The Proportional Coloring Problem: Optimizing Buffers in Radio Mesh Networks

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Abstract

In this paper, we consider a new edge coloring problem to model call scheduling optimization issues in wireless mesh networks: the proportional coloring. It consists in finding a minimum cost edge coloring of a graph which preserves the proportion given by the weights associated to each of its edges. We show that deciding if a weighted graph admits a proportional coloring is pseudo-polynomial while determining its *proportional chromatic index* is NP-hard. We then give lower and upper bounds for this parameter that can be computed in pseudo-polynomial time. We finally identify a class of graphs and a class of weighted graphs for which the proportional chromatic index can be exactly determined.

Keywords: edge coloring, proportional coloring, mesh networks, call scheduling.

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1 Introduction

Given a weighted graph (G, w), where w is a non-negative weight function over E, several distinct coloring problems on G have been defined. In [4], one wants to color the vertices of G using a given number of colors while minimizing the sum of the weights of the edges whose extremities receive the same color. On the other hand, in [10], one wants to color the vertices of G so that for each edge uv, $|c(v) - c(u)|$ is at least the weight of uv, where $c(u)$ and $c(v)$ are the colors assigned to u and v. There is also the problem of finding a partition of the edges of G into matchings (colors), each one of weight equal to the maximum weight of its edges, so that the total weight of the partition is minimized [14].

In this paper, we consider a new edge coloring problem: the proportional edge coloring. Given a weighted simple graph (G, w), where w is a function called *weight function* defined on the edges of G and taking value between 0 and 1 (i.e. $w : E \rightarrow [0, 1]$), we want to find a proper coloring, assigning one or more colors to each edge, which preserves the proportion given by the weights associated to each edge. If such a coloring exists, we want to find one using a minimum number of colors. We proved that deciding if a weighted graph admits a proportional coloring is pseudo-polynomial while determining its proportional chromatic index is NP-hard. We then give a lower bound and an upper bound for this parameter, both polynomial time computable. Finally, we exhibit a class of graphs and a class of weighted graphs for which we can exactly determine the proportional chromatic index.

The remainder of this section is dedicated to introduce the terminology used in this paper as well as some classical results on the proper edge coloring problem and the fractional edge coloring problem which will be useful in what follows.

Every graph in this paper is simple, which means that it is finite, with no loops nor multiple edges, except when it is clearly stated. In this case, it will be called a multigraph. We denote a weighted graph by (G, w), where $w : E \rightarrow [0, 1]$ is the weight function. The maximum degree of G is denoted by $\Delta(G)$ (or simply Δ, if G is clear from the context). A proper edge coloring

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of G is an assignment of colors to the edges of G such that no adjacent edges have the same color. Observe that if G is properly edge colored, each set of edges which are assigned a same color induces a matching of G.

The classical edge coloring problem is to determine the edge chromatic index of a simple graph G, $\chi'(G)$, that is, the minimum integer k such that G admits a proper edge coloring using k colors.

Equivalently, the edge chromatic index can be stated as an integer linear program. Let $\mathcal{M}$ be the set of all the matchings M of G. A proper edge coloring of G is an assignment of a non-negative integer weight p to every matching $M \in \mathcal{M}$, such that for every edge $e \in E(G)$, $\sum_{M:e\in M} p(M) \geq 1$. Then, the edge chromatic index $\chi'(G)$ is the minimum value given by $\sum_{\forall M\in\mathcal{M}} p(M)$ over all proper edge colorings of G.

In 1964, Vizing proved that $\chi'(G)$ is at most $\Delta + 1$ [16]. Since it is at least Δ, we can classify every graph: a graph is *Class 1* if its edge chromatic index is Δ and *Class 2* otherwise. Surprisingly, deciding if a graph is Class 1 or 2 is hard [6], even for cubic graphs [8]. In contrast, Padberg and Rao [12] have shown that the fractional relaxation of the integer linear program stated before can be solved in polynomial time, despite the fact that it has an exponential number of variables. The optimal solution of the fractional relaxation is known as the fractional edge chromatic index of G, denoted by $\chi'^*(G)$, whose formal definition follows.

