Time optimal boundary controls for the heat equation

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Abstract

The fact that the time optimal controls for parabolic equations have the bang-bang property has been recently proved for controls distributed inside the considered domain (interior control). The main result in this article asserts that the *boundary controls* for the heat equation have the same property, at least in rectangular domains. This result is proved by combining methods from traditionally distinct fields: the Lebeau-Robbiano strategy for null controllability and estimates of the controllability cost in small time for parabolic systems, on one side, and a Remez-type inequality for Müntz spaces and a generalization of Turán's inequality, on the other side.

1 Introduction and main result

Let m be a positive integer, let $\Omega \subset \mathbb{R}^m$ be an open and bounded set and let Γ be a non-empty open subset of $\partial\Omega$. We consider the heat equation

$$\frac{\partial z}{\partial t}(x,t) = \Delta z(x,t) \text{ for } (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

with the initial and boundary conditions

$$z(x,t) = u(x,t) \quad \text{on } \Gamma \times (0,\infty), \tag{1.2}$$

$$z(x,t) = 0 \quad \text{on } (\partial\Omega \setminus \Gamma) \times (0,\infty), \tag{1.3}$$

$$z(x,0) = z_0(x) \text{ for } x \in \Omega. \tag{1.4}$$

It is known (see, for instance, Tucsnak and Weiss [24, Section 10.7]) that if $\partial\Omega$ is of class C^2 or Ω is a rectangular domain then, for every $u \in L^2([0,\infty), L^2(\Gamma))$ and $z_0 \in H^{-1}(\Omega)$, there exists a unique solution $z \in C([0,\infty), H^{-1}(\Omega))$ of (1.1)-(1.4). It is also known that the system defined by (1.1)-(1.4) is null controllable in any time $\tau > 0$, in the sense that for every $z_0 \in H^{-1}(\Omega)$ there exists an input $u \in L^2([0,\tau], L^2(\Gamma))$ such that the corresponding solution of (1.1)-(1.4) verifies

$$z(\cdot,\tau) = 0. \tag{1.5}$$

Our aim consists in studying the associated time optimal control problems in an L^∞ setting. To state the problem, we set, given $M > 0$,

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Gamma \times [0,\infty)) \ \mid \ |u(x,t)| \leqslant M \text{ a. e. in } \Gamma \times [0,\infty)\}. \tag{1.6}$$

Given $z_0 \in H^{-1}(\Omega)$, we define the set of reachable states from z_0 as

$$\mathcal{R}(z_0,\mathcal{U}_{ad}) = \{z(\tau) \mid \tau > 0 \text{ and } z \text{ is the solution of (1.1)-(1.4) with } u \in \mathcal{U}_{ad}\}.$$

For $z_0 \in H^{-1}(\Omega)$ and $z_1 \in \mathcal{R}(z_0,\mathcal{U}_{ad})$, *the time optimal control problem* for (1.1)-(1.4) consists in determining an input $u^* \in \mathcal{U}_{ad}$ such that the corresponding solution z^* of (1.1)-(1.4) satisfies

$$z^*(\tau^*(z_0,z_1)) = z_1, \tag{1.7}$$

where $\tau^*(z_0,z_1)$ is the minimal time needed to steer the initial data z_0 towards target z_1 with controls in $\mathcal{U}_{ad}$,

$$\tau^*(z_0,z_1) = \inf_{u\in\mathcal{U}_{ad}} \{\tau \ \mid \ z(\cdot,\tau) = z_1\}. \tag{1.8}$$

General conditions ensuring the existence of at least one solution for the above time optimal control problem (i.e., of at least one input such that the inf in (1.8) is attained) will be recalled in Section 2. The main result in this work asserts that, if Ω is a rectangular domain, then this solution is bang-bang and it is unique. More precisely, we have:

Theorem 1.1. *Let $m \geqslant 2$. Suppose that Ω is a rectangular domain in $\mathbb{R}^m$ and that Γ is a nonempty open set of $\partial\Omega$. Then, for every $z_0 \in H^{-1}(\Omega)$ and $z_1 \in \mathcal{R}(z_0,\mathcal{U}_{ad})$, there exits a unique solution u^* of the time optimal control problem* (1.8)*. This solution u^* has the bang-bang property:*

$$|u^*(x,t)| = M \qquad a.\ e.\ in \quad \Gamma \times [0,\tau^*(z_0,z_1)]. \tag{1.9}$$

Time optimal control problems for linear parabolic partial differential equations and the bang-bang property of the corresponding controls have been intensively studied during the last decades, beginning with Fattorini's paper [5]. The progress made in this field has been successively reported in the books of Lions [14] and of Fattorini [6]. The bang-bang property of time optimal controls has been quite rapidly established for invertible input operators (which means, roughly speaking, that the control is active in the entire spatial domain where the parabolic equation is considered).

Several important extensions of the classical results of Fattorini have been obtained during the last decades. We first recall those corresponding to the heat equation, in the case of an input operator which is active only in a proper subset of the domain where the heat equation holds. Firstly, in Wang [26], the set of admissible inputs is defined (unlike in (1.6)) by bounding the $L^\infty([0,\tau];L^2(\Omega))$ norm of u. A strategy which has been introduced by Lebeau and Robbiano in [12] is adapted in [26] to establish a bang-bang property of the time optimal controls. This property is different from the one in Theorem 1.1, in the sense that, instead of (1.9), it is shown that $\|u^*(\cdot,t)\|_{L^2(\Omega)} = 1$ for almost every $t \in [0,\tau^*]$.

The strategy in [26] does not seem directly applicable to the boundary control case. The results in [26] have been recently extended by Phung and Wang [20] to a system governed by a perturbed heat equation with internal controls. In the case in which the target is an open ball in the state space instead of a point, the corresponding time optimal control problem, with control distributed inside the domain and pointwise control constraints, has been studied in Kunisch and Wang [11]. The main tools of their approach are Pontryagin's maximum principle and a special kind of property concerning the measure of the set where a nontrivial solution of the linear heat equation vanishes.

In the case of boundary control, with the control constraint $|u(x,t)| \leqslant M$, the first result establishing the bang-bang property has been obtained by Schmidt [21], under a *slackness* condition on the target state. More precisely, the assumption in [21] is that there exists $M' < M$ such that the target is actually reachable (in some time) subject to $|u| \leqslant M'$. In the case of the heat equation in one space dimension, this condition has been removed by Mizel and Seidman in [18], by using in an essential manner previous results of Borwein and Erdelyi [2].

The main novelty of Theorem 1.1 consists in showing that, in the case of rectangular domains in several space dimensions, the bang-bang property holds for the time optimal *boundary control* for the heat equation. The only requirement for the target points is to be reachable in some time. Our methodology is partially inspired by the fact, remarked in Tenenbaum and Tucsnak [23], that a well-known inequality of Turán can be successfully used in control theory. More precisely, two of the most important ingredients of the proof of Theorem 1.1 are Nazarov's generalization of the Turán's inequality and the above mentioned results on Müntz spaces of Borwein and Erdelyi.

The outline of the paper is as follows. In Section 2 we present some background on null controllability and time optimal controls for infinite dimensional systems. Most of the included material is well-known, although not necessarily in the L^∞ setting presented here. Proposition 2.6 gives a general sufficient condition for the existence, uniqueness and bang-bang property of time optimal controls. In Section 3 the Lebeau-Robbiano strategy (see, for instance, [12, 13]) to study the null controllability of the heat equation is adapted to prove the L^∞ null controllability over a positive measure set property. The main novelty we bring in into this section is that we replace the Lebeau-Robbiano assumption on the observability of finite combinations of eigenvectors with an assumption on the observability of the dynamical system's truncation to a finite number of modes (see inequality (3.8) in Theorem 3.2). The latter property involves the time variable and it is, in general, weaker than the observability of finite combinations of eigenvectors. The last two sections of the paper are devoted to prove that, in our case, this observability property holds. In Section 4, following an idea from Nazarov [19], we give an estimate for finite combinations of eigenfunctions of the Dirichlet Laplacian in a rectangular domain. Finally, in Section 5, by combining a result from [2, 3] for real exponential functions defined on a measurable set with the one obtained in the previous section, we provide the proof of our main result in Theorem 1.1.

2 Some background on null controllability and time optimal controls for infinite dimensional systems

We first introduce some notation. If $P \in \mathcal{L}(X;Y)$ then the *null-space* and the *range* of P are the subspaces of X and Y respectively defined by

$$\text{Ker } P = \{x \in X \ : \ Px = 0\}, \qquad \text{Ran } P = \{Px \ : \ x \in X\}.$$

Given $m \in \mathbb{N}$, the notation

$$\|f\|_e = \sup_{x \in e} |f(x)|,$$

is used throughout this paper for continuous complex valued functions f defined on a measurable set $e \subset \mathbb{R}^m$.

Throughout this section, X and U are complex Hilbert spaces, identified with their duals. The inner product and the norm in X are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We denote by $\mathbb{T} = (\mathbb{T}_t)_{t \geqslant 0}$ a strongly continuous semigroup on X generated by an operator $A : \mathcal{D}(A) \to X$ with resolvent set $\varrho(A)$. The notation X_1 stands for $\mathcal{D}(A)$ equipped with the norm $\|z\|_1 := \|(\beta I - A)z\|$, where $\beta \in \varrho(A)$ is fixed, while X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} := \|(\beta I - A)^{-1} z\|$. We use the notation A and $\mathbb{T}$ also for the extensions of the original generator to X and of the original semigroup to X_{-1}. It is known that X_{-1} is the dual of $\mathcal{D}(A^*)$ with respect to the pivot space X. The semigroup $\mathbb{T}$ can be extended to X_{-1}, and then its generator is an extension of A, defined on X. We use the same notation for all these extensions as for the original operators.

Let $B \in \mathcal{L}(U; X_{-1})$ be a control operator, let $z_0 \in X$ and let $u \in L^2([0,\infty), U)$. We consider the infinite dimensional system described by the equation

$$\dot{z}(t) = Az(t) + Bu(t) \qquad (t \geqslant 0), \qquad z(0) = z_0. \tag{2.10}$$

With the above notation, the solution z of (2.10) is defined by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \qquad \big(t \geqslant 0\big), \tag{2.11}$$

where $\Phi_t \in \mathcal{L}(L^2([0,t], U); X_{-1})$ is given by

$$\Phi_t u = \int_0^t \mathbb{T}_{t-\sigma} B u(\sigma) \mathrm{d}\sigma. \tag{2.12}$$

Recall the following classical definition (see, for instance, [24, Section 4.2]):

Definition 2.1. With the above notation, the operator $B \in \mathcal{L}(U; X_{-1})$ is called an *admissible control operator* for $\mathbb{T}$ if the operator Φ_τ defined by (2.12) satisfies Ran $\Phi_\tau \subset X$ for some $\tau > 0$.

