Symmetric Games with networking applications

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***Abstract*—In their seminal paper [1], Orda, Rom and Shimkin have already studied fully symmetric routing games, i.e. games in which all players have the same sources, destinations, demands and costs. They established the uniqueness of an equilibrium in these games. We extend their result to weaker forms of symmetry, which does not require a common source or destination. Considering routing games, we provide conditions under which whenever there is some symmetry between some players, then any equilibrium necessarily has these symmetry property as well. We then extend the symmetry result to general games.**

I. Introduction

Symmetry is a phenomena frequently encountered in game applications, such as in telecommunication networks; often there are some agents (users) that have similar utilities and similar "views" of the network. Such potential symmetry can have various implications. First it can be used to conclude some qualitative properties of the game: existence and/or uniqueness of equilibria, and even symmetry in the equilibrium strategy. It can further be exploited for computation purposes in order to obtain simple algorithms for computing equilibria. This is indeed desirable as it is well known that the determination either analytically or numerically of a Nash equilibrium can be a very challenging problem due to complexity issues.

Already in [1], the authors have highlighted the role of symmetry in routing games and showed that networks with general topologies have unique symmetric equilibria if the game is symmetric and the cost satisfies some convexity properties. The notion of symmetry was stronger than the one we are interested in: it required in particular, all players to have the same source and destination. We begin by establishing uniqueness results for routing games under more general concepts of symmetry that do not have the above restrictions. We then extend the scope to symmetric properties of equilibria in arbitrary games.

II. Routing games

We consider here an atomic game, in the sense that there are a finite (or perhaps a discrete) number of decision makers. Yet the game is splitable: each decision maker can decide what fraction of its demand would be routed on each link. So far, uniqueness was established for these games either for some very simple topologies (the case of parallel links and its generalizations [2],[3]) or under specific conditions on the cost functions (see e.g. [4]). In case of general topologies and costs, Orda et al. have shown that the equilibrium is unique if all players are equal (same source, destination, demand etc). The general set of conditions that are sufficient and necessary for the uniqeness of equilibrium remain an open problem, alghough some considerable progress has been done since [1], see [2].

Let $G = (\mathcal{N}, \mathcal{L}, \mathcal{I}, \mathcal{P})$ be a network routing game with $\mathcal{N}$ the set of nodes and $\mathcal{L}$ the set of links, $\mathcal{I}$ is the set of classes (e.g. players), and $\mathcal{P} = (s^i, d^i, \phi^i)$ is a set that characterizes class i: s^i is the source, d^i is the destination and ϕ^i is the demand related to player i.

We describe the system with respect to the variables x_l^i which denote the amount of flow that player i sends over link l. They are restricted by the non-negativity constraints for each link l and player i: $x_l^i \geq 0$ and by the conservation constraints for each player i and each node v:

$$r_v^i + \sum_{j \in In(v)} x_j^i = \sum_{j \in Out(v)} x_j^i \qquad (1)$$

where $r_v^i = \phi^i$ if v is the source node for player i, $r_v^i = -\phi^i$ if v is its destination node, and $r_v^i = 0$

otherwise; $In(v)$ and $Out(v)$ are respectively all ingoing and outgoing links of node v. (ϕ^i is the total demand of player i).

A player i determines the routing decisions for all the traffic that corresponds to the corresponding class i. The cost of player i is assumed to be additive over links

$$J^i(\mathbf{x}) = \sum_l J^i_l(\mathbf{x}_l). \tag{2}$$

Here $\mathbf{x}_l = (x^i_l,\ i \in \mathcal{I})$ and $\mathbf{x} = (\mathbf{x}_l,\ l \in \mathcal{L})$.

We shall assume that
(i) $K^i_l := \frac{\partial J^i_l(\mathbf{x})}{\partial x^i_l}$ exist and are continuous in x^i_l (for all i and l),
(ii) J^i_l are convex in x^i_l (for all i and l),

We shall often make the following assumption for each link l and player i:

A1: J^i_l depends on $\mathbf{x}_l$ only through the total flow $x_l = \sum_i x^i_l$ and the flow of x^i_l of player i over the link.
A2: J^i_l is increasing in both arguments
A3: Whenever J^i_l is finite, $K^i_l(x_l, x^i_l)$ is strictly increasing in both arguments.

