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RANDOM PARKING, EUCLIDEAN FUNCTIONALS, AND RUBBER ELASTICITY

ANTOINE GLORIA AND MATHEW D. PENROSE

ABSTRACT. We study subadditive functions of the random parking model previously analyzed by the second author. In particular, we consider local functions S of subsets of \mathbb{R}^d and of point sets that are (almost) subadditive in their first variable. Denoting by ξ the random parking measure in \mathbb{R}^d , and by ξ^R the random parking measure in the cube $Q_R = (-R, R)^d$, we show, under some natural assumptions on S , that there exists a constant $\bar{S} \in \mathbb{R}$ such that

$$\lim_{R \rightarrow +\infty} \frac{S(Q_R, \xi)}{|Q_R|} = \lim_{R \rightarrow +\infty} \frac{S(Q_R, \xi^R)}{|Q_R|} = \bar{S}$$

almost surely. If $\zeta \mapsto S(Q_R, \zeta)$ is the counting measure of ζ in Q_R , then we retrieve the result by the second author on the existence of the jamming limit. The present work generalizes this result to a wide class of (almost) subadditive functions. In particular, classical Euclidean optimization problems as well as the discrete model for rubber previously studied by Alicandro, Cicalese, and the first author enter this class of functions. In the case of rubber elasticity, this yields an approximation result for the continuous energy density associated with the discrete model at the thermodynamic limit, as well as a generalization to stochastic networks generated on bounded sets.

Keywords: random parking, subadditive ergodic theorem, Euclidean optimization problems, stochastic homogenization, polymer-chain networks, thermodynamic limit.

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CONTENTS

1. Introduction and informal statement of the results	1
2. Random parking and subadditive ergodic theorems	4
2.1. Random parking	4
2.2. Main result	6
3. Application to classical Euclidean optimization problems	9
4. Application to the derivation of rubber elasticity	14
4.1. A discrete model for rubber and its thermodynamic limit	14
4.2. Isotropic homogenized energy density and approximation result	16
4.3. Rubber elasticity and random parking in a bounded set	26
Acknowledgements	27
References	27

1. INTRODUCTION AND INFORMAL STATEMENT OF THE RESULTS

Rényi's model of random parking (also known as random sequential adsorption or random sequential packing), is defined in d dimensions as follows. A parameter $\rho_0 > 0$ is specified, and open balls $B_{1,R}, B_{2,R}, \dots$ of radius ρ_0 , arrive sequentially and uniformly at random in the d -dimensional cube $Q_R := (-R, R)^d$, subject to non-overlap until saturation occurs. It is known [Pen01] that the random parking measure ξ^R in Q_R (i.e. the point measure representing the

locations of the balls at saturation) converges weakly in the sense of measures to a measure ξ in \mathbb{R}^d , called the random parking measure in \mathbb{R}^d , and that there exists $\lambda \in \mathbb{R}^+$ such that

$$\lim_{R \rightarrow \infty} \frac{\xi(Q_R)}{|Q_R|} = \lambda$$

almost surely — which yields the existence of a deterministic averaged density of packed balls in \mathbb{R}^d . Perhaps of greater interest, however, is the existence of the limit of $\frac{\xi^R(Q_R)}{|Q_R|}$, which would be the thermodynamic limit of the averaged density of packed balls in the domains Q_R . Using quantitative properties of “stabilization” of the random parking measures, it is proved in [Pen01] that in probability,

$$(1) \quad \lim_{R \rightarrow \infty} \frac{\xi^R(Q_R)}{|Q_R|} = \lambda.$$

That is, not only does the measure ξ^R converge weakly to ξ , but also the averaged density of packed balls converges to λ (the jamming limit). Moreover, if the arrivals processes for different R are coupled by being all derived from a single Poisson process in space-time, the convergence (1) holds almost surely, as does the convergence of ξ^R to ξ (in fact, almost surely for all $r > 0$ we have $\xi^R \cap Q_r = \xi \cap Q_r$ for all large enough R).

The first part of this paper is concerned with the extension of (1) to more general functions of ξ^R , besides the total measure of ξ^R . We consider *local subadditive* functions S of bounded open sets and point sets, that is such that for all open bounded disjoint subsets D, D_1, \dots, D_n of \mathbb{R}^d , and for every point set ζ in some fixed class (i. e. satisfying the non-empty space and hard-core conditions, see Section 2.2), we have (see Theorem 2.3 for milder conditions)

$$\begin{aligned} S(D, \zeta) &= S(D, \zeta|_D), \\ S(\cup_{i=1}^n D_i, \zeta) &\leq \sum_{i=1}^n S(D_i, \zeta). \end{aligned}$$

Under the following further assumptions on S :

- uniform boundedness: there is a constant C such that $|S(D, \zeta)| \leq C|D|$ for all D, ζ ;
- insensitivity to boundary effects: there exists $0 < \alpha < 1$ such that $\frac{S(Q_R, \zeta) - S(Q_{R-R^\alpha}, \zeta)}{|Q_R|} = o(1)$;

we shall prove that there exists deterministic $\bar{S} \in \mathbb{R}^+$ such that almost surely

$$(2) \quad \lim_{R \rightarrow +\infty} \frac{S(Q_R, \xi^R)}{|Q_R|} = \lim_{R \rightarrow +\infty} \frac{S(Q_R, \xi)}{|Q_R|} = \bar{S}.$$

Note that (1) is a particular case of (2) for the *additive* function $S(D, \zeta) := \zeta(D) = \int_D d\zeta(x)$. Our proof of (2) differs from the proof of (1) in [Pen01] in two respects. First, the additivity of the counting measure allows one to appeal to an ergodic theorem, whereas subadditivity requires the use of subadditive ergodic theorems (such that of Akcoglu and Krengel [AK81]), and the ergodicity of the point process itself (which is used in a weaker form in [Pen01]). Second, and more importantly, the additivity of $D \mapsto \zeta(D)$ and the uniform bound $0 \leq \zeta(D) \leq C|D|$ imply that the contribution of any subset D' of D to $S(D, \zeta)$ is uniformly bounded by $C|D'|$ — which is crucial in [Pen01]. In the subadditive case, this does not hold: The contribution of a subset D' of D to $S(D, \zeta)$ is *not* a priori bounded by $C|D'|$. This compels us to appeal to the stabilization properties introduced by Schreiber, Yukich, and the second author in [SPY] — which are complementary to the ones in [Pen01]. The rest of the proof relies on a percolation argument and the Borel-Cantelli lemma.

In the second part of this work we apply this general result to classical Euclidean optimization problems on the random parking measure. In particular, we shall prove a so-called umbrella

theorem, which allows to cover at once a wide class of problems, including the traveling salesman, the minimum spanning tree, and the minimal matching. This is one of the first examples where subadditive arguments are combined with stabilization properties. We refer to the recent survey of Yukich [Yuk10], where both types of arguments are presented and used — although not combined.

The third part of this work is concerned with another application of our general result: the discrete model for rubber introduced in [GLTV], whose thermodynamic limit is studied in [ACG]. In particular, as recalled in Subsection 4.1, the model under investigation is based on the notion of stochastic lattices (in the sense of random point processes) and on the associated Delaunay tessellations. In a nutshell, every edge of the tessellation represents a polymer chain (the vertices are then permanent cross-links). As argued in [GLTV] and [ACG], it is reasonable at first order to consider cross-links at zero temperature and the polymer chains at finite temperature. The free energy of the ε -rescaled network of polymer chains in a domain D of \mathbb{R}^3 is then given by the sum of the free energies of the edges, plus a volumetric term accounting for the incompressibility of the network (the volume of each simplex is conserved). Under some assumptions on the energy terms (which are standard in the theory of homogenization of integral functionals), and under assumptions on the stochastic lattice compatible with the random parking measure, it is proved in [ACG] that the free energy functional of the ε -rescaled network in D (seen as a function of the position of the cross-links) Γ -converges (as ε goes to zero) to a continuous energy functional of the type

$$u \mapsto \int_D W_{\text{hom}}(\nabla u(x)) \, dx,$$

where the “homogenized energy density” W_{hom} is a quasiconvex frame-invariant function, as encountered in continuum mechanics. The map $u : D \rightarrow \mathbb{R}^3$ is a deformation. In particular, this type of convergence ensures that minimizers of the free energy of the discrete system converge (up to extraction) to minimizers of the continuous energy functional — which yields a rigorous derivation of nonlinear elasticity compatible with minimization. In addition, if the stochastic lattice is statistically isotropic, then the associated homogenized energy density W_{hom} is isotropic, as expected in rubber elasticity. The stochastic lattices considered in [ACG] satisfy the following three properties: ergodicity, non-empty space condition, and hard-core condition. Two questions were left open in [ACG]:

- Do there exist such stochastic lattices which are indeed statistically isotropic ?
- What happens to the thermodynamic limit in D if the stochastic lattice on \mathbb{R}^d is replaced by an approximation on D (in which case boundary effects may appear and rule out stationarity) ?

This article gives a clear answer to both questions. As we quickly show in Subsection 2.1, the random parking measure in \mathbb{R}^d is an example of stochastic lattice which is statistically isotropic, ergodic, and satisfies the non-empty and hard-core conditions. In Subsection 4.2, we shall prove that the model studied in [ACG] can be recast in terms of a subadditive function satisfying the assumptions introduced above, so that the identity corresponding to (2) will hold true. This allows us to complete the program initiated in [ACG] both in terms of lattices generated in D instead of \mathbb{R}^d (which seems more reasonable from a physical point of view), and in terms of numerical approximations (effective computations for W_{hom} in [GLTV] are based on random parking in bounded domains Q_R).

This paper is organized as follows. Section 2 is dedicated to the study of the qualitative properties of the random parking measure, and to the proof of (2). In Section 3 we apply the main result to classical Euclidean optimization problems. Finally, Section 4 is concerned with the application of the main result to the discrete model for rubber studied in [ACG]. In

particular, we show that this model satisfies the assumptions of the main result, which completes the analysis of the model when the stochastic lattice is the random parking measure.

We make use of the following notation throughout the article:

- $d, n \geq 1$ denote dimensions;
- $\mathbb{R}^+ = [0, +\infty)$;
- $\mathcal{O}(\mathbb{R}^d)$ is the set of bounded nonempty Lipschitz subsets of \mathbb{R}^d (a bounded measurable set D , whose boundary is denoted by ∂D , is Lipschitz if there exists a finite collection of relatively open sets $U_k \subset \partial D$ such that $\partial D = \cup_k U_k$, and U_k is the graph of a Lipschitz function on \mathbb{R}^{d-1} up to a rotation);
- For all $D \in \mathcal{O}(\mathbb{R}^d)$, $|D|$ denotes its d -dimensional Lebesgue measure, ∂D denotes its boundary, and \bar{D} its closure;
- For all $D \in \mathcal{O}(\mathbb{R}^d)$ we set $\mathcal{O}(D) = \{A \in \mathcal{O}(\mathbb{R}^d) \mid A \subset D\}$ (note that $D \in \mathcal{O}(D)$);
- For all $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm of x , and d_2 denotes the Euclidean distance, i.e. $d_2(x, A) := \inf\{|x - y| : y \in A\}$ for nonempty $A \subset \mathbb{R}^d$.
- diam denotes the Euclidean diameter of a subset of \mathbb{R}^d ;
- $\mathbb{M}^{n \times d}$ is the space of $n \times d$ matrices, that we simply denote by \mathbb{M}^d when $n = d$;
- For all $\Lambda \in \mathbb{M}^{n \times d}$, we define $\varphi_\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^n$ by $x \mapsto \Lambda x$, and denote by $|\Lambda|$ the Frobenius norm $\sqrt{\text{Trace}(\Lambda^T : \Lambda)}$ (which is the operator norm of φ_Λ associated with the Euclidean norms of \mathbb{R}^d and \mathbb{R}^n);
- SO_d is the set of rotations in \mathbb{R}^d ;
- \mathcal{A}_f^d and \mathcal{A}_{lf}^d denote the spaces of finite and locally finite point sets in \mathbb{R}^d , respectively;
- For all open $D \subset \mathbb{R}^d$, $C(D, \mathbb{R}^n)$ and $C_0^\infty(D, \mathbb{R}^n)$ respectively denote the space of continuous functions from D into \mathbb{R}^n , and the space of smooth functions which have support in $D \setminus \partial D$;
- For all $D \in \mathcal{O}(\mathbb{R}^d)$ open, $n \geq 1$, and $p \in [1, \infty]$, $L^p(D, \mathbb{R}^n)$, $W^{1,p}(D, \mathbb{R}^n)$, and $W_0^{1,p}(D, \mathbb{R}^n)$ denote the Lebesgue space of p -integrable (or essentially bounded if $p = \infty$) functions from D into \mathbb{R}^n , the Sobolev space of p -integrable functions on D whose distributional derivatives are p -integrable (or essentially bounded if $p = \infty$), and the closure of $C_0^\infty(D, \mathbb{R}^n)$ in $W^{1,p}(D, \mathbb{R}^n)$, respectively;
- When no confusion occurs we use the short-hand notation $C(D)$, $L^p(D)$ etc. for $C(D, \mathbb{R}^n)$ in $L^p(D, \mathbb{R}^n)$ etc.;
- For all $D \in \mathcal{O}(\mathbb{R}^d)$ open and $u \in W^{1,\infty}(D)$, we denote by $\|u\|_{\text{Lip}}$ the Lipschitz constant of u on D .

2. RANDOM PARKING AND SUBADDITIVE ERGODIC THEOREMS

2.1. Random parking. We first quickly recall the graphical construction of the random parking measure ξ^A in a Borel set $A \subset \mathbb{R}^d$. Let \mathcal{P} be a homogeneous Poisson process of unit intensity in $\mathbb{R}^d \times \mathbb{R}^+$. An *oriented graph* is a special kind of directed graph in which there is no pair of vertices $\{x, y\}$ for which both (x, y) and (y, x) are included as directed edges. We shall say that x is a *parent* of y and y is an *offspring* of x if there is an oriented edge from x to y . By a *root* of an oriented graph we mean a vertex with no parent.

