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HAL Id: hal-00647231

https://hal.inria.fr/hal-00647231

Submitted on 1 Dec 2011

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ABSTRACT

This article address the identification problem of the natural frequency and the damping ratio of a second order continuous system where the input is a sinusoidal signal. An algebra based approach for identifying parameters of a Mass Spring Damper (MSD) system is proposed and compared to the Kalman-Bucy filter. The proposed estimator uses the algebraic parametric method in the frequency domain yielding exact formula, when placed in the time domain to identify the unknown parameters. We focus on finding the optimal sinusoidal exciting trajectory which allow to minimize the variance of the identification algorithms. We show that the variance of the estimators issued from the algebraic identification method introduced by Fliess and Sira-Ramirez is less sensitive to the input frequency than the ones obtained by the classical recursive Kalman-Bucy filter. Unlike conventional estimation approach, where the knowledge of the statistical properties of the noise is required, algebraic method is deterministic and non-asymptotic. We show that we don't need to know the variance of the noise so as to perform these algebraic estimators. Moreover, as they are non-asymptotic, we give numerical results where we show that they can be used directly for online estimations without any special setting.

Keywords: Parameter estimation; Recursive algorithm; Kalman-Bucy algorithm; Forgetting factor; Algebraic approach; Laplace transform; Operational calculus; Leibniz formula; Integral rules; Filtering.

1. INTRODUCTION

Since a wide large of mechanical systems are modeled through coupled or isolated Mass Springer Damper systems, the estimation problem of the MSD parameters is classic in nature. Moberg et al [1] have modeled a 2 link of an ABB industrial robot based on serial MSD system for each axis. This is done aiming to simplify the related elastic dynamic equations. Also, a double mass model of an elastic cam mechanism was described in [2], that gives a more realistic idea of the relationship in mass distribution in the process. The method introduced in this article concerns the parameters estimation problem based on a new algebraic method introduced by Fliess and Ramirez [3] and compare to a conventional algorithm proposed through the Kalman-Bucy filter in parameter estimation. These algebraic parametric estimation techniques for linear systems [4] have been extended for various problems in signal processing for example [5],[6],[7],[8],[9],[10]. Let us emphasize that those methods, which are non-asymptotic, exhibit excellent robustness properties with respect to corrupting noises, without the need of knowing their statistical properties [19]. We propose to apply this algebra based approach for identifying parameters of a Mass Spring Damper (MSD) excited by a sinusoidal input. Similar approach is proposed in [8], however the novelty of this article, is to compare two types of identification algorithms based on finding the optimal input solution in order to well and quickly identify the mechanical system parameters. We perform a numerical study to obtain the optimal solution in case when a wave generator is used as excitation signal. The optimal input signal design depends on two parameters : frequency ω_1 and the amplitude that gives the best training exciting trajectory. We compare the results to the ones obtained via a classical recursive approach [11], [12], [13], [14]. In particular, this method is compared to a weighted Kalman-Bucy filter [13] in order to show the robustness and the efficiency of the proposed technique where measurements are corrupted by a noise. We study the effect of a Gaussian noise added to the output on the estimators variance [15]. This is performed by taking the sampling period into account. We focus on optimal input excitation in order to maximize the convergence rate of estimators based on minimum variance analysis [16]. Hence, we compare the algebraic method variance with the one of the Kalman-Bucy filter. This variance analysis allows us to show that contrary to the recursive approach [17], [18], the algebraic method is less sensitive to the value of the exiting frequency input and more robust to the corrupted noise. Moreover, the Kalman-Bucy approach needs the knowledge of the statistical properties of the noise that is not required for the algebraic method [19]. We show that this method is also robust even for a high frequency sinusoidal disturbance. Online identification of the unknowns undamped angular frequency and the damping ratio is devoted. The identified parameters are obtained in finite time and the noise effect is attenuated by the iterated integrals. Numerical results show the accuracy of the estimation and the best training signal design.

The outline of this paper is as follows. Section 2 describes the problem statement. Section 3 contains a variance analysis of identified parameters thorough the Kalman-Bucy algorithm and the corresponding exciting trajectory design. Mathematical framework for algebraic parameters estimation is presented in Section 4 which also contains the estimation methods following from the rules of operational calculus. Simulations and comparative analysis of estimators are proposed in Section 5, while Section 6 concludes the paper.

2. PROBLEM STATEMENT

Consider a Mass Spring Damper (MSD) system defined by the following continuous-time second-order system :

$$\ddot{x}(t) + 2\zeta\omega_0\dot{x}(t) + \omega_0^2 x(t) = u(t), \qquad (1)$$

where $u(t)$ is the input, ω_0 is the natural undamped frequency and $\zeta \in [0,1]$ is the damping ratio. One can note that physically $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped angular frequency of the mechanical system and $\zeta = \frac{c}{2\sqrt{km}}$ be the damping ratio; Where c is the viscous damping coefficient, k denotes the spring constant and m the mass of the load. We set $\theta_1 = 2\zeta\omega_0$, $\theta_2 = \omega_0^2$ and $u(t) = A_1\sin(\omega_1 t)$. Let $\tilde{x}_i = x_i + \varpi_i$ be a noisy observation of the "true" position $x_i = x(t_i)$ of the system at $t_i = iT_s$ for $i = 0,\ldots,N$. The real value T_s denotes the sampling period. We assume that ϖ is an additive noise corruption which is a second order continuous stochastic process with zero-mean and a known variance σ^2. Consequently, we search the values of ω_1 which allows us to estimate θ_1 and θ_2 with the minimum variance for a given time estimation.