Remark 2.2. *System* (1.1)-(1.4) *can be written in the form* (2.10)*. Indeed, let* $X = H^{-1}(\Omega)$*,* $U = L^2(\Gamma)$*,* $D(A) = H_0^1(\Omega)$ *and* $A = \Delta$*.*

The control operator $B \in \mathcal{L}(L^2(\Gamma), X_{-1})$ *is defined by* $B = AD$*, where* $D : L^2(\partial\Omega) \to L^2(\Omega)$ *is the "Dirichlet map". This map is defined by* $Dv = z$*, where* $z \in L^2(\Omega)$ *is the unique solution of the nonhomogeneous elliptic equation*

$$\left\{ \begin{array}{ll} \Delta z = 0 & \text{in } \Omega \\ z = v & \text{on } \partial\Omega. \end{array} \right. \tag{2.13}$$

With the above notation, (1.1)-(1.4) *is equivalent to* (2.10) *and B is an admissible control operator for the semigroup $\mathbb{T}$ generated by A.*

Moreover, the operator $B^* \in \mathcal{L}(X_1, L^2(\partial\Omega))$ is given by

$$B^*\varphi = -\frac{\partial(-A)^{-1}\varphi}{\partial\nu}|_\Gamma \qquad (\varphi \in L^2(\Omega)). \tag{2.14}$$

We refer the interested reader to [24, Sections 10.6-10.7] for a detailed description of the above functional analytic setting.

The null controllability of the pair (A, B) in some time $\tau > 0$ is usually defined by the property Ran $\Phi_\tau \supset$ Ran $\mathbb{T}_\tau$. In this work we will mainly use a different concept of null controllability, which makes sense in the case $U = L^2(\Gamma)$, where Γ is a measurable set endowed with a measure μ.

Definition 2.3. Given $\tau > 0$, $e \subset \Gamma \times [0, \tau]$ a set of positive measure and an admissible control operator $B \in \mathcal{L}(U; X_{-1})$ for $\mathbb{T}$, consider the operator

$$\Phi_{\tau,e} \in \mathcal{L}(L^\infty(\Gamma \times [0, \tau]); X_{-1}),$$

defined by

$$\Phi_{\tau,e}u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B\chi_e(\sigma)u(\sigma)\mathrm{d}\sigma \qquad (u \in L^\infty(\Gamma \times [0, \tau]), \tag{2.15}$$

where χ_e is the characteristic function of e. The pair (A, B) is said L^∞ *null controllable in time τ over e* if Ran $\Phi_{\tau,e} \supset$ Ran $\mathbb{T}_\tau$. For $e = \Gamma \times [0, \tau]$, the above property is simply called L^∞ *null controllability in time τ*. Given $z_0 \in X_{-1}$, a function $u \in L^\infty(\Gamma \times [0, \tau])$ such that $\Phi_{\tau,e}u = \mathbb{T}_\tau z_0$ is called L^∞ *null control for z_0*.

If the pair (A, B) is L^∞ *null controllable in time τ over e* then, for every $z_0 \in X$, the set

$$\mathcal{C}_{\tau,e,z_0} := \left\{u \in L^\infty(\Gamma \times [0, \tau]) \ \mid \Phi_{\tau,e}u + \mathbb{T}_\tau z_0 = 0\right\}$$

is non empty. The quantity

$$C_{\tau,e} := \sup_{\|z_0\|=1} \inf_{u \in \mathcal{C}_{\tau,e,z_0}} \|u\|_{L^\infty(e)}, \tag{2.16}$$

is then called the *control cost in time τ over e*. If $e = \Gamma \times [0, \tau]$ the control cost will be simply denoted by C_τ.

Let $C \in \mathcal{L}(X_1, U)$ be an admissible observation operator for $\mathbb{T}$. The admissibility assumption means that for some $\tau > 0$, the operator Ψ_τ defined by

$$(\Psi_\tau z_0)(t) = C\mathbb{T}_t z_0 \qquad (z_0 \in X_1),$$

has an extension to an operator $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau], U))$. Equivalently, there is a positive number k such that $\int_0^\tau \|C\mathbb{T}_t z_0\|^2 \mathrm{d}t \leqslant k^2\|z_0\|^2$ for all $z_0 \in \mathcal{D}(A)$. We refer to [24, 27, 28] for more material on this concept. Here we only mention that it follows from the admissibility assumption that $\Psi_\tau \in \mathcal{L}(X, L^2([0, \tau]; U))$ holds for all $\tau \geqslant 0$. The operators Ψ_τ are called output maps corresponding to the pair (A, C). Denote by Ψ_τ^d the output maps corresponding to the pair (A^*, B^*). If $e \subset \Gamma \times [0, \tau]$ is a set of positive measure, we consider the map

$$\Psi_{\tau,e}^d \in \mathcal{L}(X, L^1(\Gamma \times [0, \tau])), \qquad \Psi_{\tau,e}^d = \chi_e \Psi_\tau^d.$$

We have the following duality result

Proposition 2.4. *Suppose that $B \in \mathcal{L}(U, X_{-1})$. Then B is an admissible control operator for $\mathbb{T}$ if and only if B^* is an admissible observation operator for the adjoint semigroup $\mathbb{T}^*$. If B is admissible, then*

$$\Phi_{\tau,e} = (\Psi^d_{\tau,e'})^* \mathbf{Я}_\tau\,, \tag{2.17}$$

where $e' = \{(x, \tau - t) \mid (x,t) \in e\}$, $(\Psi^d_{\tau,e'})^* \in \mathcal{L}(L^\infty([0,\tau];U), X)$ is the dual operator of $\Psi^d_{\tau,e'}$ and $\mathbf{Я}_\tau$ is the reflection operator on $L^2([0,\tau];U)$, defined by $\mathbf{Я}_\tau u(t) = u(\tau - t)$ (Notice that $\mathbf{Я}_\tau$ is self-adjoint and also unitary.).

Proof. The first assertion in the statement of the proposition is well-known (see, for instance, [24, Section 4.4]). To check the second one we first note that for every $v \in L^\infty(\Gamma \times [0,\tau])$ and $\varphi \in X_1$ we have

$$\int_0^\tau \int_\Gamma v\overline{\Psi^d_{\tau,e'}\varphi}\,\mathrm{d}x\,\mathrm{d}t = \int_0^\tau \langle \chi_{e'}(\cdot,t)v(\cdot,t), B^*\mathbb{T}^*_t\varphi\rangle_U\,\mathrm{d}t = \int_0^\tau \langle \mathbb{T}_t B\chi_{e'}(\cdot,t)v(\cdot,t), \varphi\rangle_{X_{-1},X_1}\,\mathrm{d}t.$$

By making the change of variable $t = \tau - \sigma$ in the above integral and using the fact that B is admissible for $\mathbb{T}$, we obtain

$$\int_0^\tau \int_\Gamma v\overline{\Psi^d_{\tau,e'}\varphi}\,\mathrm{d}x\,\mathrm{d}t = \langle \Phi_{\tau,e}\mathbf{Я}_\tau v, \varphi\rangle_{X_{-1},X_1} = \langle \Phi_{\tau,e}\mathbf{Я}_\tau v, \varphi\rangle \quad (v \in L^\infty(\Gamma \times [0,\tau]),\ \varphi \in X_1).$$

The above formula implies the conclusion by simply using the density of X_1 in X. $\square$

The following result shows the equivalence between the concepts of controllability and observability. Although this is a rather known property, since our framework escapes from the usual hilbertian setting, we have chosen to include it here (we refer to [26] for the proof of a quite close statement).

Proposition 2.5. *Let $e \subset \Gamma \times [0,\tau]$ be a set of positive measure and $K_{\tau,e} > 0$. The following two properties are equivalent*

1. *The inequality*
$$K_{\tau,e}\left\|\Psi^d_{\tau,e'}\varphi\right\|_{L^1(\Gamma\times[0,\tau])} \geqslant \|\mathbb{T}^*_\tau\varphi\|, \tag{2.18}$$
holds for any $\varphi \in X$, where $e' = \{(x,\tau - t) \mid (x,t) \in e\}$.

2. *The pair (A,B) is L^∞ null controllable in time τ over e at cost not larger than $K_{\tau,e}$.*

Proof. "1. $\Rightarrow$ 2." Consider the subspace $\mathcal{X}$ of $L^1(\Gamma \times [0,\tau])$ defined by

$$\mathcal{X} = \{\Psi^d_{\tau,e'}\varphi \ \mid\ \varphi \in X\}.$$

Given $z_0 \in X$, consider the linear functional $\mathcal{F}$ on $\mathcal{X}$ defined by

$$\mathcal{F}(\Psi^d_{\tau,e'}\varphi) = -\langle z_0, \mathbb{T}^*_\tau\varphi\rangle \qquad (\varphi \in X).$$

The fact that this functional is well defined follows from (2.18). Moreover, using again (2.18), it follows that

$$|\mathcal{F}v| \leqslant K_{\tau,e}\|z_0\|\ \|v\|_{L^1(\Gamma\times[0,\tau])} \qquad (v \in \mathcal{X}).$$

By the Hahn-Banach Theorem, $\mathcal{F}$ can be extended to a bounded linear functional $\widetilde{\mathcal{F}}$ on $L^1(\Gamma \times [0,\tau])$ such that

$$|\widetilde{\mathcal{F}}v| \leqslant K_{\tau,e}\|z_0\| \, \|v\|_{L^1(\Gamma\times[0,\tau])} \qquad (v \in L^1(\Gamma \times [0,\tau])).$$

By the Riesz representation theorem it follows that there exists $u \in L^\infty(\Gamma \times [0,\tau])$ such that $\|u\|_{L^\infty(\Gamma\times[0,\tau])} \leqslant K_{\tau,e}\|z_0\|$ and

$$\int_0^\tau \int_\Gamma u(\tau-\sigma,x)\overline{\Psi^d_{\tau,e'}\varphi} + \langle z_0, \mathbb{T}^*_\tau \varphi\rangle = 0 \qquad (\varphi \in X).$$

By using (2.17) in the above formula, it follows that

$$\langle \Phi_{\tau,e}u, \varphi\rangle + \langle \mathbb{T}_\tau z_0, \varphi\rangle = 0 \qquad (\varphi \in X),$$

which is equivalent to

$$\Phi_{\tau,e}u + \mathbb{T}_\tau z_0 = 0.$$

Since the above construction holds for every $z_0 \in X$, we get the desired result.