The graphical construction goes as follows. Let $\rho_0 > 0$ and let B denote the Euclidean ball in \mathbb{R}^d of radius ρ_0 centred at the origin. Make the points of the Poisson process \mathcal{P} on $\mathbb{R}^d \times \mathbb{R}^+$ into the vertices of an infinite oriented graph, denoted by \mathcal{G} , by putting in an oriented edge $(X, T) \rightarrow (X', T')$ whenever $(X' + B) \cap (X + B) \neq \emptyset$ and $T < T'$. For completeness we also put an edge $(X, T) \rightarrow (X', T')$ whenever $(X' + B) \cap (X + B) \neq \emptyset$, $T = T'$, and X precedes X' in the lexicographical order — although in practice the probability that \mathcal{P} generates such an edge is zero. It can be useful to think of the oriented graph as representing the spread of

an ‘‘epidemic’’ through space over time; each time an individual is ‘‘born’’ at a Poisson point in space-time, it becomes (and stays) infected if there is an earlier infected point nearby in space (in the sense that the translates of B centred at the two points overlap). This graph determines which items have to be accepted.

For $(X, T) \in \mathcal{P}$, let $C_{(X, T)}$ (the ‘‘cluster at (X, T) ’’) be the (random) set of ancestors of (X, T) , that is, the set of $(Y, U) \in \mathcal{P}$ such that there is an oriented path in \mathcal{G} from (Y, U) to (X, T) . As shown in [Pen01, Corollary 3.1], the ‘‘cluster’’ $C_{(X, T)}$ is finite for $(X, T) \in \mathcal{P}$ with probability 1. It represents the set of all items that can potentially affect the acceptance status of the incoming particle represented by the Poisson point (X, T) . The set of accepted items may be reconstructed from the graph \mathcal{G} , as follows. Let $A \subset \mathbb{R}^d$ be a (possibly unbounded) Borel set, and let \mathcal{P}_A denote the set $\mathcal{P} \cap (A \times \mathbb{R}^+)$, i. e. the set of Poisson points that lie in $A \times \mathbb{R}^+$. Let $\mathcal{G}|_A$ denote the restriction of \mathcal{G} to the vertex set \mathcal{P}_A . Recursively define subsets $F_i(A)$, $G_i(A)$, $H_i(A)$ of A , $i = 1, 2, 3, \dots$ as follows. Let $F_1(A)$ be the set of roots of the oriented graph $\mathcal{G}|_A$, and let $G_1(A)$ be the set of offspring of roots. Set $H_1(A) = F_1(A) \cup G_1(A)$. For the next step, remove the set $H_1(A)$ from the vertex set, and define $F_2(A)$ and $G_2(A)$ the same way; so $F_2(A)$ is the set of roots of the restriction of \mathcal{G} to vertices in $\mathcal{P}_A \setminus H_1(A)$, and $G_2(A)$ is the set of vertices in $\mathcal{P}_A \setminus H_1(A)$ which are offspring of $F_2(A)$. Set $H_2(A) = F_2(A) \cup G_2(A)$, remove the set $H_2(A)$ from $\mathcal{P}_A \setminus H_1(A)$, and repeat the process to obtain $F_3(A)$, $G_3(A)$, $H_3(A)$. Continuing ad infinitum gives us subsets $F_i(A)$, $G_i(A)$ of \mathcal{P}_A defined for $i = 1, 2, 3, \dots$. These sets are disjoint by construction. In the case when $A = \mathbb{R}^d$, we drop the reference to A and use the abbreviation F_i and G_i for $F_i(\mathbb{R}^d)$ and $G_i(\mathbb{R}^d)$.

As proved in [Pen01, Lemma 3.2], for every bounded nonnull Borel set A in \mathbb{R}^d , the sets $F_1(A)$, $G_1(A)$, $F_2(A)$, $G_2(A)$, \dots form a partition of \mathcal{P}_A , and F_1 , G_1 , F_2 , G_2 , \dots form a partition of \mathcal{P} with probability 1.

Definition 2.1. The random parking measure ξ^A in A is given by the projection of the union $\cup_{i=1}^{\infty} F_i(A)$ on \mathbb{R}^d . Likewise, the random parking measure ξ in \mathbb{R}^d is given by the projection of the union $\cup_{i=1}^{\infty} F_i$ on \mathbb{R}^d .

Definition 2.2. Let $\Sigma \subset \mathbb{R}^d$ be a locally finite set of points. We say that Σ is *general* if no $d + 1$ points lie in the same hyperplane and if no $d + 2$ points lie in the same hypersphere.

Definition 2.3. Let $0 < \rho_1 \leq \rho_2 \leq \infty$. Given $D \subset \mathbb{R}^d$, suppose Σ is a subset of D . Then Σ is said to be (ρ_1, ρ_2) -*admissible* in D iff for all $x \neq y \in \Sigma$, $|x - y| \geq \rho_1$, and for all $z \in D$, $B(z, \rho_2) \cap \Sigma \neq \emptyset$, where $B(z, \rho_2)$ denotes the open ball centred in z of radius ρ_2 (and $B(z, \infty) := \mathbb{R}^d$). Let $\mathcal{A}_{\rho_1, \rho_2}$ denote the class of general point sets that are (ρ_1, ρ_2) -admissible in \mathbb{R}^d .

In other words, an admissible point set in D is one which satisfies the hard-core and empty space conditions. We shall sometimes write simply ‘admissible’ for ‘admissible in \mathbb{R}^d ’.

Definition 2.4. A random subset Σ of \mathbb{R}^d is called a *point process*. We say that a point process Σ in \mathbb{R}^d is *isotropic* if the distribution of Σ and of $\mathbf{R}\Sigma$ are the same for every rotation $\mathbf{R} \in SO_d$.

We are now in position to prove that the random parking measure ξ in \mathbb{R}^d (which is admissible by construction) is an isotropic admissible stochastic lattice in the sense of [ACG].

Proposition 2.1. *The random measure ξ is stationary (under real shifts), ergodic, isotropic, and almost surely general.*

Proof of Proposition 2.1. We prove each property separately.

Step 1. Stationarity.

By definition, the Poisson point process \mathcal{P} on $\mathbb{R}^d \times \mathbb{R}^+$ and its translation $(x_1, \dots, x_d, 0) + \mathcal{P}$

have the same distribution for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Hence the graphical construction is stationary, and therefore also the random parking measure ξ .

Step 2. Isotropy.

The proof of the isotropy of ξ is similar to the proof of the stationarity. For all $\mathbf{R} \in SO_d$, the Poisson point process \mathcal{P} and its rotated version $\mathbf{R}\mathcal{P} := \{(\mathbf{R}x, t) : x \in \mathbb{R}^d, t \in \mathbb{R}^+, (x, t) \in \mathcal{P}\}$ have the same distribution, which implies that the random parking measure ξ is isotropic.

Step 3. General position.

For all $t > 0$ we set $\overline{\mathcal{P}}_t = \{x \in \mathbb{R}^d : \exists \tau \in [0, t], (x, \tau) \in \mathcal{P}\}$. Since $\xi \subset \cup_{n \in \mathbb{N}} \overline{\mathcal{P}}_n$, and since $\overline{\mathcal{P}}_n \subset \overline{\mathcal{P}}_{n+1}$ for all $n \in \mathbb{N}$, the event that ξ is not in general position is contained in the union over $n \in \mathbb{N}$ of the events that $\overline{\mathcal{P}}_n$ is not in general position. Since $\overline{\mathcal{P}}_n$ is a Poisson process of intensity n in \mathbb{R}^d , the probability that $\overline{\mathcal{P}}_n$ is not in general position is zero, so that the union of this countable set of events has also probability zero, and ξ is almost surely general.

Step 4. Ergodicity.

Let us view ξ and \mathcal{P} as elements of \mathcal{A}_{lf}^d and \mathcal{A}_{lf}^{d+1} respectively. Also let us extend \mathcal{P} to a homogeneous Poisson process of unit intensity on the whole of \mathbb{R}^{d+1} (also denoted \mathcal{P}). Let T_x denote translation by an element x of \mathbb{R}^d (acting either on \mathcal{A}_{lf}^d or \mathcal{A}_{lf}^{d+1} according to context). Then as described earlier in this section, ξ is the image of \mathcal{P} under a certain mapping h from \mathcal{A}_{lf}^{d+1} to \mathcal{A}_{lf}^d , which commutes with T_x for any $x \in \mathbb{R}^d$, that is $T_x \circ h = h \circ T_x$.

Suppose A is a measurable subset of \mathcal{A}_{lf}^d which is shift-invariant, meaning $T_x(A) = A$ for all $x \in \mathbb{R}^d$. Then for $x \in \mathbb{R}^d$,

$$h^{-1}(A) = h^{-1}(T_x(A)) = T_x(h^{-1}A)$$

so $h^{-1}(A)$ is invariant under the mapping T_x acting on \mathcal{A}_{lf}^{d+1} . Now, \mathcal{P} is ergodic under translations, that is if $B \subset \mathcal{A}_{lf}^{d+1}$ satisfies $T_x(B) = B$ for some non-zero $x \in \mathbb{R}^{d+1}$, then $P[B] \in \{0, 1\}$. See for example the proof of Proposition 2.6 of [MR]. Therefore with A as above,

$$P[\xi \in A] = P[\mathcal{P} \in h^{-1}(A)] \in \{0, 1\}.$$

Thus ξ is ergodic. □

2.2. Main result. For $i \in \mathbb{Z}^d$, let $C^i := i + [-1/2, 1/2]^d$ be the closed rectilinear unit cube centred at i .

Let $0 < \rho_1 < \rho_2$. Given $D \in \mathcal{O}(\mathbb{R}^d)$, and $R > 0$, let $D_R := \{Rx, x \in D\}$ be the dilation of D by a factor $R > 0$. For $r > 0$ set $D_{R,r} := \{x \in D_R : d_2(x, \mathbb{R}^d \setminus D_R) > r\}$. Also set $D_{R,0} := D_R$, and let $\mathcal{D}(D) := \{D_{R,r} : R > 0, r \geq 0, D_{R,r} \in \mathcal{O}(\mathbb{R}^d)\}$. The next result shows that $D_{R,r} \in \mathcal{D}(D)$ whenever r/R is small enough.

Lemma 2.2. *Suppose $D \in \mathcal{O}(\mathbb{R}^d)$. Then there exists $\delta > 0$ such that $D_{1,t} \in \mathcal{O}(\mathbb{R}^d)$ for all $t \in]0, \delta[$.*

Proof. Assume without loss of generality that $d \geq 2$ and D is closed. For $r > 0$ let B'_r denote the closed Euclidean ball of radius r centred at the origin in \mathbb{R}^{d-1} . Suppose $x \in \partial D$. By the definition of $\mathcal{O}(\mathbb{R}^d)$ we may assume after translation and rotation that $x = 0$ and that there exist $r > 0$ and $a > 0$, and Lipschitz function $g : B'_{2r} \rightarrow \mathbb{R}$ such that $|g(u)| \leq a$ for all $u \in B'_{2r}$ and

$$D \cap (B'_{2r} \times [-4a, 4a]) = \{(u, y) | u \in B'_{2r}, y \in [-4a, g(u)]\}.$$

Suppose $0 < t < \min(r, a)$. Then

$$D_{1,t} \cap (B'_r \times [-2a, 2a]) = \{(u, y) | u \in B'_r, -2a \leq y < \inf_{w \in B'_t} g_w(u)\}$$

where for $w \in B_t'$ we set $g_w(u) = g(u + w) - \sqrt{t^2 - |w|^2}$. Since the infimum of Lipschitz functions with common Lipschitz constant is itself Lipschitz, it follows that the intersection of $\partial D_{1,t}$ with the interior of $B_r' \times [-2a, 2a]$ is the graph of a Lipschitz function of $d - 1$ variables. We may then use a compactness argument to deduce that there exists $\delta > 0$ such that for $0 < t < \varepsilon$ the set $D_{1,t}$ is Lipschitz. \square

It follows from Lemma 2.2 that given any $D \in \mathcal{O}(\mathbb{R}^d)$ and $0 < \alpha < 1$, the set D_{R,R^α} is in $\mathcal{O}(\mathbb{R}^d)$ for all large enough R . We now define some properties of functions parametrized by point sets (or restrictions of point sets on bounded domains).

Definition 2.5. Let $D \in \mathcal{O}(\mathbb{R}^d)$. A measurable function $S : \mathcal{D}(D) \times \mathcal{A}_{\rho_1, \rho_2} \rightarrow \mathbb{R}$ is said to be:

- *local on $\mathcal{D}(D)$* if for all $\hat{D} \in \mathcal{D}(D)$, and all $\zeta, \tilde{\zeta} \in \mathcal{A}_{\rho_1, \rho_2}$ such that $\zeta \cap \hat{D} = \tilde{\zeta} \cap \hat{D}$, we have

$$S(\hat{D}, \zeta) = S(\hat{D}, \tilde{\zeta});$$

- *insensitive to boundary effects on $\mathcal{D}(D)$* if there exists $0 < \alpha < 1$ such that we have

$$(3) \quad \limsup_{R \rightarrow +\infty} \sup_{\zeta \in \mathcal{A}_{\rho_1, \rho_2}} \left\{ \frac{|S(D_R, \zeta) - S(D_{R, R^\alpha}, \zeta)|}{R^d} \right\} = 0$$

- Also, S is said to have the *averaging property* on $\mathcal{A}_{\rho_1, \rho_2}$ with respect to D if for any stationary and ergodic random lattice ζ whose realization almost surely belongs to $\mathcal{A}_{\rho_1, \rho_2}$, there exists a deterministic $\bar{S} = \bar{S}(\zeta) \in \mathbb{R}$ such that almost surely

$$(4) \quad \lim_{R \rightarrow +\infty} \frac{S(D_R, \zeta)}{|D_R|} = \bar{S}.$$

If S is local, then we can properly define $S(D, \zeta)$ for a point set ζ only defined in $D \in \mathcal{O}(\mathbb{R}^d)$, provided it is general and (ρ_1, ρ_2) -admissible in D . We do this as follows: let ζ^* be a choice of general and (ρ_1, ρ_2) -admissible set in \mathbb{R}^d such that $\zeta^* \cap D = \zeta$, and set $S(D, \zeta) := S(D, \zeta^*)$. By locality this definition does not depend on the choice of ζ^* , and there does exist at least one such choice of ζ^* because if we choose $\rho_0 \in (\rho_1, \rho_2)$ and perform the random parking process in \mathbb{R}^d as described in section 2.1, but rejecting any point that lies within distance ρ_0 of any point of ζ , then the resulting set of accepted points, together with the point set ζ itself, will be (ρ_1, ρ_2) -admissible on the whole of \mathbb{R}^d , and almost surely general.

In Sections 3 and 4 we shall give two examples — or rather two classes of functionals — which satisfy the above properties. Both examples exhibit subadditivity, and we shall make use of the Akcoglu-Krengel subadditive ergodic theorem to show they satisfy the averaging property.

