Hitting time for Bessel processes - walk on moving spheres algorithm (WoMS)

Madalina Deaconu, Samuel Herrmann

▶ **To cite this version:**

HAL Id: hal-00636056

https://hal.archives-ouvertes.fr/hal-00636056v2

Submitted on 16 Nov 2011

HITTING TIME FOR BESSEL PROCESSES - WALK ON MOVING SPHERES ALGORITHM (WOMS) *

By Madalina Deaconu[†] and Samuel Herrmann[‡]

INRIA Nancy Grand-Est & *Institut Elie Cartan de Nancy* [†] *and Institut de Mathématiques de Bourgogne, Université de Bourgogne* [‡]

In this article we investigate the hitting time of some given boundaries for Bessel processes. The main motivation is coming from mathematical finance when dealing with volatility models but the results can also be used in optimal control problems. The aim is here to construct a new and efficient algorithm in order to approach this hitting time. As an application we will consider the hitting time of a given level for the Cox-Ingersoll-Ross process. The main tools we use are on one side an adaptation of the method of images to this particular situation and on the other side the connection existing between Cox-Ingersoll-Ross processes and Bessel processes.

1. Introduction. The aim of this paper is to study the hitting time of some curved boundaries for the Bessel process. Our main motivations come from mathematical finance, optimal control and neurosciences. In finance Cox-Ingersoll-Ross processes are widely used to model interest rates. As an application, in this article we will consider the simulation of the first hitting time of a given level for the CIR by using its relation with the Bessel process. In neurosciences the firing time of a neuron is usually modelled as the hitting time of a stochastic process associated with the membrane potential behaviour (for introduction of noise in neuron systems see Part I Chapter 5 in [7]). The literature proposes different continuous stochastic models like for instance the family of integrate-and-fire models (see Chapter 10 in [6]). Most of them are related to the Ornstein-Uhlenbeck process which appears in a natural way as extension of Stein's model, a classical discrete model. In Feller's model, generalized Bessel processes appear as a more realistic alternative to the Ornstein-Uhlenbeck process, see for instance [10] for a comparison of these models. Therefore the interspike interval, which is interpreted as the first passage time of the membrane potential through a given threshold is closely related to the first hitting time of a curved boundary for some Bessel processes.

Our main results and the main algorithm are obtained for the case of Bessel processes. We use in our numerical method the particular formula that we obtain for the hitting time of some curved boundaries for the Bessel process and the connection that exists between a Bessel process and the Euclidean norm of a Brownian motion when calculating the hitting position. As an application we consider the hitting time of a given level for the Cox-Ingersoll-Ross process. In order to obtain this, we use first of all the connection that exists between CIR processes and Bessel processes and secondly the method of images for this particular situation.

The study of Bessel processes and their hitting times occupies a huge mathematical literature. Let us only mention few of them: A. Göing-Jaeschke and M. Yor [8] consider a particular case of CIR processes which are connected with radial Ornstein-Uhlenbeck processes and their hitting

1

times; L. Alili and P. Patie [1] investigate as a special situation the Bessel processes via some boundary crossing identities for diffusions having the time inversion property; recently P. Salminen and M. Yor consider the hitting time of affine boundaries for the 3-dimensional Bessel process [16].

In a recent paper Y. Hamana and H. Matsumoto [9] give explicit expressions for the distribution functions and the densities of the first hitting time of a given level for the Bessel process. Their results cover all the cases. Let us also mention a recent work of T. Byczkowski, J. Malecki and M. Ryznar [2]. By using an integral formula for the density of the first hitting time of a given level of the Bessel process, they are able to obtain uniform estimates of the hitting time density function.

In all these papers the formulas are explicit and are hard to use for a numerical purpose as they exhibit Bessel functions. The main idea of this paper is to get rid of this difficulty by using two important tools: first of all the method of images that allows to obtain, for some particular boundaries, an explicit form for the density of the hitting time and secondly the connection between δ-dimensional Bessel processes and the Euclidean norm of a δ-dimensional Brownian motion in order to get the simulated exit position. By coupling these ingredients we are able to construct a numerical algorithm easy to implement and very efficient which approaches the desired hitting time.

We will use here a modified version of the *random walk on spheres* method which was first introduced by Muller [13] in 1956. This procedure allows to solve a Dirichlet boundary value problem. The idea is to simulate iteratively, for the Brownian motion, the exit position from the largest sphere included in the domain and centered in the starting point. This exit position becomes the new starting point and the procedure goes on until the exit point is close enough to the boundary. Let us notice that the simulation of the exit time from a sphere is numerically costly.

The method of images was introduced in 1969 by H.E. Daniels [4] as a tool to construct nonlinear boundaries for which explicit calculations for the exit distribution for the Brownian motion are possible. The method was developed also in H. R. Lerche [11]. We adapt this method for the Bessel process by using the explicit form of its density. For some particular curved boundaries we can explicitly evaluate the density of the Bessel hitting time.

The paper is organized as follows. First we present some new results on hitting times for Bessel processes. Secondly we construct the new algorithm for approaching the hitting time, the so called Walk on Moving Spheres algorithm. Finally we present some numerical results and as a particular application the evaluation of the hitting time of a given level for the Cox-Ingersoll-Ross process.

2. Hitting time for Bessel processes.

Bessel processes play an important role both in the study of stochastic processes like Brownian motion and in various theoretical and practical applications as for example in finance.

Let us consider the δ-dimensional Bessel process starting from y, solution of the following stochastic differential equation:

$$(2.1) \qquad \begin{cases} Z_t^{\delta,y} & = Z_0^{\delta,y} + \dfrac{\delta - 1}{2} \displaystyle\int_0^t (Z_s^{\delta,y})^{-1} \mathrm{d}s + B_t, \\ Z_0^{\delta,y} & = y, \qquad y \geq 0, \end{cases}$$

where $(B_t)_{t \geq 0}$ is a one dimensional Brownian motion. We denote

$$(2.2) \qquad \nu = \frac{\delta}{2} - 1$$

the *index* of this process. We call δ the *dimension* of the process. This terminology is coming from the fact that, in the case of positive integer $\delta \in \mathbb{N}$, a δ-dimensional Bessel process can be

represented as the Euclidean norm of a δ-dimensional Brownian motion. This will be a key point in our numerical method.

The density of this process starting from y is given by:

$$(2.3) \qquad p_y(t, x) = \frac{x}{t} \left(\frac{x}{y}\right)^\nu \exp\left(-\frac{x^2 + y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right), \quad \text{for } t > 0,\, y > 0,\, x \geq 0,$$

where $I_\nu(z)$ is the Bessel function whose expression writes:

$$(2.4) \qquad I_\nu(z) = \sum_{n=0}^\infty \left(\frac{z}{2}\right)^{\nu+2n} \frac{1}{n! \Gamma(\nu + n + 1)}.$$

When starting from $y = 0$, the density of $Z_t^{\delta,0}$ is:

$$(2.5) \qquad p_0(t, x) = \frac{1}{2^\nu} \frac{1}{t^{\nu+1}} \frac{1}{\Gamma(\nu+1)} x^{\delta-1} \exp\left(-\frac{x^2}{2t}\right), \quad \text{for } t > 0,\, x \geq 0.$$

2.1. *The method of images for Bessel processes.* In this section, we investigate the first hitting time of a curved boundary for the Bessel process starting from the origin. Let $\psi(t)$ denote the boundary, and introduce the following stopping time:

$$(2.6) \qquad \tau_\psi = \inf\{t \geq 0;\ Z_t^{\delta,0} \geq \psi(t)\}.$$

For some suitable choice of the boundary, the distribution of τ_ψ can be explicitly computed. The idea is given by the following remark on the method of images (see for instance [4] for the origin of this method and [11] for a complete presentation):

Fundamental idea. Suppose that F is a positive σ-finite measure satisfying some integrability conditions (to be specified later on) and define

$$(2.7) \qquad u(t, x) = p_0(t, x) - \frac{1}{a} \int_{\mathbb{R}_+} p_y(t, x) F(\mathrm{d}y),$$

for some real constant $a > 0$. Let

$$\psi(t) = \inf\{x \in \mathbb{R};\, u(t, x) < 0\}, \quad \text{for all } t > 0.$$

Then $u(t, x)$ is solution of the partial differential equation:

$$(2.8) \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) = \dfrac{1}{2} \dfrac{\partial^2 u}{\partial x^2}(t, x) - \dfrac{\delta - 1}{2} \dfrac{\partial}{\partial x}\left(\dfrac{1}{x} u(t, x)\right), & \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ u(t, \psi(t)) = 0, & \text{for all } t > 0, \\ u(0, .) = \delta_0(.) & \text{on } (-\infty, \psi(0+)]. \end{cases}$$

From this remark we deduce an interesting expression for the hitting time. We can prove that:

$$\tau_\psi = \inf\{t > 0;\ u(t, Z_t^{\delta,0}) = 0\}.$$

This means simply that in order to obtain informations on the hitting time it suffices to look for $u(t, Z_t^{\delta,0}) = 0$.

Let us express this in a general result

3

THEOREM 2.1. *Let $F(\mathrm{d}y)$ be a positive σ-finite measure such that $\int_0^\infty p_0(t, \sqrt{\varepsilon}y)F(\mathrm{d}y) < \infty$ for all $\varepsilon > 0$. Let $a > 0$ and define the function:*

$$(2.9) \qquad u(t,x) = p_0(t,x) - \frac{1}{a}\int_{\mathbb{R}_+} p_y(t,x)F(\mathrm{d}y).$$

Consider $\psi(t)$ such that $u(t, \psi(t)) = 0$. Then the probability density function of τ_ψ is given by

$$(2.10) \qquad \mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \left[-\frac{1}{2}\frac{\partial u}{\partial x}(t,x)\Big|_{x=\psi(t)} + \frac{1}{2}\frac{\partial u}{\partial x}(t,x)\Big|_{x=0} - \frac{\delta-1}{2x}u(t,x)\Big|_{x=0} \right]\mathrm{d}t.$$

PROOF. We will only point out the main ideas for the proof in this case as it follows mainly the ideas introduced in [11]. A complete description of the method and this result for the Brownian motion case can be found in [11].