Definition 1.1(i) Let $\mathcal{M}$ be the set of all the matchings M of G. A fractional edge coloring is an assignment of a non-negative weight p to each matching $M \in \mathcal{M}$, so that for each edge we have: $\sum_{M:e\in M} p(M) \geq 1$. Thus, $\chi'^*(G)$ is the minimum $\sum_{\forall M\in\mathcal{M}} p(M)$ over all fractional edge colorings of G;

(ii) For any natural number $k \geq 1$, let G^k be the graph obtained from G by replacing each edge by k parallel edges. Then

$$\chi'^*(G) = \min_{k\geq 1} \frac{\chi'(G^k)}{k}.$$

Observe that in an optimal solution of the relaxed linear program, the weights of the matchings may not be integers. Therefore, since for every edge e, $\sum_{M:e\in M} p(M) \geq 1$, it may force an edge to belong to several weighted matchings. In terms of coloring, it means that an edge could be assigned several colors. Alternatively, the fractional edge chromatic index of a graph G can also be seen as the problem of determine integers a and b, such that we can assign a colors to each edge of G of a set of b colors, respecting the requirement that there are no two adjacent edges with a same color and such that $\frac{b}{a}$ is minimum.

Putting these observations together with Vizing's result we have that:

$$\Delta(G) \leq \chi'^{*}(G) \leq \chi'(G) \leq \Delta(G) + 1$$

A characterization stating that bipartite graphs may be decomposed into perfect matchings, due to Frobenius [5], has as an immediate corollary: for every bipartite graph G, $\Delta(G) = \chi'^{*}(G) = \chi'(G)$. In fact, Reed proved that this equality holds also for nearly bipartite graphs (graphs containing a vertex whose removal produces a bipartite graph) [13].

The upper bound for the proportional chromatic index presented in this work is closely related to the fractional edge chromatic index. To the reader interested in fractional edge coloring theory, we recommend [15].

2 Proportional coloring

The *proportional coloring problem* is formally presented in Definition 2.1 and illustrated in Figure 1. Before giving it, we define $\mathcal{P}(S)$ to be the set of all subsets of a given set S.

Definition 2.1 [c-proportional coloring] Given a weighted graph (G, w), a *c-proportional coloring* of (G, w) is a function $\mathcal{C} : E \to \mathcal{P}(\{1, \dots, c\})$ such that for all $e \in E$, we have: $|\mathcal{C}(e)| \geq cw(e)$ and for all pairs of edges $e = (uv), f = (vw)$ for some nodes u, v, w, we have $\mathcal{C}(e) \cap \mathcal{C}(f) = \emptyset$. We call the *proportional chromatic index* of G, $\chi'_{\pi}(G, w)$, the minimum number c of colors for which a c-proportional coloring of G exists. If it does not exist, then $\chi'_{\pi}(G, w) = \infty$. By proportional coloring, we mean a c-proportional coloring for some c.

To link the proportional chromatic index to the fractional edge chromatic index, we will use a multigraph, the m-graph, constructed from a weighted graph (G, w) and a positive integer m, whose definition follows.

Definition 2.2 [m-graph] Given a weighted graph (G, w) and a positive integer m, the m-graph G_m is obtained from G by replacing each edge $e = (uv) \in E(G)$ by $\lceil mw(e) \rceil$ multiple edges uv.

Definition 2.3 [$mcd(w)$, mcd] Given a weighted graph (G, w), with w taking value in $\mathbb{Q}$, we set $mcd(w)$ as the minimum common denominator of all the values taken by w. If there are no ambiguities, we call it mcd. It can be equivalently defined as the lowest common multiple of the denominators of all the values taken by w.

Remark 2.4 Given a weighted graph (G, w) and a positive integer m. Suppose there exists a proper edge coloring of G_m using a multiple of m colors, say km colors for some integer k, such that every edge receives the same number of colors. Then, this proper edge coloring can be easily transformed into a proportional coloring of (G, w) with km colors (cf Figure 1).

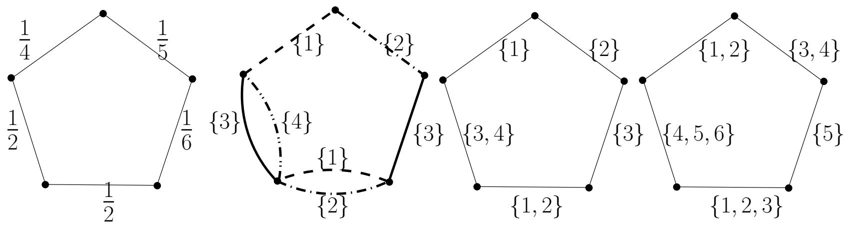

Fig. 1. A weighted C_5, a proper edge coloring of $(C_5)_4$ using 1×4 colors, the same coloring transformed into an optimal proportional coloring of the weighted C_5 and a suboptimal proportional coloring of the weighted C_5 using 6 colors.