Let $0 < \rho_1 < \rho_0 < \rho_2 < \infty$, and for $D \in \mathcal{O}(\mathbb{R}^d)$, let ξ (respectively ξ^D) be the random parking measure on \mathbb{R}^d (respectively on D) with parameter ρ_0 (as in Definition 2.1). The main result of this section is the following:

Theorem 2.3. *Let $D \in \mathcal{O}(\mathbb{R}^d)$. If the measurable function $S : \mathcal{D}(D) \times \mathcal{A}_{\rho_1, \rho_2} \rightarrow \mathbb{R}^+$ is local on $\mathcal{D}(D)$, insensitive to boundary effects on $\mathcal{D}(D)$, and has the averaging property on $\mathcal{A}_{\rho_1, \rho_2}$ with respect to D , then with \bar{S} given by (4), almost surely*

$$(5) \quad \lim_{R \rightarrow +\infty} \frac{S(D_R, \xi^{D_R})}{|D_R|} = \lim_{R \rightarrow +\infty} \frac{S(D_R, \xi)}{|D_R|} = \bar{S}.$$

Theorem 2.3 can be seen as a version of the ‘subadditive ergodic theorem’ for random parking measures on homothetic sets. It is proved using the following result due to Schreiber, Yukich and the second author [SPY], which gives an ‘exponential stabilization’ property of ξ . Let us say $\mathcal{Y} \subset \mathbb{R}^d \times \mathbb{R}^+$ is *temporally locally finite* if $\mathcal{Y} \cap (\mathbb{R}^d \times [0, t])$ is finite for all $t \in (0, \infty)$. Also, for $r > 0$ let B_r denote the Euclidean ball $\{x \in \mathbb{R}^d : |x| \leq r\}$.

Lemma 2.4. *There exists a nonnegative random variable V such that (i) V has an exponentially decaying tail, i.e. for some constant $c > 0$ the probability that V exceeds t is at most $c \exp(-c^{-1}t)$ for all $t \geq 0$; and (ii) for all temporally locally finite $\mathcal{Y} \subset (\mathbb{R}^d \setminus B_V) \times \mathbb{R}^+$ the measures ξ^{B_V} and $\xi^{B_V, \mathcal{Y}}$ coincide on $[-1/2, 1/2]^d$, where $\xi^{B_V, \mathcal{Y}}$ is the parking measure induced by the union of $\mathcal{P} \cap (B_V \times \mathbb{R}^+)$ and \mathcal{Y} .*

Proof. Essentially this result is Lemma 3.5 of [SPY]. That result says only that the total number of points in $[-1/2, 1/2]^d$ coincide for $\xi^{B_V, \mathcal{Y}}$ and ξ^{B_V} , not that the measures themselves coincide on $[-1/2, 1/2]^d$. However, the proof in [SPY] in fact demonstrates the stronger statement given here. The proof in [SPY] assumes that $\rho_0 < 1/4$ (see [SPY], p.173) but once we have the result for this case, we can deduce it for the general case by a scaling argument. \square

In the next lemma, we use Lemma 2.4 to show that $S(D_{R, R^\alpha}, \xi)$ is a good approximation to $S(D_{R, R^\alpha}, \xi^{D_R})$.

Lemma 2.5. *Suppose $D \in \mathcal{O}(\mathbb{R}^d)$ and $S : \mathcal{D}(D) \times \mathcal{A}_{\rho_1, \rho_2} \rightarrow \mathbb{R}^+$ is local on $\mathcal{D}(D)$. Suppose $0 < \alpha < 1$. Then there exists an almost surely finite random variable R_0 such that for $R \geq R_0$ we have $S(D_{R, R^\alpha}, \xi^{D_R}) = S(D_{R, R^\alpha}, \xi)$.*

Proof. For $i \in \mathbb{Z}^d$ and $r > 0$, let B_r^i denote the closed Euclidean ball of radius r centred at i , and let $\overline{Q}_{1/2}^i$ denote the rectilinear unit cube centred at i . By Lemma 2.4, there exists a family of random variables $(V(i), i \in \mathbb{Z}^d)$ which are identically distributed with exponentially decaying tails, such that $\xi^{B_{V(i)}^i, \mathcal{Y}}$ coincides with $\xi^{B_{V(i)}^i}$ on $\overline{Q}_{1/2}^i$, for all temporally locally finite $\mathcal{Y} \subset (\mathbb{R}^d \setminus B_{V(i)}^i) \times \mathbb{R}^+$, where $\xi^{B_{V(i)}^i, \mathcal{Y}}$ denotes the parking measure induced by the union of $\mathcal{P} \cap (B_{V(i)}^i \times \mathbb{R}^+)$ and \mathcal{Y} .

Let $D_{R, R^\alpha}^* := \{i \in \mathbb{Z}^d : \overline{Q}_{1/2}^i \cap D_{R, R^\alpha} \neq \emptyset\}$. For $k \in \mathbb{N}$, let $E(k)$ denote the event that there exists $R \in [k, k+1[$ and $i \in D_{R, R^\alpha}^*$ such that $B_{V(i)}^i \setminus D_R \neq \emptyset$. If $E(k)$ does not occur, then for all $R \in [k, k+1[$ and all $i \in D_{R, R^\alpha}^*$ we have $B_{V(i)}^i \subset D_R$, and therefore both ξ and ξ^{D_R} coincide with $\xi^{B_{V(i)}^i}$ on $\overline{Q}_{1/2}^i$. Hence ξ and ξ^{D_R} coincide on the whole of D_{R, R^α} .

Choose K such that $D \subset B_K$. Suppose $k \in \mathbb{N}$ and $R \in [k, k+1[$. Then for $i \in D_{R, R^\alpha}^*$, since i is distant at least $R^\alpha - d$ from $\mathbb{R}^d \setminus D_R$ by the definition of D_{R, R^α}^* , we have that $B_{k^\alpha - d}^i \subset B_{R^\alpha - d}^i \subset D_R$. Also, we have $D_{R, R^\alpha}^* \subset B_{KR+d} \subset B_{K(k+1)+d}$.

Therefore, for large enough $k \in \mathbb{N}$ the probability that $E(k)$ occurs is bounded by the cardinality of $\mathbb{Z}^d \cap B_{K(k+1)+d}$, multiplied by the probability that $V(0) \geq k^\alpha - d$, and by Lemma 2.4 this is bounded by $ck^d \exp(-c^{-1}k^\alpha)$ for some universal constant $c > 0$. Hence, the probability of $E(k)$ is summable over $k \in \mathbb{N}$, and from the Borel-Cantelli lemma we deduce that almost surely, $E(k)$ occurs for only finitely many k so that there exists R_0 such that for all $R \geq R_0$ the measures ξ and ξ^{D_R} coincide on D_{R, R^α} , so that $S(D_{R, R^\alpha}, \xi) = S(D_{R, R^\alpha}, \xi^{D_R})$ by locality. \square

Proof of Theorem 2.3. Since ξ is ergodic by Proposition 2.1, and S has the averaging property in the sense of Definition 2.5, there exists $\overline{S} \in \mathbb{R}^+$ such that (4) holds, i.e. the second equality of (5) holds. Choose $\alpha \in (0, 1)$ such that (3) holds. To prove the first equality of (5), for all $R \geq 1$, we write ξ^R for ξ^{D_R} and rewrite the averaged energy as

$$(6) \quad \frac{S(D_R, \xi^R)}{|D_R|} = \frac{S(D_R, \xi)}{|D_R|} + e_1(R) + e_2(R) + e_3(R),$$

with

$$\begin{aligned} e_1(R) &:= \frac{S(D_{R,R^\alpha}, \xi)}{|D_R|} - \frac{S(D_R, \xi)}{|D_R|}, \\ e_2(R) &:= \frac{S(D_{R,R^\alpha}, \xi^R)}{|D_R|} - \frac{S(D_{R,R^\alpha}, \xi)}{|D_R|}, \\ e_3(R) &:= \frac{S(D_R, \xi^R)}{|D_R|} - \frac{S(D_{R,R^\alpha}, \xi^R)}{|D_R|}. \end{aligned}$$

Now $e_2(R)$ tends to zero almost surely by Lemma 2.5, and by the insensitivity to boundary effects $e_1(R)$ and $e_3(R)$ tend to zero almost surely as $R \rightarrow \infty$, so that the first equality of (5) follows. \square

3. APPLICATION TO CLASSICAL EUCLIDEAN OPTIMIZATION PROBLEMS

A measurable functional $S : \mathcal{O}(\mathbb{R}^d) \times \mathcal{A}_{lf}^d \rightarrow \mathbb{R}$ is said to be *translation invariant* if for every $y \in \mathbb{R}^d$, $D \in \mathcal{O}(\mathbb{R}^d)$, and every $\zeta \in \mathcal{A}_{lf}^d$, we have $S(D + y, \zeta + y) = S(D, \zeta)$, where for any subset U of \mathbb{R}^d , $U + y := \{z + y : z \in U\}$.

Definition 3.1. Let $S : \mathcal{O}(\mathbb{R}^d) \times \mathcal{A}_{lf}^d \rightarrow \mathbb{R}$ be a translation invariant measurable functional, and let $p \geq 0$. We say that S is

- *localized* if there exists $\tilde{S} : \mathcal{A}_{lf}^d \rightarrow \mathbb{R}$ such that for all $\zeta \in \mathcal{A}_{lf}^d$ and $D \in \mathcal{O}(\mathbb{R}^d)$, $S(D, \zeta) = \tilde{S}(\zeta \cap D)$,
- *almost subadditive* of order p if there exists $C > 0$ such that for all $\zeta_1, \zeta_2 \in \mathcal{A}_{lf}^d$ and all $D \in \mathcal{O}(\mathbb{R}^d)$,

$$S(D, \zeta_1 \cup \zeta_2) \leq S(D, \zeta_1) + S(D, \zeta_2) + C \text{diam}(D)^p;$$

- (strongly) *superadditive* on rectangles if for any partition of a semi-open rectangle D into a finite number k of semi-open rectangles $\{D_i\}_{i \in \{1, \dots, k\}}$,

$$S(D, \zeta) \geq \sum_{i=1}^k S(D_i, \zeta);$$

- *smooth* of order p if there exists $C > 0$ such that for every rectilinear cube D and all $\zeta_1, \zeta_2 \in \mathcal{A}_{lf}^d$,

$$(7) \quad |S(D, \zeta_1 \cup \zeta_2) - S(D, \zeta_1)| \leq C \text{diam}(D)^p (\text{card } \zeta_2 \cap D)^{(d-p)/d}.$$

- *homogeneous* of order p if for all $\alpha > 0$, $D \in \mathcal{O}(\mathbb{R}^d)$, and $\zeta \in \mathcal{A}_{lf}^d$, we have $S(D_\alpha, \alpha\zeta) = \alpha^p S(D, \zeta)$, where $\alpha\zeta := \{\alpha x : x \in \zeta\}$.

Definition 3.2. Two measurable functionals S_1 and S_2 are said to be pointwise close of order $p \geq 0$ if for every rectilinear cube $D \subset \mathbb{R}^d$ and every $\zeta \in \mathcal{A}_{lf}^d$, we have

$$(8) \quad \frac{|S_1(D, \zeta) - S_2(D, \zeta)|}{\text{diam}(D)^p} = o\left((\text{card } \zeta \cap D)^{(d-p)/d}\right)$$

in the sense that there is a function $g : \mathbb{Z}^+ \rightarrow \mathbb{R}^+$ with $\lim_{k \rightarrow \infty} k^{(p-d)/d} g(k) = 0$, such that the left side of (8) is bounded by $g(\text{card } \zeta \cap D)$ uniformly over D and ζ .

Regarding terminology, note that the condition that S be localized (in the sense of Definition 3.1) is stronger than the condition that S be local (in the sense of Definition 2.5).

A translation invariant measurable function S that is also homogeneous of order $p \geq 0$ and satisfies $S(D, \emptyset) = 0$ for all D , is called a *Euclidean functional*. See [Yuk98] for examples of Euclidean functionals.

We are now in position to state an umbrella theorem for translation invariant functionals on the random parking measure. Let ξ and ξ^D be as in Definition 2.1

Theorem 3.1. *Let $\rho_0 > 0$. Let $1 \leq p < d$, and let S and T be measurable translation invariant smooth functionals of order p . Assume that S is localized and almost subadditive of order p , and T is superadditive. If S and T are pointwise close of order p , then there exists deterministic $\bar{S} \in \mathbb{R}^+$ such that for all $D \in \mathcal{O}(\mathbb{R}^d)$, we have almost surely*

$$\lim_{R \rightarrow \infty} \frac{S(D_R, \xi^{D_R})}{|D_R|} = \lim_{R \rightarrow \infty} \frac{S(D_R, \xi)}{|D_R|} = \bar{S}.$$

Remark 1. We do not assume homogeneity of S and T in the statement of this theorem. However, if in fact we do assume that S and T are homogeneous of order p , then to check S is smooth of order p we need to check (7) only for $D = [0, 1]^d$, and to check S and T are pointwise close we need to check (8) only for $D = [0, 1]^d$.

Remark 2. By [Yuk98, Lemmas 3.2, 3.5, and 3.7], the p -power weighted total length version of the traveling salesman problem, the minimal spanning tree, and the minimal matching all satisfy the assumptions of Theorem 3.1 with exponent p . In particular, these examples are homogeneous and the associated superadditive Euclidean functions T are the so-called canonical boundary functionals (denoted by S_B in [Yuk98]).

We believe there could be interesting examples where homogeneity fails but our theorem is still applicable, for example to do with the number of percolation components. See [Yuk98], page 47 for a discussion of examples where homogeneity fails.

Remark 3. The proof of Theorem 3.1 shows that its conclusion holds for any bounded connected subset D of \mathbb{R}^d whose boundary has zero d -Lebesgue measure.

Remark 4. In the proof of Theorem 3.1 we show that S has the averaging property on $\mathcal{A}_{\rho_1, \infty}$ with respect to D for any $D \in \mathcal{O}(\mathbb{R}^d)$ and for any $\rho_1 > 0$ (in particular we need only the hard-core property).

Proof of Theorem 3.1. We split the proof into five steps. We first prove that S satisfies the insensitivity to boundary effects. In the second step, we address the averaging property of T on cubes with vertices in \mathbb{Z}^d , then that of S on cubes with vertices in \mathbb{R}^d , and finally on any domain in $\mathcal{O}(\mathbb{R}^d)$. Appealing then to Theorem 2.3 we will conclude the proof.