Sometimes we further restrict the cost to satisfy the following:

B1: For each link l there is a nonnegative cost density $T_l(x_l)$ T_l is a function of the total flow through the link and $J^i_l = x^i_l T_l(x_l)$.
B2: T_l is positive, strictly increasing and convex, and is continuously differentiable.

The Lagrangian $L^i(\mathbf{x}, \lambda)$ with respect to the constraints on the conservation of flow is

$$\sum_{l\in\mathcal{L}} J^i_l(\mathbf{x}_l) + \sum_{v\in\mathcal{N}} \lambda^i_v \left(r^i_v + \sum_{j\in In(v)} x^i_j - \sum_{j\in Out(v)} x^i_j \right),$$

for each player i.

Below we shall use uv to denote the link defined by node pair u, v.

Thus a vector $\mathbf{x}$ with nonnegative components satisfying (1) for all i and v is an equilibrium if and only if the following Karush-Kuhn-Tucker (KKT) condition holds:

There exist Lagrange multipliers λ^i_u for all nodes u and all players, i, such that for each pair of nodes u, v connected by a directed link uv,

$$K^i_{uv}(x^i_{uv}, x_{uv}) \geq \lambda^i_u - \lambda^i_v, \tag{3}$$

with equality if $x_{uv} > 0$.

Assume cost structure B. Then, the Lagrangian $L^i(\mathbf{x}, \lambda)$ is given by

$$\sum_{l\in\mathcal{L}} x^i_l T_l(x_l) + \sum_{v\in\mathcal{N}} \lambda^i_v \left(r^i_v + \sum_{j\in In(v)} x^i_j - \sum_{j\in Out(v)} x^i_j \right),$$

for each player i.

(3) can be written as

$$T_{uv}(x_{uv}) + x^i_{uv} \frac{\partial T_{uv}(x_{uv})}{\partial x^i_{uv}} \geq \lambda^i_u - \lambda^i_v. \tag{4}$$

Definition 1. *(Symmetry in Routing games). Introduce a map $\Theta : \mathcal{N} \to \mathcal{N}$ and some permutaiton π of $\mathcal{I}$.*

Define the game $G' = \Gamma(G, \Theta, \pi) = (\mathcal{N}', \mathcal{L}', \mathcal{I}', \mathcal{P}')$ as follows.

- *$\mathcal{N}' = \mathcal{N}$, $\mathcal{I}' = \mathcal{I}$.*
- *For each player i in the game G, characterized by $\mathcal{P}^i = (s^i, d^i, \phi^i)$ we have for the corresponding player $i' = \pi(i)$ in the game G' characterized by $\mathcal{P}^{i'} = (s^{i'}, d^{i'}, \phi^{i'})$:*
  $s^{i'} = \Theta(s^i)$, $d^{i'} = \Theta(d^i)$, and $\phi^{i'} = \phi^i$.
- *A link $u'v'$ is in $\mathcal{L}'$ if and only if $u' = \Theta(u)$, $v' = \Theta(v)$ for some $uv \in \mathcal{L}$, and $J^{i'}_{u'v'} = J^i_{uv}$ for $i' = \pi(i)$, $i \in \mathcal{I}$.*

If for a given game G, there exists a permutation π such that $G = \Gamma(G, \Theta, \pi)$ then we say that G is symmetric under Θ.
A routing policy $\mathbf{x}$ is symmetric under Θ if for all i, l, $x^i_l = x^{\pi(i)}_{\Theta(l)}$.

A. Uniqueness result

We shall show that symmetric games have only symmetric equilibria. We use here type A cost.

Indeed, let $\mathbf{x}$ be an equilibrium in a symmetric game with Θ and π. and assume that for some i, l, $x^i_l \neq x^{\pi(i)}_{\Theta(l)}$.