3. KALMAN-BUCY FILTER ESTIMATORS

A. Introduction

This section aims to use the Kalman-Bucy filter [13] so as to estimate the vector $\Theta = (\theta_1, \theta_2)^T$ which is involved in the motion equation (1). In order to quickly identify these parameters through an optimal designed sinusoidal input, a variance analysis of the estimator is described in the following. This will allow us to optimally choose the values of A_1 and ω_1. The input sequence $(u_i)_{i=1,\ldots,N}$ and the output sequence $(\tilde{x}_i)_{i=1,\ldots,N}$ are measured synchronously at the sampling period T_s. Consequently, we obtain the following linear relations from these measurements :

$$Y_k = X_k\Theta + \rho_k, \text{ with } m < k \leq N, \qquad (2)$$

where the regression matrix $X_k = \left(\begin{array}{cc} (\dot{x}_i)_e & \tilde{x}_i \end{array} \right)_{i=m+1,\ldots,k}$, the observed signal vector $Y_k = (u_i - (\ddot{x}_i)_e)_{i=m+1,\ldots,k}$ and $(\dot{x}_i)_e$ (resp. $(\ddot{x}_i)_e$) is the velocity estimation (resp. acceleration estimation) at $t_i = iT_s$. We assume that ρ_k is a sequence of independent Gaussian variables with zero mean and known variance σ_ρ^2 issued from the variance estimators due to both of the measurement noises ϖ and the derivative estimation errors. Moreover, the integer m is the minimum value needed so as to calculate $(\dot{x}_i)_e$ and $(\ddot{x}_i)_e$. Usually, these estimators are computed through a filtered finite numerical differentiator [22],[23].

From now on, the problem is to estimate Θ based on the measurements and the observed signal vector. We consider the situation when the observations are obtained one-by-one from the process. We would like to update the parameters estimate whenever new observation to the previous set of observations. In what follows, a recursive formulation is derived. Instead of recomputing the estimates with all available data, the previous parameters estimate are updated with the new data sample. In order to do this, the Kalman-Bucy filter is written in the form of a recursive algorithm. The recursive algorithm is given by the following structure:

$$\begin{cases} K_{k+1} &= P_k X_{k+1}^T (R_{k+1} + X_{k+1} P_k X_{k+1}^T)^{-1}, \\ \alpha_{k+1} &= Y_{k+1} - X_{k+1}\hat{\Theta}_k, \\ \hat{\Theta}_{k+1} &= \hat{\Theta}_k + K_{k+1}\alpha_{k+1}, \\ P_{k+1} &= \lambda^{-1}(P_k - K_{k+1}X_{k+1}P_k), \end{cases} \qquad (3)$$

where $\hat{\Theta}_k$ is the parameters estimation vector after the first $k-$samples and $\lambda \in]0,1]$ is a forgetting factor which reduces the influence of old data. In particular, if $\lambda = 1$, then all the data are taken into account in the same manner. In this algorithm (3), one notes that the vector Θ_k and the matrix P_k are involved in the recursions. In order to initialize the algorithm, we must provide initial values for these variables. We choose to apply the Ordinary Least Square solution of this identification problem by using a "small" samples of the first m-measures $(\tilde{x}_i)_{i=1,\ldots,k}$ to compute

$$\hat{\Theta}_m = P_m B_m, \quad \text{where } \begin{cases} P_m = (X_m^T R_m^{-1} X_m)^{-1}, \\ B_m = X_m^T R_m^{-1} Y_m. \end{cases} \qquad (4)$$

Let us denote

$$\alpha(i) = k - \max\{i - m, k\} \text{ for } i \in \{m+1,\ldots,k\} \qquad (5)$$

After $k \geq m$ stacked samples, by applying recursions (3) initialized with (4), one can recursively obtain the following estimation

$$\hat{\Theta}_k = \frac{\sum_{i=m+1}^k \lambda^{\alpha(i)} X_i Y_i}{\sum_{i=m+1}^k \lambda^{\alpha(i)} X_i^2} \qquad (6)$$

with $K_k = \dfrac{X_k}{\sum_{i=m+1}^k \lambda^{\alpha(i)} X_i^2}$ and $P_k = \dfrac{\sigma_\rho^2}{\sum_{i=m+1}^k \lambda^{\alpha(i)} X_i^2}$. $\qquad (7)$

B. Variance analysis

In this subsection, we are interested in the variance analysis of the estimation (6), aiming to find the input trajectory $u(t)$ i.e. the values of $(A_1)_{opt}$, and $(\omega_1)_{opt}$, which allow to minimize the variance of (6). The value $(\omega_1)_{opt}$ is investigated in term of the optimal ratio $Z_{opt} = \frac{(\omega_1)_{opt}}{\omega_0}$.

Besides, for small values of ζ, the dynamic equation (1) can be simplified by neglecting the damping effect based on a numerical simulation of the differential equation. For example, in Fig 1, we compare the difference between the exact solutions of (1) with $\zeta = 0.0021$ and $\zeta = 0$.