Step 1. Insensitivity of S to boundary effects.

Let $D \in \mathcal{O}(\mathbb{R}^d)$ and $0 \leq \alpha < 1$. Let \hat{Q} be a cube aligned with the canonical basis of \mathbb{R}^d and which contains D . By localization and smoothness of S , for all $\zeta \in \mathcal{A}_{\rho_1, \infty}$,

$$\begin{aligned} |S(D_R, \zeta) - S(D_{R, R^\alpha}, \zeta)| &= |S(\hat{Q}_R, \zeta \cap D_R) - S(\hat{Q}_R, \zeta \cap D_{R, R^\alpha})| \\ &\leq CR^p (\text{card } \zeta \cap D_R \setminus D_{R, R^\alpha})^{(d-p)/d}. \end{aligned}$$

Since the boundary of D has zero d -Lebesgue measure, we can cover the boundary of D_R by $o(R^{(1-\alpha)d})$ balls of radius R^α . Provided $R \geq 1$, the corresponding balls of radius $3R^\alpha$ cover $D_R \setminus D_{R, R^\alpha}$ and by the hard-core condition, each such ball contains $O(R^{d\alpha})$ points of ζ . Hence $\text{card } \zeta \cap D_R \setminus D_{R, R^\alpha}$ is $o(R^d)$, and hence $|S(D_R, \zeta) - S(D_{R, R^\alpha}, \zeta)| = o(R^d)$. Thus (3) holds, so S is insensitive to boundary effects on $\mathcal{D}(D)$.

Step 2. Superadditive ergodic theorem on rectangles with vertices in \mathbb{Z}^d .

Let ζ be a stationary ergodic random lattice taking values in $\mathcal{A}_{\rho_1, \infty}$. Let \mathcal{R} denote the collection of half-open rectilinear rectangles in \mathbb{R}^d with vertices in \mathbb{Z}^d , and let \mathcal{C} be the class of all cubes

in \mathcal{R} . By localization $S(D, \emptyset)$ is a constant $\tilde{S}(\emptyset)$ independent of D . By pointwise closeness, there is a constant C such that for every $D \in \mathcal{C}$ we have that

$$T(D, \emptyset) \leq S(D, \emptyset) + C(\text{diam}(D))^p$$

and hence by taking $\zeta_1 = \emptyset$ and $\zeta_2 = \zeta$ in the definition of the smoothness of T , we have for a possibly different C that for all $D \in \mathcal{C}$, almost surely

$$T(D, \zeta) \leq C \text{diam}(D)^p (1 + \text{card } \zeta \cap D)^{(d-p)/d}.$$

The hard-core condition combined with an elementary geometric argument imply there exists $C > 0$ depending only on ρ_1 such that $\text{card } \zeta \cap D \leq C|D|$ for every $D \in \mathcal{R}$. Hence, there is a further C such that if $D \in \mathcal{C}$ then

$$(9) \quad \mathbb{E}T(D, \zeta) \leq C|D|.$$

However, this implies that (9) also holds whenever $D \in \mathcal{R}$, since if not, then we could take a large cube in \mathcal{C} which was the union of disjoint translates of D and using stationarity of ζ and translation-invariance and superadditivity of T , we would have a violation of (9) for this cube.

We may now apply the Akcoglu-Krengel superadditive ergodic theorem [AK81] to the functional $D \mapsto T(D, \zeta)$ which is stationary under integer shifts. By the Akcoglu-Krengel theorem, for any unit cube $\hat{Q} \in \mathcal{C}$ there is a random variable $\overline{S}(\hat{Q})$ such that almost surely

$$(10) \quad \lim_{R \rightarrow +\infty, R \in \mathbb{N}} \frac{T(\hat{Q}_R, \zeta)}{R^d} = \overline{S}(\hat{Q}).$$

Indeed, if \hat{Q} is contained in the positive orthant then the sequence of cubes \hat{Q}_R is regular in the sense of Akcoglu and Krengel. If not, then (since it is a unit cube) \hat{Q} is contained in some other orthant and the argument is similar.

By the pointwise closeness of S and T , and the hard-core condition on ζ , (10) implies that for $\hat{Q} \in \mathcal{C}$ a unit cube we have almost surely

$$(11) \quad \lim_{R \rightarrow +\infty, R \in \mathbb{N}} \frac{S(\hat{Q}_R, \zeta)}{R^d} = \overline{S}(\hat{Q}).$$

We claim that $\overline{S}(\hat{Q})$ is deterministic and the same for all unit cubes $\hat{Q} \in \mathcal{C}$ (hence subsequently denoted by simply \overline{S}). Indeed, given such a \hat{Q} and given $z \in \mathbb{R}^d$, we may deduce by localization and smoothness of S that

$$|S(\hat{Q}_R, \zeta) - S(\hat{Q}_R \cap (\hat{Q}_R + z), \zeta)| \leq CR^p (\text{card } \zeta \cap \hat{Q}_R \setminus (\hat{Q}_R + z))^{(d-p)/d}.$$

By the hard-core property of ζ , there is a constant C depending on ρ_1 and z such that $\text{card } \zeta \cap \hat{Q}_R \setminus (\hat{Q}_R + z)$ is asymptotically bounded by CR^{d-1} , so that

$$(12) \quad \limsup_{R \rightarrow \infty, R \in \mathbb{N}} \frac{|S(\hat{Q}_R, \zeta) - S(\hat{Q}_R \cap (\hat{Q}_R + z), \zeta)|}{R^d} = 0$$

and likewise $R^{-d}|S(\hat{Q}_R + z, \zeta) - S(\hat{Q}_R \cap (\hat{Q}_R + z), \zeta)| \rightarrow 0$ as $R \rightarrow \infty$ through \mathbb{N} . Hence by (11) we have

$$\lim_{R \rightarrow +\infty, R \in \mathbb{N}} \frac{S(\hat{Q}_R + z, \zeta)}{R^d} = \overline{S}(\hat{Q}).$$

Hence $\overline{S}(\hat{Q})$ is shift invariant, and therefore constant by ergodicity. Also, by translation invariance of S and stationarity of ζ , the distribution of $\overline{S}(\hat{Q})$ is the same for all unit cubes $\hat{Q} \in \mathcal{C}$, so we have justified our claim.

Step 3. Extension of (11) to general cubes.

Let ζ be as in Step 2. Let $\hat{Q} \in \mathcal{C}$ be a cube of side k with $k \in \mathbb{N}$. Let \hat{Q}^i , $1 \leq i \leq k^d$, be disjoint unit cubes in \mathcal{C} , whose union is \hat{Q} . Then by superadditivity of T ,

$$(13) \quad \liminf_{R \rightarrow \infty, R \in \mathbb{N}} \frac{T(\hat{Q}_R, \zeta)}{R^d} \geq \liminf_{R \rightarrow \infty, R \in \mathbb{N}} \frac{1}{R^d} \sum_{i=1}^{k^d} T((\hat{Q}^i)_R, \zeta) = k^d \bar{S}.$$

Also, by repeated use of the almost subadditivity and localization of S , as in [Yuk98, Remark 1 (3.5)], there is a constant $C(k)$ such that

$$S(\hat{Q}_R, \zeta) \leq \sum_{i=1}^{k^d} S((\hat{Q}^i)_R, \zeta) + C(k)R^p,$$

so that

$$(14) \quad \limsup_{R \rightarrow \infty, R \in \mathbb{N}} \frac{S(\hat{Q}_R, \zeta)}{R^d} \leq \limsup_{R \rightarrow \infty, R \in \mathbb{N}} \frac{1}{R^d} \sum_{i=1}^{k^d} S((\hat{Q}^i)_R, \zeta) = k^d \bar{S}.$$

Moreover by closeness of T and S , and the hard-core property of ζ ,

$$(15) \quad \limsup_{R \rightarrow \infty, R \in \mathbb{N}} \frac{1}{R^d} |T(\hat{Q}_R, \zeta) - S(\hat{Q}_R, \zeta)| = 0.$$

Combining (13), (14) and (15) we may deduce that (10) and (11) hold for all $\hat{Q} \in \mathcal{C}$, with $\bar{S}(\hat{Q}) = |\hat{Q}| \bar{S}$.

Next, for $\hat{Q} \in \mathcal{C}$ we relax the assumption that $R \in \mathbb{N}$ in the asymptotic formula (11). Denoting by $[R]$ the integer part of R , by using the localization and smoothness of S and the hard-core property of ζ as in the proof of (12) above, we have

$$\limsup_{R \rightarrow \infty} \frac{|S(\hat{Q}_R, \zeta) - S(\hat{Q}_{[R]}, \zeta)|}{R^d} = 0.$$

Combined with (11) this shows that

$$(16) \quad \lim_{R \rightarrow +\infty} \frac{S(\hat{Q}_R, \zeta)}{|\hat{Q}_R|} = \bar{S}.$$

Now let \hat{Q} be a general rectilinear cube in \mathbb{R}^d . For all $\tilde{R} > 0$ large enough, there exist $k_{\tilde{R}} \in \mathbb{N}$ with $k_{\tilde{R}} \geq \tilde{R} - 2 > 0$ and a rectilinear cube $\hat{Q}^{k_{\tilde{R}}}$ with vertices in \mathbb{Z}^d and sidelength $k_{\tilde{R}}$ such that $\hat{Q}^{k_{\tilde{R}}} \subset \hat{Q}_{\tilde{R}}$. As above, the hard-core property of ζ yields the estimate

$$\text{card } \zeta \cap \hat{Q}_{R\tilde{R}} \setminus (\hat{Q}^{k_{\tilde{R}}})_R = \text{card } \zeta \cap (\hat{Q}_{\tilde{R}} \setminus \hat{Q}^{k_{\tilde{R}}})_R \leq CR^d \tilde{R}^{d-1}$$

so that localization and smoothness of S yields

$$\limsup_{R \rightarrow \infty} \frac{|S((\hat{Q}^{k_{\tilde{R}}})_R, \zeta) - S(\hat{Q}_{R\tilde{R}}, \zeta)|}{|\hat{Q}_{R\tilde{R}}|} \leq C \tilde{R}^{(p-d)/d},$$

so using (16) applied to $\hat{Q}^{k_{\tilde{R}}}$, and the arbitrariness of \tilde{R} , we obtain

$$(17) \quad \lim_{R \rightarrow +\infty} \frac{S(\hat{Q}_R, \zeta)}{|\hat{Q}_R|} = \bar{S}.$$

Step 4. Extension of (17) to general domains $D \in \mathcal{O}(\mathbb{R}^d)$.

Let ζ be as in Step 2. Let D^+ be a cube aligned with the canonical basis of \mathbb{Z}^d , containing D , and such that its sidelength δ is bounded by $\text{diam}(D)$.

We proceed by approximation. For every integer $j \geq 1$, let $\varepsilon_j = 2^{-j}\delta$, and split D^+ into 2^{jd} cubes $\{\hat{Q}^{j,i}\}_{i \in \{1, \dots, 2^{jd}\}}$ of sidelength ε_j . Define $D^j = \cup_{i \in I^j} \hat{Q}^{j,i}$ as the union of those cubes $\hat{Q}^{j,i}$ (indexed by $i \in I^j$) whose intersection with D is not empty, and define $\underline{D}^j = \cup_{i \in \underline{I}^j} \hat{Q}^{j,i}$ as the union of those cubes $\hat{Q}^{j,i}$ (indexed by $i \in \underline{I}^j$) which are contained in D . In particular, I^j is finite and $\underline{D}^j \subset D \subset D^j \subset D^+$.

Let $\eta > 0$. Using the fact that the boundary of D has zero Lebesgue measure, choose j such that $|D^j \setminus \underline{D}^j| \leq \eta$, so that $D^j \setminus \underline{D}^j$ consists of at most $\eta \varepsilon_j^{-d}$ cubes of side ε_j . This j will be fixed for a while. By the hard-core constraint there is a constant C depending only on d and ρ_1 , such that for R so large that $R\varepsilon_j > 1$ we have

$$(18) \quad \text{card } \zeta \cap (D^j \setminus \underline{D}^j)_R \leq C\eta \varepsilon_j^{-d} (R\varepsilon_j)^d = CR^d \eta.$$

Hence by localization and smoothness of S ,

$$(19) \quad \limsup_{R \rightarrow \infty} \frac{|S(D_R, \zeta) - S((D^j)_R, \zeta)|}{R^d} = \limsup_{R \rightarrow \infty} \frac{|S(D_R^+, \zeta \cap D_R) - S(D_R^+, \zeta \cap (D^j)_R)|}{R^d} \leq C\eta^{(d-p)/d}.$$

Using localization and repeatedly using almost-subadditivity, as in [Yuk98, Remark 1 (3.5)], yields for some constant $C(j)$ that

$$S((D^j)_R, \zeta) = S(D_R^+, \zeta \cap (D^j)_R) \leq \sum_{i=1}^{2^{jd}} S((\hat{Q}^{j,i})_R, \zeta \cap (D^j)_R) + C(j)R^p.$$

Since $S((\hat{Q}^{j,i})_R, \zeta \cap (D^j)_R)$ is a constant $\tilde{S}(\emptyset)$ for $i \notin I^j$, this turns into

$$S((D^j)_R, \zeta) \leq \sum_{i \in I^j} S((\hat{Q}^{j,i})_R, \zeta) + C(j)R^p.$$

Hence, using (19) and using (17) for each cube $\hat{Q}^{j,i}$, $i \in I^j$, this yields

$$(20) \quad \limsup_{R \rightarrow +\infty} R^{-d} S(D_R, \zeta) \leq C\eta^{\frac{d-p}{d}} + \sum_{i \in I^j} |\hat{Q}^{j,i}| \bar{S} \leq C\eta^{\frac{d-p}{d}} + (|D| + \eta)\bar{S}.$$

We now turn to the lower bound. By almost subadditivity and localization of S ,

$$(21) \quad S(D_R^+, \zeta) \leq S(D_R, \zeta) + S(D_R^+ \setminus D_R, \zeta) + CR^p.$$

By localization and smoothness of S ,

$$|S(D_R^+ \setminus D_R, \zeta) - S(D_R^+ \setminus (\underline{D}^j)_R, \zeta)| \leq CR^p (\text{card } \zeta \cap D_R \setminus (\underline{D}^j)_R)^{(d-p)/d},$$

and by (18) this gives us

$$(22) \quad \limsup_{R \rightarrow \infty} \frac{|S(D_R^+ \setminus D_R, \zeta) - S(D_R^+ \setminus (\underline{D}^j)_R, \zeta)|}{R^d} \leq C\eta^{1-p/d}.$$

We now treat $S(D_R^+ \setminus (\underline{D}^j)_R, \zeta)$: by localization and almost-subadditivity,

$$(23) \quad \begin{aligned} S(D_R^+ \setminus (\underline{D}^j)_R, \zeta) &= S(D_R^+, \zeta \cap D_R^+ \setminus (\underline{D}^j)_R) \\ &\leq \sum_{i=1}^{2^{jd}} S((\hat{Q}^{j,i})_R, \zeta \cap D_R^+ \setminus (\underline{D}^j)_R) + C(j)R^p \\ &= \sum_{i \notin \underline{I}^j} S((\hat{Q}^{j,i})_R, \zeta) + C(j)R^p. \end{aligned}$$

Applying (21), (22) and (23) in turn, and using (17) for each cube $\hat{Q}^{j,i}$ and for D^+ , we obtain

$$\begin{aligned}
\liminf_{R \rightarrow +\infty} R^{-d} S(D_R, \zeta) &\geq |D^+| \bar{S} - \limsup_{R \rightarrow +\infty} R^{-d} S(D_R^+ \setminus D_R, \zeta) \\
&\geq |D^+| \bar{S} - C \eta^{1-p/d} - \limsup_{R \rightarrow +\infty} R^{-d} S(D_R^+ \setminus (\underline{D}^j)_R, \zeta) \\
&\geq |D^+| \bar{S} - C \eta^{1-p/d} - |D^+ \setminus \underline{D}^j| \bar{S} \\
&\geq (|D| - \eta) \bar{S} - C \eta^{1-p/d}.
\end{aligned}$$

Combined with (20) and using the arbitrariness of η , this concludes the proof of the averaging property (4) for S .