Construct a network $\hat{G}$ as following. The nodes are the same as in G. If for some link $m = uv$, $D_m \neq 0$ where

$$D_m = x^i_m - x^{\pi(i)}_{\Theta(m)}$$

then m is included in $\hat{G}$. It is assigned an amount $|D_m|$ of flow that goes from u to v if $D_m > 0$, and from v to u otherwise.

Note that the flow conservation constraints hold. Indeed, at any node u which is not a source nor a destination, since the conservation constraints hold in the original game for that node, they also hold for node $\Theta(u)$ and thus also to their difference. If node u is the source or destination node for player i then so is node $\Theta(u)$ for player $\pi(i)$. Therefore the difference between the flows corresponding to these two players at u and $\Theta(u)$ also satisfy the flow conservation.

It follows from the assumption in the beginning of the proof that the flow on link l in the new network is strictly nonnegative. This, together with the fact that the new graph has no exogeneous source nor destination, imply that there is a closed circuit (sequence of links) ξ that contains link l such that all links on that circuit carry strictly positive flow. For any link $m = uv$ in this circuit we have

$$0 < K^i_{uv}(x^i_{uv}, x_{uv}) - K^{\pi(i)}_{\Theta(uv)}(x^{\pi(i)}_{\Theta(uv)}, x_{i\Theta(uv)})$$

$$= \lambda_u^i - \lambda_v^i - (\lambda_{\Theta(u)}^{\pi(i)} - \lambda_{\Theta(v)}^{\pi(i)})$$

The first inequality holds since K is strictly monotone in its first component. Taking the sum over all links in ξ, we get $0 < 0$ which is impossible. This concludes the proof by contradiction. ∎

III. Routing Game on the circle: transmit or relay

We shall introduce an example of symmetric routing games, in which symmetry holds in a weaker sense than in [1]. They are related to a circular network in which there is symmetry under a rotation as depicted in Figure 1.

Consider a circle with nodes on the integers $1, 2, ..., N$. With each node we associate a distinct player. Node i has a demand of ϕ^i to ship to a destination common to all nodes. It has two paths for shipping traffic: either directly to the destination, using a link whose cost function is f, or use the next node, $i+1$, on the circle. Forwarding it to the next node prior to the final destination costs an aditional fixed amount of d. We denote by g the derivative of f. $i+1$ will be understood below in the cyclic sense so that $i+1 = 1$ when $i = N$.

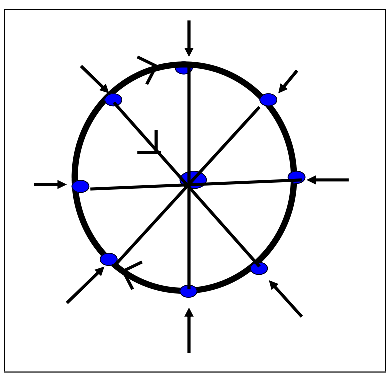

Fig. 1. Competitive routing on the circle: a common destination

The cost for player i is

$$J^i(\mathbf{x}) = x_i^i f(x_i) + (\phi^i - x_i^i)(f(x_{i+1}) + d) \qquad (5)$$

where $i = 1, ..., N$. If x_i^i is an interior point, i.e. it is neither on the boundary ϕ nor on that of 0, then a necessary condition for x_i^i to be a minimum is that the following partial derivative equals zero:

$$\frac{\partial J^i(\mathbf{x})}{\partial x_i^i} = f(x_i) + x_i^i g(x_i) - (\phi^i - x_i^i) g(x_{i+1}) - f(x_{i+1}) - d$$

Hence

$$x_i^i = \frac{d + f(x_{i+1}) - f(x_i) + \phi^i g(x_{i+1})}{g(x_i) + g(x_{i+1})} \qquad (6)$$

so that

$$\phi^i - x_i^i = \frac{-d - f(x_{i+1}) + f(x_i) + \phi^i g(x_i)}{g(x_i) + g(x_{i+1})}$$

and also

$$x_{i+1}^{i+1} = \frac{d + f(x_{i+2}) - f(x_{i+1}) + \phi^{i+1} g(x_{i+2})}{g(x_{i+1}) + g(x_{i+2})}$$