"2. $\Leftarrow$ 1." Let $\varphi \in X$ and $z_0 = \mathbb{T}^*_\tau\varphi \in X$. Our assumption implies that there exists $u \in L^\infty(\Gamma \times [0,\tau])$ such that $\|u\|_{L^\infty(\Gamma\times[0,\tau])} \leqslant K_{\tau,e}\|z_0\|$ and $\Phi_{\tau,e}u + \mathbb{T}_\tau z_0 = 0$. It follows that

$$\|\mathbb{T}^*_\tau\varphi\|^2 = -\langle \Phi_{\tau,e}u, \varphi\rangle = -\int_0^\tau \int_\Gamma \boldsymbol{\mathcal{R}}_\tau u \overline{\Psi^d_{\tau,e'}\varphi}$$
$$\leqslant \|u\|_{L^\infty(\Gamma\times[0,\tau])}\|\Psi^d_{\tau,e'}\varphi\|_{L^1(\Gamma\times[0,\tau])} \leqslant K_{\tau,e}\|\mathbb{T}^*_\tau\varphi\| \, \|\Psi^d_{\tau,e'}\varphi\|_{L^1(\Gamma\times[0,\tau])},$$

which ends the proof. □

We are now in a position to state the time optimal problem. Define the set of admissible controls

$$\mathcal{U}_{ad} = \{u \in L^\infty(\Gamma \times [0,\infty)) \;\mid\; |u(x,t)| \leqslant M \text{ a. e. in } \Gamma \times [0,\infty)\}.$$

Given $z_0 \in X$, we define the set of targets which are reachable from z_0

$$\mathcal{R}(z_0, \mathcal{U}_{ad}) = \cup_{t>0}\{z_1 = \mathbb{T}_t z_0 + \Phi_t u \in X \;\mid\; u \in \mathcal{U}_{ad}\}.$$

We consider the time optimal control problem which consists in determining, for every $z_0 \in X$ and $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, a control $u^* \in \mathcal{U}_{ad}$ such that

$$\mathbb{T}_{\tau^*(z_0,z_1)}z_0 + \Phi_{\tau^*(z_0,z_1)}u^* = z_1, \tag{2.19}$$

where $\tau^*(z_0,z_1)$ is the minimal time needed to steer the initial data z_0 towards target z_1 with controls in $\mathcal{U}_{ad}$

$$\tau^*(z_0,z_1) = \inf_{u\in\mathcal{U}_{ad}} \{t > 0 \;\mid\; \mathbb{T}_t z_0 + \Phi_t u = z_1\}. \tag{2.20}$$

As shown in [6, 14], the above problem admits at least one solution for every $z_0 \in X$ and $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$ (see also the proof of Proposition 2.6 below).

The function $(z_0,z_1) \mapsto \tau^*(z_0,z_1)$ is called the *minimal time function*. A natural question consists in investigating if the time optimal control u^* is bang-bang, in the sense that $|u^*(x,t)| = M$ almost everywhere. A sufficient condition for this property is given in the following known proposition, which is an abstract version of a result for the heat equation from [20]. For the sake of convenience, we provide the detailed proof below.

Proposition 2.6. *With the notation in Proposition 2.5, assume that the pair (A, B) is L^∞ null controllable in time τ over e for every $\tau > 0$ and for every set of positive measure $e \subset \Gamma \times [0, \tau]$. Then, for every $z_0 \in X$ and $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, the time optimal problem (2.19) has a unique solution u^* which is bang-bang.*

Proof. Let us first prove the existence of a solution u^* of the time optimal problem (2.19). Since $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, there exists a minimizing sequence $(\tau_n, u_n)_{n\geqslant 1}$ such that $\lim_{n\to\infty} \tau_n = \tau^*(z_0, z_1)$ and $(u_n)_{n\geqslant 1} \subset \mathcal{U}_{ad}$ has the property that $\mathbb{T}_{\tau_n} z_0 + \Phi_{\tau_n} u_n = z_1$ for each $n \geqslant 1$.

Since $(u_n)_{n\geqslant 1} \subset \mathcal{U}_{ad}$, it follows that $(u_n)_{n\geqslant 1}$ tends weakly-$*$ to some $u^* \in \mathcal{U}_{ad}$ in $L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])$. We define $\tilde{z}_1 = \mathbb{T}_{\tau^*(z_0,z_1)} z_0 + \Phi_{\tau^*(z_0,z_1)} u^*$ and we note that u^* is a time optimal control if $\tilde{z}_1 = z_1$. The latter equality follows if we prove that $\langle \tilde{z}_1, \varphi\rangle = \langle z_1, \varphi\rangle$ for each $\varphi \in X$ which, in view of the facts that $\lim_{n\to\infty} \mathbb{T}_{\tau_n} z_0 = \mathbb{T}_{\tau^*(z_0,z_1)} z_0$, $\lim_{n\to\infty} \langle \Phi_{\tau_n} u_n - \Phi_{\tau^*(z_0,z_1)} u_n, \varphi\rangle = 0$ and $\mathbb{T}_{\tau_n} z_0 + \Phi_{\tau_n} u_n = z_1$ for each $n \geqslant 1$, is reduced to prove that

$$\langle \Phi_{\tau^*(z_0,z_1)} u, \varphi\rangle = \lim_{n\to\infty} \langle \Phi_{\tau^*(z_0,z_1)} u_n, \varphi\rangle \qquad (\varphi \in X). \tag{2.21}$$

By noting that

$$\langle \Phi_{\tau^*(z_0,z_1)} u_n, \varphi\rangle = \int_0^{\tau^*(z_0,z_1)} \langle \boldsymbol{\mathcal{R}}_{\tau^*(z_0,z_1)} u_n(t), \Psi^d_{\tau^*(z_0,z_1)} \varphi\rangle_U \, \mathrm{d}t$$

and taking into account that $\Psi^d_{\tau^*(z_0,z_1)} \varphi \in L^1([0, \tau^*(z_0, z_1)]; U)$, we deduce from the weak-$*$ convergence of the sequence $(u_n)_{n\geqslant 1}$ that (2.21) holds and the existence of a solution u^* for the time optimal problem is proved.

Now, let us show that $u^* \in L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])$ is bang-bang. We denote by z^* the corresponding state trajectory. Assume that there exist $\varepsilon > 0$ and a set of positive measure $E \subset \Gamma \times [0, \tau^*(z_0, z_1)]$ such that

$$|u^*(x, t)| < M - \varepsilon \qquad ((x, t) \in E). \tag{2.22}$$

Let $\delta_0 > 0$ be small enough such that

$$\begin{cases} \tau_0 = \tau^*(z_0, z_1) - \delta_0 > 0, \\ \\ \text{the set } e_0 = \{(x, t) \in \Gamma \times [\delta_0, \tau_0] \ \mid \ (x, t) \in E\} \text{ has positive measure.} \end{cases} \tag{2.23}$$

Since $\lim_{t\to 0} z^*(t) = z_0$, there exists $\delta \in (0, \delta_0)$ such that

$$\|z_0 - z^*(\delta)\| \leqslant \frac{\varepsilon}{2C_{\tau_0,e_0}}. \tag{2.24}$$

Moreover, the L^∞ null controllability of (A, B) in time τ_0 over e_0 implies that there exists $v \in L^\infty(\Gamma \times [0, \tau^*(z_0, z_1)])$ with

$$\begin{cases} \operatorname{supp} v \subset e_0, \\ \\ \mathbb{T}_{\tau_0}(\mathbb{T}_{\delta_0-\delta}(z_0 - z^*(\delta))) + \displaystyle\int_{\delta_0}^{\tau^*(z_0,z_1)} \mathbb{T}_{\tau^*(z_0,z_1)-s} B v(s) \, \mathrm{d}s \\ = \mathbb{T}_{\tau^*(z_0,z_1)-\delta}(z_0 - z^*(\delta)) + \displaystyle\int_{\delta}^{\tau^*(z_0,z_1)} \mathbb{T}_{\tau^*(z_0,z_1)-s} B v(s) \, \mathrm{d}s = 0, \\ \\ \|v\|_{L^\infty(e_0)} \leqslant 2C_{\tau_0,e_0} \|z_0 - z^*(\delta)\|, \end{cases} \tag{2.25}$$

where C_{τ_0,e_0} is the cost constant defined in (2.16). From (2.24), together with the first and the third conditions in (2.25), it follows that

$$\|v\|_{L^\infty(\Gamma\times[0,\tau^*(z_0,z_1)])} \leqslant \varepsilon. \tag{2.26}$$

Now, let $\widetilde{u} \in L^\infty(\Gamma \times [0,\tau_0])$ be defined by

$$\widetilde{u}(t) = u^*(t+\delta) + v(t+\delta) \qquad (t \in [0,\tau_0]).$$

By combining (2.22), (2.26) and the fact that $\operatorname{supp} v \subset e_0$, it follows that

$$\|\widetilde{u}\|_{L^\infty(\Gamma\times[0,\tau_0])} \leqslant M. \tag{2.27}$$

Finally, the semi-group property, the above definition of $\widetilde{u}$ and (2.25), imply that

$$\begin{aligned}
\mathbb{T}_{\tau^*(z_0,z_1)-\delta}\, z_0 + \Phi_{\tau^*(z_0,z_1)-\delta}\, \widetilde{u} &= \mathbb{T}_{\tau^*(z_0,z_1)-\delta}(z_0 - z^*(\delta)) + \mathbb{T}_{\tau^*(z_0,z_1)-\delta}\, z^*(\delta) \\
&\quad + \Phi_{\tau^*(z_0,z_1)-\delta}\, v(\,\cdot\, + \delta) + \Phi_{\tau^*(z_0,z_1)-\delta}\, u^*(\,\cdot\, + \delta) \\
&= \mathbb{T}_{\tau^*(z_0,z_1)-\delta}(z_0 - z^*(\delta)) + \mathbb{T}_{\tau^*(z_0,z_1)}\, z_0 + \mathbb{T}_{\tau^*(z_0,z_1)-\delta} \int_0^\delta \mathbb{T}_{\delta-s} Bu(s)\,\mathrm{d}s \\
&\quad + \int_\delta^{\tau^*(z_0,z_1)} \mathbb{T}_{\tau^*(z_0,z_1)-s} Bv(s)\,\mathrm{d}s + \int_\delta^{\tau^*(z_0,z_1)} \mathbb{T}_{\tau^*(z_0,z_1)-s} Bu(s)\,\mathrm{d}s \\
&= \mathbb{T}_{\tau^*(z_0,z_1)-\delta}(z_0 - z^*(\delta)) + \int_\delta^{\tau_0} \mathbb{T}_{\tau^*(z_0,z_1)-s} Bv(s)\,\mathrm{d}s + \mathbb{T}_{\tau^*(z_0,z_1)}\, z_0 + \Phi_{\tau^*(z_0,z_1)}\, u^* = z_1.
\end{aligned}$$

Hence, $\tilde{u} \in \mathcal{U}_{ad}$ is a control which drives z_0 to z_1 in time $\tau^*(z_0,z_1) - \delta$. This contradicts the definition of $\tau^*(z_0,z_1)$ and the bang-bang property is proved.