Let us consider

$$(2.11) \qquad u(t,x) = p_0(t,x) - \frac{1}{a}\int_{\mathbb{R}_+} p_y(t,x)F(\mathrm{d}y),$$

where $F(\mathrm{d}y)$ is a measure on $\mathbb{R}_+$. We consider $\psi(t)$ the solution of $u(t, \psi(t)) = 0$. Let us define as before $\tau_\psi = \inf\{t \geq 0; Z_t^{\delta,0} \geq \psi(t)\}$. Then $u(t,x)\mathrm{d}x = \mathbb{P}(\tau_\psi > t, \ Z_t^{\delta,0} \in \mathrm{d}x)$ and

$$(2.12) \qquad \mathbb{P}_0(\tau_\psi > t) = \int_0^{\psi(t)} u(t,x)\mathrm{d}x.$$

In order to get the distribution of τ_ψ we have to evaluate the derivative of $\mathbb{P}_0(\tau_\psi > t)$. By using the equality (2.12) we obtain:

$$(2.13)\begin{aligned}\mathbb{P}_0(\tau_\psi \in \mathrm{d}t) &= \left(-\psi'(t)u(t,\psi(t)) - \int_0^{\psi(t)} \frac{\partial u}{\partial t}(t,x)\mathrm{d}x \right)\mathrm{d}t \\ &= \left(-\frac{1}{2}\int_0^{\psi(t)} \frac{\partial^2 u}{\partial x^2}(t,x)\mathrm{d}x + \frac{\delta-1}{2}\int_0^{\psi(t)} \frac{\partial}{\partial x}\left(\frac{1}{x}u(t,x)\right)\mathrm{d}x \right)\mathrm{d}t,\end{aligned}$$

as $u(t,x)$ is solution of the partial differential equation (2.8). We obtain thus:
(2.14)

$$\mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \left(-\frac{1}{2}\frac{\partial u}{\partial x}(t,x)\Big|_{x=\psi(t)} + \frac{\delta-1}{2\psi(t)}u(t,\psi(t)) + \left(\frac{1}{2}\frac{\partial u}{\partial x}(t,x) - \frac{\delta-1}{2x}u(t,x)\right)\Big|_{x=0} \right)\mathrm{d}t.$$

As $\frac{\delta-1}{2\psi(t)}u(t,\psi(t)) = 0$ and this ends the proof of the theorem. □

The idea behind the method of images is that for some particular forms of $F(\mathrm{d}y)$ we can derive explicit formulas of the hitting time distribution. More precisely:

PROPOSITION 2.2. *Let, for $\delta = 2\nu + 2 > 0$ and $a > 0$*

$$(2.15) \qquad \psi_a(t) = \sqrt{2t\log\frac{a}{\Gamma(\nu+1)t^{\nu+1}2^\nu}}, \qquad for \ t \leq \left[\frac{a}{\Gamma(\nu+1)2^\nu}\right]^{\frac{1}{\nu+1}}.$$

Then the probability density of τ_ψ is given by

$$(2.16) \qquad \mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \frac{1}{2at}\left(2t\log\frac{a}{\Gamma(\nu+1)t^{\nu+1}2^\nu} \right)^{\nu+1}\mathrm{d}t.$$

4

PROOF. By using the expression in (2.3) we remark first that

$$(2.17) \qquad y^{2\nu+1} p_y(t, x) = x^{2\nu+1} p_x(t, y).$$

Let us consider, as in Theorem 2.1

$$(2.18) \qquad u(t, x) = p_0(t, x) - \frac{1}{a} \int_{\mathbb{R}_+} p_y(t, x) F(\mathrm{d}y),$$

with $F(\mathrm{d}y) = y^{2\nu+1} \mathbb{1}_{\{y>0\}} \mathrm{d}y$. In this situation the function u defined in (2.18) writes

$$(2.19) \qquad \begin{aligned} u(t, x) &= p_0(t, x) - \frac{1}{a} x^{2\nu+1} \\ &= \left(\frac{1}{2^\nu} \frac{1}{t^{\nu+1}} \frac{1}{\Gamma(\nu+1)} \exp\left(-\frac{x^2}{2t} \right) - \frac{1}{a} \right) x^{2\nu+1}. \end{aligned}$$

For simplicity we will write ψ instead of ψ_a. Following the result in the Theorem 2.1, we are looking for $\psi(t)$ such that $u(t, \psi(t)) = 0$. This yields :

$$(2.20) \qquad x = \psi(t) = \sqrt{2t \log \frac{a}{\Gamma(\nu+1) t^{\nu+1} 2^\nu}}$$

under the obvious condition $t^{\nu+1} \le \frac{a}{\Gamma(\nu+1)2^\nu}$.
We can now notice that:

$$p_0(t, \psi(t)) = \frac{1}{a} (\psi(t))^{2\nu+1},$$

and we can prove easily that:

$$\frac{\partial u}{\partial x}(t, x) = (\delta - 1) \frac{u(t, x)}{x} - \frac{x}{t} p_0(t, x).$$

We obtain, after replacing in (2.10) and after applying the Theorem 2.1, for this particular case:

$$\begin{aligned} \mathbb{P}_0(\tau_\psi \in \mathrm{d}t) &= \frac{1}{2t} \psi(t) p_0(t, \psi(t)) \mathrm{d}t \\ &= \frac{1}{2at} \psi^{2\nu+2}(t) \mathrm{d}t \\ &= \frac{1}{2at} \left(2t \log \frac{a}{\Gamma(\nu+1) t^{\nu+1} 2^\nu} \right)^{\nu+1} \mathrm{d}t, \end{aligned}$$

which gives the desired result. $\qquad \square$

The second boundary which allows to express explicit results is obtained by using the Markov property for the Bessel process.

PROPOSITION 2.3. *Let, for $\delta = 2\nu + 2 > 0$, $s > 0$ and $a > 0$ fixed,*

$$(2.21) \qquad \psi_a(t) = \sqrt{\frac{2t(t+s)}{s} \left[(\nu+1) \log\left(1 + \frac{s}{t} \right) + \log a \right]},$$

for all $t \ge 0$ if $a \ge 1$ and for $t \le \frac{s}{(\frac{1}{a})^{\frac{1}{\nu+1}} - 1}$ if $0 < a < 1$. Then the probability density function of the hitting time τ_ψ is given by:
$$(2.22)$$

$$\mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \frac{1}{\Gamma(\nu+1)} \frac{1}{t} \left(\frac{t+s}{s} \right)^\nu \left[\log\left(a \left(\frac{t+s}{t} \right)^{\nu+1} \right) \right]^{\nu+1} \exp\left[-\frac{t+s}{s} \log\left(a \left(\frac{t+s}{t} \right)^{\nu+1} \right) \right] \mathrm{d}t.$$

PROOF. We will only sketch the proof as it follows the same ideas as the one of the Theorem 2.1. Let us consider the measure $F(\mathrm{d}y) = p_0(s, y)\mathrm{d}y$ for $s > 0$ fixed. Then, when evaluating the corresponding $u(t, x)$ we have:

$$\begin{aligned} u(t, x) &= p_0(t, x) - \frac{1}{a} \int_{\mathbb{R}_+} p_0(s, y) p_y(t, x)\mathrm{d}y \\ &= p_0(t, x) - \frac{1}{a} p_0(t + s, x) \\ &= \frac{1}{2^\nu} \frac{1}{\Gamma(\nu + 1)} x^{2\nu+1} \left[\frac{1}{t^{\nu+1}} \exp\left(-\frac{x^2}{2t}\right) - \frac{1}{a} \frac{1}{(t+s)^{\nu+1}} \exp\left(-\frac{x^2}{2(t+s)}\right) \right], \end{aligned}$$

by using the Markov property. We obtain the form of $\psi(t)$ by the condition $u(t, \psi(t)) = 0$ which gives:

$$(2.23) \qquad \psi(t) = \sqrt{\frac{2t(t+s)}{s} \left[(\nu+1) \log\left(1 + \frac{s}{t}\right) + \log a\right]}, \quad \begin{cases} \text{for } t \geq 0 \text{ if } a \geq 1 \\ \text{for } t \leq \frac{s}{(\frac{1}{a})^{\frac{1}{\nu+1}} - 1} \text{ if } a < 1. \end{cases}$$

In order to obtain the distribution of τ_ψ one has only to evaluate:

$$(2.24) \qquad \frac{\partial u}{\partial x}(t, x) = (\delta - 1) \frac{u(t, x)}{x} - \frac{s}{t(t+s)} x\, p_0(t, x),$$

and $\frac{u(t,x)}{x}$ for $x = 0$ and $x = \psi(t)$ and replace the values in the general form (2.14). The expression (2.22) follows. $\qquad \square$

REMARK 2.4.   *We can notice that the function $\psi_a(t)$ defined by (2.21) satisfies for large times*

$$\begin{cases} \psi_a(t) &\simeq& \sqrt{t} & \text{for } a = 1, \\ \psi_a(t) &\simeq& t & \text{for all } a > 1. \end{cases}$$

In particular, we can approach large times by considering this kind of boundary.

A new boundary can be obtained by using the Laplace transform of the square of the δ-dimensional Bessel process starting from x. More precisely:

PROPOSITION 2.5.   *Let, for $\delta = 2\nu + 2 > 0$, $\lambda > 0$ and $a > 0$ fixed,*

$$(2.25) \qquad \psi_a(t) = \frac{\lambda t}{1 + 2\lambda t} + t\sqrt{\left(\frac{\lambda}{1 + 2\lambda t}\right)^2 + \frac{2}{t} \log \frac{a(1 + 2\lambda t)^{\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)}}$$

for

$$(2.26) \qquad \begin{cases} t \leq \frac{1}{(2^\nu \Gamma(\nu+1)/a)^{1/(\nu+1)} - 2\lambda} & \text{if } \lambda < \frac{1}{2}\left(\frac{2^\nu \Gamma(\nu+1)}{a}\right)^{\frac{1}{\nu+1}}, \\ t \geq 0 & \text{if } \lambda \geq \frac{1}{2}\left(\frac{2^\nu \Gamma(\nu+1)}{a}\right)^{\frac{1}{\nu+1}}. \end{cases}$$

Then the probability density function of the hitting time is given by:

$$(2.27) \qquad \mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \sqrt{\left(\frac{\lambda}{1 + 2\lambda t}\right)^2 + \frac{2}{t} \log \frac{a(1 + 2\lambda t)^{\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)}}\, p_0(t, \psi(t))\mathrm{d}t.$$

6

PROOF. We present only the main ideas as the result follows as above from the general method in Theorem 2.1 applied to the measure $F(\mathrm{d}y) = y^{2\nu+1}e^{-\lambda y^2}\mathbb{1}_{\{y\geq 0\}}\mathrm{d}y$. For this measure $u(t,x)$ takes the form:

$$
\begin{aligned}
u(t,x) &= p_0(t,x) - \frac{1}{a}\int_{\mathbb{R}_+} p_y(t,x)F(\mathrm{d}y) \\
&= p_0(t,x) - \frac{1}{a}\int_{\mathbb{R}_+} p_y(t,x)y^{2\nu+1}e^{-\lambda y^2}\mathrm{d}y \\
&= p_0(t,x) - \frac{1}{a}\int_{\mathbb{R}_+} x^{2\nu+1}p_x(t,y)e^{-\lambda y^2}\mathrm{d}y \\
&= p_0(t,x) - \frac{x^{2\nu+1}}{a}\mathbb{E}\left(e^{-\lambda Z_t^{\delta,x}}\right).
\end{aligned}
$$

By using the expression of the Laplace transform for $Z_t^{\delta,x}$ we obtain

(2.28)
$$
u(t,x) = p_0(t,x) - \frac{x^{2\nu+1}}{a}\frac{1}{(1+2\lambda t)^{\delta/2}}\exp\left(-\frac{\lambda x}{1+2\lambda t}\right).
$$

We consider first the equality $u(t,\psi(t)) = 0$ in (2.28) and this gives the form of $\psi(t)$ in (2.25). Afterwards, we can evaluate once again in this particular situation

$$
\frac{\partial u}{\partial x}(t,x) = (\delta-1)\frac{u(t,x)}{x} - \left(\frac{x}{t} - \frac{\lambda t}{1+2\lambda t}\right)p_0(t,x).
$$

For this particular case, there is only one no vanishing term in the expression (2.10) of $\mathbb{P}_0(\tau_\psi \in \mathrm{d}t)$, that is the term $-\left(\frac{x}{t} - \frac{\lambda t}{1+2\lambda t}\right)p_0(t,x)$ of $\frac{\partial u}{\partial x}(t,x)$ for $x = \psi(t)$ and this is exactly given by the right side of the formula (2.27). $\square$

COROLLARY 2.6. *The previous results writes for $\delta = 2$*

(1) *For $a > 0$, $t \leq a$ and $\psi(t) = \sqrt{2t\log\dfrac{a}{t}}$, the density of the hitting time τ_ψ is*

$$
\mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \frac{1}{2a}\log\frac{a}{t}\mathrm{d}t.
$$

(2) *For $s > 0$, $a > 0$, $t \leq \frac{sa}{1-a}$ and $\psi(t) = \sqrt{\dfrac{2t(t+s)}{s}\log\left(a\dfrac{t+s}{t}\right)}$, the probability density function of τ_ψ is given by:*

$$
\mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \frac{t+s}{t}\log\left(a\frac{t+s}{t}\right)\exp\left[-\frac{t+s}{t}\log\left(a\frac{t+s}{t}\right)\right]\mathrm{d}t.
$$