Step 5. Conclusion.

We can now apply Theorem 2.3 to S , since in Step 1 we have shown the insensitivity to boundary effects, and in Step 4 we have shown the averaging property. \square

4. APPLICATION TO THE DERIVATION OF RUBBER ELASTICITY

4.1. A discrete model for rubber and its thermodynamic limit. Let $0 < \rho_1 < \rho_2 < \infty$. Suppose that $\mathcal{L} \in \mathcal{A}_{\rho_1, \infty}$ and \mathcal{L} has at least $d + 1$ elements (so its convex hull has strictly positive d -Lebesgue measure, since \mathcal{L} is assumed general). Then there is a unique Delaunay triangulation of the convex hull of \mathcal{L} , by simplices with edges given by the edges of the Delaunay graph of \mathcal{L} , which is itself the dual graph of the Voronoi tessellation (see [OBSNC], and [Del34] for Delaunay tessellations of \mathbb{R}^d , and for instance [For95] for Delaunay tessellations of a bounded domain). By convention we consider simplices to be open sets. Let $\mathcal{T}(\mathcal{L})$ denote the Delaunay triangulation of the convex hull of \mathcal{L} . For any $d + 1$ points of \mathcal{L} , the simplex generated by these points is in $\mathcal{T}(\mathcal{L})$ if and only if no point of \mathcal{L} lies in the interior of the circumsphere of these points. We denote by $\mathcal{N}(\mathcal{L})$ the associated neighbour pairs, that is, those unordered pairs of points $\{x, y\}$ such that (x, y) is an edge of $\mathcal{T} := \mathcal{T}(\mathcal{L})$. For all $\varepsilon > 0$ and all open sets $D \in \mathcal{O}(\mathbb{R}^d)$, we define a space of continuous piecewise-affine functions $\mathfrak{S}_\varepsilon^D(\mathcal{L})$ on D , by

$$(24) \quad \mathfrak{S}_\varepsilon^D(\mathcal{L}) := \{u \in C(D, \mathbb{R}^n) \mid \forall T \in \mathcal{T}(\mathcal{L}), \text{ with } \varepsilon T \cap D \neq \emptyset, u|_{\varepsilon T \cap D} \text{ is affine}\}.$$

Let \bar{D} denote the closure of D . From now on, we identify $u : \varepsilon \mathcal{L} \cap \bar{D} \rightarrow \mathbb{R}^n$ with its class of piecewise-affine interpolations (still denoted by u) in $\mathfrak{S}_\varepsilon^D(\mathcal{L}) \subset W^{1, \infty}(D, \mathbb{R}^n)$. Note that the extension of $u : \varepsilon \mathcal{L} \cap \bar{D} \rightarrow \mathbb{R}^n$ to $D \setminus \cup_{T \in \mathcal{T}, T \subset D} T$ is not uniquely defined — as we shall see, the energy under consideration does not depend on the extension. In order to define an energy functional on the set $\mathfrak{S}_\varepsilon^D(\mathcal{L})$, we first introduce the following energy functions:

Definition 4.1. Let $p > 1$. We denote by \mathcal{U}_p the subset of functions f_{nn} of $C(\mathbb{R}^d \times \mathbb{R}^n, \mathbb{R}^+)$ for which there exists $C > 0$ such that, for all $z \in \mathbb{R}^d$ and $s \in \mathbb{R}^n$,

$$(25) \quad \frac{1}{C} |s|^p - C \leq f_{nn}(z, s) \leq C(|s|^p + 1).$$

We denote by \mathcal{V}_p the subset of functions W_{vol} of $C(\mathbb{M}^{n \times d}, \mathbb{R}^+)$ for which there exists $C > 0$ such that for all $\Lambda \in \mathbb{M}^{n \times d}$,

$$(26) \quad W_{\text{vol}}(\Lambda) \leq C(|\Lambda|^p + 1).$$

Let $p > 1$ and $f_{nn} \in \mathcal{U}_p, W_{\text{vol}} \in \mathcal{V}_p$. For all $u \in L^1(D, \mathbb{R}^n)$ and open $D \in \mathcal{O}(\mathbb{R}^d)$ we then set

$$(27) \quad F_\varepsilon^D(\mathcal{L}, u) := \begin{cases} F_{nn, \varepsilon}^D(\mathcal{L}, u) + F_{\text{vol}, \varepsilon}^D(\mathcal{L}, u) & \text{if } u \in \mathfrak{S}_\varepsilon^D(\mathcal{L}), \\ +\infty & \text{else,} \end{cases}$$

where we set

$$(28) \quad F_{nn,\varepsilon}^D(\mathcal{L}, u) = \sum_{\{x,y\} \in \mathcal{N}(\varepsilon\mathcal{L}) : (x,y) \subset \overline{D}} \varepsilon^d f_{nn} \left(\varepsilon^{-1}(y-x), \frac{u(y)-u(x)}{|y-x|} \right),$$

and

$$(29) \quad F_{\text{vol},\varepsilon}^D(\mathcal{L}, u) = \sum_{T \in \mathcal{T}(\varepsilon\mathcal{L}) : T \subset D} |T| W_{\text{vol}}(\nabla u|_T).$$

As announced, if $u_1, u_2 \in \mathfrak{S}_\varepsilon^D(\mathcal{L})$ are such that $u_1 = u_2$ on $\cup_{T \in \mathcal{T}(\varepsilon\mathcal{L}), T \subset D} T$, then $F_\varepsilon^D(\mathcal{L}, u_1) = F_\varepsilon^D(\mathcal{L}, u_2)$.

A suitable notion to study the convergence of the functional $F_\varepsilon^D(\mathcal{L}) := F_\varepsilon^D(\mathcal{L}, \cdot)$ is Γ -convergence. We briefly recall its definition for the unfamiliar reader. We say that a functional $F_\varepsilon : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ Γ -converges to some functional $F : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ if the following two statements hold:

- (i) (liminf inequality) For all $v \in L^p(D, \mathbb{R}^n)$ and every sequence $v_k \in L^p(D, \mathbb{R}^n)$ which converges to v in $L^p(D, \mathbb{R}^n)$ and every sequence $\varepsilon_k \rightarrow 0$, we have

$$F(v) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}(v_k);$$

- (ii) (recovery sequence) For all $v \in L^p(D, \mathbb{R}^n)$ and every sequence $\varepsilon_k \rightarrow 0$, there exists a sequence $v_k \in L^p(D, \mathbb{R}^n)$ which converges to v in $L^p(D, \mathbb{R}^n)$ and such that

$$F(v) = \lim_{k \rightarrow \infty} F_{\varepsilon_k}(v_k).$$

The notion of Γ -convergence is natural for minimization problems since it ensures the convergence of minima and minimizers (recall that a set is precompact if its closure is compact):

Lemma 4.1. [BD98, Section 7.1] *Suppose that $F_{\varepsilon_k} : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ Γ -converges to some functional $F : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ as $k \rightarrow \infty$. If there exists a precompact sequence of minimizers u_k of F_{ε_k} in $L^p(D, \mathbb{R}^n)$, then*

$$\lim_{k \rightarrow \infty} \inf_{L^p(D, \mathbb{R}^n)} F_{\varepsilon_k} = \inf_{L^p(D, \mathbb{R}^n)} F.$$

Moreover, if $u_k \rightarrow u$ for some $u \in L^p(D, \mathbb{R}^n)$, then u is a minimum point for F .

For an introduction to Γ -convergence and its application to homogenization of integral functionals we refer the reader to [Br02, BD98], whence come the following useful definitions and properties of integral functionals:

Definition 4.2. We say that a function $W : \mathbb{M}^{n \times d} \rightarrow \mathbb{R}$ is *quasiconvex* if for all $\Lambda \in \mathbb{M}^{n \times d}$,

$$W(\Lambda) = \inf_u \left\{ \int_Q W(\Lambda + \nabla u(x)) dx : u \in W_0^{1,\infty}(D, \mathbb{R}^n) \right\}.$$

We say that it is *isotropic* if $W(\Lambda \mathbf{R}) = W(\Lambda)$ for all $\Lambda \in \mathbb{M}^{n \times d}$, $\mathbf{R} \in SO_d$. We say that it satisfies a *standard growth condition* of order $1 < p < \infty$ if there exists a positive constant C such that for all $\Lambda \in \mathbb{M}^{n \times d}$

$$(30) \quad C^{-1}|\Lambda|^p - C \leq W(\Lambda) \leq C(1 + |\Lambda|^p).$$

Let $D \in \mathcal{O}(\mathbb{R}^d)$ be an open set and let $W : D \times \mathbb{M}^{n \times d} \rightarrow \mathbb{R}$ be a measurable function. We say that W is *Carathéodory* if for almost every $x \in D$, $\Lambda \mapsto W(x, \Lambda)$ is continuous. In particular, this implies that for all $u \in W^{1,1}(D)$, the function $x \mapsto W(x, \nabla u(x))$ is measurable on D .

If in addition there exist $1 < p < \infty$ and $C > 0$ such that for almost every $x \in D$, $\Lambda \mapsto W(x, \Lambda)$ is quasiconvex and satisfies the standard growth condition (30), then the integral functional $F : W^{1,p}(D) \rightarrow \mathbb{R}$, $u \mapsto F(u) = \int_D W(x, \nabla u(x)) dx$ is finite and lower-semicontinuous for the weak topology of $W^{1,p}(D)$.

For $\Lambda \in \mathbb{M}^{n \times d}$, define the function $\varphi_\Lambda : \mathbb{R}^d \rightarrow \mathbb{R}^n$ by $x \mapsto \Lambda x$.

Recall that $Q_R := (-R, R)^d$ for all $R > 0$. In [ACG], Alicandro, Cicalese and the first author proved the following Γ -convergence (or discrete homogenization) result:

Theorem 4.2. [ACG, Theorem 5] *Let $0 < \rho_1 < \rho_2 < \infty$. Let \mathcal{L} be a stationary and ergodic stochastic lattice in $\mathcal{A}_{\rho_1, \rho_2}$. Let $p > 1$ and let f_{nn} and W_{vol} be of class \mathcal{U}_p and \mathcal{V}_p , respectively. Let $D \in \mathcal{O}(\mathbb{R}^d)$ be an open set, and let $F_\varepsilon^D(\mathcal{L})$ be the energy functional given by (27). Then the functionals $F_\varepsilon^D(\mathcal{L})$ almost surely Γ -converge as $\varepsilon \rightarrow 0$ to the deterministic integral functional $F_{\text{hom}}^D : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ defined by*

$$(31) \quad F_{\text{hom}}^D(u) = \begin{cases} \int_D W_{\text{hom}}(\nabla u(x)) dx & \text{if } u \in W^{1,p}(D, \mathbb{R}^n), \\ +\infty & \text{otherwise,} \end{cases}$$

where $W_{\text{hom}} : \mathbb{M}^{n \times d} \rightarrow \mathbb{R}^+$ is a quasiconvex function which depends only on f_{nn} , W_{vol} , and on the stochastic lattice, and which satisfies a standard growth condition (30) of order p . Also for all $\Lambda \in \mathbb{M}^{n \times d}$, W_{hom} satisfies the following asymptotic homogenization formula almost surely:

$$(32) \quad W_{\text{hom}}(\Lambda) = \lim_{R \rightarrow \infty} \frac{1}{|Q_R|} \inf_u \left\{ F_1^{Q_R}(\mathcal{L}, u) \mid u \in \mathfrak{S}_1^{Q_R}(\mathcal{L}), u \equiv \varphi_\Lambda \text{ on } \mathcal{L} \cap Q_R \setminus Q_{R-2\rho_2} \right\}.$$

For the link between the above result and the derivation of rubber elasticity from the statistical physics of interacting polymer-chains, we refer the reader to [ACG, Section 4.1], [GLTV], and the references therein. Note that the combination with Lemma 4.1 yields the convergence of minimum problems.

Note that in (32), the lattice \mathcal{L} and the domain Q_R correspond to ξ and D_R in Theorem 2.3, respectively. Yet, as we shall see, the function $(\mathcal{L}, D_R) \mapsto S(\mathcal{L}, D_R)$ defined by

$$S(\mathcal{L}, D_R) := \inf_u \left\{ F_1^{D_R}(\mathcal{L}, u) \mid u \in \mathfrak{S}_1^{D_R}(\mathcal{L}), u \equiv \varphi_\Lambda \text{ on } \mathcal{L} \cap D_R \setminus D_{R-2\rho_2} \right\},$$

is not local in the sense of Definition 2.5.

4.2. Isotropic homogenized energy density and approximation result. Let $0 < \rho_1 < \rho_0 < \rho_2 < \infty$, and let ξ denote the random parking measure of parameter ρ_0 in \mathbb{R}^d . Also, let $\xi^R := \xi^{Q_R}$ be the random parking measure of parameter ρ_0 in the bounded domain Q_R . Let $p > 1$ and let f_{nn} and W_{vol} be of class \mathcal{U}_p and \mathcal{V}_p , respectively.