This will be used so as to simplify the variance analysis. Also, this approximation will take place only in order to
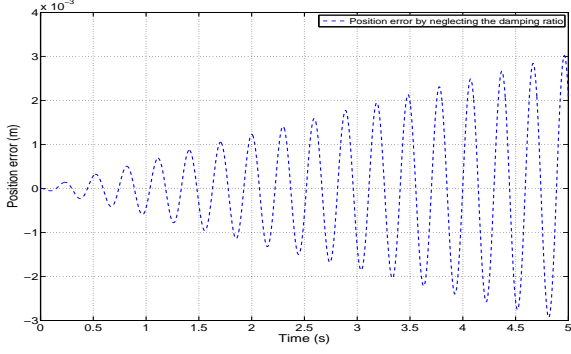
Fig. 1. Error of the exact solution of $x(t)$ with $\zeta = 0.0021$ (real damping ratio) and $\zeta = 0$ and a given sinusoidal exciting trajectory

perform the Kalman-Bucy filter variance of Θ, $Var(\Theta)$, in term of the ratio $Z = \dfrac{\omega_1}{\omega_0}$. This is done in order to find a variance expression of the recursive estimator. However, Kalman-Bucy algorithm in parameter estimation will be rebuilt, by means of (2) and (3), in order to estimate the unknowns parameters θ_1 and θ_2 based on the calculated variance expression. Under this assumption, in order to perform the variance expression, Θ is limited to the scalar variable θ_2. Moreover, the regression matrix X_k can be rewritten $X_k = (\tilde{x}_i)_{i=m+1,\ldots,k}$. The explicit solution of this reduced differential equation becomes :

$$x(t) = \frac{A_1 \left[\omega_1 \sin(\omega_0 t) - \omega_0 \sin(\omega_1 t) \right]}{\omega_0 (\omega_1^2 - \omega_0^2)}. \tag{8}$$

We denote $P_k = ((X_k R_k^{-1} X_k)^T)^{-1}$, where R_k is a diagonal matrix

$$R_k = diag(\underbrace{r_1, \ldots, r_{k-m}}_{k-m \text{ times}}), \tag{9}$$

with the $r_j > 0$ and $e_k = Y_k - X_k \hat{\Theta}_{k-1}$ is the a priori error of estimation. Consequently, the Kalman-Bucy filter consists of two stages. The first part employs an estimate $\hat{\Theta}_k$ using the information already available at time k and the second part provides the main time-update made by the innovation process (a priori errors), denoted α_{k+1} in (3), in order to estimate $\hat{\Theta}_{k+1}$ from measurements Y_{k+1}, regression X_{k+1} and $\hat{\Theta}_k$.

In fact, ρ_k depicts a white noise vector with zero mean and it is defined by the following autocorrelation function

$$\mathbb{E}[\rho(t)\rho^*(t-\tau)] = \begin{cases} \sigma_\rho^2, & \tau = 0, \\ 0, & \tau \neq 0. \end{cases} \tag{10}$$

Concerning the matrix P_k, it represents the variance-covariance matrix of the estimation error.

$$P_k = cov[e_k] = \mathbb{E}[(\hat{\Theta}_k - \Theta)^T (\hat{\Theta}_k - \Theta)].$$

At this stage, the developments below, will be based on the Kalman-Bucy algorithm with a fixed variance, i.e., for any $k \geq m$, $r_{k-m} = \sigma_\rho^2$.

Therefore, by applying the linearity property of the variance, we obtain the above variance expression of (6)

$$Var(\hat{\Theta}_k) = \frac{\sigma_\rho^2 \sum\limits_{i=m+1}^{k} \lambda^{2\alpha(i)} X_i^2}{\left(\sum\limits_{i=m+1}^{k} \lambda^{\alpha(i)} X_i^2 \right)^2}. \tag{11}$$

Relation (11) can be expressed by using the explicit solution (8), as follows

$$Var(\hat{\Theta}_k) = \frac{\sigma_\rho^2}{A_1^2} K(Z, \lambda, \omega_0, T_s, m, k)$$

where

$$K(Z, \lambda, \omega_0, T_s, m, k) =$$
$$\frac{(\omega_0^2 (Z^2 - 1))^2 \sum\limits_{i=m+1}^{k} \lambda^{2\alpha(i)} (Z \sin(\omega_0 t_i) - w_0 \sin(Z\omega_0 t_i))^2}{\left(\sum\limits_{i=m+1}^{k} \lambda^{\alpha(i)} (Z \sin(\omega_0 t_i) - \omega_0 \sin(Z\omega_0 t_i))^2 \right)^2}. \tag{12}$$

Hence, the minimization of the variance of the Kalman-Bucy estimator may be obtained by increasing the magnitude A_1 of the input force. However, this strategy is naturally restricted by some physical limits. Concerning the variable ω_1 i.e. the ratio $Z = \dfrac{\omega_1}{\omega_0}$, it will be explained in next subsection.

C. Influence of the forgetting factor λ

In a first series of experiments, we investigate the influence of the forgetting factor λ on the value of $K(Z, \lambda, \omega_0, T_s, m, k)$, Fig 2. In fact, Fig 3 shows the logarithm value of $K(Z, \lambda, \omega_0, T_s, m, k)$ according to a discretized value of Z belonging to $[0.01, 2]$ where the sampling period $T_s = 0.001$ s, $k = 100$ and $m = 3$. A set of different values of the forgetting factor $\lambda = \{0.95, 0.98, 0.99, 1\}$ is choosen. As we can see, $\lambda = 1$ is always the optimal value for our application.
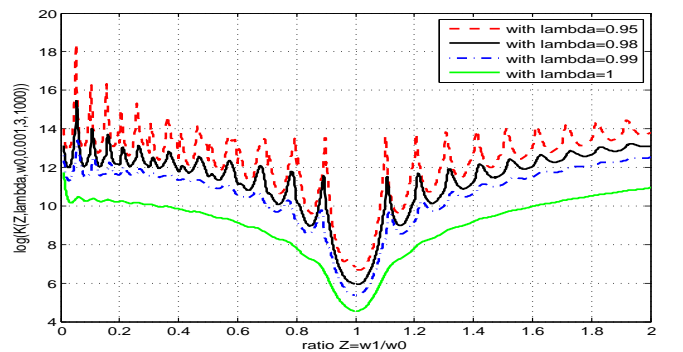