(3) *For $a > 0$ and $\psi(t) = \dfrac{\lambda t}{1+2\lambda t} + t\sqrt{\left(\dfrac{\lambda}{1+2\lambda t}\right)^2 + \dfrac{2}{t}\log\dfrac{a(1+2\lambda t)}{t}}$, for*

(2.29)
$$
\begin{cases}
t \geq 0, & \text{if } \lambda \geq \dfrac{1}{2a}, \\[2mm]
t \leq \dfrac{a}{1-2\lambda a} & \text{if } \lambda < \dfrac{1}{2a},
\end{cases}
$$

the probability density function of τ_ψ is:

$$
\mathbb{P}_0(\tau_\psi \in \mathrm{d}t) = \sqrt{\left(\frac{\lambda}{1+2\lambda t}\right)^2 + \frac{2}{t}\log\frac{a(1+2\lambda t)}{t}}\,p_0(t,\psi(t))\mathrm{d}t.
$$

7

2.2. Approximation of the first hitting time for Bessel processes starting from the origin. In this section we will construct a stepwise procedure, the so-called *random Walk on Moving Spheres (WoMS)* algorithm, which allows to approach the first time the standard Bessel process hits a given level $l > 0$. Of course, this stopping time $\tau_l = \inf\{t > 0; Z_t^{\delta,x} = l\}$ can be characterized by its well-known Laplace transform computed by solving an eigenvalue problem. Indeed if $(Z_t^{\delta,x}, t \geq 0)$ is the Bessel process starting from x, of index $\nu = \frac{\delta}{2} - 1$ then, for $\nu > 0$ and $x \leq l$, we get

$$\mathbb{E}_x[e^{-\lambda\tau_l}] = \frac{x^{-\nu}I_\nu(x\sqrt{2\lambda})}{l^{-\nu}I_\nu(l\sqrt{2\lambda})}, \ x > 0, \quad \text{and } \mathbb{E}_0\left[e^{-\lambda\tau_l}\right] = \frac{(l\sqrt{2\lambda})^\nu}{2^\nu\Gamma(\nu+1)}\frac{1}{I_\nu(l\sqrt{2\lambda})}.$$

Here I_ν denotes the modified Bessel function. This Laplace transform can be used to describe the tail distribution: Ciesielski and Taylor [3] proved that, for $\delta = 2\nu + 2 \in \mathbb{N}^*$,

$$\mathbb{P}_0(\tau_l > t) = \frac{1}{2^{\nu-1}\Gamma(\nu+1)}\sum_{k=1}^\infty \frac{j_{\nu,k}^{\nu-1}}{\mathcal{J}_{\nu+1}(j_{\nu,k})}e^{-\frac{j_{\nu,k}^2}{2l^2}t}.$$

where $\mathcal{J}_\nu$ is the Bessel function of the first kind and $(j_{\nu,k})_{\nu,k}$ is the associated sequence of its positive zeros.

We are looking for a numerical approach for the hitting time and these formulas are not easy to handle and approach, in particular we can not compute the inverse cumulative function ! The aim of this section is to construct an easy and efficient algorithm without need of inverting the Laplace transform and without using discretization schemes since the hitting times are unbounded. In the next step we will extend this procedure to the hitting time of time-dependent boundaries like straight lines, useful in the description of the hitting time of a given level for the CIR process (the Laplace transform is then unknown).

2.2.1. Hitting time of a given level for the Bessel process with positive integer dimension δ. Let us consider δ independent one-dimensional Brownian motions $(B_t^{(k)}, t \geq 0)$, $1 \leq k \leq \delta$. Then the Bessel process of index δ starting from 0, satisfies the following property:

$$(Z_t^{\delta,0}, t \geq 0) \quad \text{has the same distribution as} \quad \left(\sqrt{\left(B_t^{(1)}\right)^2 + \ldots + \left(B_t^{(\delta)}\right)^2}, t \geq 0\right).$$

Let

(2.30)
$$\tau_l = \inf\{t \geq 0; Z_t^{\delta,0} \geq l\}.$$

In particular, we can express τ_l by using the first time when the δ-dimensional Brownian motion $\mathbf{B} = (B^{(1)}, \ldots, B^{(\delta)})$ exits from the Euclidean ball D centered in the origin with radius l. Approximating the exit time and the exit position for the 2-dimensional Brownian motion of a convex domain was already studied by Milstein [12]. He used the so-called random walk on spheres algorithm which allows to approach the exit location and the exit time through an efficient algorithm. The exit position is really simple to obtain (as it is uniformly distributed on the circle) while the exit time is much more difficult to approach. That is why we will construct an adaptation of this initial procedure in order to obtain nice and efficient results concerning the Bessel process exit time.

Let us introduce now our *Walk on Moving Spheres (WoMS)* algorithm. We first define a continuous function $\rho : \mathbb{R}^2 \to \mathbb{R}_+$ which represents the distance to the boundary of D:

(2.31)
$$\rho(x) = \inf\{\|x - y\|; \ y \in D^c\} = l - \|x\|.$$

8

For any small enough parameter $\varepsilon > 0$, we will denote by D^ε the sphere centered at the origin with radius $l - \varepsilon$

$$(2.32) \qquad D^\varepsilon = \{x \in D; \|x\| \leq l - \varepsilon\} = \{x \in D; \rho(x) \geq \varepsilon\}.$$

Algorithm (A1) for $\delta = 2$. *Let us fix a parameter $0 < \gamma < 1$.*
Initialization: *Set $X(0) = (X_1(0), X_2(0)) = 0$, $\theta_0 = 0$, $\Theta_0 = 0$, $A_0 = \gamma^2 l^2 e / 2$.*
First step: *Let (U_1, V_1, W_1) be a vector of three independent random variables uniformly distributed on $[0, 1]$. We set*

$$\begin{cases} \theta_1 = A_0 U_1 V_1, \quad \Theta_1 = \Theta_0 + \theta_1, \\[2mm] X(1)^\mathsf{T} = (X_1(1), X_2(1))^\mathsf{T} = X(0)^\mathsf{T} + \psi_{A_0}(\theta_1) \begin{pmatrix} \cos(2\pi W_1) \\ \sin(2\pi W_1) \end{pmatrix}, \end{cases}$$

where

$$(2.33) \qquad \psi_a(t) = \sqrt{2t \log \frac{a}{t}}, \quad t \leq a, \ a > 0.$$

At the end of this step we set $A_1 = \gamma^2 \rho(X(1))^2 e / 2$.
The n-th step: *While $X(n-1) \in D^\varepsilon$, simulate (U_n, V_n, W_n) a vector of three independent random variables uniformly distributed on $[0, 1]$ and define*

$$(2.34) \qquad \begin{cases} \theta_n = A_{n-1} U_n V_n, \quad \Theta_n = \Theta_{n-1} + \theta_n, \\[2mm] X(n)^\mathsf{T} = (X_1(n), X_2(n))^\mathsf{T} = X(n-1)^\mathsf{T} + \psi_{A_{n-1}}(\theta_n) \begin{pmatrix} \cos(2\pi W_n) \\ \sin(2\pi W_n) \end{pmatrix}. \end{cases}$$

At the end of this step we set $A_n = \gamma^2 \rho(X(n))^2 e / 2$.
When $X(n-1) \notin D^\varepsilon$ the algorithm is stopped: we set $\theta_n = 0$, $\Theta_n = \Theta_{n-1}$ and $X(n) = X(n-1)$.
Outcome: *The hitting time Θ_n and the exit position $X(n)$.*

REMARK 2.7. *The WoMS algorithm describes a D-valued Markov chain $(X(n), n \geq 0)$. Each step corresponds to an exit problem for the 2-dimensional Brownian motion. If $X(n) = x$ then we focus our attention to the exit problem of the ball centered in x and of radius $(\psi_a(t), t \geq 0)$: the exit location corresponds to $X(n+1)$ and the exit time to θ_{n+1}. Of course the choice of the parameter a plays a crucial role since the moving sphere has to belong to the domain D as time elapses... When the Markov chain X is close to the boundary ∂D, we stop the algorithm and obtain therefore a good approximation of the exit problem of D.*

Comparison with the classical (WoS) algorithm: at each step, the n-th step of the classical walk on spheres (WoS) is based on the exit location and exit time, which are mutually independent, for the Brownian paths exiting from a ball centered in $X(n-1)$ and with radius $\gamma \rho(X(n-1))$. The exit location is uniformly distributed on the sphere while the exit time is characterized by its Laplace transform. Therefore, if one knows $X(n-1)$ then the diameter of the sphere is deterministic. For the WoMS the center of the ball should also be $X(n-1)$ but the radius is random, smaller than $\gamma \rho(X(n-1))$. The exit location will also be uniformly distributed on the sphere but the exit time will be much easier to simulate: in particular, you don't need to evaluate the Bessel functions.

The stochastic process $(X(n), n \geq 0)$ is an homogeneous Markov chain stopped at the first time it exits from D^ε. In the following, we shall denote N^ε this stopping time which represents in fact the number of steps in the algorithm:

$$N^\varepsilon = \inf\{n \geq 0; \ X(n) \notin D^\varepsilon\}.$$

We just notice that $X(N^\varepsilon) \in D^\varepsilon$ by its definition.

The algorithm (A1) is presented in the 2-dimensional case. Of course we can construct a generalization of this procedure for the δ-dimensional Bessel process when $\delta \in \mathbb{N}^*$. For notational simplicity, we use a slightly different method: instead of dealing with a Markov chain $(X(n), n \in \mathbb{N})$ living in $\mathbb{R}^\delta$ we shall consider its squared norm, which is also (surprisingly) a Markov chain. At each step, we shall construct a couple of random variables $(\xi_n, \chi(n))$ associated to an exit problem, the first coordinate corresponds to an exit time and the second one to the norm of the exit location. We introduce some notations: $\mathcal{S}^\delta$ represents the unit ball in $\mathbb{R}^\delta$, and $\pi_1 : \mathbb{R}^\delta \to \mathbb{R}$ the projection on the first coordinate.

Algorithm (A2). *Let us fix a parameter $0 < \gamma < 1$.*
Initialization: *Set $\chi(0) = 0$, $\xi_0 = 0$, $\Xi_0 = 0$, $A_0 = (\gamma^2 l^2 e/(\nu+1))^{\nu+1} \frac{\Gamma(\nu+1)}{2}$.*
The n-th step: *While $\sqrt{\chi(n-1)} < l - \varepsilon$, we choose U_n a uniform distributed random vector on $[0,1]^{\lfloor \nu \rfloor + 2}$, G_n a standard gaussian random variable and V_n an uniformly distributed random vector on $\mathcal{S}^\delta$. Consider U_n, G_n and V_n independent. We set*

(2.35)
$$\begin{cases} \xi_n = \left(\frac{A_{n-1}}{\Gamma(\nu+1)2^\nu} U_n(1) \ldots U_n(\lfloor \nu \rfloor + 2) \right)^{\frac{1}{\nu+1}} \exp\left\{ -\frac{\nu - \lfloor \nu \rfloor}{\nu+1} G_n^2 \right\}, \quad \Xi_n = \Xi_{n-1} + \xi_n, \\ \chi(n) = \chi(n-1) + 2\pi_1(V_n)\sqrt{\chi(n-1)}\psi_{A_{n-1}}(\xi_n) + \psi_{A_{n-1}}^2(\xi_n), \end{cases}$$

where

(2.36)
$$\psi_a(t) = \sqrt{2t \log \frac{a}{\Gamma(\nu+1)t^{\nu+1}2^\nu}}, \quad t \leq t_{\max}(a) := \left[\frac{a}{\Gamma(\nu+1)2^\nu} \right]^{\frac{1}{\nu+1}}, \quad a > 0.$$

At the end of this step we set

$$A_n = \left(\gamma^2 (l - \sqrt{\chi(n)})^2 e/(\nu+1) \right)^{\nu+1} \frac{\Gamma(\nu+1)}{2}.$$

When $\sqrt{\chi(n)} \geq l - \varepsilon$ the algorithm is stopped: we then set $\xi_n = 0$, $\Xi_n = \Xi_{n-1}$ and $\chi(n) = \chi(n-1)$.
Outcome: *The hitting time Ξ_n and the value of the Markov chain $\chi(n)$.*

It is obvious that for the particular dimension $\delta = 2$, that is $\nu = 0$, the stopping times obtained by the algorithm (A1) and (A2) have the same distribution. Moreover, for each n, $\chi(n)$ has the same distribution as $\|X(n)\|^2$. In other words, the number of steps of (A1) and (A2) are identical in law, the number of steps will be denoted in both cases N^ε.