To show the uniqueness, let u and v be two time optimal controls in $\mathcal{U}_{ad}$. Note that in this case $w = \frac{1}{2}(u+v)$ is also a time optimal control. From the proof above it follows that $|u(x,t)| = |v(x,t)| = |w(x,t)| = M$ a. e. in $\Gamma \times [0,\tau^*(z_0,z_1)]$. If $u(x,t) \neq v(x,t)$ in a set of positive measure $E \subset \Gamma \times [0,\tau^*(z_0,z_1)]$ then

$$0 = u(x,t) + v(x,t) = 2w(x,t) \qquad (x,t) \in E,$$

which contradicts the fact that $|w(x,t)| = M$ a. e. in $\Gamma \times [0,\tau^*(z_0,z_1)]$. $\square$

3 A modified Lebeau-Robbiano strategy

In this section we propose a version of a method introduced by Lebeau and Robbiano [12] to study the null controllability of the heat equation. Roughly speaking, the Lebeau-Robbiano strategy combines the observability of finite combinations of eigenvectors (which is a property not involving the time variable) with the exponential decay of the heat semigroup to obtain the null controllability. The fact that this method can be adapted to null controllability with inputs in $L^\infty([0,\tau];L^2(\Gamma))$ over a positive measure set $e \subset [0,\tau]$ has been remarked in Wang [26]. The main novelties we bring in into this section consist in the facts that this strategy is adapted to null controllability with inputs in $L^\infty([0,\tau]\times\Gamma)$ over a positive measure set $e \subset [0,\tau]\times\Gamma$ and that we replace the Lebeau-Robbiano assumption on the observability of finite combinations of eigenvectors with an assumption of controllability of the truncation of the dynamical system to a finite number of modes. This latter property involves the time variable and it is, in general, weaker than the observability of finite combinations of eigenvectors.

We continue to use in this section the notation and assumptions in Section 2 on the spaces X, U and on the operators A and B. Moreover, we add some new notation and assumptions.

The operator $A : \mathcal{D}(A) \to X$ is supposed to be a self-adjoint (possibly unbounded) operator on X such that

$$\langle A\psi, \psi\rangle \leqslant 0 \qquad (\psi \in \mathcal{D}(A)).$$

Such an operator will be briefly called a *negative operator*. We also assume that A is diagonalizable with an orthonormal basis of eigenvectors $\{\varphi_k\}_{k\geqslant 1}$ and corresponding family of eigenvalues $\{-\lambda_k\}_{k\geqslant 1}$, where the sequence $\{\lambda_k\}$ is positive, non decreasing and satisfies $\lambda_k \to \infty$ as k tends to infinity. According to classical results, this holds, in particular, if A has compact resolvents. With the above assumptions on A, we have

$$A\psi = -\sum_{k\geqslant 1} \lambda_k \langle \psi, \varphi_k\rangle \varphi_k \qquad (\psi \in \mathcal{D}(A)), \tag{3.1}$$

so that the semigroup $\mathbb{T}$ generated by A is a contraction semigroup on X satisfying

$$\mathbb{T}_t z = \sum_{k\geqslant 1} \mathrm{e}^{-\lambda_k t} \langle z, \varphi_k\rangle \varphi_k \qquad (t \geqslant 0,\ z \in X). \tag{3.2}$$

Moreover, the sets

$$X_\beta := \left\{ z \in X : \textstyle\sum_{k\geqslant 1} \left(1+\lambda_k^2\right)^\beta |\langle z, \varphi_k\rangle|^2 < \infty \right\} \qquad (\beta > 0), \tag{3.3}$$

endowed with the inner product

$$\langle y, z\rangle_\beta = \sum_{k\geqslant 1} \left(1+\lambda_k^2\right)^\beta \langle y, \varphi_k\rangle \ \overline{\langle z, \varphi_k\rangle} \qquad (z,\, y \in X_\beta), \tag{3.4}$$

are Hilbert spaces. The scale $\{X_\beta\}_{\beta\geqslant 0}$ of Hilbert spaces can be extended to a scale $\{X_\beta\}_{\beta\in\mathbb{R}}$ by defining, for every $\beta < 0$, X_β as the completion of X with respect to the norm associated to the inner product (3.4). Alternatively, $X_{-\beta}$ may be defined, for $\beta > 0$, as the dual of X_β with respect to the pivot space X. For every $\beta > 0$, formulas (3.1) and (3.2), with $\langle\cdot,\cdot\rangle$ standing this time for the duality between $X_{-\beta}$ and X_β, provide canonical extensions for the operator A and the semigroup $\mathbb{T}$ to a negative operator and a contraction semigroup on $X_{-\beta}$, respectively. These extensions will be still denoted by A and $\mathbb{T}$. Note that, for every $\beta \in \mathbb{R}$, the family $\{(1+\lambda_k^2)^{\beta/2}\varphi_k\}_{k\geqslant 1}$ is an orthonormal basis in X_β. Finally, in the sequel we need the following lemma which represents the dual version of Proposition 5.1.3 from [24]. However, to make precise the admissibility constant, we provide a short proof below.

Lemma 3.1. *Assume that U is a Hilbert space and $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$. Then B is an admissible control operator for $\mathbb{T}$ with admissibility constant $\frac{1}{\sqrt{2}}\|B\|_{\mathcal{L}(U,X_{-\frac{1}{2}})}$.*

Proof. Let $u \in L^2([0,\tau];U)$ and let $z \in C([0,\tau];X_{-1})$ be the mild solution of the equation

$$\dot{z}(t) = Az(t) + Bu(t), \qquad z(0) = 0. \tag{3.5}$$

If $(F_n)_{n\geqslant 1} \subset W^{1,1}([0,\tau];X_1)$ is a sequence convergent to Bu in $L^2([0,\tau];X_{-\frac{1}{2}})$, then let $z_n \in C([0,\tau];X_1) \cap C^1([0,\tau];X)$ be the solution of

$$\dot{z}_n(t) = Az_n(t) + F_n(t), \qquad z_n(0) = 0. \tag{3.6}$$

It follows that z_n verifies

$$\frac{1}{2}\frac{d}{dt}\|z_n(t)\|^2 = -\|z_n(t)\|^2_{\frac{1}{2}} + \langle F_n(t), z_n(t)\rangle = -\|z_n(t)\|^2_{\frac{1}{2}} + \langle F_n(t), z_n(t)\rangle_{-\frac{1}{2},\frac{1}{2}} \leqslant \frac{1}{4}\|F_n(t)\|^2_{-\frac{1}{2}}.$$

By integrating the last inequality from 0 to τ, we obtain that

$$\limsup_{n\to\infty} \|z_n(\tau)\|^2 \leqslant \frac{1}{2}\int_0^\tau \|Bu(s)\|^2_{-\frac{1}{2}}\,\mathrm{d}s \leqslant \frac{1}{2}\|B\|^2_{\mathcal{L}(U,X_{-\frac{1}{2}})}\,\|u\|^2_{L^2([0,\tau];U)}.$$

We deduce that $(z_n(\tau))_{n\geqslant 1}$ converges weakly to some $\widetilde{z}$ in X. Since $(z_n(\tau))_{n\geqslant 1}$ converges to $z(\tau)$ in X_{-1}, it follows that $z(\tau) = \widetilde{z} \in X$ and verifies

$$\|z(\tau)\| \leqslant \frac{1}{\sqrt{2}}\|B\|_{\mathcal{L}(U,X_{-\frac{1}{2}})}\,\|u\|_{L^2([0,\tau];U)}. \tag{3.7}$$

The proof of the Lemma is complete. □

For $\gamma,\ \varsigma > 0$ we denote by

$$V_{\varsigma,\gamma} = \operatorname{span}\{\varphi_k \ \mid\ \lambda_k^\gamma \leqslant \varsigma\},$$

and we denote by $P_{\varsigma,\gamma}$ the orthogonal projection from X onto $V_{\varsigma,\gamma}$.

We recall that through this paper $U = L^2(\Gamma)$ where Γ is a measurable set with respect to a measure μ. In the sequel, for every $k \in \mathbb{N}^*$ we denote by μ_k the Lebesgue measure in $\mathbb{R}^k$ and by $\widetilde{\mu}$ the product of measures μ and μ_1. We now state the main result in this section.

Theorem 3.2. *Let $\tau > 0$ and let $e \subset \Gamma \times [0,\tau]$ be a set of positive measure. Assume $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$. Moreover, assume that there exist positive constants $\gamma \in (0,1)$, $a \in (0,\tau)$, d_0, d_1 and κ such that for every $\varsigma > 0$, $s,t > 0$, with $a \leqslant s < t \leqslant \tau$, and $\mathcal{E} = \{(x,\sigma) \in e \ \mid\ s \leqslant \sigma \leqslant t\}$ of positive measure, we have*

$$\|\mathbb{T}^*_\tau \varphi\| \leqslant d_0 e^{d_1\left[1+\ln\left(\frac{1}{\widetilde{\mu}(\mathcal{E})}\right)\right]\varsigma + \frac{\kappa}{\widetilde{\mu}(\mathcal{E})}} \|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])} \quad (\varphi \in V_{\varsigma,\gamma}), \tag{3.8}$$

where $\mathcal{E}' = \{(x, \tau - \sigma) \ \mid\ (x,\sigma) \in \mathcal{E}\}$. Then the pair (A,B) is L^∞ null controllable in time τ over e.

In order to prove Theorem 3.2 we need the following measure theoretic result whose proof may be found, for instance, in Lions [14, p. 275].

Lemma 3.3. *Let F be a set of positive measure. Then there exist positive constants ρ and c such that for almost every $t \in F$ there exists an increasing sequence $(t_n)_{n\geqslant 0}$ such that $\lim_{n\to\infty} t_n = t$ and*

$$\mu_1([t_n, t_{n+1}] \cap F) \geqslant \rho(t_{n+1} - t_n), \qquad \frac{t_{n+1} - t_n}{t_{n+2} - t_{n+1}} \leqslant c \qquad (n \geqslant 0). \tag{3.9}$$

The following simple lemma will help us to separate the time and space variables in our estimates.