Figure 1 is an example of a weighted graph and its proportional coloring. The proportional coloring problem is divided into two subproblems: the first one consists in proving that, for some integer c, there exists a c-proportional coloring of (G, w). The second one is to determine the proportional chromatic index of (G, w).

We now give simple facts that enable us to say if a weighted graph has a finite or an infinite proportional chromatic index and eventually to bound it.

Fact 2.5 *Let (G, w) be a weighted graph.*

i If there is a vertex u of G with $\sum_{uv \in E} w(uv) > 1$, then $\chi'_\pi(G, w) = \infty$.

ii If for all $uv \in E$, $w(uv) \leq 1/(\Delta + 1)$, then $\chi'_\pi(G, w) \leq \Delta + 1$.

iii Similarly, if for all $uv \in E$, $w(uv) \leq 1/\chi'^*(G_{mcd})$, then $\chi'_\pi(G, w) \leq q$ where q is the numerator of $\chi'^*(G_{mcd})$.

Proof.

i If a vertex u is such that $\sum_{uv \in E(G)} w(uv) > 1$, any coloring satisfying the weights have to repeat at least one color at the edges incident to u. Hence, no proportional coloring exists, since in a proportional coloring any two adjacent edges have to receive different colors. Therefore, $\chi'_\pi(G, w) = \infty$.

ii Recall that G is $\Delta + 1$-colorable ([16]). If for all $uv \in E(G)$, $w(uv) \leq 1/(\Delta+1)$, then any proper edge coloring using $\Delta+1$ colors is a proportional coloring of (G, w).

iii If for all $uv \in E(G)$, $w(uv) \leq 1/\chi'^*(G_{mcd}) = p/q$, then a proper edge coloring assigning p colors out of q to each edge of G gives a proportional coloring of (G, w). Such a proper edge coloring exists by definition of $\chi'^*(G_{mcd})$. □

However the conditions given in Facts 2.5 are not sufficient. This is illustrated by Figure 2.

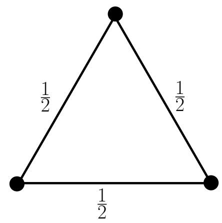

Fig. 2. Example of a weighted graph satisfying $\forall v$, $\sum_{uv \in E(G)} w(uv) \leq 1$, which has infinite proportional chromatic index.

Before moving to the next section, we prove a theorem which shows that we can suppose the value of the weight function of our instance (G, w) to be rational. Indeed, any weight function for which a proportional coloring exists can be replaced by a weight function taking only rational values and that accepts the same optimal proportional coloring. However, this theorem is only an existence theorem. Given a weight function for which we do not know an optimal proportional coloring, it does not state how to compute a rational weight function accepting the same optimal proportional coloring.

Theorem 2.6 *Given a weighted graph (G, w), w not necessarily taking values in $\mathbb{Q}$, there is a weight function w' taking values in $\mathbb{Q}$, such that for every $e \in E$, $w(e) \leq w'(e)$ and $\chi'_{\pi(G,w)} = \chi'_{\pi(G,w')}$*

Proof. If $\chi'_\pi(G, w) = \infty$, just take w' as the constant weight function equals to 1 on every edge. Suppose $\chi'_\pi(G, w) = c$. Given an edge $e \in E$, we set $w'(e) = \frac{\lceil c \times w(e) \rceil}{c}$. It is clear that any optimal proportional coloring of (G, w) is also an optimal proportional coloring of (G, w') and vice versa. □

From now on, we suppose that *all the weight functions will take rational values.*

3 A practical motivation from networking

The proportional coloring problem is a resource sharing problem similar in this sense to the one proposed in [2,17]. It is motivated by the following telecommunication problem. Wireless mesh networks (WMNs) are cost-effective solutions for ubiquitous high-speed services [1]. They are self-organized networks with a fixed infrastructure of interconnected wireless mesh routers whose purpose is to provide Internet access to mobile network users. This infrastructure, forming a wireless backhaul network, is connected to the internet by special routers called mesh gateways, as depicted in Figure 3.

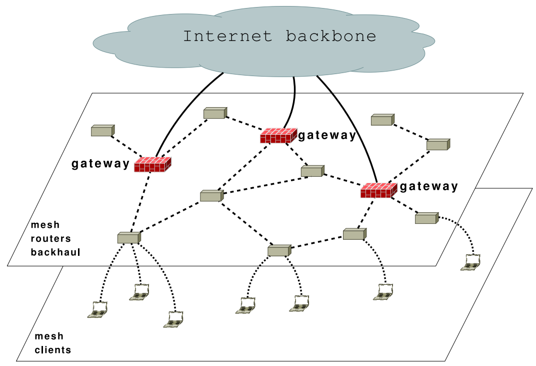

Fig. 3. A WMN topology: mobile clients access Internet through a multi-hop wireless backhaul network of routers and gateways.