THEOREM 2.8. *Set $\delta \in \mathbb{N}^*$. The number of steps N^ε of the algorithm $WoMS$ (A2) is almost surely finite. Moreover, there exist a constant $C_\delta > 0$ and $\varepsilon_0(\delta) > 0$, such that*

$$\mathbb{E}[N^\varepsilon] \leq C_\delta |\log \varepsilon|, \quad \text{for all} \ \ \varepsilon \leq \varepsilon_0(\delta).$$

10

THEOREM 2.9.   *Set $\delta \in \mathbb{N}^*$. As ε goes to zero, Ξ_{N^ε} converges in distribution towards τ_l the hitting time of the δ-dimensional Bessel process (with cumulative distribution function F), which is almost surely finite. Moreover, for any $\alpha > 0$ small enough,*

$$
(2.37) \qquad \left(1 - \frac{\varepsilon}{\sqrt{2\alpha\pi}}\right) F^\varepsilon(t - \alpha) \leq F(t) \leq F^\varepsilon(t), \text{ for all } t > 0,
$$

where $F^\varepsilon(t) := \mathbb{P}(\Xi_{N^\varepsilon} \leq t)$.

These results and the key ideas of the proofs are adapted from the classical random walk on spheres (WoS), see [12].

Proof of Theorem 2.8.

Step 1. Let us estimate the number of steps. Since $(\chi(n),\, n \geq 0)$ is a homogeneous Markov chain, we introduce the operator $P_x f$ defined, for any non-negative function $f : \mathbb{R}_+ \to \mathbb{R}_+$, by

$$
P_x f := \int_{\mathbb{R}_+} f(y) \mathbb{P}(x, \mathrm{d}y),
$$

where $\mathbb{P}(x, \mathrm{d}y)$ is the transition probability of the Markov chain. By definition, $\chi(n+1)$ depends only on $\chi(n)$, V_n and ξ_n. Let us note that, by construction, V_n and ξ_n are independent. Moreover using the result developed in the Appendix, the density of $\xi_n \left(\frac{2^\nu \Gamma(\nu+1)}{A_{n-1}}\right)^{\frac{1}{\nu+1}}$ is given by:

$$
(2.38) \qquad \mu(r) = \frac{(\nu+1)^{\nu+2}}{\Gamma(\nu+2)} r^\nu \left(-\log r\right)^{\nu+1} \mathbb{1}_{[0,1]}(r).
$$

If we denote σ^d the uniform surface measure on the unit sphere in $\mathbb{R}^d$, we get

$$
(2.39) \qquad P_x f = \int_0^1 \int_{\mathcal{S}^\delta} f\left(x + 2\pi_1(u)\sqrt{x}K(x,r) + K^2(x,r)\right) \mu(r)\mathrm{d}r\, \sigma_1^\delta(\mathrm{d}u),
$$

with $K(x,r)$ defined by

$$
(2.40) \qquad K(x,r) = \psi_A\left(\left[\frac{A}{2^\nu \Gamma(\nu+1)}\right]^{\frac{1}{\nu+1}} r\right),
$$

and A depending on x in the following way:

$$
A = \left(\frac{\gamma^2(l - \sqrt{x})^2 e}{\nu+1}\right)^{\nu+1} \frac{\Gamma(\nu+1)}{2}.
$$

We can observe the following scaling property: $\psi_A(A^{\frac{1}{\nu+1}}t) = A^{\frac{1}{2\nu+2}}\psi_1(t)$. Therefore the definition of ψ_1 leads to

$$
(2.41) \qquad K(x,r) = \gamma(l - \sqrt{x})\sqrt{\frac{er}{\nu+1}\log\frac{1}{r^{\nu+1}}} = \gamma(l - \sqrt{x})\sqrt{er(-\log r)}.
$$

Step 2. Using classical potential theory for discrete time Markov chains (see, for instance, Theorem 4.2.3 in [14]), we know that

$$
\phi(x) = \mathbb{E}_x\left(\sum_{n=0}^{N^\varepsilon - 1} g(\chi(n))\right)
$$

11

satisfies, for any non-negative function g,

(2.42)
$$\begin{cases} \phi(x) = P_x\phi + g(x), & 0 \le x < (l - \varepsilon)^2, \\ \phi((l - \varepsilon)^2) = 0. \end{cases}$$

In particular, for $g = 1$, we obtain that $\phi(x) = \mathbb{E}_x[N^\varepsilon]$. In order to get an upper-bound for the averaged number of steps, it suffices to apply a comparison result. We choose the function

(2.43)
$$U^\varepsilon(x) = \{\log((l - \sqrt{x})/\varepsilon) - \log(1 - \gamma)\}/(C_\delta \gamma^2)$$

which satisfies $U^\varepsilon(x) \ge P_x U^\varepsilon + 1$, for all $0 < x < (l - \varepsilon)^2$ (see Lemma 2.10 for the definition of the constant and for the inequality) and $U^\varepsilon(x) \ge 0$ for all $0 < x < (l - \varepsilon)^2$. A classical comparison result related to the potential theory (see, for instance, Theorem 4.2.3 in [14]) implies that $\mathbb{E}_x[N^\varepsilon] \le U^\varepsilon(x)$ for all $x \in [0, (l - \varepsilon)^2]$ and consequently leads to the announced statement. $\qquad\square$

LEMMA 2.10. *Let us define, for small $\varepsilon > 0$, $U^\varepsilon(x) = \{\log((l - \sqrt{x})/\varepsilon) - \log(1 - \gamma)\}/(C_\delta \gamma^2)$ for $x \in [0, l^2[$, where γ is related to the definition of the WoMS. There exists a constant $C_\delta > 0$, such that, for any $x \in]0, (l - \varepsilon)^2[$, the following inequality yields*

$$P_x U^\varepsilon - U^\varepsilon(x) \le -1.$$

We recall that $P_x U^\varepsilon$ is defined by (2.39) and (2.41).

PROOF. We will split the proof in several steps.

Step 1. First of all, we observe that $U^\varepsilon \ge -\log(1 - \gamma)/(C_\delta \gamma^2)$ in the domain $[0, (l - \varepsilon)^2]$. Let us consider now $\chi(0) = x \in [0, (l - \varepsilon)^2]$ and y in the support of the law of $\chi(1)$ and let us prove that $U^\varepsilon(y) \ge 0$. By the definition of $\chi(1)$ we obtain

$$\chi(1) \le \sup_{y \in [-1,1], t \in [0, t_{\max}(A)]} (x + 2y\sqrt{x}\psi_A(t) + \psi_A^2(t)),$$

where $A = \left(\gamma^2(l - \sqrt{x})^2 e/(\nu + 1)\right)^{\nu+1} \frac{\Gamma(\nu+1)}{2}$ and both ψ_A and $t_{\max}$ are defined by (2.36). The right hand side of the preceding inequality is increasing with respect to y so that:

$$\chi(1) \le \left(\sqrt{x} + \sup_{t \in [0, t_{\max}(A)]} \psi_A(t)\right)^2.$$

Furthermore, for $a > 0$ the maximum of the function ψ_a is reached for $t_{\max}(a) = \frac{1}{e}\left(\frac{a}{\Gamma(\nu+1)2^\nu}\right)^{\frac{1}{\nu+1}}$ and is equal to

(2.44)
$$\sup_{t \in [0, t_{\max}(a)]} \psi_a(t) = \left\{\frac{2(\nu + 1)}{e}\left(\frac{a}{\Gamma(\nu + 1)2^\nu}\right)^{\frac{1}{\nu+1}}\right\}^{1/2}.$$

Finally using the definition of A and the inequality $x \le (l - \varepsilon)^2$, we find the following lowerbound:

$$l - \sqrt{\chi(1)} \ge (l - \sqrt{x})(1 - \gamma) \ge \varepsilon(1 - \gamma).$$

We can therefore conclude that, for any y in the support of the law of $\chi(1)$ (even for $y \ge (l - \varepsilon)^2$), $U^\varepsilon(y) \ge 0$ which ensures that U^ε is well defined and non-negative in the domain of the operator

12

P_x.

Step 2. Furthermore the Taylor expansion yields

$$(2.45) \qquad U^\varepsilon(y) \le U^\varepsilon(x) + \frac{\sqrt{x} - \sqrt{y}}{C_\delta \gamma^2(l - \sqrt{x})} - \frac{(\sqrt{x} - \sqrt{y})^2}{2C_\delta \gamma^2(l - \sqrt{x})^2} + \frac{(\sqrt{x} - \sqrt{y})^3}{3C_\delta \gamma^2(l - \sqrt{x})^3}, \qquad x, \ y \in [0, l^2[.$$

If $\chi(0) = x$ and y is in the support of the random variable $\chi(1)$, then

$$\sqrt{y} - \sqrt{x} = \sqrt{x + 2\pi_1(u)\sqrt{x}K(x,r) + K^2(x,r)} - \sqrt{x}$$
$$\ge \pi_1(u)K(x,r).$$

By the expansion (2.45) and the definition of the operator P_x given by (2.39), the following upper-bound for the operator P_x holds

$$P_x U^\varepsilon = \int_0^1 \int_{\mathcal{S}^\delta} U^\varepsilon\Big(x + 2\pi_1(u)\sqrt{x}K(x,r) + K^2(x,r)\Big)\mu(r)\mathrm{d}r\,\sigma_1^\delta(\mathrm{d}u),$$
$$\le U^\varepsilon(x) - \int_0^1 \int_{\mathcal{S}^\delta} \frac{\pi_1(u)K(x,r)}{C_\delta \gamma^2(l - \sqrt{x})}\,\mu(r)\mathrm{d}r\,\sigma_1^\delta(\mathrm{d}u)$$
$$- \int_0^1 \int_{\mathcal{S}_+^\delta} \frac{\pi_1^2(u)K^2(x,r)}{2C_\delta \gamma^2(l - \sqrt{x})^2}\,\mu(r)\mathrm{d}r\,\sigma_1^\delta(\mathrm{d}u)$$
$$- \int_0^1 \int_{\mathcal{S}^\delta} \frac{\pi_1^3(u)K^3(x,r)}{3C_\delta \gamma^2(l - \sqrt{x})^3}\,\mu(r)\mathrm{d}r\,\sigma_1^\delta(\mathrm{d}u),$$

where

$$(2.46) \qquad\qquad\qquad \mathcal{S}_+^\delta := \{u \in \mathcal{S}^\delta : \ \pi_1(u) > 0\}.$$

Due to symmetry properties, the first and the third integral terms vanish. Then (2.41) leads to

$$P_x U^\varepsilon \le U^\varepsilon(x) - \frac{I}{C_\delta} \int_{\mathcal{S}_+^\delta} \pi_1^2(u)\,\sigma_1^\delta(\mathrm{d}u)$$

with

$$I = \frac{(\nu + 1)^{\nu+2}e}{2\Gamma(\nu + 2)} \int_0^1 r^{\nu+1}(-\log r)^{\nu+2}\,\mathrm{d}r.$$

The description of the probability density function in the Appendix leads to the following explicit value

$$I = \left(\frac{\nu + 1}{\nu + 2}\right)^{\nu+2} \frac{e}{\Gamma(\nu + 2)}.$$

In order to conclude the proof, it suffices to choose the particular constant

$$C_\delta = I \int_{\mathcal{S}_+^\delta} \pi_1^2(u)\,\sigma_1^\delta(\mathrm{d}u).$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Proof of Theorem 2.9

The proof is splited in two parts. First the steps of the algorithm and the hitting time of the Bessel process of index ν shall be related to stopping times of a δ-dimensional Brownian motion ($\nu = \frac{\delta}{2} - 1$). Secondly we point out that the corresponding stopping times are close together by evaluating deviations of the Brownian paths.