Proposition 2.1 answers the first question of the introduction: there does exist an ergodic admissible stochastic lattice which is statistically isotropic, namely ξ :

Theorem 4.3. *Suppose $n = d$. Suppose there exists $\tilde{f}_{nn} : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that*

$$(33) \quad f_{nn}(z_1, z_2) = \tilde{f}_{nn}(|z_1|, z_2) \quad \forall z_1, z_2 \in \mathbb{R}^d,$$

and suppose also that W_{vol} is isotropic. Then the energy density W_{hom} defined by the asymptotic formula (32) (with $\mathcal{L} = \xi$ and $F_\varepsilon^{Q_R}$ given by (27)) is well-defined and isotropic.

Proof. Putting $\mathcal{L} = \xi$ gives \mathcal{L} satisfying the conditions of Theorem 4.2, so the existence of W_{hom} is ensured by that result. The isotropy of W_{hom} is a direct consequence of the isotropy of ξ in Proposition 2.1 and of [ACG, Theorem 9]. \square

We now turn to a first version of the second question: in the case of the random parking measure, does the asymptotic formula (32) hold if \mathcal{L} is replaced by ξ^R ? This question is particularly relevant for numerical approximations (see [GLTV]), since the approximation of W_{hom} on a bounded domain Q_R also requires an approximation of the random measure ξ on the domain Q_R . The answer is positive.

Before we state the result proper, let us provide more details on the approximation procedure. In Theorem 4.2, the infimum in (32) uses the Delaunay tessellation associated with the point process \mathcal{L} , which is slightly nonlocal (the restriction of the Delaunay tessellation to Q_R depends on points of \mathcal{L} that lie outside Q_R , although in some bounded annulus around Q_R due to the non-empty space condition). Hence, for the approximation process we need to define a local version of (32) through the introduction of some tessellation depending only on \mathcal{L} in Q_R . A natural choice would be to consider the Delaunay tessellation associated with $\mathcal{L} \cap Q_R$. This is however not suitable for the following reason: the edge lengths of this Delaunay tessellation are a priori only bounded by R — the Delaunay tessellation of a point set is a tessellation of the convex hull of the point set into d -simplices, so that there exist with positive probability configurations which have long edges. Arbitrarily large edge lengths are incompatible with the modeling of polymer chains (see [ACG, Section 4.1] and [GLTV]: in the case of unbounded edges, the associated functions f_{nn} are not of class \mathcal{U}_p , see Definition 4.1). We deal with this issue by considering our energy functional on a slightly smaller region than the cube Q_R .

We now introduce an approximation of W_{hom} on Q_R using ξ^R . The main result of this subsection is the following approximation result:

Theorem 4.4. *For all $\Lambda \in \mathbb{M}^{n \times d}$, the energy density W_{hom} defined in (32) (with $\mathcal{L} = \xi$) satisfies almost surely the identity*

$$W_{\text{hom}}(\Lambda) = \lim_{R \rightarrow \infty} \frac{1}{|Q_R|} \inf_u \left\{ F_1^{Q_{R-2\rho_2}}(\xi^R, u) \mid u \in \mathfrak{S}_1^{Q_R}(\xi^R), u \equiv \Lambda \text{ on } Q_{R-2\rho_2} \setminus Q_{R-4\rho_2} \right\}.$$

As we shall see, Theorem 4.4 follows from Theorem 2.3 applied to (some function related to) the infimum in (32), seen as a function of random measures and sets.

Remark 5. Other approximations of W_{hom} using the random parking measure on a bounded domain can be used instead of the one presented in Theorem 4.4. Let us mention two alternatives.

First, instead of restricting the energy to the domain $Q_{R-2\rho_2}$, one may consider the energy on the whole domain Q_R provided we modify the random parking measure on $Q_R \setminus Q_{R-2\rho_2}$, typically by taking $\tilde{\xi}^R := (\xi^R \cap Q_{R-2\rho_2}) \cup \zeta^R$ where ζ^R is a suitable point set in ∂Q_R (say deterministic). The point set ζ^R can be chosen so that the convex envelope of $\tilde{\xi}^R$ is Q_R , and such that any associated Delaunay triangulation of Q_R has edge lengths bounded by $4\rho_2$. A result similar to Theorem 4.4 can be proved in this case, when the boundary conditions are imposed on ζ^R only (which greatly simplifies the practical implementation, see [GLTV]).

The second alternative is an adaptation of the popular “periodization method” in homogenization. Recall that \mathcal{P} denotes a space-time Poisson process. For all $R > 0$ we define the Q_R -periodization of \mathcal{P} as $\mathcal{P}_{\#}^R := \cup_{k \in \mathbb{Z}^d} (kR + \mathcal{P} \cap (Q_R \times \mathbb{R}^+))$, and let $\xi_{\#}^R$ be the output of the graphical construction of Subsection 2.1 associated with $\mathcal{P}_{\#}^R$ in place of \mathcal{P} . We consider a Q_R -periodic tessellation $\mathcal{T}_{\#}^R$ of \mathbb{R}^d associated with $\xi_{\#}^R$. This allows us to define a set of Q_R -periodic functions on \mathbb{R}^d :

$$\mathfrak{S}_{\#}^R(\xi_{\#}^R) := \{u \in C(\mathbb{R}^d, \mathbb{R}^n) \mid u \text{ is } Q_R\text{-periodic, } \forall T \in \mathcal{T}(\xi_{\#}^R), u|_T \text{ is affine}\}.$$

We then let $C_{\#}^R$ be a minimal connected periodic cell obtained as the union of simplices of $\mathcal{T}(\xi_{\#}^R)$. Although we have not checked all the details, one should have that

$$W_{\text{hom}}(\Lambda) = \lim_{R \rightarrow \infty} \frac{1}{|Q_R|} \inf_u \left\{ F_1^{C_{\#}^R}(\xi_{\#}^R, u) \mid u \in \mathfrak{S}_{\#}^R(\xi_{\#}^R) \right. \\ \left. \text{such that } \frac{1}{|C_{\#}^R|} \int_{C_{\#}^R} \nabla u(x) dx = \Lambda \right\}$$

almost surely. In view of the results of [GNO] on the periodization method on a simplified model (discrete linear elliptic equation on \mathbb{Z}^d with independent and identically distributed conductances), this approach (although it is much more delicate to implement in practice) should yield better convergence rates than the first two.

The proof of Theorem 4.4 requires a series of lemmas. The first of these provides elementary control of energy functionals for homogeneous deformations:

Lemma 4.5. *There exists $C > 0$ (depending only on $d, n, \rho_1, \rho_2, p, f_{nn}$ and W_{vol}), such that for all $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, $D \in \mathcal{O}(\mathbb{R}^d)$, all $\varepsilon > 0$ and all $\Lambda \in \mathbb{M}^{n \times d}$, F_{ε}^D (given by (27)) satisfies*

$$0 \leq F_{\varepsilon}^D(\zeta, \varphi_{\Lambda}) \leq C|D|(1 + |\Lambda|^p).$$

Proof. For $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, no edge of $\mathcal{T}(\zeta)$ has length greater than $2\rho_2$. Hence the number of neighbours of vertices of $\mathcal{T}(\zeta)$ is bounded by some finite constant depending only on ρ_1, ρ_2 and d . This implies that the number of edges of the Delaunay tessellation $\mathcal{T}(\varepsilon\zeta) \cap D$ is bounded by a constant times the volume $\varepsilon^{-d}|D|$. Using Definition 4.1 and (27), this yields the desired result. \square

Our next lemma shows that sequences of minimizers are precompact in $L^p(A, \mathbb{R}^n)$. This precompactness is a consequence of an argument à la Fréchet-Kolmogorov. We give this argument for completeness in the following lemma since it does not explicitly appear in [ACG].

Lemma 4.6. *Let $D \in \mathcal{O}(\mathbb{R}^d)$ be an open set, let $(u_k)_{k \geq 1}$ be a bounded sequence in $L^p(D, \mathbb{R}^n)$, and let $A \in \mathcal{O}(D)$ be an open set whose closure is contained in D . If there exist an increasing continuous function $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $f(0) = 0$ and a sequence ε_k converging to zero such that for all $h \in \mathbb{R}^d$ with $|h|$ small enough, and for all k ,*

$$(34) \quad \|\tau_h u_k - u_k\|_{L^p(A, \mathbb{R}^n)} \leq f(|h|) + \varepsilon_k,$$

where $\tau_h u_k : x \mapsto u_k(x + h)$ (whenever it is defined), then there exists $u \in L^p(D, \mathbb{R}^n)$ such that for a subsequence $u_k \rightarrow u$ in $L^p(A, \mathbb{R}^n)$.

Proof. Since the sequence u_k is bounded in $L^p(D, \mathbb{R}^n)$, by the Banach-Alaoglu theorem, there exists $u \in L^p(D, \mathbb{R}^n)$ such that u_k converges weakly to u for a subsequence (not relabeled). By the lower-semicontinuity of the norm for the weak convergence, the weak convergence of $\tau_h u_k - u_k$ to $\tau_h u - u$ in $L^p(A, \mathbb{R}^n)$ for all h small enough, and by (34)

$$(35) \quad \|\tau_h u - u\|_{L^p(A, \mathbb{R}^n)} \leq \liminf_{k \rightarrow \infty} \|\tau_h u_k - u_k\|_{L^p(A, \mathbb{R}^n)} \leq f(|h|).$$

Let ρ_j be a sequence of smoothing kernels of unit mass with support in the ball $B(0, 1/j)$ centred at zero and of radius $1/j$. For j large enough and for all k , the convolution $\rho_j \star u_k$ of u_k with ρ_j is in $L^p(A, \mathbb{R}^n)$. In addition, by Jensen's inequality with measure $\rho_j(y)dy$, and Fubini's theorem,

$$\int_A |\rho_j \star u_k - u_k|^p \leq \int_A \left(\int_{B(0, 1/j)} |u_k(x - y) - u_k(x)| \rho_j(y) dy \right)^p dx \\ \leq \int_{B(0, 1/j)} \int_A |u_k(x - y) - u_k(x)|^p dx \rho_j(y) dy$$

so that by (34),

$$(36) \quad \|\rho_j \star u_k - u_k\|_{L^p(A, \mathbb{R}^n)} \leq f(j^{-1}) + \varepsilon_k.$$

Likewise, by (35)

$$(37) \quad \|\rho_j \star u - u\|_{L^p(A, \mathbb{R}^n)} \leq f(j^{-1}).$$

By construction, $\{\rho_j \star u_k\}_k$ is an equi-continuous sequence of bounded functions in $C(A)$. Indeed the functions are bounded since

$$\|\rho_j \star u_k\|_{L^\infty(A)} \leq \|\rho_j\|_{L^\infty} \|u_k\|_{L^1(D, \mathbb{R}^n)}.$$

Also for all $x_1, x_2 \in A$,

$$|\rho_j \star u_k(x_1) - \rho_j \star u_k(x_2)| \leq |x_1 - x_2| \|\rho_j\|_{\text{Lip}} \|u_k\|_{L^1(D, \mathbb{R}^n)},$$

so that the equi-continuity follows by the uniform boundedness of u_k in $L^1(D, \mathbb{R}^n)$. Hence, by Ascoli's theorem, for all j there exists $\tilde{u}_j \in C(A)$ such that $\rho_j \star u_k$ converges up to extraction to \tilde{u}_j uniformly in A as $k \rightarrow \infty$, and therefore in $L^p(A, \mathbb{R}^n)$. Noting that for all $x \in A$,

$$\lim_{k \rightarrow \infty} \rho_j \star u_k(x) = \lim_{k \rightarrow \infty} \int_D u_k(y) \rho_j(x-y) dy = \int_D u(y) \rho_j(x-y) dy$$

by weak convergence of u_k to u , we have $\tilde{u}_j = \rho_j \star u$. Since the limit is unique, the entire sequence $\{\rho_j \star u_k\}_k$ converges to $\rho_j \star u$ in $L^p(A, \mathbb{R}^n)$. Hence, for all j , there exists k_j such that for all $k \geq k_j$,

$$(38) \quad \|\rho_j \star u_k - \rho_j \star u\|_{L^p(A, \mathbb{R}^n)} \leq f(j^{-1}).$$

By the triangle inequality, for j large enough and for all k ,

$$\|u_k - u\|_{L^p(A, \mathbb{R}^n)} \leq \|u_k - \rho_j \star u_k\|_{L^p(A, \mathbb{R}^n)} + \|\rho_j \star u_k - \rho_j \star u\|_{L^p(A, \mathbb{R}^n)} + \|\rho_j \star u - u\|_{L^p(A, \mathbb{R}^n)}.$$

Hence, by (36), (38), and (37), for all j large enough and all $k \geq k_j$,

$$\|u_k - u\|_{L^p(A, \mathbb{R}^n)} \leq 3f(j^{-1}) + \varepsilon_k.$$

This implies that $u_k \rightarrow u$ in $L^p(A, \mathbb{R}^n)$. \square

In the proof of Theorem 4.4 we shall make use of the following Γ -compactness and convergence of infima results, for which we use terminology from Definition 4.2.

Lemma 4.7. *Let $\{\zeta_k\}_{k \in \mathbb{N}}$ be a sequence of point sets in $\mathcal{A}_{\rho_1, \rho_2}$. Let $D \in \mathcal{O}(\mathbb{R}^d)$ be an open set. Then for any sequence $\{\varepsilon_k\}_{k \in \mathbb{N}}$ of positive numbers converging to zero, there exist a subsequence (not relabeled) and a Carathéodory function $W : D \times \mathbb{M}^{n \times d} \rightarrow \mathbb{R}^+$ such that for all open sets $A \in \mathcal{O}(D)$ the functionals $F_{\varepsilon_k}^A(\zeta_k)$ (given by (27)) satisfy*

$$(39) \quad \Gamma - \lim_{k \rightarrow \infty} F_{\varepsilon_k}^A(\zeta_k) = F^A$$

where the integral functional $F^A : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ is defined by

$$(40) \quad F^A(u) = \begin{cases} \int_A W(x, \nabla u(x)) dx & \text{if } u \in W^{1,p}(A, \mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

In addition, W is quasiconvex in its second variable and satisfies a standard growth condition (30) of order p .

In this subsection we will only make use of this result for $D = Q$.

