** (2011) 512–520.
- 410 [10] H. Enomoto and M. Horňák, *A general upper bound for the cyclic chromatic*  
411 *number of 3-connected plane graphs*, J. Graph Theory **62** (2009) 1–25.
- 412 [11] H. Enomoto, M. Horňák, and S. JendroŮ, *Cyclic chromatic number of 3-*  
413 *connected plane graphs*, SIAM J. Discrete Math. **14** (2001) 121–137.
- 414 [12] M. Horňák and J. Zlámalová, *Another step towards proving a conjecture by*  
415 *Plummer and Toft*, Discrete Math. **310** (2010) 442–452.
- 416 [13] T. Kaiser, O. Rucký, M. Stehlík, and R. Škrekovski, *Strong parity vertex col-*  
417 *oring of plane graphs*, IMFM, Preprint series 49 (2011), 1144.
- 418 [14] O. Ore, *The Four-color Problem* (Academic Press, New York, 1967).
- 419 [15] J. Pach and G. Tóth, *Graphs drawn with few crossings per edge*, Combinatorica  
420 **17** (1997) 427–439.
- 421 [16] D.P. Sanders and Y. Zhao, *A new bound on the cyclic chromatic number*, J.  
422 Combin. Theory (B) **83** (2001) 102–111.

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