Step 1. Let $\mathbf{B} = (B^{(1)}, B^{(2)}, \ldots, B^{(\delta)})$ be a δ-dimensional Brownian motion. Then the norm of $\mathbf{B}$ has the same distribution as a Bessel process of index ν (see, for instance [15]). Hence the first hitting time τ_l is identical in law to the stopping time

$$\mathbb{T}_l = \inf\{t \geq 0; \ \mathbf{B}_t \notin D\},$$

where D is the Euclidean ball centered at the origin and of radius l. We introduce then a procedure in order to come close to $\mathbb{T}_l$. For the first step we shall focus our attention to the first exit time of a moving sphere centered at the origin and of radius $\psi_a(t)$ defined by (2.36), we denote $\hat{\xi}_1$ this stopping time. Of course this moving sphere should always stay in D, so we choose a such that the maximum of ψ_a stays smaller than l. By (2.44), we get

$$\sup_{t \leq a} \psi_a(t) < l \iff a < \frac{\Gamma(\nu + 1)}{2} \left(\frac{el^2}{\nu + 1} \right)^{\nu + 1}.$$

For $a = A_0 = \frac{\Gamma(\nu+1)}{2} \left(\frac{e\gamma^2 l^2}{\nu+1} \right)^{\nu+1}$ with a parameter $\gamma < 1$, the condition is satisfied: $\sup_{t \leq a} \psi_a(t) = \gamma^{2\nu+2} l < l$. Let us describe the law of $(\hat{\xi}_1, \mathbf{B}_{\hat{\xi}_1})$. The norm of the Brownian motion is identical in law with the Bessel process; therefore Proposition 2.2 implies that the density function of $\hat{\xi}_1$ is given by (2.16) with a replaced by A_0. Using the law described in Proposition A.1, we can prove that $\hat{\xi}_1$ has the same distribution as

$$\left(\frac{A_0}{\Gamma(\nu+1)2^\nu} \right)^{\frac{1}{\nu+1}} e^{-Z},$$

where Z is Gamma distributed with parameters $\alpha = \nu + 2$ and $\beta = \frac{1}{\nu+1}$. By construction we deduce that $\hat{\xi}_1 \overset{(d)}{=} \xi_1$ where ξ_1 is defined in the algorithm $WoMS$ (A2). Knowing the stopping time, we can easily describe the exit location since the Brownian motion is rotationnaly invariant: $\mathbf{B}_{\hat{\xi}_1}$ is then uniformly distributed on the sphere of radius $\psi_{A_0}(\hat{\xi}_1)$. Hence

$$(\hat{\xi}_1, \|\mathbf{B}_{\hat{\xi}_1}\|) \overset{(d)}{=} (\xi_1, \chi(1)), \quad \text{and } \hat{\xi}_1 < \mathbb{T}_l.$$

By this procedure we can construct a sequence of stopping times $(\hat{\xi}_n, n \geq 1)$ and define $\hat{\Xi}_n = \hat{\xi}_1 + \ldots + \hat{\xi}_n$; $\hat{\Xi}_n$ is the first time after $\hat{\Xi}_{n-1}$ such that the Brownian motion exits from a sphere centered in $\mathbf{B}_{\hat{\Xi}_{n-1}}$ of radius ψ_{a_n} initialized at time $\hat{\Xi}_{n-1}$. The moving sphere should stay in the domain D, so we choose

$$a_n = \left(\gamma^2 \left(l - \sqrt{\mathbf{B}_{\hat{\Xi}_{n-1}}} \right)^2 e/(\nu+1) \right)^{\nu+1} \frac{\Gamma(\nu+1)}{2}.$$

Using the same arguments as before and by the Markov property for the Brownian motion, we obtain the identities in law

$$(a_n, n \geq 1) \overset{(d)}{=} (A_n, n \geq 1), \quad \left(\hat{\Xi}_n, \|\mathbf{B}_{\hat{\Xi}_n}\| \right)_{n \geq 1} \overset{(d)}{=} \left(\Xi_n, \chi(n) \right)_{n \geq 1}$$
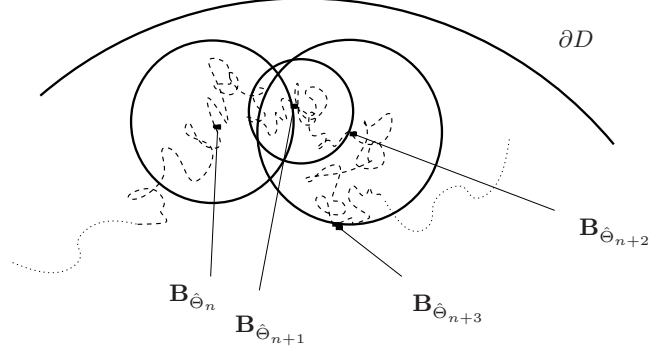
14

FIG 1. *Walk on Moving Spheres*

with $\hat{\bar{\Xi}}_n < \mathbb{T}_l$ and Ξ_n, A_n, $\chi(n)$ defined in the algorithm $WoMS$ (A2). Consequently defining $\hat{N}^\varepsilon = \inf\{n \geq 0; \mathbf{B}_{\hat{\bar{\Xi}}_n} \notin D^\varepsilon\}$, the following identity yields

$$(2.47) \qquad \left(\hat{\bar{\Xi}}_{\hat{N}^\varepsilon}, \|\mathbf{B}_{\hat{\bar{\Xi}}_{\hat{N}^\varepsilon}}\|\right) \overset{(d)}{=} \left(\Xi_{N^\varepsilon}, \chi(N^\varepsilon)\right) \quad \text{and} \quad \hat{\bar{\Xi}}_{\hat{N}^\varepsilon} < \mathbb{T}_l.$$

Step 2. Let us now estimate the difference between $\hat{\bar{\Xi}}_{\hat{N}^\varepsilon}$ and $\mathbb{T}_l$. By (2.47) we first deduce

$$(2.48) \qquad F(t) := \mathbb{P}(\tau_l \leq t) = \mathbb{P}(\mathbb{T}_l \leq t) \leq F^\varepsilon(t) := \mathbb{P}(\Theta_{N^\varepsilon} \leq t), \quad t > 0.$$

Furthermore, for any small $\alpha > 0$,

$$\begin{aligned}
1 - F(t) &= \mathbb{P}(\mathbb{T}_l > t, \hat{\bar{\Xi}}_{\hat{N}^\varepsilon} \leq t - \alpha) + \mathbb{P}(\mathbb{T}_l > t, \hat{\bar{\Xi}}_{\hat{N}^\varepsilon} > t - \alpha) \\
&\leq \mathbb{P}(\mathbb{T}_l > t, \hat{\bar{\Xi}}_{\hat{N}^\varepsilon} \leq t - \alpha) + \mathbb{P}(\hat{\bar{\Xi}}_{\hat{N}^\varepsilon} > t - \alpha) \\
(2.49) \qquad &\leq \mathbb{P}(\mathbb{T}_l > t, \hat{\bar{\Xi}}_{\hat{N}^\varepsilon} \leq t - \alpha) + 1 - F^\varepsilon(t - \alpha).
\end{aligned}$$

At time $\hat{\bar{\Xi}}_{\hat{N}^\varepsilon}$ the Brownian motion is in the ε-neighborhood of the boundary ∂D, hence $l - \|\mathbf{B}_{\hat{\bar{\Xi}}_{\hat{N}^\varepsilon}}\| \leq \varepsilon$. Using the strong Markov property, we obtain

$$(2.50) \qquad \mathbb{P}(\mathbb{T}_l > t, \hat{\bar{\Xi}}_{\hat{N}^\varepsilon} \leq t - \alpha) \leq F^\varepsilon(t - \alpha) \sup_{y \in \partial D^\varepsilon} \mathbb{P}_y(\mathbb{T}_l > \alpha).$$

Since the Brownian motion is rotationnaly invariant, it suffices to choose $y = (l - \varepsilon, 0, \ldots, 0)$. Due to the convexity of ∂D, the following upper-bound holds

$$(2.51) \qquad \mathbb{P}_y(\mathbb{T}_l > \alpha) \leq \mathbb{P}_0(\sup_{0 \leq t \leq \alpha} \overline{B}_t^{(1)} < \varepsilon) = \mathbb{P}_0(2|\overline{B}_\alpha^{(1)}| < \varepsilon) \leq \frac{\varepsilon}{\sqrt{2\alpha\pi}}.$$

Combining (2.48) for the upper-bound and (2.49), (2.50) and (2.51) for the lower-bound yields the announced estimation (2.37). $\qquad \square$

2.2.2. *The first time the Bessel process of index ν hits a decreasing curved boundary.* The algorithm developed in the previous paragraph can be adapted to the problem of hitting a deacreasing curved boundary. Let us define :

$$(2.52) \qquad \tau = \inf\{t \geq 0 : Z_t^{\delta,0} = l(t)\}, \quad \text{where } l \text{ is decreasing and } l(0) > 0.$$

15

ASSUMPTION 2.11. *There exists a constant $\Delta_{\min} > 0$ which bounds the derivative of l*

$$l'(t) \geq -\Delta_{\min}, \quad \forall t \geq 0.$$

The procedure then also consists in building a $WoMS$ which reaches a neighborhood of the boundary. But instead of dealing with a fixed boundary as in Section 2.2.1, that is a ball of radius l, we shall in this section introduce the following moving boundary: the ball centered in the origin and of radius $l(t)$. The arguments developed in order to prove Theorem 2.8 and Theorem 2.9 will be adapted to this new context.

Algorithm (A3):
Let us define the following positive constants

(2.53) $$L = \max\left(l(0), \Delta_{\min}, \sqrt{\nu + 1}\right), \quad \kappa = \frac{2^\nu}{5^{\nu+1} L^{2\nu+2}} \Gamma(\nu + 1).$$

Initialization: *Set $\chi(0) = 0$, $\xi_0 = 0$, $\Xi_0 = 0$, $A_0 = \kappa\left(l(0) - \sqrt{\chi(0)}\right)^{2(\nu+1)}$.*
The n-th step: *While the following condition holds*

$$l(\Xi_{n-1}) - \sqrt{\chi(n - 1)} > \varepsilon$$

(denoted by $\mathbb{C}(n - 1)$), we simulate U_n an uniform distributed random vector on $[0, 1]^{\lfloor \nu \rfloor + 2}$, G_n a standard gaussian random variable and V_n a uniformly distributed random vector on $\mathcal{S}^\delta$. U_n, G_n and V_n have to be independent. We then construct $(\xi_n, \Xi_n, \chi(n))$ using (2.35). At the end of this step we set $A_n = \kappa\left(l(\Xi_n) - \sqrt{\chi(n)}\right)^{2(\nu+1)}$.
The algorithm stops when $\mathbb{C}(n - 1)$ is not longer satisfied: we set $\xi_n = 0$ and so $\Xi_n = \Xi_{n-1}$ and $\chi(n) = \chi(n - 1)$.
Outcome *The exit position $\chi(n)$ and the exit time.*

Let us note that the stochastic process $(\chi(n), n \geq 0)$ is not a Markov chain since the sequence $(A_n)_{n \geq 0}$ depends on both $(\Xi_n, \chi(n))$. That is why we define the following Markov chain

$$R_n := (\Xi_n, \chi(n)) \in \mathbb{R}_+^2$$

stopped at the first time the condition $\mathbb{C}(n)$ is not satisfied. In the following, we shall denote N^ε this stopping time (number of steps of the algorithm):

$$N^\varepsilon = \inf\{n \geq 0; \ l(\Xi_n) - \sqrt{\chi(n)} \leq \varepsilon\}.$$

THEOREM 2.12. *The number of steps N^ε of the algorithm $WoMS$ (A3) is almost surely finite. Moreover, there exist a constant $C_\delta > 0$ and $\varepsilon_0(\delta) > 0$, such that*

$$\mathbb{E}[N^\varepsilon] \leq C_\delta |\log \varepsilon|, \ \text{for all} \ \varepsilon \leq \varepsilon_0(\delta).$$

THEOREM 2.13. *As ε goes to zero, Ξ_{N^ε} converges in distribution towards τ defined by (2.52) (with cumulative distribution function F), which is almost surely finite. Moreover, for any $\alpha > 0$ small enough,*

(2.54) $$\left(1 - \frac{\varepsilon}{\sqrt{2\alpha\pi}}\right) F^\varepsilon(t - \alpha) \leq F(t) \leq F^\varepsilon(t), \ \text{for all} \ t > 0,$$

where $F^\varepsilon(t) := \mathbb{P}(\Xi_{N^\varepsilon} \leq t)$.