Each device has a single interface allowing to send or receive packets on the only available channel. This channel is therefore shared between all the nodes [9]. We denote by *call* two antennas communicating together. We suppose that a call is achievable in both directions. Therefore, the network topology defines a graph $G = (V, E)$ whose vertices are the routers and edges are the achievable calls.

At any time, an eligible state of the network implies that a set of simultaneously achievable calls are activated. Such sets strongly depend on the radio propagation and interference models. A quite broadly used family of models, denoted as *binary models*, assumes that two achievable calls are either interfering or not. In these settings, an eligible set of calls is a set of pairwise non interfering calls. In this application, we assume that there is a radio mesh network connecting routers through directional antennas. Hence, radio interferences and near-far effects impose that each router is involved in at most one call at a time. In consequence, a set of simultaneously achievable calls induces a matching on the graph.

In the following, the network is assumed to be synchronous and operating a slotted time division multiplexing (TDM). During each time slot, a set of simultaneously achievable calls is activated. We focus on the steady state of the network, which is therefore periodic, with a period consisting in a sequence of T time slots.

Therefore, the classical *time slot assignment* problem consists, for a given set of calls to be achieved periodically, in decomposing this set into a minimum number of subsets of simultaneously achievable calls. The number of subsets used is the size of the period, which is a key factor of the quality of service provided to the users of the network [11]. This problem is equivalent to the proper edge coloring problem [3].

The proportional coloring problem arises when we consider *Constant Bit Rates* (CBR) requests. In these settings, we are given a set of communication requests, each request being a source-destination path in the network, and a bit rate. Sending this amount of data on the paths induces that each call has to be periodically activated a given proportion of the time. This is modeled by a weight function w, which is computed in the following way: for each link e, one has to sum, over all connections whose paths use edge e, their bit rates, and divide the total by the capacity of the link.

The problem is now, if possible, to find a *periodical schedule* of the calls satisfying the CBR requests.

An assignment of calls to the time slots satisfies the CBR requests if:

- each slot is assigned an eligible set of calls;
- each call e is assigned to a number of slots $k(e)$ such that $k(e) = \frac{w(e)}{T}$, where T is the number of time slots.

After one time slot, each node creates a number of packets equals to its request and places them in its router queue, while each destination extracts the packets that have reached its router queue.

Observe that during each period, all the packets present at the beginning of the period make exactly one hop. Consequently, a packet will take as many periods to reach its destination as there are hops in its route.

Therefore, the total number of packets stored in the router queues of the nodes is proportional to the length of the period.

However, the size of these queues induces a cost in terms of memory to install in the routers as well as QoS parameters, like end-to-end delay. Therefore, decreasing the cost of the network and improving the QoS depends on minimizing the length of the longest queue, i.e., the length of the period. To illustrate this situation, we give an example in Figure 4.

Figure 4 represents four devices positioned on a line. Nodes s_1, s_2 and s_3 send one packet at destination of d every five units of time. The routing we want to use is the shortest path (and unique path in our example) routing. It implies that a call $s_1 s_2$ is made one fifth of the time, a call $s_2 s_3$ two fifths of

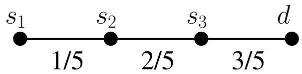

Fig. 4. An example of constant requests on a network composed of 4 antennas

the time and a call s_3d three fifths of the time. For this set of CBR requests and this routing, we can use the following period: node s_1 and s_3 send the packet generated during previous period (one slot), then node s_2 sends the packet generated during the previous period plus the one received (two slots). Then node s_3 sends the two packets received during the previous period (two slots). This period of length five is optimal in the sense that no period can length less. In this case, we need two buffers, one of size one at s_2 and one of size two at s_3.

Our problem is therefore to find a proper edge coloring of G that preserves the proportions given by the weights of the edges, with a minimum number of colors, which will be equivalent the optimum *length of the period*.

We finish this section with the remark that the assumption "w taking value in $\mathbb{Q}$", that we did before as a consequence of Theorem 2.6, makes sense because of the origin of our practical problem. In fact, any instance of the original telecommunication problem would induce rational weights over the edges since they represent a ratio of a number of time slots over the length of the period. As for an example the weights could be computed from an initial time slot assignment that we want to improve, hence yielding rational weights with, more precisely, rather small operands.

4 Complexity results

In this section, we prove Theorem 4.1 which states the complexity results.