Taking the sum, we obtain the load of link $i+1$,

$$x_{i+1} = \frac{-d - f(x_{i+1}) + f(x_i) + \phi^i g(x_i)}{g(x_i) + g(x_{i+1})} + \frac{d + f(x_{i+2}) - f(x_{i+1}) + \phi^{i+1} g(x_{i+2})}{g(x_{i+1}) + g(x_{i+2})}$$

where $i = 1, ..., N$. One of the equations can be replaced by $\sum_{i=1}^N (x_i - \phi^i) = 0$.

Next assume that all players send their flow on their direct link. We check whether player i can do better. Differentiating the cost function (5) w.r.t. x_i^i and substituting $x_i^i = \phi^i$ yields

$$\frac{\partial J^i(\mathbf{x})}{\partial x_i^i} = f(\phi^i) + \phi^i g(\phi^i) - f(\phi^{i+1}) - d$$

Sending all flow on the direct link is an equilibrium if this is non-positive.

In particular, if ϕ^i are equal, then sending all traffic through direct links is an equilibrium provided that $\phi g(\phi) \le d$.

The homogeneous case.

Assume $\phi^i = \phi$ does not depend on i. At equilibrium we always have $x_i = \phi$. Indeed, we know from the last section that only symmetric equilibria exist. Therefore x_i do not depend on i. Combining it with the flow constraint results in $x_i = \phi$. This then implies that

$$x_i^i - (\phi - x_i^i) - \frac{d}{g(\phi)} = 0$$

So that

$$x_i^i = \frac{\phi}{2} + \frac{d}{2g(\phi)}$$

Note that a necessary condition for this to be an equilibrium is that $x_i^i \ge \phi$ or equivalently that

$$d \le \phi g(\phi).$$

IV. Symmetry in general games

Consider a game with a set $\mathcal{I}$ of I players. Let $\mathcal{A}^i \subset \mathbb{R}^{n(i)}$ be a convex set of strategies available to player i. Denote by $\mathcal{A}$ the set of multistrategies $\{\mathbf{a} = (a^1, ..., a^I), i \in \mathcal{A}^i\}$. Let J^i be the cost for player i, and let $\mathbf{J}$ be the vector of the I payoffs of the players.

Next, we define symmetry. Introduce a permutation π on the set $\mathcal{I}$. Define the correspoding operator $\Theta(\pi)$ that maps $\mathcal{A}$ to $\mathcal{A}' = (\mathcal{A}^{\pi(1)}, \cdots, \mathcal{A}^{\pi(I)})$ as follows:

$$\Theta(\pi)\mathbf{a} = (a^{\pi(1)}, ..., a^{\pi(I)}).$$

A game is said to be symmetric under a permutation π if for each player i and multistrategy $\mathbf{a}$,

$$J^i(\mathbf{a}) = J^{\pi(i)}(\Theta(\pi)\mathbf{a}).$$

We have the following obvious Theorem

Theorem 1. *Suppose that we have a game that is symmetric under a permutation π.*
Then if $\mathbf{a}$ is an equilibrium then so is $\Theta(\pi)\mathbf{a}$.

A. Two symmetric players

Consider a game which is symmetric under a permutation of a single pair of players, i.e. the permutation that only alters between i and j, i.e. $\pi(i) = j$, $\pi(j) = i$ and $\pi(k) = k$ for $k \notin \{i, j\}$.

Assumption 1. *Consider a game which is symmetric under a permutation of a single pair of players, i and j. Assume that for any multi-strategy $\mathbf{a}$, the following holds. If the best response for player i against $\mathbf{a}^{-i}$) is a^i and $a^i \neq a^j$, then the best response of player j to the strategy $\mathbf{a}^{-j}$ is different than a^j.*

Remark 1. *Assume that J are differentiable and that $\mathcal{A}^i$ and $\mathcal{A}^j$ are open. A sufficient condition for Assumption 1 to be satisfied is that*

$$\frac{\partial J^i(\mathbf{a})}{\partial a^i} \neq \frac{\partial J^j(\mathbf{a})}{\partial a^j} \tag{7}$$

This condition appears to be easier to verify.