Lemma 3.4. *Let $e \subset \Gamma \times [0, \tau]$ be a set of positive measure. For each $t \in [0, \tau]$, we define the $t-$section of e as $e_t = \{x \in \Gamma \mid (x, t) \in e\}$. If $F = \left\{t \in [0, \tau] \;\middle|\; \mu(e_t) > \frac{\widetilde{\mu}(e)}{4\tau}\right\}$, then*

$$\mu_1(F) > \frac{\widetilde{\mu}(e)}{4\mu(\Gamma)}. \tag{3.10}$$

Proof. Let $F = \left\{t \in [0, \tau] \;\middle|\; \mu(e_t) > \frac{\widetilde{\mu}(e)}{4\tau}\right\}$ and suppose that $\mu_1(F) \leqslant \frac{\widetilde{\mu}(e)}{4\mu(\Gamma)}$. It follows that

$$\widetilde{\mu}(e) = \int_0^\tau \mu(e_t)\,\mathrm{d}t = \int_F \mu(e_t)\,\mathrm{d}t + \int_{[0,\tau]\setminus F} \mu(e_t)\,\mathrm{d}t \leqslant \frac{\widetilde{\mu}(e)}{4\mu(\Gamma)}\mu(\Gamma) + \tau\frac{\widetilde{\mu}(e)}{4\tau} = \frac{\widetilde{\mu}(e)}{2},$$

which is a contradiction. □

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. Since $e \subset \Gamma \times [0, \tau]$ is a set of positive measure, we deduce from Lemma 3.4 that there exists $F \subset [0, \tau]$ of measure greater than $\frac{\widetilde{\mu}(e)}{4\mu(\Gamma)}$ such that, for any $t \in F$, the sections e_t has measure greater than $\frac{\widetilde{\mu}(e)}{4\tau}$. Let $t' \in F$ be one of the points for which one can find a sequence $(t_k)_{k\geqslant 0}$ as in Lemma 3.3. Without loss of generality we may suppose that $t_0 := a > 0$. In the remaining part of the proof, for each $k \geqslant 0$, we denote $r_{k+1} = t_{k+1} - t_k$. Define the sequence $(\varsigma_k)_{k\geqslant 0}$ by

$$\varsigma_k = \frac{1}{r_{2k+2}^p} \qquad (k \geqslant 0), \tag{3.11}$$

where $p = p(\gamma)$ will be conveniently chosen latter on.

For every $k \geqslant 0$ we define the functions $z_k \in C([t_{2k}, t_{2k+2}], X)$ and $u_k \in L^\infty(\Gamma \times [t_{2k}, t_{2k+2}])$ recursively by the formulae

$$z_{-1}(t_0) = \mathbb{T}_{t_0} z_0, \tag{3.12}$$

$$u_k(x, t) = \begin{cases} \widetilde{u}_k(x, t) & \text{for } (x, t) \in \Gamma \times [t_{2k}, t_{2k+1}], \\ 0 & \text{for } (x, t) \in \Gamma \times [t_{2k+1}, t_{2k+2}], \end{cases} \qquad (k \geqslant 0), \tag{3.13}$$

$$z_k(\xi) = \mathbb{T}_{\xi - t_{2k}} z_{k-1}(t_{2k}) + \int_{t_{2k}}^{\xi} \mathbb{T}_{\xi-\sigma} B\chi_e u_k(\sigma)\,\mathrm{d}\sigma \quad (k \geqslant 0,\ \xi \in [t_{2k}, t_{2k+2}]), \tag{3.14}$$

where $\widetilde{u}_k \in L^\infty(\Gamma \times [t_{2k}, t_{2k+1}])$ is chosen such that, for each $k \geqslant 0$,

$$P_{\varsigma_k,\gamma} z_k(t_{2k+1}) = 0, \quad \operatorname{supp} \widetilde{u}_k \subset e \cap \Gamma \times [t_{2k}, t_{2k+1}], \tag{3.15}$$

$$\|\widetilde{u}_k\|_{L^\infty(\Gamma\times[t_{2k},t_{2k+1}])} \leqslant d_0 e^{d_1\left[1+\ln\left(\frac{4\tau}{\rho\widetilde{\mu}(e)\, r_{2k+1}}\right)\right]\varsigma_k + \frac{4\tau\kappa}{\rho\widetilde{\mu}(e)\, r_{2k+1}}} \|z_{k-1}(t_{2k})\|. \tag{3.16}$$

The existence of $\widetilde{u}_k$ with the above properties follows from (3.8) with $\mathcal{E} = e \cap \Gamma \times [t_{2k}, t_{2k+1}]$ by applying Proposition 2.5 and taking into account that

$$\widetilde{\mu}(\mathcal{E}) = \int_{t_{2k}}^{t_{2k+1}} \int_{e_\sigma} \mathrm{d}x\, \mathrm{d}\sigma \geqslant \int_{F\cap[t_{2k},t_{2k+1}]} \int_{e_\sigma} \mathrm{d}x\, \mathrm{d}\sigma \geqslant \frac{\widetilde{\mu}(e)}{4\tau} \rho r_{2k+1}.$$

Denoting $\tilde{\rho} = \frac{\widetilde{\mu}(e)}{4\tau}\rho$ and using (3.14) and (3.16) it follows that, for each $k \geqslant 0$,

$$\|z_k(t_{2k+1})\| \leqslant \left(1 + d_0\|B\| \sqrt{r_{2k+1}}\, e^{d_1\left[1+\ln\left(\frac{1}{\tilde{\rho}\, r_{2k+1}}\right)\right]\varsigma_k + \frac{\kappa}{\tilde{\rho}\, r_{2k+1}}}\right) \|z_{k-1}(t_{2k})\|. \tag{3.17}$$

In the above formula and in the remaining part of the proof, $\|B\|$ denotes the norm in $\mathcal{L}(U, X_{-\frac{1}{2}})$. Note that formula (3.14) for $z_k(t_{2k+2})$ and (3.15) imply that

$$\langle z_k(t_{2k+2}), \varphi_n \rangle = 0 \qquad (k \geqslant 0,\ \lambda_n^\gamma \leqslant \varsigma_k) \tag{3.18}$$

and yield that

$$\|z_k(t_{2k+2})\| \leqslant e^{-\varsigma_k^{1/\gamma}\, r_{2k+2}} \|z_k(t_{2k+1})\| \qquad (k \geqslant 0). \tag{3.19}$$

From the above relation and (3.17) it follows that there exists a positive constant $\widetilde{\kappa} > \kappa$ such that, for each $k \geqslant 0$, we have

$$\|z_k(t_{2k+2})\| \leqslant e^{-\varsigma_k^{1/\gamma}\, r_{2k+2} + d_1\left[1+\ln\left(\frac{1}{\tilde{\rho}\, r_{2k+1}}\right)\right]\varsigma_k + \frac{\widetilde{\kappa}}{\tilde{\rho}\, r_{2k+1}}} \|z_{k-1}(t_{2k})\|. \tag{3.20}$$

In order to show that the right-hand side of (3.16) forms a bounded sequence, we denote

$$a_k = d_0 e^{d_1\left[1+\ln\left(\frac{1}{\tilde{\rho}\, r_{2k+1}}\right)\right]\varsigma_k + \frac{\widetilde{\kappa}}{\tilde{\rho}\, r_{2k+1}}} \|z_{k-1}(t_{2k})\| \qquad (k \geqslant 0).$$

From the above definition of a_k and (3.20) we deduce that

$$\frac{a_{k+1}}{a_k} \leqslant e^{-\varsigma_k^{1/\gamma}\, r_{2k+2} + d_1 \varsigma_{k+1} \ln\left(\frac{1}{\tilde{\rho}\, r_{2k+3}}\right) + \frac{\widetilde{\kappa}}{\tilde{\rho}\, r_{2k+3}}} \qquad (k \geqslant 0). \tag{3.21}$$

Using Lemma 3.3, formula (3.11) and the fact that $\gamma \in (0, 1)$ it follows that there exists $k_1 > 0$ and a constant $C > 0$ such that

$$\begin{aligned} -\varsigma_k^{1/\gamma}\, r_{2k+2} + d_1 \varsigma_{k+1} \ln\left(\frac{1}{\tilde{\rho}\, r_{2k+3}}\right) + \frac{\widetilde{\kappa}}{\tilde{\rho}\, r_{2k+3}} & \\ \leqslant -\left(\frac{1}{r_{2k+2}}\right)^{\frac{p}{\gamma}-1} + d_1 c^{2p} \left(\frac{1}{r_{2k+2}}\right)^p \ln\left(\frac{c}{\tilde{\rho}\, r_{2k+2}}\right) + \frac{c\widetilde{\kappa}}{\tilde{\rho}\, r_{2k+2}} & \\ \leqslant -C\left(\frac{1}{r_{2k+2}}\right)^{\frac{p}{\gamma}-1} \qquad (k \geqslant k_1), & \end{aligned}$$

for any p which verifies $\frac{p}{\gamma} - 1 > \max\{1, p\}$. Note that the last inequality is equivalent to $p > \max\left\{2\gamma, \frac{\gamma}{1-\gamma}\right\}$ which has at least a solution for each $\gamma \in (0, 1)$.

The above estimate, (3.21) and (3.16) show that the function

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t)\chi_{[t_{2k},t_{2k+2}]} \qquad ((x,t) \in \Gamma \times [0,\tau]), \tag{3.22}$$

belongs to $L^\infty(\Gamma \times [0,\tau])$ and $\operatorname{supp} u \subset \mathcal{E}$. Moreover, the function

$$z(t) = \sum_{k=0}^{\infty} z_k(t)\chi_{[t_{2k},t_{2k+2}]} \qquad (t \in [0,\tau]) \tag{3.23}$$

belongs to $C([0,\tau]; X)$ and verifies

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0$$

and $z(\tau) = \mathbb{T}_\tau z_0 + \Phi_{\tau,\mathcal{E}} u = 0$. The proof of Theorem 3.2 is now complete. $\square$

4 A Turán type inequality

In this section we give an $m-$dimensional version of an inequality originally proved by Turán [25] for intervals and extended by Nazarov [19] for sets of positive measure in $\mathbb{R}$. Let $m \geqslant 1$ be given and, for each $1 \leqslant k \leqslant m$, let $I_k = \prod_{i=1}^{k}[0, l_i]$ be a $k-$dimensional rectangle of volume $\upsilon_k = l_1 ... l_k$. For $m \geqslant 2$ and any $\alpha \in (\mathbb{N}^*)^m$ we write $\alpha = (\alpha', \alpha_m)$ with $\alpha' = (\alpha_1, ..., \alpha_{m-1}) \in (\mathbb{N}^*)^{m-1}$, $\alpha_m \in \mathbb{N}$ and we denote

$$\Phi_\alpha(x) = \Phi_\alpha(x_1, ..., x_m) = \sqrt{\frac{2^m}{\upsilon_m}} \prod_{j=1}^{m} \sin\left(\frac{\alpha_j x_j \pi}{l_j}\right), \tag{4.24}$$

$$\Phi_{\alpha'}(x') = \Phi_{\alpha'}(x_1, ..., x_{m-1}) = \sqrt{\frac{2^{m-1}}{\upsilon_{m-1}}} \prod_{j=1}^{m-1} \sin\left(\frac{\alpha_j x_j \pi}{l_j}\right). \tag{4.25}$$

Now, let $E \subset I_m$ be a set of positive Lebesgue measure. The aim of this section is to estimate from below the $L^1(E)$-norm of the finite linear combinations of functions Φ_α. We begin with the following simple variant of Lemma 3.4 on sets of positive measure in product spaces.