Theorem 4.1 *Let (G, w) be a weighted graph on n vertices.*

i) Determining if there exists a proportional coloring for (G, w) is pseudo-polynomial in $\max(n, mcd)$.

ii) Determining the proportional chromatic index of (G, w) is NP-hard.

Proof.

i) Let (G, w) be any weighted graph and G_{mcd} be its *mcd*-graph. Observe that G_{mcd} can be computed in pseudo-polynomial time. Therefore, we will prove this item by showing that this problem is equivalent to determining the fractional edge chromatic index of G_{mcd}, which can also be done in pseudo-polynomial time [15]. In fact, we claim that $\chi'^{*}(G_{mcd}) \leq mcd \Leftrightarrow$

$\chi'_\pi(G,w) < \infty$.

Indeed, suppose that $\chi'_\pi(G,w) = k < \infty$. Then there exists a proportional coloring using k colors, such that each edge receives at least the correct proportion of colors. If we reproduce this proportional coloring mcd times (renaming colors appropriately), we still have a proportional coloring (which may not be minimal). The number of colors assigned to $uv \in E$ is at least $k \times mcd \times w(uv)$. This proportional coloring gives a proper edge coloring of G_{mcd} using $k \times mcd$ colors and each edge receives at least k colors. Hence, $\chi'^*(G_{mcd}) \leq \frac{k \times mcd}{k}$.

On the other hand, suppose that $\chi'^*(G_{mcd}) = \frac{q}{p} \leq mcd$ and consider an optimal fractional edge coloring of G_{mcd} using $q \times c$ colors for some c (such a coloring is computed in pseudo-polynomial time during the process of computing the fractional edge chromatic index [15]). This fractional edge coloring gives at least $p \times c$ colors to each edge. Furthermore, this fractional edge coloring can be extended to an edge coloring of G that uses $q \times c$ colors and gives $p \times c \times mcd \times w(e)$ to each edge e (c.f. Remark 2.4). This proper edge coloring is a proportional coloring, since, by assumption, $\chi'^*(G_{mcd}) \leq mcd \Leftrightarrow \frac{q}{p} \leq mcd \Leftrightarrow mcd \times w(e) \times p \times c \geq q \times c \times w(e)$. In consequence, $\chi'_\pi(G,w) = k \leq q \times c < \infty$.

ii) Let G be any graph of maximum degree Δ. Now, set the weights of all the edges to $\frac{1}{\Delta+1}$. Since G is $(\Delta+1)$-colorable, G is proportionally $(\Delta+1)$-colorable. But now observe that if (G,w) admits a proper edge coloring with Δ colors, then it admits a proportional coloring with Δ colors. Since determining if G is Δ or $(\Delta+1)$-colorable is NP-hard [6], it is NP-hard to determine the proportional chromatic index of an instance (G,w).

□

5 Partial characterization

Despite the difficulty of computing the proportional chromatic index of a weighted graph (G,w), we can deduce polynomially computable lower and upper bounds. Clearly, for a weighted graph (G,w), $\chi'_\pi(G,w) \geq \Delta$. We can get a better lower bound by trying to satisfy the proportions at every vertex of G.

Theorem 5.1 (Lower bound) *Let m be the minimum positive integer sat-*

isfying for all $\mathbf{u} \in V$:

$$\sum_{\mathbf{u}v \in E} \lceil m w(\mathbf{u}v) \rceil \leq m \tag{1}$$

If no positive integer m *satisfies all the above equations, then no proportional coloring exists. Otherwise, if there is a solution* m *and* (G, w) *admits a proportional coloring, then* $\chi'_\pi(G, w)$ *is at least* m.

Proof. If there is no integer m satisfying all the Equations (1), then for every integer m, there is a vertex $\mathbf{u}$ such that $\sum_{\mathbf{u}v \in E} \lceil m w(\mathbf{u}v) \rceil > m$. In particular, this holds when $m = 1$ which from Fact 2.5 i), implies that no proportional coloring using m colors exists.

On the other hand, if a proportional coloring exists which uses c colors, then c needs to satisfy all the Equations (1). Therefore, $m \leq c$. □

If (G, w) admits no proportional coloring, its proportional chromatic index is infinite. However, if it admits a proportional coloring, we can deduce an upper bound from the fractional edge coloring of the corresponding *mcd*-graph.

Corollary 5.2 (Upper bound) *Let* (G, w) *be a weighted graph. If a proportional coloring of* (G, w) *exists, then a proportional coloring of* (G, w) *using* q *colors exists, where* q *is the number of colors used in a fractional edge coloring giving* p *colors to each edge with* $\frac{q}{p} = \chi'^*(G_{mcd(w)})$ *and* $\gcd(p, q) = 1$.