Fig. 2. Influence of the forgetting factor λ in variance analysis

D. The optimal input trajectory

Consequently, when $\lambda = 1$, we have

$$K(Z, \omega_0, T_s, m, k) = \frac{\omega_0^4 (Z^2 - 1)^2}{\left(\sum\limits_{i=m+1}^{k} (Z \sin(\omega_0 t_i) - \sin(Z\omega_0 t_i))^2 \right)}. \tag{13}$$

A Taylor series at $Z = 1$ allows us to conclude that the minimum value is obtained for $Z = 1$ i.e. $(\omega_1)_{opt} = \omega_0$. Figure 3 depicts the value of $K(Z, \omega_0, T_s, m, k)$ (13), according to Z for different numbers k of samples. The other parameters are the same than those used in the previous subsection.
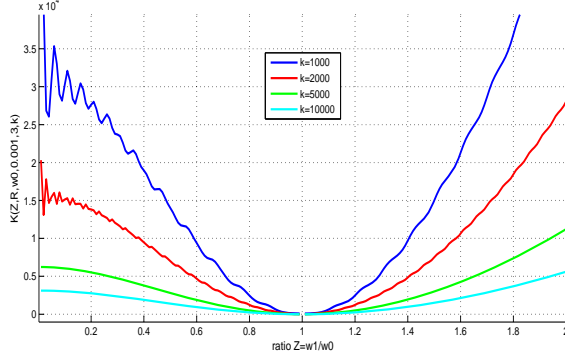

Fig. 3. Influence of the ratio $Z = \frac{\omega_1}{\omega_0}$ for optimal trajectory design

Figure 3 shows that the sensibility of the variance is quite important in the neighborhood of $Z_{opt} = 1$. In conclusion using an input trajectory as closed as possible to the natural frequency of the system, we can consequently minimize the variance of the Kalman-Bucy estimator.

4. ALGEBRAIC PARAMETRIC ESTIMATOR

In this section, we provide the interested reader with rigorous mathematical development in which the algebraic parameter estimation technique, used in this article for the estimation problem, is based. The fundamental developments are based on the module theoretic approach to linear systems [3], [5], [6].

A. Mathematical framework: Generalized expressions of parameters estimation

Set $k = k_0\{\Theta\}$, where k_0 is considered as real, or complex differential field and $\Theta = (\theta_1, ..., \theta_r)$ a finite set of *unknowns parameters* which might not be constant. The unknown parameters $\Theta = (\theta_1, ..., \theta_r)$ are said to be linearly identifiable if, and only if,

$$P(t) \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} = Q(t) + R(t), \tag{14}$$

where

- the entries of the matrices $P(t)$ and $Q(t)$, of respective sizes $r \times r$ and $r \times 1$ belong to $span_{k_0(t)[\frac{d}{dt}]}(u, y)$ where (u, y) denotes respectively the vector of inputs and outputs of systems;
- $det(P(t)) \neq 0$;
- R is a $r \times 1$ matrix with entries in $span_{k_0(t)[\frac{d}{dt}]}(\pi)$ which designed the disturbance contribution.

B. Algorithm

We consider the dynamic of a system satisfies the inputs-outputs relation:

$$\sum_{i=0}^{n} a_i \, y^{(i)} = \sum_{i=0}^{m} b_i \, u^{(i)} \tag{15}$$

where $a_n = 1$, $m < n$. The algebraic parametric estimator is derived using the following steps:

- We apply the Laplace transform of (15):

$$\sum_{i=0}^{n} a_i \, (s^i y(s) - s^{i-1} y_0 - ... - y_0^{(i-1)})$$
$$= \sum_{i=0}^{m} b_i \, (s^i u(s) - s^{i-1} u_0 - ... - u_0^{(i-1)}) \,. \tag{16}$$

One note the onset of the initial conditions to the $(n-1)^{th}$ order involved in (16).

- By applying n times the derivative operator with respect to s, we can annihilate the initial conditions. This step will be of a great advantage by eliminating these conditions since they are usually unknown. Hence, we obtain algebraic parametric estimator independent from initial conditions. It is aimed to estimate the parameters a_i and b_i in a fast way and on the basis possibly noisy measurements. For this exact expressions of the parameters are derived as a function of the integral of the output and the input, through the following inverse Laplace transform :

Proposition 4.1 *Let Γ be a causal real continuous function and t_0 be a strictly positive real value then for any positive real $T \leq t_0$, $m \in \mathbb{N}^*$ and $n \in \mathbb{N}$ we have*

$$\mathscr{L}^{-1}\left(\frac{1}{s^m} \frac{d^n \hat{\Gamma}(s)}{ds^n}\right)(T) =$$
$$\frac{(-1)^n T^{m+n}}{(m-1)!} \int_0^1 (1-\tau)^{m-1} \tau^n \gamma(t_0 - \tau T) \, d\tau, \tag{17}$$

where $\hat{\Gamma}(s)$ is the Laplace transform of the continuous function $\Gamma(t) = \gamma(t_0 - t)$.