Lemma 4.1. *Let $E \subset I_m$ be a set of positive measure. For each $k < m$ and $(\xi_1, \xi_2, ..., \xi_k) \in I_k$, we define the $(\xi_1, \xi_2, ..., \xi_k)-$section of E as the set*

$$E_{(\xi_1,\xi_2,...,\xi_k)} = \left\{ (\xi_{k+1}, ..., \xi_m) \in \prod_{i=k+1}^{m} [0, l_i] \;\middle|\; (\xi_1, \xi_2, ..., \xi_k, \xi_{k+1}, ..., \xi_m) \in E \right\}.$$

If, for each $k < m$, F_k denotes the set

$$F_k = \left\{ (\xi_1, \xi_2, ..., \xi_k) \in \prod_{i=1}^{k}[0, l_i] \;\middle|\; \mu_{m-k}(E_{(\xi_1,...,\xi_k)}) > \frac{\mu_m(E)}{4 l_1 ... l_k} \right\},$$

then

$$\mu_k(F_k) > \frac{\mu_m(E)}{4\, l_{k+1} ... l_m}. \tag{4.26}$$

Proof. It is similar to the proof of Lemma 3.4 and we omit it. $\square$

The following theorem, proved in [19, Theorem I], will play an essential role in our study.

Theorem 4.2. *Let $N \in \mathbb{N}$ be a nonnegative integer and $p(x) = \sum_{|k|\leqslant N} a_k e^{i\nu_k x}$ ($a_k \in \mathbb{C}$, $\nu_k \in \mathbb{R}$) be an exponential polynomial. Let $I \subset \mathbb{R}$ be an interval and E a measurable subset of I of positive measure. Then*

$$\sup_{x\in I} |p(x)| \leqslant \left(\frac{C\mu_1(I)}{\mu_1(E)}\right)^{2N} \sup_{x\in E} |p(x)|, \tag{4.27}$$

where $C > 0$ is an absolute constant.

The following result is a consequence of Theorem 4.2.

Corollary 4.3. *With the notations from Theorem 4.2 we have that the following inequality holds for any sequence $(a_k)_{|k|\leqslant N} \subset \mathbb{C}$*

$$\left(\sum_{|k|\leqslant N} |a_k|^2\right)^{\frac{1}{2}} \leqslant \frac{1}{C\mu_1(I)} \left(\frac{2C\,\mu_1(I)}{\mu_1(E)}\right)^{2N+1} \int_E \left|\sum_{|k|\leqslant N} a_k e^{i\nu_k x}\right| \mathrm{d}x, \tag{4.28}$$

where $C > 0$ is the constant from (4.27)*.*

Proof. By denoting $p(x) = \sum_{|k|\leqslant N} a_k e^{i\nu_k x}$ and by using the orthogonality of $\left(e^{i\nu_k x}\right)_{|k|\leqslant N}$ in $L^2(I)$, we have that

$$\mu_1(I) \sum_{|k|\leqslant N} |a_k|^2 = \int_I |p(x)|^2 \,\mathrm{d}x \leqslant \mu_1(I) \sup_{x\in I} |p(x)|^2 . \tag{4.29}$$

In order to bound $\sup_{x\in I} |p(x)|$ with an integral over E we use (4.27) and an idea from [2, Theorem 5.6]. If we denote $\mathcal{E} = \left\{x \in E \;\mid\; |p(x)| \geqslant \frac{2}{\mu_1(E)} \|p\|_{L^1(E)}\right\}$, we remark that $\mu_1(\mathcal{E}) \leqslant \frac{\mu_1(E)}{2}$. Thus $\mu_1(E \setminus \mathcal{E}) \geqslant \frac{\mu_1(E)}{2}$ and, by applying (4.27) in $E \setminus \mathcal{E}$, we deduce that

$$\sup_{x\in I} |p(x)| \leqslant \left(\frac{C\mu_1(I)}{\mu_1(E \setminus \mathcal{E})}\right)^{2N} \sup_{x\in E\setminus\mathcal{E}} |p(x)| \leqslant \frac{2}{\mu_1(E)} \left(\frac{2C\mu_1(I)}{\mu_1(E)}\right)^{2N} \|p\|_{L^1(E)}$$

and the proof is complete. $\square$

The following result generalizes Corollary 4.3 to multi-dimensional domains. For other similar results and extensions the interested reader is referred to [4, 8].

Corollary 4.4. *Let $m, N \geqslant 1$ and $\mathcal{D} \subset [1, N]^m \cap \mathbb{N}^m$. Let $E \subset I_m = \prod_{k=1}^m [0, l_i]$ be a set of positive measure. If $C > 0$ is the constant from* (4.27)*, then the following inequality holds for any sequence $(b_\alpha)_{\alpha\in\mathcal{D}} \subset \mathbb{C}$*

$$\left(\sum_{\alpha\in\mathcal{D}} |b_\alpha|^2\right)^{\frac{1}{2}} \leqslant \left(\frac{N^{m-1}}{2^m\, C^{2m}\, \upsilon_m}\right)^{\frac{1}{2}} \left(\frac{2^{p_m} C\, \upsilon_m}{\mu_m(E)}\right)^{m\,(2N+1)} \int_E \left|\sum_{\alpha\in\mathcal{D}} b_\alpha \Phi_\alpha\right| \mathrm{d}x, \tag{4.30}$$

where the sequence $(p_m)_{m\geqslant 1}$ is defined by $p_1 = 1$ and $p_m = \frac{2m+1+(m-1)p_{m-1}}{m}$ for $m \geqslant 2$.

Proof. We prove (4.30) by induction over m. For $m = 1$, (4.30) follows from Proposition 4.3. Indeed, it is sufficient to take in (4.28) $a_0 = 0$, $a_k = \frac{b_k}{2\,i}$ and $a_{-k} = -a_k$ for each $k \geqslant 1$ to obtain (4.30) in the case $m = 1$.

Now, let us suppose that (4.30) holds in any dimension less or equal than $m-1$ and prove it for dimension $m \geqslant 2$. Let $p(x) = \sum_{\alpha\in\mathcal{D}} b_\alpha \Phi_\alpha(x)$, where $\mathcal{D} \subset [1,N]^m \cap \mathbb{N}^m$. With the notations from Lemma 4.1 note that

$$\int_E |p(x)|\,\mathrm{d}x = \int_{I_{m-1}} \int_0^{l_m} \chi_E |p(x', x_m)|\,\mathrm{d}x_m\,\mathrm{d}x' \geqslant \int_{F_{m-1}} \int_{E_{x'}} |p(x', x_m)|\,\mathrm{d}x_m\,\mathrm{d}x'.$$

We have that $p(x) = \sum_{k=1}^N \left(\sum_{\alpha'\in\mathcal{D}_k} a_{(\alpha',k)} \Phi_{\alpha'}(x') \right) \sqrt{\frac{2}{l_m}} \sin\left(\frac{k\pi x_m}{l_m}\right)$, where $\mathcal{D}_k = \{\alpha' \in [1,N]^{m-1} \;|(\alpha',k)\in\mathcal{D}\}$ for each $1 \leqslant k \leqslant N$. Our recurrence assumption implies that

$$\int_{E_{x'}} |p(x', x_m)|\,\mathrm{d}x_m \geqslant \sqrt{2l_m}C \left(\frac{2Cl_m}{\mu_1(E_{x'})}\right)^{-(2N+1)} \left[\sum_{k=1}^N \left|\sum_{\alpha'\in\mathcal{D}_k} a_{(\alpha',k)}\Phi_{\alpha'}(x')\right|^2\right]^{\frac{1}{2}},$$

for each $x' \in I_{m-1}$. Integrating the above formula over F_m and taking into account that $\mu_1(E_{x'}) \geqslant \frac{\mu_m(E)}{4\upsilon_{m-1}}$ for every $x' \in F_m$, we deduce that

$$\begin{aligned}\int_E |p(x)|\,\mathrm{d}x &\geqslant \sqrt{2l_m}C \left(\frac{8\,C\,\upsilon_{m-1}l_m}{\mu_m(E)}\right)^{-(2N+1)} \int_{F_m} \left[\sum_{k=1}^N \left|\sum_{\alpha'\in\mathcal{D}_k} a_{(\alpha',k)}\Phi_{\alpha'}(x')\right|^2\right]^{\frac{1}{2}} \mathrm{d}x' \\ &\geqslant \sqrt{2l_m}C \left(\frac{8\,C\,\upsilon_m}{\mu_m(E)}\right)^{-(2N+1)} \frac{1}{\sqrt{N}} \sum_{k=1}^N \int_{F_m} \left|\sum_{\alpha'\in\mathcal{D}_k} a_{(\alpha',k)}\Phi_{\alpha'}(x')\right| \mathrm{d}x'.\end{aligned}$$