Proof. Corollary 5.2 is a consequence of the proof of Theorem 4.1 i). Indeed, to prove the fact that deciding if a proportional coloring of G exists is pseudo-polynomial, we construct a proportional coloring of G using $c \times q$ colors from a fractional edge coloring of $G_{mcd(w)}$ using $c \times q$ colors and where q is the upper bound of the theorem. This construction is valid for any value of $c \geq 1$. □

In general, given a weighted graph (G, w), the *mcd* of the values taken by w is not an upper bound, even if the graph has a finite proportional chromatic index. Indeed, consider $(P, \mathbf{\frac{1}{3}})$: the Petersen graph P with a weight function uniformly equal to $\frac{1}{3}$ on all its edges, as depicted in Figure 5. Since $\chi'^*(P) = 3$, we have $\chi'_\pi(P, \mathbf{\frac{1}{3}}) < \infty$. However $\chi'_\pi(P, \mathbf{\frac{1}{3}}) > 3 = mcd(\mathbf{\frac{1}{3}})$, since P does not have any proper edge coloring using only 3 colors.

The proportional chromatic index may differ from the upper bound given by the fractional edge chromatic index, since an edge can receive proportionally more colors than what it asks for. This is illustrated by the example of Figure 1 (C_5, w). Indeed, in this example, we have $mcd(w) = 60$, and so in $(C_5)_{mcd(w)}$, an edge with weight $1/2$ is replaced by 30 parallel edges.

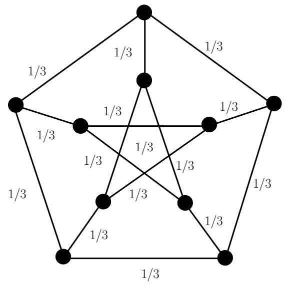

Fig. 5. The Petersen graph with the weight function uniformly equal to $\frac{1}{3}$ on all edges.

Hence, in $(C_5)_{mcd(w)}$ there is a vertex whose degree is 60, which implies that $\chi'^*((C_5)_{mcd(w)}) \geq 60 > 4 = \chi'_\pi(C_5, w)$. This can also be illustrated by the triangle of Figure 2, or by any odd cycle with constant weight function w=**1/2**, as no proportional colouring of these weighted graphs using two colours exist ; two being the *mcd* of the values taken by w. However these examples have infinite proportional chromatic index, which is not the case with the weighted Petersen graph given above.

One can also wonder on which classes of graphs the proportional coloring problem can be solved in polynomial time. The next theorems announce two positive results.

Theorem 5.3 *Let (G, w) be a weighted graph. If there is a solution to the set of Equations (1), for m the minimum solution, we have:*

$$\chi'_\pi(G, w) = m \Leftrightarrow \chi'(G_m) = m.$$

Proof. We start to prove $\chi'_\pi(G, w) = m \Rightarrow \chi'(G_m) = m$. Observe that a proportional coloring using m colors is a proper edge coloring of G, which gives at least $\lceil m \times w(e) \rceil$ colors to each edge e. Additionally, in G_m, each edge e of G is replaced by $\lceil m \times w(e) \rceil$ edges, so the previous proper edge coloring gives a proper edge coloring of G_m. By this, we proved that $\chi'(G_m) \leq m$. It remains to prove that $\chi'(G_m) \geq m$. Suppose $\chi'(G_m) < m$. Then, by the definition of a proper edge coloring, we would have, for each vertex v, $\sum_{v\ st\ uv \in E} \lceil m \times w(uv) \rceil \leq \chi'(G_m) \leq m - 1$. This is stronger than $\sum_{uv \in E} \lceil (m - 1) \times w(uv) \rceil \leq \chi'(G_m) \leq m - 1$, which contradicts the hypothesis of the minimality of m. Hence $\chi'(G_m) = m$.

We now prove $\chi'_\pi(G, w) = m \Leftarrow \chi'(G_m) = m$. If G_m has a proper edge coloring using m colors, it gives a proportional coloring of (G, w) using m colors. Hence $\chi'_\pi(G, w) \leq m$. This fact together with Theorem 5.1 implies that $\chi'_\pi(G, w) = m$. $\square$

This theorem gives the proportional chromatic index of some simple classes of weighted graphs. A sufficient condition satisfied by these classes is that, given an integer m, the corresponding m-graph is of Class 1. It includes trees, grids and more generally any bipartite graph as stated by Corollary 5.5.