Proof: By applying the Cauchy formula, we obtain for any $T \geq 0$,

$$\mathscr{L}^{-1}\left(\frac{1}{s^m} \frac{d^n \hat{\Gamma}(s)}{ds^n}\right)(T) = \frac{(-1)^n}{(m-1)!} \int_0^T (T-u)^{m-1} u^n \Gamma(u) \, du.$$

If we assume that for any $u \leq t_0$, $\Gamma(u) = \gamma(t_0 - u)$ then by substituting u by τT we obtain

$$\int_0^T (T-u)^{m-1} u^n \Gamma(u) \, du = T^{m+n} \int_0^1 (1-\tau)^{m-1} \tau^n \gamma(t_0 - \tau T) \, d\tau.$$

Consequently (17) holds. ∎

By now on, we set

$$P_{m,n,T}^{(\gamma)}(t_0) = \frac{(-1)^n T^{m+n}}{(m-1)!} \int_0^1 (1-\tau)^{m-1} \tau^n \gamma(t_0 - \tau T) \, d\tau, \tag{18}$$

where γ is either the system output $y(t)$ or the input $u(t)$ and $T > 0$ is the time length of the sliding window estimation. Let us denote t_0 is the initial step time for each sliding window i.e. time estimation T along the simulation time vector t. This estimation time T may be small especially in the absence of noise. Meanwhile, T cannot obviously be taken arbitrary small even in a noise-free context. A lower bound for T has been formally characterized in [[19], Prop. 3.2], within the framework of nonstandard analysis.

C. Application

Applying the previous rules to (1), we obtain an explicit formula for the estimates $\hat{\theta}_1$ of θ_1 and $\hat{\theta}_2$ of θ_2, as a function of the estimation time t_0 on a sliding windows of length T. We firstly apply the Laplace transform to the differential equation (1). This gives the following equation:

$$s^2 X(s) - sx(0^+) - \dot{x}(0^+) + \theta_1(sX(s) - x(0^+))$$
$$+ \theta_2 X(s) = u(s).$$

In order to eliminate the initial conditions $x(0^+)$ and $\dot{x}(0^+)$, we apply the derivative operator with respect to s two times. It leads to:

$$2X(s) + 4s\frac{dX(s)}{ds} + s^2\frac{d^2X(s)}{ds^2} + \theta_1\left[s\frac{d^2X(s)}{ds^2} + 2\frac{dX(s)}{ds}\right]$$
$$+ \theta_2\frac{d^2X(s)}{ds^2} = \frac{d^2U(s)}{ds^2}. \tag{19}$$

Multiplying the equation (19) by $s^{-\mu}$, $\mu \geq 3$, and applying the inverse Laplace transform (18), we obtain a set of linear equations in the unknown parameters $\Theta = (\theta_1 \quad \theta_2)^T$ in the time domain. It is expressed in terms of a linear combination of iterated convolution integrals over $x(t)$ and $u(t)$. Consequently

$$\begin{pmatrix} \hat{\theta}_1(t_0) \\ \hat{\theta}_2(t_0) \end{pmatrix} = \begin{pmatrix} 2P^{(x)}_{\mu,1,T}(t_0) + P^{(x)}_{\mu-1,2,T}(t_0) & P^{(x)}_{\mu,2,T}(t_0) \\ 2P^{(x)}_{\mu+1,0,T}(t_0) + P^{(x)}_{\mu,1,T}(t_0) & P^{(x)}_{\mu+1,2,T}(t_0) \end{pmatrix}^{-1}$$
$$\begin{pmatrix} -2P^{(x)}_{\mu,0,T}(t_0) - 4P^{(x)}_{\mu-1,1,T}(t_0) - P^{(x)}_{\mu-2,2,T}(t_0) + P^{(u)}_{\mu,2,T}(t_0) \\ -2P^{(x)}_{\mu+1,0,T}(t_0) - 4P^{(x)}_{\mu,1,T}(t_0) - P^{(x)}_{\mu-1,2,T}(t_0) + P^{(u)}_{\mu+1,2,T}(t_0) \end{pmatrix} \tag{20}$$

As in [6], we could get another estimators by applying a derivation to (19) before applying the $\frac{1}{s}$ operator. For an experimental design of the algebraic estimator, a discretization of the integral in (18) will be held by using the Simpson's rule * [20].

$$P^{(\gamma)}_{m,n,T}(t_0) = \frac{(-1)^n}{(m-1)!}T^{m+n}\int_0^1 (1-\tau)^{m-1}\tau^n\gamma(t_0 - \tau T)d\tau$$
$$\approx \frac{(-1)^n}{(m-1)!}T^{n+m}\frac{Ts}{3}\left[\left((1-\tau)^{m-1}\tau^n\gamma(t_0 - \tau T)\right)_{(0)}\right.$$
$$+ 2\sum_{j=1}^{L/2-1}\left((1-\tau)^{m-1}\tau^n\gamma(t_0 - \tau T)\right)_{(2j)}$$
$$+ 4\sum_{j=1}^{L/2}\left((1-\tau)^{m-1}\tau^n\gamma(t_0 - \tau T)\right)_{(2j-1)}$$
$$\left. + \left((1-\tau)^{m-1}\tau^n\gamma(t_0 - \tau T)\right)_{(n)}\right] \tag{21}$$

where L represents the sampling window T length in samples: $L = \frac{T}{T_s}$. Therefore, let us quote the following remarks:

- The estimation time T may be small, resulting in fast estimation.

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*Simpson's rule is employed aiming to reduce the numerical integration error compared to the trapezium rule and not to complicate the numerical implementation.

- The noise effect is attenuated by the iterated integrals (low pass filter).
- The computational complexity is low.

5. SIMULATIONS AND COMPARATIVE ANALYSIS

Computer simulations were carried out with the Matlab-Simulink software. Simulations are achieved on the dynamic equation

$$\ddot{x}(t) + \theta_1\dot{x}(t) + \theta_2 x(t) = A_1 sin(\omega_1 t) \tag{22}$$

where $x(t)$ is corrupted by a noise with zero-mean and a known variance. This stochastic signal is built by means of sequence of random variables by the instruction *awgn* in the Matlab package which adds white Gaussian noise to the vector signal $x(t)$. A step sampling of $T_s = 0.001s$ is used. The noise level is measured by the signal to noise ratio in dB, i.e., $SNR = 10\ log_{10}\ (\frac{\sum_i |x(t_i)|^2}{\sum_i |\rho(t_i)|^2})$. Simulations are achieved for a spring value $k = 400\ N/m$, a damping coefficient $c = 0.05\ N.s/m$ and a load mass $m = 3\ kg$. Concerning the sinusoidal input $u(t)$, the signal amplitude A_1 was chosen so that it will be the maximum allowed with respect to the limited physical properties of the system. In our case, A_1 is set to $333.3333N$. Fig 4 shows the noisy position for t_i ranging from 0 to 5 seconds. Although, the linear time invariant (LTI) MSD system (1) is discretized in order to perform the identification algorithms for each sampling time.