16

Proof of Theorem 2.12.
The proof is based mainly on arguments already presented in Theorem 2.8. So we let the details of the proof to the reader and focus our attention to the main ideas.
(1) The process $(\Xi_n, \chi(n))$ is a homogeneous Markov chain and the associated operator is given by

$$(2.55) \qquad P_{t,x}f := \int_{(s,y)\in\mathbb{R}_+^2} f(s,y)\mathbb{P}\Big((t,x),(\mathrm{d}s,\mathrm{d}y)\Big),$$

where f is a non-negative function and $\mathbb{P}\Big((t,x),(\mathrm{d}s,\mathrm{d}y)\Big)$ is the transition probability of the chain. The chain starts with $(\Xi_0,\chi(0)) = (0,0)$ and is stopped the first time when $l(\Xi_n) - \sqrt{\chi(n)} \leq \varepsilon$. Classical potential theory ensures that

$$\phi(t,x) = \mathbb{E}_{t,x}\left(\sum_{n=0}^{N^\varepsilon-1} g(\Xi_n,\chi(n))\right)$$

is solution of the following equation

$$(2.56) \qquad \begin{cases} \phi(t,x) = P_{t,x}\phi + g(t,x), & (t,x)\in D^\varepsilon, \\ \phi(t,x) = 0, & \forall(t,x)\in\partial D^\varepsilon, \end{cases}$$

with $D^\varepsilon = \{(t,x)\in\mathbb{R}_+^2 : l(t) - \sqrt{x} \leq \varepsilon\}$. For the particular choice $g = 1$, we obtain $\phi(t,x) = \mathbb{E}_{t,x}[N^\varepsilon]$, therefore the averaged number of step is given by $\phi(0,0)$.
(2) In order to point out an upper-bound for the averaged number of steps, we use a comparison result: we are looking for a function $U(t,x)$ such that

$$(2.57) \qquad \begin{cases} U(t,x) \geq P_{t,x}U + 1, & \forall(t,x)\in D^\varepsilon \\ U(t,x) \geq 0, & \forall(t,x)\in\partial D^\varepsilon. \end{cases}$$

For such a particular function, we can deduce $\phi(t,x) \leq U(t,x)$. Let us define

$$U(t,x) = c\log\left(\frac{l(t)-\sqrt{x}}{\varepsilon}\right)\mathbf{1}_{\{l(t)-\sqrt{x}\geq 0\}},$$

with some constant $c > 0$ which shall be specified later on. The positivity assumption on the boundary ∂D^ε is trivial. Moreover since l is a decreasing function, (2.55) implies

$$(2.58) \qquad P_{t,x}U = \int_{(s,y)\in\mathbb{R}_+^2} U(s,y)\mathbb{P}\Big((t,x),(\mathrm{d}s,\mathrm{d}y)\Big) \leq \int_{(s,y)\in\mathbb{R}_+^2} U(t,y)\mathbb{P}\Big((t,x),(\mathrm{d}s,\mathrm{d}y)\Big).$$

By using the Taylor expansion, we get

$$(2.59) \qquad U(t,y) \leq U(t,x) - c\,\frac{\sqrt{y}-\sqrt{x}}{l(t)-\sqrt{x}} - \frac{c}{2}\frac{(\sqrt{y}-\sqrt{x})^2}{(l(t)-\sqrt{x})^2} - \frac{c}{3}\frac{(\sqrt{y}-\sqrt{x})^3}{(l(t)-\sqrt{x})^3}, \quad (x,y)\in\mathbb{R}_+^2.$$

Using similar arguments and similar bounds as those presented in Lemma 2.10, the odd powers in the Taylor expansion don't play any role in the integral (2.58). Therefore we obtain

$$P_{t,x}U \leq U(t,x) - \frac{c}{2}\int_{(s,y)\in\mathbb{R}_+^2} \frac{(\sqrt{y}-\sqrt{x})^2}{(l(t)-\sqrt{x})^2}\mathbb{P}\Big((t,x),(\mathrm{d}s,\mathrm{d}y)\Big)$$

$$\leq U(t,x) - \frac{c}{2}\int_0^1\int_{S_+^\delta} \frac{\pi_1^2(u)K^2(x,r)}{(l(t)-\sqrt{x})^2}\,\mu(r)\mathrm{d}r\,\sigma_1^\delta(\mathrm{d}u)$$

17

where S_+^δ is given in (2.46) and K is defined by (2.40) with $A = \kappa(l(s) - \sqrt{x})^{2(\nu+1)}$. We have now

$$P_{t,x}U \le U(t,x) - \frac{c(\nu+1)}{2}\left(\frac{2K}{\Gamma(\nu+1)}\right)^{\frac{1}{\nu+1}}\left(\int_{S_+^\delta}\pi_1^2(u)\,\sigma_1^\delta(\mathrm{d}u)\right)\left(\int_0^1 r(-\log r)\mu(r)\mathrm{d}r\right).$$

An appropriate choice of the constant c leads to (2.57). Finally we get

$$\mathbb{E}[N^\varepsilon] \le U(0,0) = c\,\log(l(0)/\varepsilon).$$

$\square$

Proof of Theorem 2.13.
The arguments are similar to those developped for Theorem 2.9, the extension of the convergence result to curved boundaries is straightforward. That is why we shall not repeat the proof but just focus our attention on the only point which is quite different. We need to prove that the Markov chain $R_n = (\Xi_n, \chi(n))$ stays in the domain $D^0 = \{(t,x) : 0 \le x \le l^2(t)\}$ so that the hitting time τ defined by (2.52) satisfies $\tau > \Xi_{N^\varepsilon}$. In other words, if the Markov chain $R_n = (\Xi_n, \chi(n))$ for the n-th step is equal to (s,x), then R_{n+1} should belong to $\{(t,x) : t \ge s,\ x \le l^2(t)\}$. In the $WoMS$ setting, for $t \ge s$, this means that the ball centered in x and of time-dependent radius $\psi_A(t-s)$ always belongs as time elapses to the ball centered in 0 of radius $l(t)$. We recall that

$$A = \kappa(l(s) - \sqrt{x})^{2(\nu+1)}.$$

Therefore we shall prove that

(2.60) $$\forall t \ge s, \quad \psi_A(t-s) + \sqrt{x} \le l(t).$$

In fact, due to Assumption 2.11 and the definition of ψ_A, it suffices to obtain

(2.61) $$\psi_A(t-s) \le l(s) - \sqrt{x} - \Delta_{\min}(t-s), \quad \forall s \le t \le s + W^2,$$

where

$$W = \left(\frac{A}{\Gamma(\nu+1)2^\nu}\right)^{\frac{1}{2\nu+2}} = \left(\frac{\kappa}{\Gamma(\nu+1)2^\nu}\right)^{\frac{1}{2\nu+2}}(l(s)-\sqrt{x}) = \frac{1}{L\sqrt{5}}(l(s)-\sqrt{x}).$$

Due to the definition of the constant L, we have

$$0 \le W \le \frac{1}{2\Delta_{\min}}\frac{2(l(s)-\sqrt{x})\Delta_{\min}}{\sqrt{\frac{2\nu+2}{e}+4(l(s)-\sqrt{x})\Delta_{\min}}} \le \frac{1}{2\Delta_{\min}}\left\{\sqrt{\frac{2\nu+2}{e}+4(l(s)-\sqrt{x})\Delta_{\min}} - \sqrt{\frac{2\nu+2}{e}}\right\}.$$

The right hand side of the preceding inequality is the positive root of the polynomial function $P(X) = \Delta_{\min}X^2 + \sqrt{2(\nu+1)/e}X - (l(s) - \sqrt{x})$. We deduce that $P(W) \le 0$. By (2.44) and $P(W) \le 0$, we obtain

$$\sup_{t \ge s}\psi_A(t-s) = \left(\frac{2(\nu+1)}{e}\right)^{1/2}W$$
$$\le l(s) - \sqrt{x} - \Delta_{\min}W^2$$
$$\le l(s) - \sqrt{x} - \Delta_{\min}(t-s), \quad \forall s \le t \le s + W^2.$$

18

Finally we have proved (2.61) and so (2.60). □

If Assumption 2.11 is not satisfied then it is difficult to have a general description of an iterated procedure in order to simulate hitting times. However the particular form of the function ψ_a defined by (2.36) permits to describe a $WoMS$ algorithm for the square root boundaries. Let us therefore consider the following functions:

$$(2.62) \qquad \psi_a(t) = \sqrt{2t \log \frac{a}{\Gamma(\nu+1)t^{\nu+1}2^\nu}} \quad \text{and} \quad f(t) = \sqrt{r - ut},$$

well defined for $t \leq t_0 := \min\left(\alpha^{\frac{1}{\nu+1}}, \frac{r}{u}\right)$ where $\alpha = a(\Gamma(\nu+1)2^\nu)^{-1}$.

The algorithm is essentially based on the following result (the constants r and u associated with the hitting problem of a square root boundary for the Bessel process shall be specified in the proof of Proposition 2.15)

LEMMA 2.14. *Let us define*

$$(2.63) \qquad F_\nu(r, u) = \frac{1}{2}\left(\frac{er}{\nu+1}\right)^{\nu+1} \Gamma(\nu+1)\, e^{-u/2}, \quad r > 0,\ u > 0.$$

If $a = F_\nu(r, u)$ then

$$(2.64) \qquad \psi_a(t) \leq f(t), \quad \text{for all } 0 \leq t \leq \alpha^{\frac{1}{\nu+1}}.$$

PROOF. We are looking for a particular value a depending on both r and u such that the following bound holds: $\psi_a(t) \leq f(t)$, for all $0 \leq t \leq t_0$. Since $t \leq t_0$, it suffices to prove that

$$2t \log \frac{\alpha}{t^{\nu+1}} \leq r - ut \iff g(t) := t\left(2\log \frac{\alpha}{t^{\nu+1}} + u\right) \leq r.$$

Let us compute the maximum of the function g on the interval $[0, t_0]$, with t_0 fixed,

$$g'(t) = 2\log \frac{\alpha}{t^{\nu+1}} + u - 2(\nu+1).$$

We have

$$g'(t) = 0 \iff \log \frac{\alpha}{t^{\nu+1}} = \nu + 1 - \frac{u}{2} \iff t^{\nu+1} = \alpha \exp\left\{\frac{u}{2} - \nu - 1\right\}.$$

In other words the maximum of the function g is reached for

$$t_{max} = \alpha^{\frac{1}{\nu+1}} \exp\left\{\frac{u}{2(\nu+1)} - 1\right\}$$

and is equal to

$$g(t_{max}) = g_{\max} = 2(\nu+1)\alpha^{\frac{1}{\nu+1}} e^{\frac{u}{2(\nu+1)} - 1}.$$

Choosing $g_{\max} \leq r$ we obtain in particular (2.64) that means

$$\alpha \leq \left(\frac{er}{2(\nu+1)}\right)^{\nu+1} e^{-u/2} \iff a \leq \frac{1}{2}\left(\frac{er}{\nu+1}\right)^{\nu+1} \Gamma(\nu+1)e^{-u/2}.$$

For $a_0 = \frac{1}{2}\left(\frac{er}{\nu+1}\right)^{\nu+1} \Gamma(\nu+1)e^{-u/2}$, we get (2.64) since $t_0 = \alpha^{\frac{1}{\nu+1}}$. □

19

The aim is now to construct an algorithm which permits to approximate the hitting time of the square root boundary. Therefore we consider a Bessel process of dimension δ which hits the decreasing curved boundary $f(t)$ given by (2.62).