We now give a remark that will be useful in what follows.

Remark 5.4 Let (G, w) be a weighted bipartite graph and m a solution to the set of Equations (1). Then m is finite and G_m is well defined. Observe that for every $u \in G_m$ and for edge e incident to u, in G_m, e is replaced by $\lceil m \times w(e) \rceil$ edges. Because u satisfies Equation (1), the total number of edges incident to u is less than or equal to m. Therefore, the maximum degree of G_m is at most m. In addition, observe that in the case where w is uniform and (G, w) satisfies the necessary condition of Fact 2.5, then any $m \geq \Delta(G)$ is a solution of the set of Equations (1).

Corollary 5.5 *Let (G, w) be a weighted bipartite graph. If there is a solution m to the set of Equations (1), then (G, w) accepts a proportional coloring using m colors.*

Proof. Let (G, w) be a weighted bipartite graph and m a solution to the set of Equations (1), which does exist by assumption. By Remark 5.4, the maximum degree of G_m is at most m. Observe that G_m is also bipartite and hence its edge chromatic index is equal to m, that is, $\chi'(G_m) = m$. The corollary follows as a consequence of the previous theorem. $\square$

We would like to know if it is possible to extend this corollary to other classes of graphs. After trees and bipartite graphs, a natural class of graphs to investigate is the class of outerplanar graphs. A graph is outerplanar if it admits a planar representation where all the vertices are incident to the infinite face.

If G is an outerplanar other than an odd cycle and w is an uniform function on the edges of G, Theorem 5.3 also includes (G, w). Indeed, we know that a simple outerplanar graph is of class 1 if and only if it is not an odd cycle. Hence an outerplanar graph which is not an odd cycle and such that all the edges have the same multiplicity has its chromatic index equals to its maximum degree. Therefore, given a weighted outerplanar graph (G, w) satisfying the necessary condition stated in Fact 2.5, with w being a constant, we have that, for any $m \geq \Delta(G)$, G_m is an outerplanar multigraph with all the edges having the same multiplicity and with maximum degree equal to m (see Remark 5.4. In consequence, $\chi'(G_m) = m$. Hence, m is a solution to the set of Equations (1) and Theorem 5.3 applies.

We observe that the requirement of w uniform over all the edges is not a sufficient condition, since in the proof of Theorem 4.1, we used a weighted graph with a constant weight function. Furthermore, if G is an outerplanar but w is not uniform on all its edges, then we can not apply Theorem 5.3. To see this, consider a simple outerplanar graph G containing a triangle, a weight function w and m, the minimum solution to the Equations (1). If w is such that the m-graph G_m has too many multiple edges on one of its triangle, even if its maximum degree is at most m (see Remark 5.4), it may not have edge chromatic index equal to m. Figure 6 illustrates this situation, by presenting an outerplanar weighted graph G and its outerplanar multigraph G_m, $m = 12$. In this example, we have a triangle with edge multiplicity 4, which forces the edge chromatic index to be at least 12 while its maximum degree is 10.

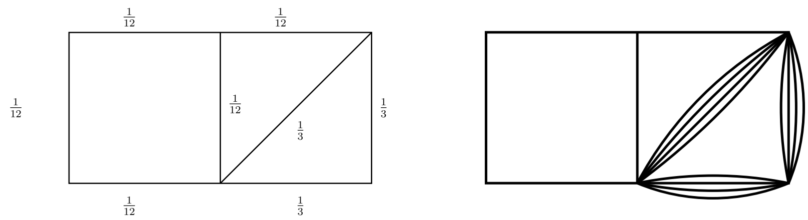

Fig. 6. Example of an outerplanar weighted graph (G, w) and its G_{12} multigraph where $\chi'(G_{12}) \neq \Delta(G_{12})$.

We can also give more information on the proportional chromatic index of a weighted graph, when its weight function verifies some properties. The next theorem goes in this direction:

Theorem 5.6 *Let (G, w) be a weighted graph. Let e be an edge with an end vertex $\mathbf{v}$ such that $\sum_{u\mathbf{v}\in \mathbf{E}} w(u\mathbf{v}) = \mathbf{1}$. Then, the denominator of $w(e)$ divides the number of colors used in a proportional coloring of (G, w). In particular, it divides $\chi'_\pi(G, w)$.*

Proof. Consider a proportional coloring of (G, w) using m colors. Let e be an edge with an end vertex $\mathbf{v}$ such that $\sum_{u\mathbf{v}\in E} w(u\mathbf{v}) = 1$. We have that $\sum_{u\mathbf{v}\in E}\lceil m \times w(u\mathbf{v})\rceil \leq m = \sum_{u\mathbf{v}\in E} m \times w(u\mathbf{v})$. Hence $\lceil m \times w(e)\rceil = m \times w(e)$. In particular, it is true for $m = \chi'_\pi(G, w)$. $\square$

In a weighted graph (G, w), a vertex v with $\sum_{uv\in E} w(uv) = 1$ represents a *saturated node*, i.e., a node through which no extra request can be routed. A weighted graph (G, w) such that every edge is incident to a saturated vertex represents a *saturated network*, i.e. a network in which no extra request can be routed.