Using again the recurrence assumption and the fact that $\mu_{m-1}(F_m) > \frac{\mu_m(E)}{4\,l_m}$, we deduce that

$$\begin{aligned}&\int_E |p(x)|\,\mathrm{d}x \\ &\geqslant \left(\frac{2^m C^{2m}\,\upsilon_m}{N^{\frac{m-1}{2}}}\right)^{\frac{1}{2}} \left(\frac{8\,C\,\upsilon_m}{\mu_m(E)}\right)^{-(2N+1)} \left(\frac{2^{2+p_{m-1}}C\,\upsilon_m}{\mu_m(E)}\right)^{-(m-1)(2N+1)} \left(\sum_{\alpha\in\mathcal{D}} |a_\alpha|^2\right)^{\frac{1}{2}} \\ &= \left(\frac{2^m C^{2m}\,\upsilon_m}{N^{\frac{m-1}{2}}}\right)^{\frac{1}{2}} \left(\frac{2^{\frac{2m+1+(m-1)p_{m-1}}{m}}C\,\upsilon_m}{\mu_m(E)}\right)^{-m\,(2N+1)} \left(\sum_{\alpha\in\mathcal{D}} |a_\alpha|^2\right)^{\frac{1}{2}},\end{aligned}$$

so that the proof is complete. □

To end this section we give a simple consequence of the above estimates to an interior controllability problem for a (possibly fractional) diffusion equation. Let m be a positive integer, let $\Omega \subset \mathbb{R}^m$ be an open and bounded set and let Γ be a non-empty open subset of Ω. For $\theta \in \left(\frac{1}{2}, \infty\right)$, we consider the diffusion equation

$$\frac{\partial z}{\partial t}(x,t) = -(-\Delta)^\theta z(x,t) + u(x,t)\chi_\Gamma(x) \text{ for } \; (x,t) \in \Omega\times(0,\infty), \tag{4.31}$$

with the boundary and initial conditions

$$z(x,t) = 0 \quad \text{on} \;\; \partial\Omega \times (0,\infty), \tag{4.32}$$

$$z(x,0) = z_0(x) \;\; \text{for} \;\; x \in \Omega. \tag{4.33}$$

Equation (4.31) involves a fractional power of the Dirichlet Laplacian, $(-\Delta)^\theta$, and it is used as mathematical model for physical processes exhibiting anomalously slow or fast diffusion (see, for instance, [10, 15]). We refer to [16, 17, 23] for some of the controllability properties of this equation. The input u acts only on the subset Γ of Ω. We have the following result for the time optimal control problem associated to (4.31)-(4.33).

Proposition 4.5. *Suppose that Ω is a rectangular domain in $\mathbb{R}^m$ and that Γ is a nonempty open subset of Ω. Then, for every $z_0 \in L^2(\Omega)$ and $z_1 \in \mathcal{R}(z_0, \mathcal{U}_{ad})$, there exits a unique solution u^* of the time optimal control problem* (2.19)-(2.20), *associated to* (4.31)-(4.33). *This solution u^* has the bang-bang property:*

$$|u^*(x,t)| = M \qquad a.\ e.\ in \quad \Gamma \times [0, \tau^*(z_0,z_1)]. \tag{4.34}$$

Proof. System (4.31)-(4.33) can be written in the form (2.10). Indeed, let $X = L^2(\Omega)$, $U = L^2(\Gamma)$ and $A = -(-\Delta)^\theta$. Let the control operator $B \in \mathcal{L}(L^2(\Gamma), X)$ be defined by

$$Bu = u\chi_\Gamma. \tag{4.35}$$

With the above notation, (4.31)-(4.33) is equivalent to (2.10) and $B \in \mathcal{L}(U,X)$ is an admissible control operator for the semigroup $\mathbb{T}$ generated by A. In this case the measure μ on Γ is the Lebesgue measure μ_m.

For any $\gamma \in (0,1)$ and $\varsigma > 0$, let $V_{\varsigma,\gamma} = \{\varphi \in X \mid \varphi = \sum_{\lambda_\alpha^\gamma \leqslant \varsigma} a_\alpha \Phi_\alpha\}$, where $(\Phi_\alpha)_{\alpha \in (\mathbb{N}^*)^m}$ are the eigenvectors of the operator $-A$ which are given by (4.24). The eigenvalue λ_α of $-A$ corresponding to the eigenvector Φ_α is given by $\lambda_\alpha = (\alpha_1^2 + \ldots + \alpha_m^2)^\theta$.

For any $0 < a \leqslant s < t \leqslant \tau$ let $\mathcal{E} = \{(x,\sigma) \in e \;\mid\; s \leqslant \sigma \leqslant t\}$ be a set of positive measure and let $\mathcal{E}' = \{(x, \tau - \sigma) \;\mid\; (x,\sigma) \in \mathcal{E}\}$. By using Lemma 3.4 we have that there exists a set $F \subset \{t \in [0,\tau] \mid (t,x) \in \mathcal{E}'\}$ such that $\mu_1(F) > \frac{\mu_{m+1}(\mathcal{E})}{4\mu_m(\Gamma)}$ and the $\sigma-$section E_σ has the property that $\mu_m(E_\sigma) > \frac{\mu_{m+1}(\mathcal{E})}{4\tau}$ for every $\sigma \in F$.

According to Proposition 2.6 and Theorem 3.2, the conclusion of our proposition follows if we show that the pair (A,B) verifies (3.8). In this particular case, (3.8) is a direct consequence of Corollary 4.4. Indeed, we have that

$$\|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])} = \int_0^\tau \int_\Gamma \chi_{\mathcal{E}'} |B^*\mathbb{T}^*_\sigma \varphi| \,\mathrm{d}x\,\mathrm{d}\sigma$$

$$\geqslant \int_F \int_{E_\sigma} |B^*\mathbb{T}^*_\sigma\varphi| \,\mathrm{d}x\,\mathrm{d}\sigma = \int_F \int_{E_\sigma} \left| \sum_{\lambda_\alpha^\gamma \leqslant \varsigma} a_\alpha \Phi_\alpha(x) e^{-\lambda_\alpha \sigma} \right| \mathrm{d}x\,\mathrm{d}\sigma.$$

By using Corollary 4.4 with $N = \varsigma^{\frac{1}{2\theta\gamma}}$, we deduce that there exists a constat $C = C(\Gamma, \theta, m)$ such that

$$\|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])} \geqslant \int_F \left\{ C\varsigma^{-\frac{m-1}{4\theta\gamma}} \left(\frac{C}{\mu_m(E_\sigma)}\right)^{-m\left(2\varsigma^{\frac{1}{2\theta\gamma}}+1\right)} \left(\sum_{\lambda_\alpha^\gamma \leqslant \varsigma} |a_\alpha|^2 e^{-2\lambda_\alpha\sigma}\right)^{\frac{1}{2}} \right\} \mathrm{d}\sigma$$

$$\geqslant C\varsigma^{-\frac{m-1}{4\theta\gamma}}\left(\frac{4C}{\mu_{m+1}(\mathcal{E})}\right)^{-m\left(2\varsigma^{\frac{1}{2\theta\gamma}}+1\right)}\mu_1(F)\left(\sum_{\lambda_\alpha^\gamma\leqslant\varsigma}|a_\alpha|^2e^{-2\lambda_\alpha\tau}\right)^{\frac{1}{2}}.$$

Now, by taking into account that there exists $\kappa>0$ such that

$$\mu_1(F)\geqslant\frac{\mu_{m+1}(\mathcal{E})}{4\mu_m(\Gamma)}\geqslant e^{-\frac{\kappa}{\mu_{m+1}(\mathcal{E})}},$$

and by choosing $\gamma=\frac{1}{2\theta}\in(0,1)$, it follows that (3.8) holds, which ends the proof. □

Remark 4.6. *The generalization of the results in Theorem 1.1 and Proposition 4.5 to a domain Ω of arbitrary shape is an interesting open question. A result which may allow to tackle this issue has been recently obtained in Apraiz and Escauriaza [1]. In this work, the authors prove the null controllability of the heat equation by means of controls supported in a subset of positive measure of the domain Ω or of its boundary. The main new tool introduced in [1] is an inequality of the same nature as* (4.30)*, in which the eigenfunctions Φ_α of the Dirichlet Laplace operator in Ω are replaced by the solutions $e^{\pm\sqrt{\lambda_\alpha}y}\Phi_\alpha(x)$ of the elliptic equation*

$$\Delta_x z(x,y)+\partial_y^2 z(x,y)=0\qquad (x,y)\in\Omega\times\mathbb{R}.$$

5 Proof of the main result

The aim of this section is to prove our main result, Theorem 1.1. In order to achieve our objective we need to show that (3.8) holds. One of the key ingredients in the proof of (3.8) is the following Remez-type inequality, which has been proved in Theorem 2.1 from Borwein and Erdélyi [3] (see, also, [2]). Recall from the previous section that, for every $k\geqslant 1$, μ_k stands for the Lebesgue measure in $\mathbb{R}^k$.

Theorem 5.1. *Let $\nu_k:=k^\eta$, $k\in\{1,2,\dots\}$, $\eta>1$. Let $\rho\in(0,1)$, $\varepsilon\in(0,1-\rho)$ and $\varepsilon\leqslant 1/2$. Then there exists a constant $c_\eta>0$, depending only on η, such that*

$$\|p\|_{[0,\rho]}\leqslant\exp\left(c_\eta\varepsilon^{1/(1-\eta)}\right)\|p\|_E,$$

for every $p\in\operatorname{span}\{x^{\nu_1},x^{\nu_2},\dots\}$ and for every set $E\subset[\rho,1]$ of Lebesgue measure at least $\varepsilon>0$.

The above result has the following consequence:

Corollary 5.2. *For every $\tau>0$ there exist two constants $C,\ \kappa>0$ such that for every $F\subset[0,\tau]$ of positive Lebesgue measure the following inequality holds*

$$Ce^{\kappa/\mu_1(F)}\int_F\left|\sum_{k\geqslant1}a_ke^{-k^2t}\right|\,\mathrm{d}t\geqslant\left[\sum_{k\geqslant1}|a_k|^2e^{-k^2\tau}\right]^{\frac{1}{2}}\qquad ((a_k)\in\ell^2(\mathbb{C})).\tag{5.36}$$

Proof. If $(a_k)_{k\geqslant1}\in\ell^2(\mathbb{C})$, let us denote $f(t)=\sum_{k\geqslant1}a_ke^{-k^2t}$. Let $\rho=e^{-\tau}$ and let

$$E=\{x=e^{-t}\ \mid\ t\in F\}\subset[\rho,1].$$

Then

$$\mu_1(E) = \int_F e^{-t}\,\mathrm{d}t \geqslant e^{-\tau}\mu_1(F).$$

We can thus apply Theorem 5.1 (with $\eta = 2$) to obtain that there exists an absolute constant $c > 0$ such that

$$\|f\|_{[0,\tau]} \leqslant \exp\left(ce^{\tau}/\mu_1(F)\right)\|f\|_F. \tag{5.37}$$

Let $\mathcal{S} = \left\{t \in F \;\mid\; |f(t)| \geqslant \frac{2}{\mu_1(F)}\|f\|_{L^1(F)}\right\}$. It is easily seen that $\mu_1(\mathcal{S}) \leqslant \frac{\mu_1(F)}{2}$. Thus $\mu_1(F \setminus \mathcal{S}) \geqslant \frac{\mu_1(F)}{2}$ so that, by applying (5.37) in $F \setminus \mathcal{S}$, we deduce that

$$\|f\|_{[0,\tau]} \leqslant \exp\left(2ce^{\tau}/\mu_1(F)\right)\|f\|_{F\setminus\mathcal{S}} \leqslant \frac{2}{\mu_1(F)}\exp\left(2ce^{\tau}/\mu_1(F)\right)\|f\|_{L^1(F)}. \tag{5.38}$$

On the other hand, by using a classical result (see, for instance, [7, 9, 22, Corollary 3.6]), we deduce that there exists a constant $C > 0$, depending only of τ, such that

$$\|f\|^2_{[0,\tau]} \geqslant \frac{1}{\tau}\int_0^{\tau}\left|\sum_{k\geqslant 1} a_k e^{-k^2 t}\right|^2 \mathrm{d}t \geqslant C\sum_{k\geqslant 1}|a_k|^2 e^{-\tau k^2}. \tag{5.39}$$

From (5.38) and (5.39) it follows that (5.36) holds with positive constants C and κ depending only of τ and the proof ends. $\square$

Now we have all the ingredients needed to prove our main result.