Algorithm (A4) – the square root boundary: $l(t) = \sqrt{\beta_0 - \beta_1 t}$ with $\beta_0 > 0$, $\beta_1 > 0$.
Let $\kappa \in]0, 1[$.
Initialization: *Set $\chi(0) = 0$, $\xi_0 = 0$, $\Xi_0 = 0$, $A_0 = \kappa F_\nu(\beta_0, \beta_1)$.*
The $(n + 1)$-th step : *While the following condition holds*

$$l(\Xi_n) - \sqrt{\chi(n)} > \varepsilon \text{(denoted by } \mathbb{C}(n)\text{)},$$

we define

(2.65)
$$A_n = \kappa F_\nu\left((l(\Xi_n) - \sqrt{\chi(n)})^2, \beta_1\left(1 - \frac{\sqrt{\chi(n)}}{l(\Xi_n)}\right)\right)$$

where F_ν is defined by (2.63), and we simulate U_{n+1} an uniform distributed random vector on $[0, 1]^{\lfloor \nu \rfloor + 2}$, G_{n+1} a standard gaussian random variable and V_{n+1} a uniformly distributed random vector on $\mathcal{S}^\delta$. U_{n+1}, G_{n+1} and V_{n+1} have to be independent. We then construct $(\xi_{n+1}, \Xi_{n+1}, \chi(n+1))$ using (2.35).
The algorithm stops when $\mathbb{C}(n)$ is not longer satisfied : we set $\xi_{n+1} = 0$ and so $\Xi_{n+1} = \Xi_n$ and $\chi(n + 1) = \chi(n)$.

PROPOSITION 2.15. *The statements of Theorem 2.12 and Theorem 2.13 are true for the Algorithm (A4) associated with the square root boundary.*

PROOF. All the arguments developped for decreasing boundaries with lower-bounded derivatives are easily adapted to the square root boundary. We let the details to the reader and focus our attention to the following fact: the stochastic process $(\Xi_n, \chi(n), n \geq 0)$ stays in the domain D^0 defined by
$$D^0 = \{(t, x) \in \mathbb{R}_+^2 : l(t) - \sqrt{x} > 0\}.$$
In the $WoMS$ setting, for $t \geq s$, this means that for $(\Xi_n, \chi(n)) = (s, x) \in D^0$ the following step leads to $\sqrt{\chi(n + 1)} < l(\Xi_{n+1})$. By (2.35), it suffices to prove that

(2.66)
$$\sqrt{x} + \psi_A(t) < l(s + t), \quad \text{for all } t \in \{u \geq 0 : \min(l(s + u), \psi_A(u)) \geq 0\},$$

with $A = \kappa F_\nu\left((l(s) - \sqrt{x})^2, \beta_1\left(1 - \frac{\sqrt{x}}{l(s)}\right)\right)$, since $\chi(n + 1) \leq (\sqrt{\chi(n)} + \psi_{A_n}(\xi_{n+1}))^2$. By Lemma 2.14 and due to the coefficient κ, we have

$$\psi_A(t) < \sqrt{(l(s) - \sqrt{x})^2 - \beta_1\left(1 - \frac{\sqrt{x}}{l(s)}\right)t}.$$

Hence

$$(l(s + t) - \sqrt{x})^2 - \psi_A(t)^2 > \left(\sqrt{l(s)^2 - \beta_1 t} - \sqrt{x}\right)^2 - (l(s) - \sqrt{x})^2 + \beta_1\left(1 - \frac{\sqrt{x}}{l(s)}\right)t$$

$$> 2\sqrt{x}\left(l(s) - \sqrt{l^2(s) - \beta_1 t}\right) - \frac{\beta_1\sqrt{x}}{l(s)}t \geq 0.$$

20

This leads directly to ([2.66](#)).

$\square$

REMARK 2.16.   *The whole study points out a new efficient algorithm in order to simulate Bessel hitting times for given levels or curved boundaries. We can use this algorithm in two generalized situation:*

(1) *We have assumed that the Bessel process starts from the origin. Of course the procedure presented here can also be applied to Bessel processes starting from $x > 0$. It suffices to change the initialization step !*

(2) *We focused our attention to the Bessel process but we linked also the initial problem to the exit time of a δ-dimensional Brownian motion from a ball of radius l. The Algorithm (A1) extended to higher dimensions can also be used in order to evaluate exit times of general compact domains whose boundary is regular.*

3. Numerical results.   In this part we will illustrate the previous results on some numerical examples. Let us figure first an outcome of our algorithm, the exit position from a sphere with radius depending on time. The figure opposite is giving this result for an radius $l = 1$ and a precision $\varepsilon = 10^{-3}$.

Let us compare our algorithm with existing results. Consider the classical Euler scheme for a Brownian motion and evaluate the first hitting time and hitting position from a disk with given radius.
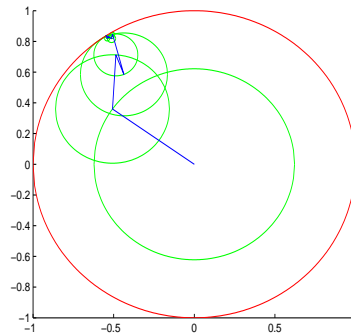
First of all we can verify that the distribution of the hitting time for the $WoMS$ algorithm matches the distribution of the hitting time of a given level for the $2-$dimensional Bessel process. Figure [2](#) gives this result for a starting disk with radius 1, a precision $\varepsilon = 10^{-3}$ and a number of simulations $N = 20000$. In the Euler scheme the time step is $\Delta t = 10^{-4}$.
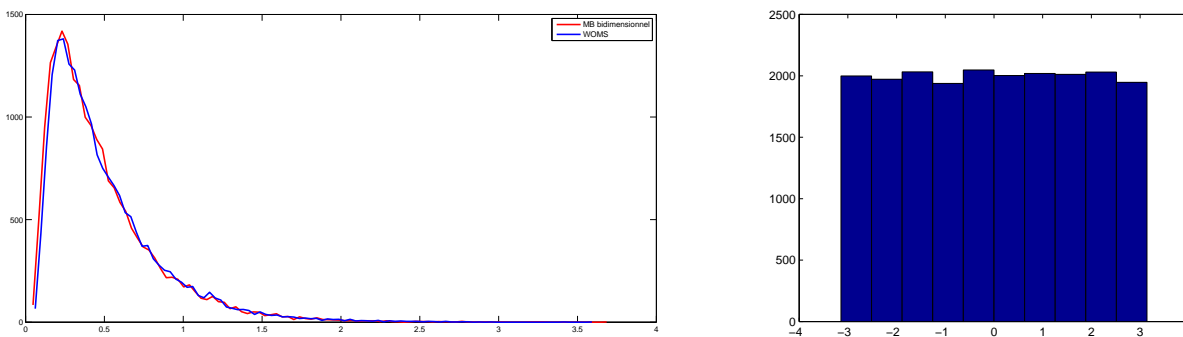

FIG 2. *Distribution of the hitting time (Euler scheme and $WoMS$ algorithm (A1)) - Histogram of the angle for the exit position*

We can also test the fact that the exit position is uniformly distributed on the circle. In order to do this we can evaluate the angle of the exit position in our $WoMS$ procedure and show that it is

an uniform distributed random variable with values in $[-\pi, \pi]$. Figure 2 also shows the histogram of the result for a disk of radius 1 an $\varepsilon = 10^{-3}$ and 20000 simulations.

Let us now present a simulation with Algorithm (A2). We consider the hitting time of the level $l = 2$ for the Bessel process of index $\nu = 2$ and we illustrate Theorem 2.8 by Figure 3. The curve represents the averaged number of steps versus the precision $\varepsilon = 10^{-k}$, $k = 1, \ldots, 7$. We can observe that the number of steps is better than suspected since the curve is sublinear. We obtain the following values (for $\gamma = 0.9$ and $100\,000$ simulations in order to evaluate the mean).
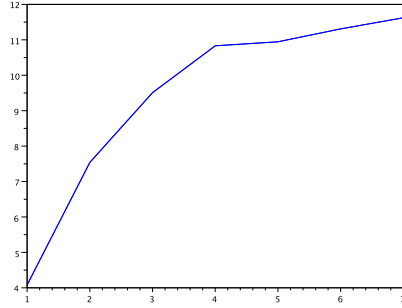
ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$\mathbb{E}[N^\varepsilon]$	4.0807	7.53902	9.50845	10.83133

ε	10^{-5}	10^{-6}	10^{-7}	
$\mathbb{E}[N^\varepsilon]$	10.94468	11.30869	11.62303	

FIG 3. *Averaged number of step of Algorithm (A2) versus ε*

Finally we present the dependence of the averaged number of steps of Algorithm (A2) with respect to the dimension of the Bessel process. For that purpose, we simulate hitting time of the level $l = 2$ with $\varepsilon = 10^{-3}$, $\gamma = 0.9$, $50\,000$ simulations for each estimation of the averaged value, and the dimension of the Bessel process takes value in the set $\{2, 3, \ldots, 18\}$.

ν	0	0.5	1	1.5	2
$\mathbb{E}[N^\varepsilon]$	6.819	7.405	8.270	8.887	9.594

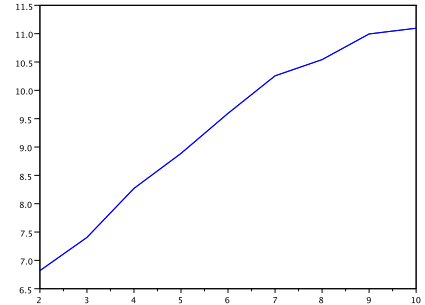| ν	2.5	3	3.5	4	
$\mathbb{E}[N^\varepsilon]$	10.256	10.542	10.995	11.096	

FIG 4. *Averaged number of step of Algorithm (A2) versus $\delta = 2\nu + 2$*

4. Application to the Cox-Ingersoll-Ross process. We now aim to estimate the hitting time of a level $l > 0$ for $(X_t^\delta,\ t \geq 0)$, a Cox-Ingersoll-Ross process. The CIR process is the solution of the following stochastic differential equation:

$$(4.1) \qquad \begin{cases} \mathrm{d}X_t^\delta = (a + bX_t^\delta)\mathrm{d}t + c\sqrt{|X_t^\delta|}\mathrm{d}B_t, \\ X_0^\delta = x_0, \end{cases}$$

22

where $x_0 \geq 0$, $a \geq 0$, $b \in \mathbb{R}$, $c > 0$ and $(B_t, t \geq 0)$ is a standard Brownian motion. We denote here $\delta = 4a/c^2$.

We will first recall a connection between this stochastic process and $(Y^\delta(t), t \geq 0)$, the square of the Bessel process BESQ(δ), solution of the equation

$$(4.2) \qquad Y^\delta(t) = y_0 + \delta t + 2 \int_0^t \sqrt{|Y^\delta(s)|} \, \mathrm{d}B_s, \quad t \geq 0.$$

LEMMA 4.1. *The CIR process has the same distribution as $(\overline{X}_t, t \geq 0)$ which is defined by*

$$(4.3) \qquad \begin{cases} \overline{X}_t = e^{bt} Y^\delta\left(\frac{c^2}{4b}(1 - e^{-bt}) \right), \\ \overline{X}_0 = Y^\delta(0), \end{cases}$$

where Y is the square of a Bessel process in dimension $\delta = 4a/c^2$ (see [15]).