The existence of a saturated node leads to a non-approximability result:

Corollary 5.7 *Let (G, w) be a weighted graph and v a vertex of G with $\sum_{uv \in E} w(uv) = 1$. We call q the mcd of all the $w(uv)$ for $uv \in E$. $\chi'_\pi(G, w)$ cannot be approximated within the constant $q - 1$.*

Proof. Theorem 5.6 implies that q divides the number of colors used in a proportional coloring. Hence no proportional coloring exists with $\chi'_\pi(G, w) + i$ colors for $0 < i < q$. □

6 Conclusion

Motivated by the applications modelled by the proportional coloring problem and its hardness, we let the following general open questions: find approximation algorithms for classes of graphs which usually occur in telecommunication, as circular-arc graphs and triangular lattices, and determine other classes of graphs (and weighted classes of graphs) for which the proportional chromatic index can be computed in polynomial time.

References

[1] I. Akyildiz, X. Wang, and W. Wang, Wireless mesh networks: a survey, *Computer Networks*, vol. 47, no. 4, 445–487, 2005.

[2] V.C. Barbosa, M.R.F. Benevides and F.M.G. França, Sharing Resources at Nonuniform Access Rates, *Theory of Computing Systems*, vol. 34, no. 1, 13–26, 2000.

[3] J. Bibb Cain, T. Billhartz, L. Foore, A link scheduling and ad hoc networking approach using directional antennas, *Military Communications Conference*, vol. 1, 643–648, 2003.

[4] R.C. Carlson and G.L. Nemhauser, Scheduling to minimize interaction cost, *Operations Research*, vol. 14, no. 1, 52–58, 1966.

[5] G. Frobenius, Über zerlegbare Determinanten, *Sitzungsberichte der KÃ¶niglich Preussischen Akademie der Wissenschaften zu Berlin*, 456–477, 1917.

[6] I. Holyer, The NP-completeness of edge colourings, *SIAM Journal on Computing*, vol. 10, 718–720, 1981.

[7] R. Klasing, N. Morales, and S. Pérennes, On the complexity of bandwidth allocation in radio networks, *Theoretical Computer Science*, Vol. 406, no. 3, 225–239, 2008.

[8] D.P. Koreas, The NP-completeness of chromatic index in triangle free graphs with maximum vertex of degree 3, *Applied mathematics and computation*, vol. 83, no. 1, 13–17, 1997.

[9] H. Liu and B. Zhao, Optimal scheduling for link assignment in traffic-sensitive STDMA wireless ad-hoc networks, *Networking and Mobile Computing*, ser. LNCS, vol. 3619, 218–228, 2005.

[10] C. McDiarmid, Discrete mathematics and radio channel assignment, *Recent advances in algorithms and combinatorics*, 27–63, 2003.

[11] C. Molle, F. Peix, and H. Rivano, An optimization framework for the joint routing and scheduling in wireless mesh networks, *19th IEEE International Symposium on Personal, Indoor and Mobile Radio Communications*, 1–5, 2008.

[12] M. Padberg and M. Rao, Odd minimum cutsets and b-matchings, *Mathematics of Operations Research*, vol. 7, no. 1, 67–80, 1982.

[13] B. Reed, Edge coloring near bipartite graphs, *Operation Research Letters*, vol. 24, no. 1–2, 11–14, 1999.

[14] F. Rendl, On the complexity of decomposing matrices arising in satellite communication, *Operation Research Letters*, vol. 4, no. 1, 5–8, 1985.

[15] E. Scheinerman and D. Ullman, Fractional Graph Theory: a rational approache to the theory of graphs, *John Wiley and Sons*, 1997.

[16] V.G. Vizing, On an estimate of a chromatic class of a p-graph (in Russian), *Diskret Analiz*, vol. 3, 25–30, 1964.

[17] H.G. Yeh, X. Zhu, Resource-sharing system scheduling and circular chromatic number, *Theoretical Computer Science*, vol. 332 no. 1-3, 447–460, 2005.