Proof. Lemma 4.7 is a corollary of [ACG, Theorem 3], which is the corresponding compactness result when the sequence ζ_k is a constant sequence which corresponds to the realization of a stochastic lattice, and for $W_{\text{vol}} \equiv 0$. Yet, there, the energy functionals are defined on piecewise constant functions (on the Voronoi tessellation associated with the point set), rather than on continuous and piecewise affine functions on a Delaunay tessellation. As pointed out in [ACG, Remark 4], the compactness result of [ACG, Theorem 3] holds as well for energy functionals defined on continuous and piecewise affine functions on an associated Delaunay tessellation. Hence, since the proof of the individual compactness result [ACG, Theorem 3] only makes use of deterministic arguments, it carries over to the case dealt with here provided $W_{\text{vol}} \equiv 0$. It remains to argue that one can add the volumetric term W_{vol} in [ACG, Theorem 3]. Since this term yields a continuous contribution (it already has the form of an integral) and since the proof of [ACG, Theorem 3] has the same structure as for the homogenization of multiple integrals (application of the De Giorgi-Letta criteria, and of the integral representation results by Buttazzo and Dal Maso), this additional contribution can be treated in a standard way (see for instance [BD98, Chapter 12]). \square

Recall notation $D_{R,r}$ from Section 2.2. For $\varepsilon > 0$, $r > 0$, $A \in \mathcal{O}(\mathbb{R}^d)$, and $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, and $\Lambda \in \mathbb{M}^{n \times d}$, define

$$(41) \quad \mathfrak{S}_\varepsilon^{A, \Lambda, r}(\zeta) := \left\{ u \in \mathfrak{S}_\varepsilon^A(\zeta) \mid u \equiv \varphi_\Lambda \text{ on } (\varepsilon\zeta \cap A \setminus A_{1, \varepsilon r}) \cup \partial A \right\}.$$

For all tessellations \mathcal{T} of \mathbb{R}^d , and all $A \in \mathcal{O}(\mathbb{R}^d)$, we define $A_{\mathcal{T}}$ as the interior of the closed set $\cup_{T \in \mathcal{T}, T \subset A} \overline{T}$.

Lemma 4.8. *Suppose $A, A', A_+ \in \mathcal{O}(\mathbb{R}^d)$ are open sets with the closure of A contained in A' and the closure of A' contained in A_+ . Let $\Lambda \in \mathbb{M}^{n \times d}$ and let $r \geq 2\rho_2$. Then there exists a constant $C > 0$ such that for all $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, $\varepsilon \in (0, 1)$, and $u \in \mathfrak{S}_\varepsilon^{A, \Lambda, r}(\zeta)$, with u extended by φ_Λ on $A_+ \setminus A$, and all $h \in \mathbb{R}^d$ with $|h| < 1/C$, we have*

$$\|\tau_h u - u\|_{L^p(A', \mathbb{R}^n)}^p \leq C[|h|^p(1 + F_\varepsilon^A(\zeta, u)) + \varepsilon],$$

where τ_h is as defined in Lemma 4.6.

Proof. By [ACG, formula (31)] (the result can be extended to the case $W_{\text{vol}} \geq 0$, since the associated contribution to the energy is non-negative), there is a constant C such that for all $h \in \mathbb{R}^d$ with $|h| < 1/C$,

$$(42) \quad \|\tau_h u - u\|_{L^p(A', \mathbb{R}^n)}^p \leq C[|h|^p(F_\varepsilon^{A_+}(\zeta, u) + 1) + \varepsilon]$$

for all ζ, ε, u . By definition of the functionals and nonnegativity of f_{nn} and W_{vol} ,

$$(43) \quad \begin{aligned} F_\varepsilon^{A_+}(\zeta, u) &\leq F_\varepsilon^{A_{\mathcal{T}(\varepsilon\zeta)}}(\zeta, u) + F_\varepsilon^{A_+ \setminus \overline{A_{\mathcal{T}(\varepsilon\zeta)}}}(\zeta, u) \\ &\leq F_\varepsilon^A(\zeta, u) + F_\varepsilon^{A_+ \setminus \overline{A_{\mathcal{T}(\varepsilon\zeta)}}}(\zeta, u). \end{aligned}$$

Since $r \geq 2\rho_2$, for all k , all cells of $\mathcal{T}(\varepsilon\zeta)$ which have a vertex in $A_{1, \varepsilon r}$ are contained in A so that for $u \in \mathfrak{S}_\varepsilon^{A, \Lambda, r}(\zeta)$ we have

$$u \equiv \varphi_\Lambda \text{ on } A_+ \setminus A_{\mathcal{T}(\varepsilon\zeta)}.$$

Hence by non-negativity of f_{nn} and W_{vol} ,

$$F_\varepsilon^{A_+ \setminus \overline{A_{\mathcal{T}(\varepsilon\zeta)}}}(\zeta, u) = F_\varepsilon^{A_+ \setminus \overline{A_{\mathcal{T}(\varepsilon\zeta)}}}(\zeta, \varphi_\Lambda) \leq F_\varepsilon^{A_+}(\zeta, \varphi_\Lambda).$$

By Lemma 4.5, the last expression is bounded uniformly in ζ, ε, u . Combined with (42) and (43), this gives us result. \square

Lemma 4.9. *In addition to the notation and assumptions of Lemma 4.7, let $\Lambda \in \mathbb{M}^{n \times d}$ and $A \in \mathcal{O}(D)$ be an open set. Let W be as in Lemma 4.7. Suppose ε_k is a sequence such that $\varepsilon_k > 0$, $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ and (39) holds. Then for $r \geq 2\rho_2$,*

$$\liminf_{k \rightarrow \infty} \{F_{\varepsilon_k}^A(\zeta_k, u) \mid u \in \mathfrak{S}_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k)\} = \inf_{u \in W_0^{1,p}(A)} \left\{ \int_A W(x, \Lambda + \nabla u(x)) dx \right\}.$$

Proof. Let $r \geq 2\rho_2$. For $\varepsilon > 0$ and $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, define the functionals $F_{\varepsilon}^{A, \Lambda, r}(\zeta, \cdot) : L^p(A, \mathbb{R}^n) \rightarrow [0, +\infty]$ and $F^{A, \Lambda} : L^p(A, \mathbb{R}^n) \rightarrow [0, +\infty]$ by

$$F_{\varepsilon}^{A, \Lambda, r}(\zeta, u) = \begin{cases} F_{\varepsilon}^A(\zeta, u) & \text{if } u \in \mathfrak{S}_{\varepsilon}^{A, \Lambda, r}(\zeta), \\ +\infty & \text{else;} \end{cases}$$

$$F^{A, \Lambda}(u) = \begin{cases} F^A(u) & \text{if } u - \varphi_{\Lambda} \in W_0^{1,p}(A, \mathbb{R}^n), \\ +\infty & \text{else.} \end{cases}$$

We split the proof into two steps. First we prove the Γ -convergence result

$$(44) \quad \Gamma - \lim_{k \rightarrow \infty} F_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k) = F^{A, \Lambda}.$$

Then we appeal to Lemma 4.1 to prove the desired result.

Step 1. Proof of (44).

Let $A' \in \mathcal{O}(\mathbb{R}^d)$ be an open set which contains the closure of A , and let $A_+ \in \mathcal{O}(\mathbb{R}^d)$ be an open set which contains the closure of A' .

To prove (44), we first address the Γ -liminf inequality. Let $u_k \in \mathfrak{S}_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k)$ be a sequence converging to some u in $L^p(A, \mathbb{R}^n)$ such that

$$(45) \quad \sup_k F_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k, u_k) < +\infty.$$

Since $F_{\varepsilon_k}^A(\zeta_k, \cdot) \leq F_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k, \cdot)$, the assumption (39) and estimate (45) imply that

$$F^A(u) \leq \liminf_{k \rightarrow \infty} F_{\varepsilon_k}^A(\zeta_k, u_k) < +\infty,$$

so that $u \in W^{1,p}(A, \mathbb{R}^n)$. For this to imply the desired Γ -liminf inequality, we need to prove that $u - \varphi_{\Lambda} \in W_0^{1,p}(A, \mathbb{R}^n)$ so that $F^{A, \Lambda}(u) = F^A(u)$.

Extend u_k and u by φ_{Λ} on $A_+ \setminus A$ for all k , so that $u_k \rightarrow u$ in $L^p(A_+)$. By Lemma 4.8, and (45), there exists some $C > 0$ such that for all k and for all $h \in \mathbb{R}^d$ with $|h| < 1/C$,

$$(46) \quad \|\tau_h u_k - u_k\|_{L^p(A', \mathbb{R}^n)}^p \leq C(|h|^p + \varepsilon_k).$$

Since $\tau_h u_k - u_k \rightarrow \tau_h u - u$ in $L^p(A', \mathbb{R}^n)$,

$$\|\tau_h u - u\|_{L^p(A', \mathbb{R}^n)}^p \leq C|h|^p$$

for all $h \in \mathbb{R}^d$ with $|h| < 1/C$. From the characterization of $W^{1,p}(A', \mathbb{R}^n)$ by difference quotients (see for instance [Eva98, Theorem 3 Section 5.8]) we thus deduce that $u \in W^{1,p}(A', \mathbb{R}^n)$. Combined with the fact that $u \equiv \varphi_{\Lambda}$ on $A' \setminus A$, and the continuity of the trace operator from $W^{1,p}(A', \mathbb{R}^n)$ to $W^{1-1/p, p}(\partial A, \mathbb{R}^n)$ (see for instance [Eva98, Theorem 1 Section 5.5]), this shows that $u = \varphi_{\Lambda}$ in $W^{1-1/p, p}(\partial A, \mathbb{R}^n)$, and therefore $u - \varphi_{\Lambda} \in W_0^{1,p}(A, \mathbb{R}^n)$. The Γ -liminf inequality is proved.

To prove the existence of recovery sequences, for every u with $u - \varphi_{\Lambda} \in W_0^{1,p}(A, \mathbb{R}^n)$ we have to construct a sequence $u_k \in \mathfrak{S}_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k)$ which converges to u in $L^p(A, \mathbb{R}^n)$ and satisfies

$$F^{A, \Lambda}(u) = \lim_{k \rightarrow \infty} F_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k, u_k),$$

which we may also rewrite as $F^A(u) = \lim_{k \rightarrow \infty} F_{\varepsilon_k}^A(\zeta_k, u_k)$. Using the Γ -convergence of $F_{\varepsilon_k}^A(\zeta_k)$, it is enough to modify a recovery sequence for u in $\mathfrak{S}_{\varepsilon_k}^A(\zeta_k)$ so that it belongs to

$\mathfrak{S}_{\varepsilon_k}^{A,\Lambda,r}(\zeta_k)$ and still satisfies the identity above. This modification can be achieved using De Giorgi's averaging method, which is a very technical argument in this discrete case. We refer the reader to [ACG, Proof of Proposition 3], where this issue is treated in detail.

Step 2. Proof of the convergence of infima.

With (44) established, we shall complete the proof by an application of Lemma 4.1. Let u_k be a sequence of minimizers of $F_{\varepsilon_k}^{A,\Lambda,r}(\zeta_k)$ on $\mathfrak{S}_{\varepsilon_k}^{A,\Lambda,r}(\zeta_k)$. We extend u_k by φ_Λ on $A_+ \setminus A$ for all k . Consider an arbitrary subsequence of the original sequence u_k , with this subsequence also denoted u_k . Since the u_k are minimizers, by Lemma 4.5 we have that

$$(47) \quad \sup_k F_{\varepsilon_k}^{A,\Lambda,r}(\zeta_k, u_k) \leq \sup_k F_{\varepsilon_k}^{A,\Lambda,r}(\zeta_k, \varphi_\Lambda) < \infty.$$

To check the conditions of Lemma 4.1, we shall apply Lemma 4.6. To check that we may do so, we need to prove that u_k is bounded in $L^p(A_+, \mathbb{R}^n)$. Since $\zeta_k \in \mathcal{A}_{\rho_1, \rho_2}$, and u_k is affine on each simplex of $\mathcal{T}(\varepsilon_k \zeta_k)$ contained in A , we claim that there exists $C > 0$ such that for all k

$$\begin{aligned} \|u_k\|_{L^p(A_{\mathcal{T}(\varepsilon_k \zeta_k)}, \mathbb{R}^n)}^p &= \sum_{T \in \mathcal{T}(\varepsilon_k \zeta_k), T \subset A} \int_T |u_k(x)|^p dx \\ &\leq \sum_{T \in \mathcal{T}(\varepsilon_k \zeta_k), T \subset A} |T| \sup_{x \text{ vertex of } T} |u_k(x)|^p \\ &\leq C \varepsilon_k^d \sum_{x \in \overline{A} \cap \varepsilon_k \zeta_k} |u_k(\varepsilon_k x)|^p. \end{aligned}$$

This claim follows from the following two facts:

- the volume of any simplex in $\mathcal{T}(\varepsilon_k \zeta_k)$ is bounded by $C_d (2\rho_2 \varepsilon_k)^d$ (where C_d denotes the volume of the unit ball in dimension d),
- the number of neighbours of any point in the tessellation is uniformly bounded (as proved already) so that each point $x \in \varepsilon_k \zeta_k \cap \overline{A}$ lies in the closure of at most a fixed number of Delaunay cells $T \in \mathcal{T}(\varepsilon_k \zeta_k)$ (independent of k).