PROOF. Let us only sketch some ideas of the proof. Let $Y^\delta(t)$ be the square of the δ-dimensional Bessel process. By applying Itô formula, we get the stochastic differential equation satisfied by the process $\overline{X}_t$

$$
\begin{aligned}
\mathrm{d}\overline{X}_t &= b\overline{X}_t \, \mathrm{d}t + e^{bt} \mathrm{d}\left(Y\left(\frac{c^2}{4b}(1 - e^{-bt}) \right) \right) \\
&= b\overline{X}_t \, \mathrm{d}t + b\delta \frac{c^2}{4b} \, \mathrm{d}t + 2e^{bt} \sqrt{|e^{-bt}\overline{X}_t|} \, \mathrm{d}B_{\frac{c^2}{4b}(1 - e^{-bt})} \\
&= (a + b\overline{X}_t) \, \mathrm{d}t + 2e^{\frac{bt}{2}} \sqrt{|\overline{X}_t|} \, \mathrm{d}B_{\frac{c^2}{4b}(1 - e^{-bt})},
\end{aligned}
$$

$$(4.4)$$

where $\delta = 4a/c^2$. Let us remark that

$$\frac{c^2}{4b}(1 - e^{-bt}) = \int_0^t \rho^2(s)\mathrm{d}s, \quad \text{with } \rho(t) = \frac{c}{2} e^{-\frac{bt}{2}}.$$

We can deduce that there exists a Brownian motion $(\beta_t, t \geq 0)$ such that

$$B_{\frac{c^2}{4b}(1 - e^{-bt})} = \int_0^t \rho(s)\mathrm{d}\beta_s,$$

for all $t \geq 0$. With this notation, the equation (4.4) writes

$$
\begin{aligned}
\mathrm{d}\overline{X}_t &= (a + b\overline{X}_t) \, \mathrm{d}t + 2e^{\frac{bt}{2}} \sqrt{|\overline{X}_t|}\rho(t) \, \mathrm{d}\beta_t \\
&= (a + b\overline{X}_t) \, \mathrm{d}t + c\sqrt{|\overline{X}_t|}\mathrm{d}\beta_t,
\end{aligned}
$$

and $\overline{X}_0 = Y(0)$. This proves that the process $(\overline{X}_t, t \geq 0)$ has the same distribution as the CIR process given by (4.1). □

Let us consider the hitting time of a given level l for the CIR process and denote it by T_l. This time is defined by:

$$T_l = \inf\{s \geq 0; X_s^\delta = l\}.$$

The previous Lemma 4.1 gives also an equivalence (in distribution) connecting the hitting time of the CIR process and the hitting time of the square of a δ-dimensional Bessel process.

PROPOSITION 4.2. *The hitting time T_l of a level $l > 0$ for a CIR process has the same distribution as* $-\frac{1}{b}\log\left(1 - \frac{4b}{c^2}\tau_\psi\right)$ *where*

$$\tau_\psi = \inf\left\{t \geq 0; Y^\delta(t) = l\left(1 - \frac{4b}{c^2}t\right)\right\},$$

and Y^δ is the square of a Bessel process of dimension $\delta = 4a/c^2$.

PROOF. By using Lemma 4.1, τ_ψ has the same distribution as $\overline{T}_l$ given by

(4.5) $$\overline{T}_l = \inf\left\{s \geq 0; \ Y^\delta\left(\frac{c^2}{4b}(1 - e^{-bs})\right) = le^{-bs}\right\}.$$

Define $t = \frac{c^2}{4b}(1 - e^{-bs})$, we have two situations:
First case: If $b < 0$ let $s = \eta(t)$ where

$$\eta(t) = -\frac{1}{b}\log\left(1 - \frac{4b}{c^2}t\right), \quad \text{for } t \geq 0.$$

The map η is a strictly non-decreasing function, and we get thus

$$\overline{T}_l = \inf\left\{\eta(t); \ t \geq 0, \ Y^\delta(t) = l\left(1 - \frac{4b}{c^2}t\right)\right\} = \eta\left(\inf\left\{t \geq 0; Y^\delta(t) = l\left(1 - \frac{4b}{c^2}t\right)\right\}\right).$$

Second case: If $b \geq 0$, let also $s = \eta(t)$. In this case the variable t takes its values only on the interval $\left[0, \frac{c^2}{4b}\right)$. So

$$\overline{T}_l = \inf\left\{\eta(t); \ 0 \leq t \leq \frac{c^2}{4b}, \ Y^\delta(t) = l\left(1 - \frac{4b}{c^2}t\right)\right\}.$$

The condition $0 \leq t \leq \frac{c^2}{4b}$ can be omitted in the estimation of the infimum as the boundary to hit : $1 - \frac{4bt}{c^2}$ is negative outside this interval and the Bessel process is always positive. Furthermore the function η is also non-decreasing for $b \geq 0$, and the result is thus obtained. $\square$

Application of Algorithm (A4):
An immediate consequence of Proposition 4.2 is that the hitting time T_l is related to the first time the Bessel process of dimension $\delta = 4a/c^2$ reaches the curved boundary: $f(t) = \sqrt{l\left(1 - \frac{4b}{c^2}t\right)}$. We are able to apply Algorithm (A4) if $4a/c^2 \in \mathbb{N}^*$ and $b > 0$ (the boundary is then decreasing). Let us denote by N^ε the number of steps of (A4) and Ξ_{N^ε} the approximated hitting time of the Bessel process associated with the particular curved boundary f. Combining Proposition 2.15 and Proposition 4.2 leads to

$$\left(1 - \frac{\varepsilon}{\sqrt{2\alpha\pi}}\right)\mathbb{P}\left(\Xi_{N^\varepsilon} \leq \frac{c^2}{4b}(1 - e^{-bt}) - \alpha\right) \leq \mathbb{P}(T_l \leq t) \leq \mathbb{P}\left(\Xi_{N^\varepsilon} \leq \frac{c^2}{4b}(1 - e^{-bt})\right),$$

for α small enough and $t > 0$.

APPENDIX A: SIMULATION OF RANDOM VARIABLES

Let us introduce simulation procedures related to particular probability density functions.

PROPOSITION A.1. *Let Z be a random variable with Gamma distribution* $\mathrm{Gamma}(\alpha, \beta)$ *i.e.*

$$\mathbb{P}(Z \in \mathrm{d}z) = \frac{1}{\Gamma(\alpha)\beta^\alpha} z^{\alpha-1} e^{-\frac{z}{\beta}} \mathbb{1}_{\{z>0\}} \, \mathrm{d}z, \quad \alpha > 0, \ \beta > 0.$$

Then $W = \exp(-Z)$ has the following distribution

$$\mathbb{P}(W \in \mathrm{d}r) = \frac{1}{\Gamma(\alpha)\beta^\alpha} (-\log r)^{\alpha-1} r^{1/\beta-1} \mathbb{1}_{[0,1]}(r) \, \mathrm{d}r.$$

In particular the stopping time τ_ψ defined by (2.16) *has the same law as* $\left[\frac{a}{\Gamma(\nu+1)2^\nu} \right]^{\frac{1}{\nu+1}} e^{-Z}$. *Here Z is a Gamma distributed random variable with parameters $\alpha = \nu + 2$ and $\beta = \frac{1}{\nu+1}$.*

PROOF. Let f be a non-negative function. Using suitable changes of variables, we obtain

$$\begin{aligned}
\mathbb{E}[f(W)] &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty f(e^{-z}) z^{\alpha-1} e^{-\frac{z}{\beta}} \, \mathrm{d}z \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^1 f(r)(-\log r)^{\alpha-1} r^{1/\beta-1} \, \mathrm{d}r.
\end{aligned}$$

In order to end the proof it suffices to multiply W by a constant and use once again a change of variables formula. □

We need to simulate Gamma distributed variables. Let us just recall some common facts.

PROPOSITION A.2.   *(i) If $\alpha \in \mathbb{N}$ (so-called Erlang distributions) then the Gamma distributed variables Z has the same law as $-\beta \log(U_1 \ldots U_\alpha)$, where $(U_i)_{1 \le i \le \alpha}$ are independent uniformly distributed random variables. Hence W defined by $W = \exp(-Z)$ can be simulated by $(U_1 U_2 \ldots U_\alpha)^\beta$.*

(ii) If $\alpha - 1/2 \in \mathbb{N}$ then Z has the same law as $-\beta \log(U_1 \ldots U_{\lfloor \alpha \rfloor}) + \frac{\beta N^2}{2}$, where $(U_i)_{1 \le i \le \lfloor \alpha \rfloor}$ are i.i.d. uniformly distributed random variables and N is an independent standard Gaussian r.v. (see, for instance, [5] chapter IX.3).

REFERENCES

[1] L. Alili and P. Patie. Boundary-crossing identities for diffusions having the time-inversion property. *J. Theoret. Probab.*, 23(1):65–84, 2010.

[2] T. Byczkowski, J. Malecki, and M. Ryznar. Hitting times of Bessel processes. *arXiv*, 2011.

[3] Z. Ciesielski and S. J. Taylor. First passage times and sojourn times for Brownian motion in space and the exact hausdorff measure of the sample path. *Trans. Amer. Math. Soc.*, 103:434–450, 1962.

[4] H. E. Daniels. The minimum of a stationary Markov process superimposed on a U-shaped trend. *J. Appl. Probability*, 6:399–408, 1969.

[5] L. Devroye. *Non-Uniform Random Variate Generation*. Springer-Verlag, New York, 1986.

[6] G.B. Ermentrout and D. Terman. *Mathematical Foundations of Neuroscience*. Interdisciplinary Applied Mathematics. Springer, 2010.

[7] W. Gerstner and W.M. Kistler. *Spiking neuron models: single neurons, populations, plasticity*. Cambridge University Press, 2002.

[8]   Anja Göing-Jaeschke and Marc Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[9]   Y. Hamana and H. Matsumoto. The probability distribution of the first hitting times of Bessel processes. *arXiv*, 2011.

[10] Petr Lànský, Laura Sacerdote, and Francesca Tomassetti. On the comparison of Feller and Ornstein-Uhlenbeck models for neural activity. *Biological Cybernetics*, 73:457–465, 1995.

[11] Hans Rudolf Lerche. *Boundary crossing of Brownian motion*, volume 40 of *Lecture Notes in Statistics*. Springer-Verlag, Berlin, 1986.

[12] G. N. Milstein. Weak approximation of a diffusion process in a bounded domain. *Stochastics Stochastics Rep.*, 62(1-2):147–200, 1997.

[13] Mervin E. Muller. Some continuous Monte Carlo methods for the Dirichlet problem. *Ann. Math. Statist.*, 27:569–589, 1956.

[14] J. R. Norris. *Markov chains*, volume 2 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 1998. Reprint of 1997 original.

[15] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999.

[16] Paavo Salminen and Marc Yor. On hitting times of affine boundaries by reflecting Brownian motion and Bessel processes. *Period. Math. Hungar.*, 62(1):75–101, 2011.

M. Deaconu

Project team TOSCA, INRIA Nancy Grand-Est

Institut Elie Cartan de Nancy - UMR 7502

Nancy-Université, CNRS, INRIA

B.P. 70239, 54506 Vandoeuvre-lès-Nancy Cedex, France

E-mail: Madalina.Deaconu@inria.fr

S. Herrmann

Institut de Mathématiques de Bourgogne (IMB) - UMR 5584

Université de Bourgogne, B.P. 47 870

21078 Dijon Cedex, France

E-mail: Samuel.Herrmann@u-bourgogne.fr