Proof of Theorem 1.1. Recall from Section 2 that (1.1)-(1.4) may be written in the form (2.10), with (A, B) defined in Remark 2.2. In this case the spaces $X_{\frac{1}{2}}$ and $X_{-\frac{1}{2}}$ defined at the beginning of Section 3 are given by $X_{\frac{1}{2}} = L^2(\Omega)$, $X_{-\frac{1}{2}} = (H^2(\Omega) \cap H^1_0(\Omega))'$ (the dual space of $H^2(\Omega) \cap H^1_0(\Omega)$ with respect to the pivot space $L^2(\Omega)$). Moreover, $B \in \mathcal{L}(L^2(\Gamma), X_{-\frac{1}{2}})$ is an admissible control operator for the semigroup $\mathbb{T}$ generated by A. Therefore, according to Proposition 2.6, it suffices to show the L^∞ null controllability of the pair (A, B). This will be done by using Theorem 3.2. We recall that $B^* \in \mathcal{L}(L^2(\Omega), L^2(\Gamma))$ is given by

$$B^*\varphi = -\frac{\partial(-A)^{-1}\varphi}{\partial\nu}\Big|_{\Gamma} \qquad (\varphi \in L^2(\Omega)). \tag{5.40}$$

Without loss of generality we may suppose that Γ is an open subset of $\{x = (x', l_m) \in \partial\Omega\}$. In this case, μ is the $m-1$ dimensional Lebesgue measure.

Note that $(\Phi_\alpha)_{\alpha\in(\mathbb{N}^*)^m}$ defined by (4.24) is the complete family of orthonormal eigenvectors of the operator $-A$ and $\lambda_\alpha = \alpha_1^2 + \ldots + \alpha_m^2$ are the corresponding eigenvalues. Now, for any $\varsigma > 0$ and $\gamma \in (0,1)$, let $V_{\varsigma,\gamma} = \{\varphi = \sum_{\lambda_\alpha^\gamma \leqslant \varsigma} a_\alpha\Phi_\alpha\}$. For any $0 < a \leqslant s < t \leqslant \tau$, let $\mathcal{E} = \{(x,\sigma) \in e \;\mid\; s \leqslant \sigma \leqslant t\}$ be a set of positive measure and let $\mathcal{E}' = \{(x, \tau - \sigma) \;\mid\; (x,\sigma) \in \mathcal{E}\}$.

By using Lemma 3.4 we have that there exists a set $F \subset \{t \in [0,\tau] \mid (t,x) \in \mathcal{E}\}$ such that

$$\mu_1(F) > \frac{\mu_m(\mathcal{E})}{4\mu_{m-1}(\Gamma)}, \tag{5.41}$$

and, if $\sigma \in F$ then the $\sigma-$section E_σ has the property that

$$\mu_{m-1}(E_\sigma) > \frac{\mu_m(\mathcal{E})}{4\tau}. \tag{5.42}$$

By using (5.40), it follows that for every $\varphi \in V_{\varsigma,\gamma}$ we have

$$\|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])} = \int_0^\tau \int_\Gamma \chi_{\mathcal{E}'} |B^* \mathbb{T}^*_\sigma \varphi| \, \mathrm{d}x' \, \mathrm{d}\sigma \geqslant \int_F \int_{E_\sigma} |B^* \mathbb{T}^*_\sigma \varphi| \, \mathrm{d}x' \, \mathrm{d}\sigma$$

$$= \int_F \int_{E_\sigma} \left| \sum_{\lambda_\alpha^\gamma \leqslant \varsigma} a_\alpha \frac{(-1)^{\alpha_m} \alpha_m \pi}{l_m \lambda_\alpha} \Phi_{\alpha'}(x') e^{-\lambda_\alpha \sigma} \right| \mathrm{d}x' \, \mathrm{d}\sigma = \int_F \int_{E_\sigma} \left| \sum_{\alpha' \in \mathcal{D}_m} b_{\alpha'}(\sigma) \Phi_{\alpha'}(x') \right| \mathrm{d}x' \, \mathrm{d}\sigma,$$

where $\mathcal{D}_m = \left\{ \alpha' \in (\mathbb{N}^*)^{m-1} \mid |\alpha'|^2 \leqslant \varsigma^{\frac{1}{\gamma}} - 1 \right\}$ and

$$b_{\alpha'}(\sigma) = \sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}} - |\alpha'|^2}} a_\alpha \frac{(-1)^{\alpha_m} \alpha_m \pi}{l_m \lambda_\alpha} e^{-\lambda_\alpha \sigma} \qquad (\alpha' \in \mathcal{D}_m). \tag{5.43}$$

From Corollary 4.4, by taking $N = \varsigma^{\frac{1}{2\gamma}}$ and using (5.42), we deduce that there exists a constant $C = C(\Gamma, m)$ such that

$$\|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])} \geqslant \int_F \left\{ C \varsigma^{-\frac{m-2}{4\gamma}} \left(\frac{C}{\mu_{m-1}(E_\sigma)} \right)^{-(m-1)\left(2\varsigma^{\frac{1}{2\gamma}}+1\right)} \left(\sum_{\alpha' \in \mathcal{D}_m} |b_{\alpha'}|^2 \right)^{\frac{1}{2}} \right\} \mathrm{d}\sigma$$

$$\geqslant C \varsigma^{-\frac{m-2}{4\gamma}} \left(\frac{4C}{\mu_m(\mathcal{E})} \right)^{-(m-1)\left(2\varsigma^{\frac{1}{2\gamma}}+1\right)} \int_F \left(\sum_{\alpha' \in \mathcal{D}_m} |b_{\alpha'}|^2 \right)^{\frac{1}{2}} \mathrm{d}\sigma.$$

We deduce that

$$\|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])} \geqslant C \varsigma^{-\frac{2m-3}{4\gamma}} \left(\frac{4C}{\mu_m(\mathcal{E})} \right)^{-(m-1)\left(2\varsigma^{\frac{1}{2\gamma}}+1\right)} \sum_{\alpha' \in \mathcal{D}_m} \int_F |b_{\alpha'}| \, \mathrm{d}\sigma. \tag{5.44}$$

In order to find a lower bound for the integral over the measurable set $F \subset [0,\tau]$, we use Corollary 5.2. We have that

$$\int_F |b_{\alpha'}| \, \mathrm{d}\sigma \geqslant e^{-|\alpha'|^2 \tau} \int_F \left| \sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}} - |\alpha'|^2}} a_\alpha \frac{(-1)^{\alpha_m} \alpha_m \pi}{l_m \lambda_\alpha} e^{-\alpha_m^2 \sigma} \right| \mathrm{d}\sigma$$

$$\geqslant \frac{1}{C} \exp\left(-\frac{\kappa}{\mu_1(F)} \right) e^{-|\alpha'|^2 \tau} \left[\sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}} - |\alpha'|^2}} \frac{|a_\alpha|^2 \alpha_m^2 \pi^2}{l_m^2 \lambda_\alpha^2} e^{-\alpha_m^2 \tau} \right]^{\frac{1}{2}}.$$

By taking into account (5.41), we deduce that

$$\int_F |b_{\alpha'}|\,\mathrm{d}\sigma \geqslant \frac{1}{C}\exp\left(-\frac{4\kappa\mu_{m-1}(\Gamma)}{\mu_m(\mathcal{E})}\right)\left[\sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}}-|\alpha'|^2}}\left|\frac{a_\alpha}{\lambda_\alpha}\right|^2 e^{-2\lambda_\alpha\tau}\right]^{\frac{1}{2}}. \tag{5.45}$$

Finally, from (5.44) and (5.45) it follows that

$$\begin{aligned}
&\|\Psi^d_{\tau,\mathcal{E}'}\varphi\|_{L^1(\Gamma\times[0,\tau])}\\
&\geqslant C\varsigma^{-\frac{m-2}{4\gamma}}\left(\frac{4C}{\mu_m(\mathcal{E})}\right)^{-(m-1)\left(2\varsigma^{\frac{1}{2\gamma}}+1\right)} e^{-\frac{4\kappa\mu_{m-1}(\Gamma)}{\mu_m(\mathcal{E})}}\varsigma^{-\frac{1}{\gamma}}\sum_{\alpha'\in\mathcal{D}_m}\left[\sum_{\alpha_m=1}^{\sqrt{\varsigma^{\frac{1}{\gamma}}-|\alpha'|^2}}\left|\frac{a_\alpha}{\lambda_\alpha^{\frac{1}{2}}}\right|^2 e^{-2\lambda_\alpha\tau}\right]^{\frac{1}{2}}\\
&\geqslant \frac{1}{d_0}e^{-d_1\left(1+\ln\left(\frac{1}{\mu_m(\mathcal{E})}\right)\right)\varsigma^{\frac{1}{2\gamma}}+\frac{\kappa}{\mu_m(\mathcal{E})}}\left[\sum_{\lambda_\alpha^\gamma\leqslant\varsigma}\left|\frac{a_\alpha}{\lambda_\alpha^{\frac{1}{2}}}\right|^2 e^{-2\lambda_\alpha\tau}\right]^{\frac{1}{2}}.
\end{aligned}\tag{5.46}$$

Choosing $\gamma = \frac{1}{2}$, we conclude that (A, B) satisfies condition (3.8) in Theorem 3.2, which concludes the proof of Theorem 1.1. □

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