We note that in this section, most of figures depict the natural frequency estimation and are limited to $\omega_0 = \sqrt{\theta_2}$.
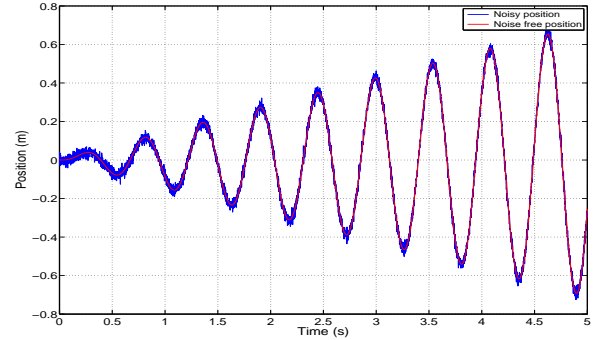

Fig. 4. Noisy Position $x(t)$ with $SNR = 25dB$ for a given sinusoidal input

A. Robustness Analysis

In order to compare the performance of the proposed algebra-based method with the Kalman-Bucy filter, we generate numerical simulations with high level noises which allows us to illustrate the robustness of the parameters estimators involved in (1) with respect to the SNR in dB and the ratio $Z = \frac{\omega_1}{\omega_0}$. Both of estimators algorithms were carried out around two important quantities that reflect the robustness and the performance of the identification methods: signal-to-noise ratio and Z. Fig 5 and 6 depict the weightiness of both SNR and Z in the convergence of each estimation algorithm.
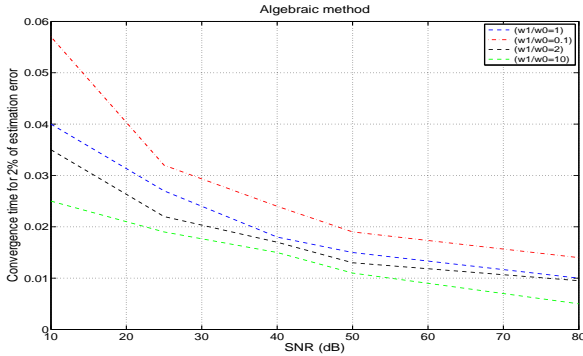
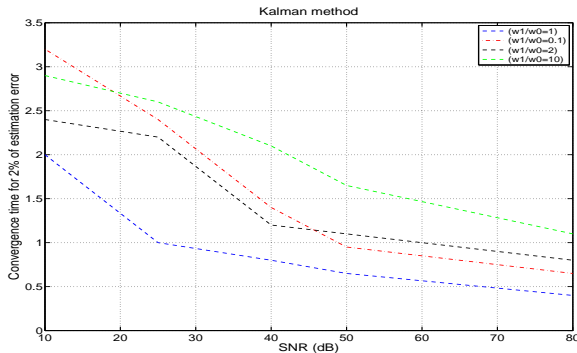Fig. 5. Convergence time (s) for 2 % of estimation error with Algebraic technique

Fig. 7. Algebraic method estimates for ω_0 with SNR = $25dB$

Fig. 6. Convergence time (s) for 2 % of estimation error with Kalman-Bucy algorithm