Theorem 2. *Suppose that Assumption 1 holds. Then any Nash equilibrium is symmetrical, under π, i.e. if $\mathbf{a}^*$ is an equilibrium then so is $\Theta(\pi)\mathbf{a}^*$.*

Proof:
This follows by contradiction, obtained by combining Theorem 1 with Assumption 1. ∎

Theorem 3. *Consider a game which is symmetric under a permutation of a single pair of players, i and j. Let $n_i = n_j = 1$ (i.e. $\mathcal{A}^i$ and $\mathcal{A}^j$ are scalars). Then the following is a sufficient condition for Assumption 1 to hold:*
J^i is a strictly convex function in the direction $d_{ij} = e_i - e_j$ where e_k is the unit vector whose kth entry is 1 and the rest are zero. In other words,

$$J^i(\mathbf{a} + \theta d_{ij}) - J^i(\mathbf{a})$$

is increasing in θ. Equivalently,

$$\frac{\partial J^i(\mathbf{a})}{\partial a^i} \geq \frac{\partial J^i(\mathbf{a})}{\partial a^j}$$

with equality holding only if $a^i = a^j$.

B. Submodular game

Consider a two player game.

A cost function J^i is said to be strictly submodular if for any different actions a and b, the utility satisfies

$$J^i(b, b) + J^i(a, a) - J^i(a, b) - J^i(b, a) > 0$$

Assume that (a, a) and (b, b) are different symmetrical equilibria, we have

$$J^i(b, b) - J^i(a, b) \leq 0, \quad \textit{and}$$

$$J^i(a, a) - J^i(b, a) \leq 0$$

Summing the two last equations gives a contradiction to the submodularity assumption. We conclude that there may not exist more than one symmetric equilibrium in a submodular game.

This is also another example of a sufficient condition for Assumption 1, and hence for Theorem 2 to hold.

C. Flow control game

Consider a network with I sources, with a potential demand of ϕ^i for class i. Let the link cost J_l^i satisfy **A** and **B**. For each player i, we consider a fixed routing policy $\mathbf{x}^i$ i.e. a set $x_l^i, l \in \mathcal{L}$ that satisfy the flow-conservation constraints (1). We assume that a player i can control its total demand which is given by $\alpha^i\phi^i$, where α^i, the decision variable of player i is assumed to lie within some interval $[\underline{\alpha}, \overline{\alpha}]$. The proportions of the demand that a player sends to each link are equal to those that are used according to $\mathbf{x}^i$; they are assumed to be fixed and not to depend on α^i. Thus player i sends $\alpha^i x_l^i$ over link l.

Let the cost for player i be given as

$$J^i(\alpha) = \sum_l \alpha^i x_l^i T_l(\sum_{i'} \alpha^{i'} x_l^{i'}) - Q(\alpha^i \phi^i)$$

The first term is the same delay cost that we had in the routing games. The second term represents a cost that is a function of the demand: Q is assumed to be a concave increasing function.

Now,

$$\frac{\partial J^i(\alpha)}{\partial \alpha^i} = \sum_l x_l^i T_l(\sum_{i'} \alpha^{i'} x_l^{i'}) + x_l^i T_l'(\sum_{i'} \alpha^{i'} x_l^{i'})$$
$$-\phi^i Q'(\alpha^i \phi^i)$$

which is strictly increasing in α^i. Using Remark 1 one can obtain the following:

Theorem 4. *Assume that the flow control game is symmetric under some Θ, for some routing policy x. Then any equilibrium is symmetric under Θ.*

Note: there is a standard way to transform a flow control game into a routing game with a fixed demand. Thus the above theorem would follow from corresponding results in routing game, such as we saw in Section II-A.

V. Concluding Section

We investigated in this paper symmetric games starting with routing games and then extending to general non-cooperative games. We provided simple conditions under which symmetric properties of the costs in a game imply the same symmetric properties in any equilibria.

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