Fig. 8. Algebraic method estimates for ω_0 with SNR = $50dB$

Fig. 9. Algebraic method estimates for ω_0 with SNR = $80dB$

We assume that the time-estimation is stopped when the absolute value error estimation is less than 2%. Consequently, one can note that the algebraic technique converges as well as possible with respect to the rapidity in time when the period of the signal $u(t)$ is 10 times less than natural period of the MSD system. Moreover, the computation time of ω_0 decreases whenever the SNR and $\frac{\omega_1}{\omega_0}$ is increased. As we can see, for a $SNR = 80$ dB and an angular frequency ratio Z = 10, ω_0 is computed in 0.005s when the sampling time T_s is 0.001s. Fig 7, 8 and 9 depict the algebraic estimation of the natural frequency ω_0 with the presence of a noise effect using the algorithm represented by equations (20). One can note that peaks in Fig 7 are generated from numerical artifacts due to the implementation of (21) and does not exceed 0.0035% of estimation error .We can conclude that this algorithm is non-asymptotic and the noise contribution is attenuated by the presence of iterated integrals after the numerical discretization through the Simpson's rule (21).
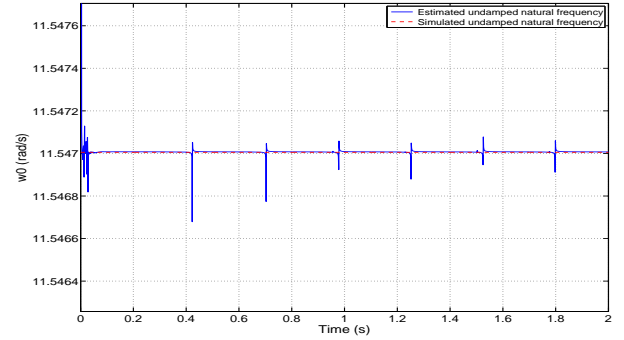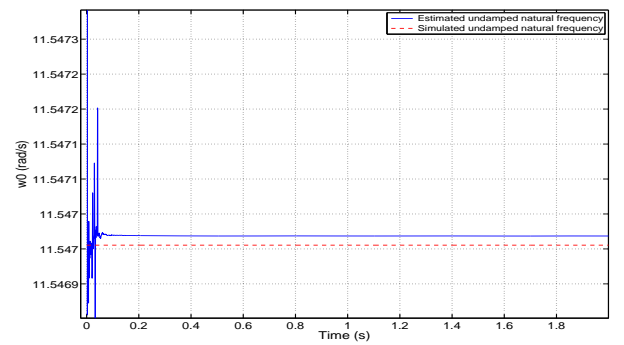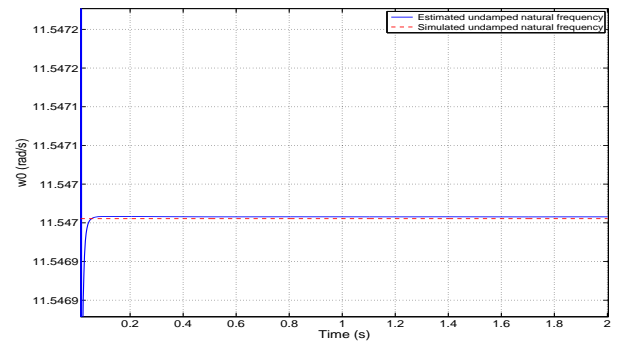
The Kalman-Bucy filter is performed based on the variance analysis as illustrated in section 3.B. This is done through the minimization of the variance expression (11). Fig 3 depicts $Var(\Theta)$ according to $\frac{\omega_1}{\omega_0}$. In fact, as it was shown in Fig 6, $Var(\Theta)$ is minimum when $\omega_1 = \omega_0$. Besides, from Fig 10, 11 and 12 we can conclude that the convergence time decreases when the signal-to-noise ratio in dB increases. However, the convergence time in case of Kalman-Bucy filter is 100 times more as compared to the algebra-based approach for a given SNR. It should be emphasized that for recursive algorithm, the convergence is made asymptotically. From this, it was noted that algebraic technique presents an online-estimator due to the quickness of the parameter computation.
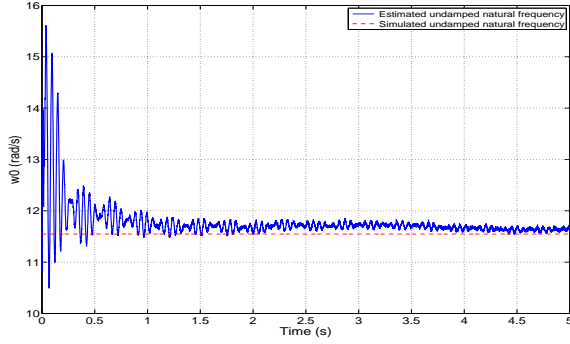
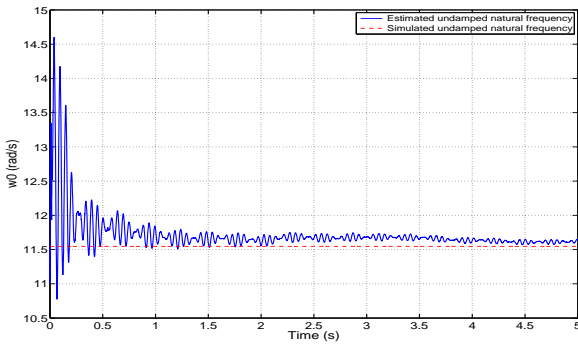Fig. 10. Kalman-Bucy algorithm estimates for ω_0 with SNR = $25dB$

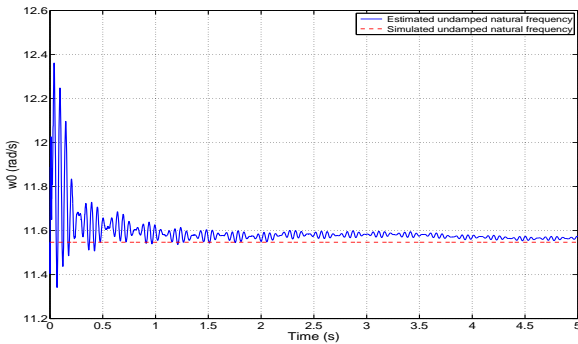Fig. 11. Kalman-Bucy algorithm estimates for ω_0 with SNR = $50dB$

Fig. 12. Kalman-Bucy algorithm estimates for ω_0 with SNR = $80dB$

B. High frequency sinusoidal perturbation

This section devoted a numerical simulation to evaluate the performance of the proposed algebraic approach compared to the Kalman-Bucy algorithm on a current perturbed position. Unfortunately, this type of experiment involves a severe tempering of the compared estimation algorithms. We consider that the measured position $x(t)$ is corrupted by another sinusoidal perturbation with a higher frequency generator (which is not satisfy the sampling limit) and a white noise process $\rho(0, 0.001)$ with a high signal-to-noise ratio. This can be expressed as $\widetilde{x}(t) = x(t) + \rho(t) + A_2 sin(\omega_2 t)$ where, $A_2 = 0.1$ and $\omega_2 = 500\omega_0$ (Fig 15). The experiments are performed with the optimal conditions for the Kalman-Bucy ($\omega_1 = \omega_0$) and with $\omega_1 = 10\ \omega_0$ for the algebra-based algorithms. Fig (14) and (15) depict the estimation of the angular frequency with presence of

a higher frequency sinusoidal. We note that, even the "true" position is highly corrupted Kalman-Bucy filter and the proposed algorithm converge. For the algebra-based technique, the estimations are achieved in about $5 \times T_s$ and $50 \times T_s$ for the recursive algorithm for a 2% of estimation error. It should be noted that the convergence time is faster than for highly Gaussian noisy measurement. Indeed, the robustness of the obtained estimations with respect to the unknown measurement noise is quite high.
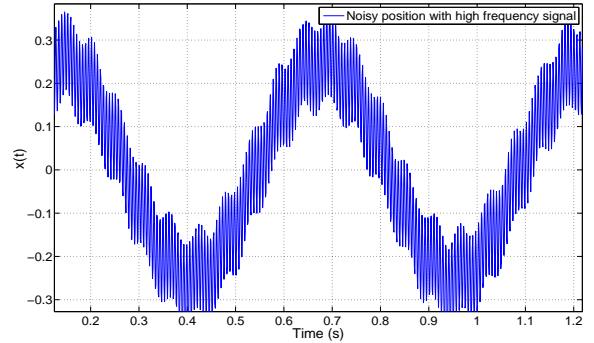

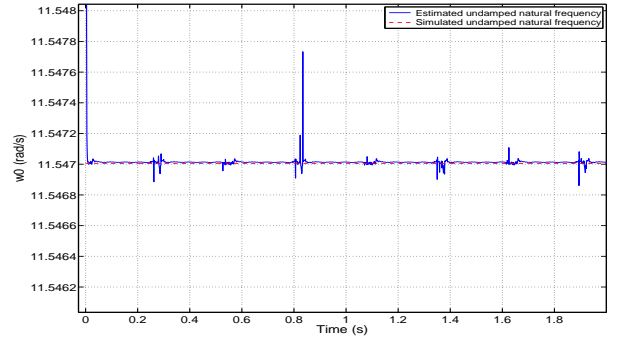Fig. 13. Position measurement with a high frequency sinusoidal noise

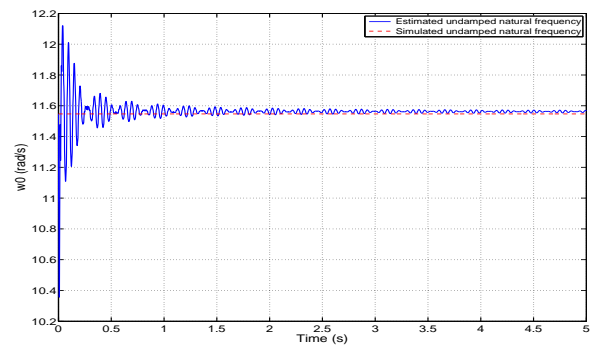Fig. 14. Algebraic method estimates for ω_0 with SNR = $80dB$ and high frequency sinusoidal noise

Fig. 15. Kalman filter estimates for ω_0 with SNR = $80dB$ and high frequency sinusoidal noise

C. Variable parameter estimation

There are many applications where the involved parameters vary in time due to behavior of the system or some physical change [21]. For example, due to the thermal effect, the angular frequency of the MSD system may change with respect to time. This is explained through the fluctuation of spring constant k or the viscous damping coefficient c. To evaluate the performance of the algebraic

algorithm, we made an interesting experience where we have simulated the variation of ω_0 with a discontinuity change-point. However, the system still LTI in that range where ω_0 is constant. Therefore, we can apply directly our algebraic algorithm so as to estimate different abrupt changes of the values of $\omega_0 = \sqrt{\frac{k}{m}}$.

Fig 16 shows the performance of the estimation approach where the first jump is carried for $t = 0.1$ s. That result proves the accuracy of the proposed algorithm even if the unknown parameter is time varying with a specific behavior.
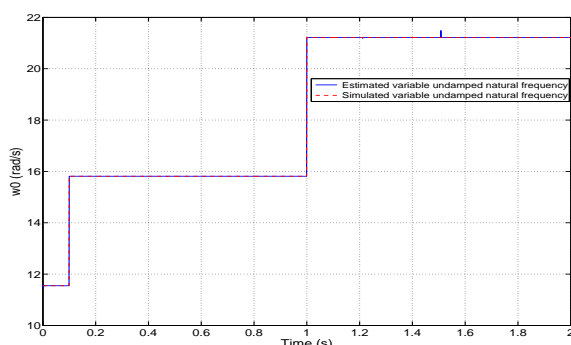

Fig. 16.   Algebraic methods estimates for a variable ω_0

6. CONCLUSION

The objective of this paper has been to study optimal sinusoidal input design so as to minimize the variance parameters estimation of a range of mechanical system. We have presented an algebraic approach to the fast and reliable identification of dynamical parameters of a Mass Spring Damper system, compared to a conventional algorithm introduced by the Kalman-Bucy filter in parameter estimation. The calculation of the characteristic parameters of these mechanical systems was interesting for many reasons touching the engineering theme. As it known, a lot of mechanical structures are modeled via coupled or isolated MSD systems aiming to simplify both of their static and dynamic behaviors . It results an important problem in the control theory such as feedback and feed-forward control where the involved parameters are unknown and should be identified for each time step.

By analyzing the variance estimators, contrary to the Kalman-Bucy filter, we show that the proposed algebraic approach is less sensitive to the choice of the input frequency and is more robust to additive noise on the output. In this study, the numerical differentiation of the output signal employed for the recursive algorithm, was a simple finite difference technique with a low pass filtering. This latter has an influence in the robustness and fastness of the identification for small SNR. Such problems can be minimized by using numerical algebraic differentiators (see [7], [22], [23]). Moreover, we can also directly address the real-time identification of the parameters, where the computational complexity is low as it shown is in the algebra-based approach.

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