On the other hand, since $r \geq 2\rho_2$, for all $v \in \mathfrak{S}_{\varepsilon_k}^{A,\Lambda,r}(\zeta_k)$, $(v - \varphi_\Lambda)|_{A_{\mathcal{T}(\varepsilon_k \zeta_k)}} \in W_0^{1,p}(A_{\mathcal{T}(\varepsilon_k \zeta_k)})$, and we claim that we have a discrete Poincaré's inequality in $\mathfrak{S}_{\varepsilon_k}^A(\zeta_k) \cap W_0^{1,p}(A_{\mathcal{T}(\varepsilon_k \zeta_k)})$, namely

$$(48) \quad \sum_{x \in \overline{A} \cap \varepsilon_k \zeta_k} \varepsilon_k^d |v(x) - \varphi_\Lambda(x)|^p \leq C \sum_{\substack{\{x, y\} \in \mathcal{N}(\varepsilon_k \zeta_k) \\ (x, y) \subset \overline{A}}} \varepsilon_k^d \left(\frac{|v(y) - \varphi_\Lambda(y) - v(x) + \varphi_\Lambda(x)|}{|y - x|} \right)^p.$$

where C depends on A but not on k . The proof of (48) uses the fact, proved in [ACG, Lemma 3], that for any $x \in \overline{A} \cap \varepsilon_k \zeta_k$, one can find a path $\gamma(x)$ on the Delaunay graph from x to some point $x_0 \in \partial A_{\mathcal{T}(\varepsilon_k \zeta_k)} \cap \varepsilon_k \zeta_k$ with $O(\varepsilon_k^{-1})$ steps, and moreover arrange that each Delaunay edge appears in at most $O(\varepsilon_k^{-1})$ of the paths $\gamma(x)$ for $x \in \overline{A} \cap \varepsilon_k \zeta_k$. Let $\gamma(x) = \{x_0, x_1, \dots, x_\ell, x\}$ for some $\ell \in \mathbb{N}$. Then, for any $w \in \mathfrak{S}_{\varepsilon_k}^A(\zeta_k) \cap W_0^{1,p}(A_{\mathcal{T}(\varepsilon_k \zeta_k)})$,

$$|w(x)| \leq |w(x_0)| + \sum_{j=1}^{\ell} |w(x_j) - w(x_{j-1})| \leq C \sum_{j=1}^{\ell} \varepsilon_k \frac{|w(x_j) - w(x_{j-1})|}{|x_j - x_{j-1}|}$$

where $x_{\ell+1} = x$, using in addition that the edge lengths are of order ε_k . Since the path has length of order $O(\varepsilon_k^{-1})$, Jensen's inequality yields

$$|w(x)|^p \leq C \sum_{j=1}^{\ell} \varepsilon_k \left(\frac{|w(x_j) - w(x_{j-1})|}{|x_j - x_{j-1}|} \right)^p.$$

Putting $w = v - \varphi_\Lambda$ and summing over $x \in \bar{A} \cap \varepsilon_k \zeta_k$, using that each Delaunay edge appears in at most $O(\varepsilon_k^{-1})$ of the paths $\gamma(x)$, yields the desired Poincaré inequality (48).

By the triangle inequality and the uniform bound $|x - y| \leq 2\rho_2$ for any edge (x, y) , the inequality (48) for $v = u_k$ yields

$$\sum_{x \in \bar{A} \cap \varepsilon_k \zeta_k} \varepsilon_k^d |u_k(x)|^p \leq C \sum_{\substack{\{x, y\} \in \mathcal{N}(\varepsilon_k \zeta_k) \\ (x, y) \subset \bar{A}}} \varepsilon_k^d \left(\frac{|u_k(y) - u_k(x)|}{|y - x|} \right)^p + C|\Lambda|^p.$$

Using the property (25) of f_{nn} and the definition of $F_{\varepsilon_k}^A(\zeta_k)$, we deduce that

$$\begin{aligned} \|u_k\|_{L^p(A_{\mathcal{T}(\varepsilon_k \zeta_k)}, \mathbb{R}^n)}^p &\leq C \sum_{\substack{\{x, y\} \in \mathcal{N}(\varepsilon_k \zeta_k) \\ (x, y) \subset \bar{A}}} \varepsilon_k^d f_{nn} \left(\varepsilon^{-1}(y - x), \frac{u_k(y) - u_k(x)}{|y - x|} \right) + C \\ &\leq C F_{\varepsilon_k}^{A, \Lambda, r}(\zeta_k, u_k) + C \end{aligned}$$

for some $C > 0$ independent of k . By (47) the last expression is bounded uniformly in k .

Using Lemma 4.8, followed by (47), we see that there are constants C, C' such that for $|h| < 1/C$ and for all k we have

$$\|\tau_h u_k - u_k\|_{L^p(A', \mathbb{R}^n)}^p \leq C[|h|^p(1 + F_{\varepsilon_k}^A(\zeta_k, u_k)) + \varepsilon_k] \leq C'(|h|^p + \varepsilon_k).$$

Hence Lemma 4.6 is applicable, showing that the sequence u_k converges along some subsequence to a limit in $L^p(A, \mathbb{R}^n)$. Thus the closure of the original sequence u_k is sequentially compact in $L^p(A, \mathbb{R}^n)$, and by the equivalence of compactness and sequential compactness in any metric space, this sequence is precompact. We are thus in position to apply Lemma 4.1, which concludes the proof. \square

Proof of Theorem 4.4. In order to apply Theorem 2.3 to the problem under consideration, we need to first define the function S ; we also define a family of auxiliary functions \tilde{S}^r , $r > 0$. Fix $\Lambda \in \mathbb{M}^{n \times d}$. For $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, and for $\varepsilon, r > 0$, recall the definition (41) of $\mathfrak{S}_\varepsilon^{A, \Lambda, r}(\zeta)$, and define

$$\begin{aligned} S(Q_R, \zeta) &= \inf_u \{F_1^{Q_{R-2\rho_2}}(\zeta, u) \mid u \in \mathfrak{S}_1^{Q_{R-2\rho_2}, \Lambda, 2\rho_2}\}; \\ \tilde{S}^r(Q_R, \zeta) &= \inf_u \{F_1^{Q_R}(\zeta, u) \mid u \in \mathfrak{S}_1^{Q_R, \Lambda, r}\}. \end{aligned}$$

Let $\zeta, \zeta' \in \mathcal{A}_{\rho_1, \rho_2}$ with $\zeta \cap Q_R = \zeta' \cap Q_R$. Then for any $T \in \mathcal{T}(\zeta)$ with $T \cap Q_{R-2\rho_2} \neq \emptyset$, the circumsphere of T contains no point of ζ in its interior by definition, so it has radius less than $2\rho_2$ and therefore is contained in Q_R ; hence it contains no point of ζ' and hence $T \in \mathcal{T}(\zeta')$ as well. In short the Delaunay tessellations of ζ and ζ' coincide on $Q_{R-2\rho_2}$.

Consequently, our S is local on $\mathcal{D}(Q)$ because changes to ζ outside Q_R do not affect $S(Q_R, \zeta)$. Also it can readily be deduced from Theorem 4.2 that S satisfies the averaging property on $\mathcal{A}_{\rho_1, \rho_2}$ with respect to Q .

It remains to prove that S is insensitive to boundary effects. Let $\alpha \in (0, 1)$. It is enough to prove that for any sequence $\{\zeta_k\}_{k \in \mathbb{N}}$ of general (ρ_1, ρ_2) -admissible point sets and any sequence

of positive numbers $\{R_k\}_{k \in \mathbb{N}}$ tending to infinity, one has

$$(49) \quad \limsup_{k \rightarrow \infty} \frac{|S(Q_{R_k}, \zeta_k) - S(Q_{R_k, R_k^\alpha}, \zeta_k)|}{|Q_{R_k}|} = 0$$

where we are using notation $D_{R,r}$ from Section 2.2. In fact, we first prove this for \tilde{S}^r , for $r = 4\rho_2$.

Set $\varepsilon_k = 1/R_k$ for all $k \in \mathbb{N}$. By Lemma 4.7, after extraction (we do not relabel k) there exists a Carathéodory function $W : Q \times \mathbb{M}^{n \times d} \rightarrow \mathbb{R}^+$ such that for every open set $A \in \mathcal{O}(Q)$, (39) holds, with F^A given by (40), and such that W is quasiconvex in its second variable and satisfies a standard growth condition (30) of order $p > 1$ (where p is as in Theorem 4.2).

Next we rescale. For every open set $A \in \mathcal{O}(Q)$, and for all $\varepsilon > 0, r > 0$, define $\mathfrak{S}_\varepsilon^{A, \Lambda, r}(\zeta_k)$ by (41) and define the rescaling $\tilde{S}^{\varepsilon, r}$ of \tilde{S}^r by

$$\tilde{S}^{\varepsilon, r}(A, \zeta_k) = \inf_u \{F_\varepsilon^A(\zeta_k, u) \mid u \in \mathfrak{S}_\varepsilon^{A, \Lambda, r}(\zeta_k)\}.$$

If $R > 0$ and $u \in C(Q_R)$ and if $v \in C(Q)$ is defined by $v(x) = R^{-1}u(Rx)$, then for any $\zeta \in \mathcal{A}_{\rho_1, \infty}$ we have

$$F_{1/R}^Q(\zeta, v) = R^{-d}F_1^{Q_R}(\zeta, u),$$

and moreover $u \in \mathfrak{S}_1^{Q_R, \Lambda, r}(\zeta)$ if and only if $v \in \mathfrak{S}_{1/R}^{Q, \Lambda, r}(\zeta)$ for any $r > 0$. Hence for any $r > 0$,

$$(50) \quad \begin{aligned} \tilde{S}^r(Q_R, \zeta) &= \inf \{R^d F_{1/R}^Q(\zeta, v) \mid v \in \mathfrak{S}_{1/R}^{Q, \Lambda, r}(\zeta)\} \\ &= R^d \tilde{S}^{1/R, r}(Q, \zeta_k). \end{aligned}$$

Next we claim that if $(r_k)_{k \in \mathbb{N}}$ is any sequence of numbers satisfying $r_k \geq 4\rho_2$ and $r_k = o(R_k)$ as $k \rightarrow \infty$, then

$$(51) \quad \limsup_{k \rightarrow \infty} \left| \tilde{S}^{\varepsilon_k, r_k}(Q, \zeta_k) - R_k^{-d} \tilde{S}^{4\rho_2}(Q_{R_k}, \zeta_k) \right| = 0.$$

To prove this, first we deduce from Lemma 4.9, that for any $r \geq 2\rho_2$ and any open set $A \in \mathcal{O}(Q)$, we have

$$(52) \quad \lim_{k \rightarrow \infty} \tilde{S}^{\varepsilon_k, r}(A, \zeta_k) = \inf_{u \in W_0^{1,p}(A)} \left\{ \int_A W(x, \Lambda + \nabla u(x)) dx \right\}.$$

Let $\eta > 0$. For all $r \geq s \geq 4\rho_2$ and all $\varepsilon > 0$, $\mathfrak{S}_\varepsilon^{Q, \Lambda, r}(\zeta_k) \subset \mathfrak{S}_\varepsilon^{Q, \Lambda, s}(\zeta_k)$, so that by definition of $\tilde{S}^{\varepsilon, r}$ and (50), for all $k \in \mathbb{N}$ we have that

$$(53) \quad R_k^{-d} \tilde{S}^{4\rho_2}(Q_{R_k}, \zeta_k) = \tilde{S}^{\varepsilon_k, 4\rho_2}(Q, \zeta_k) \leq \tilde{S}^{\varepsilon_k, r_k}(Q, \zeta_k).$$

Also, by the non-negativity of the energy functions f_{nm} and W_{vol} , for all $k \in \mathbb{N}$ such that $r_k \leq \eta R_k$,

$$(54) \quad \begin{aligned} \tilde{S}^{\varepsilon_k, r_k}(Q, \zeta_k) &\leq \tilde{S}^{\varepsilon_k, \eta R_k}(Q, \zeta_k) \\ &\leq F_{\varepsilon_k}^{Q \setminus \overline{Q_{1-2\eta}}}(\zeta_k, \varphi_\Lambda) + \tilde{S}^{\varepsilon_k, 2\rho_2}(Q_{1-\eta}, \zeta_k) \end{aligned}$$

provided all the simplices of $\mathcal{T}(\varepsilon_k \zeta_k)$ which intersect $Q_{1-\eta}$ without being contained in $Q_{1-\eta}$ are contained in $Q \setminus \overline{Q_{1-2\eta}}$ — which holds provided $2\rho_2 \varepsilon_k < \eta$ since the edge lengths of the simplices of $\mathcal{T}(\zeta_k)$ are bounded by $2\rho_2$.

Lemma 4.5 yields the bound $F_{\varepsilon_k}^{Q \setminus Q_{1-2\eta}}(\zeta_k, \varphi_\Lambda) \leq C|Q \setminus Q_{1-2\eta}|(1 + |\Lambda|^p) \leq C'\eta$. Hence by (53) and (54), and (50) and property (52) applied to Q and to $Q_{1-\eta}$, this implies that

$$\limsup_{k \rightarrow \infty} \left| \tilde{S}^{\varepsilon_k, r_k}(Q, \zeta_k) - R_k^{-d} \tilde{S}^{4\rho_2}(Q_{R_k}, \zeta_k) \right| \leq C\eta$$

$$+ \inf_{u \in W_0^{1,p}(Q_{1-\eta})} \left\{ \int_{Q_{1-\eta}} W(x, \Lambda + \nabla u(x)) dx \right\} - \inf_{u \in W_0^{1,p}(Q)} \left\{ \int_Q W(x, \Lambda + \nabla u(x)) dx \right\}.$$

We claim that the right hand side of this inequality tends to zero as η vanishes. It is enough to prove that the limsup of the difference of the infimum problems is non-positive. By the properties of W , $u \mapsto \int_Q W(x, \Lambda + \nabla u(x)) dx$ is continuous on $W_0^{1,p}(Q)$. Let η_k be a sequence of positive numbers which tends to zero. Since $C_0^\infty(Q)$ is dense in $W_0^{1,p}(Q)$ by definition, there exists a sequence $u_k \in C_0^\infty(Q)$ such that for all $k \in \mathbb{N}$, $\text{supp}(u_k) \subset Q_{1-2\eta_k}$ and which satisfies

$$\lim_{k \rightarrow \infty} \int_Q W(x, \Lambda + \nabla u_k(x)) dx = \inf_{u \in W_0^{1,p}(Q)} \left\{ \int_Q W(x, \Lambda + \nabla u(x)) dx \right\}.$$

Since W is non-negative and $u_k \in W_0^{1,p}(Q_{1-\eta_k})$ for all $k \in \mathbb{N}$, we have

$$\inf_{u \in W_0^{1,p}(Q_{1-\eta_k})} \left\{ \int_{Q_{1-\eta_k}} W(x, \Lambda + \nabla u(x)) dx \right\} \leq \int_Q W(x, \Lambda + \nabla u_k(x)) dx,$$

so that

$$\limsup_{k \rightarrow \infty} \inf_{u \in W_0^{1,p}(Q_{1-\eta_k})} \left\{ \int_{Q_{1-\eta_k}} W(x, \Lambda + \nabla u(x)) dx \right\}$$

$$\leq \inf_{u \in W_0^{1,p}(Q)} \left\{ \int_Q W(x, \Lambda + \nabla u(x)) dx \right\},$$

which is the claim. This proves (51).

Now observe that $Q_{R, R^\alpha} = Q_{R-R^\alpha}$. The test-functions in the infimum problems defining $\tilde{S}^{R^\alpha+4\rho_2}(Q_R, \zeta_k)$ and $\tilde{S}^{4\rho_2}(Q_{R-R^\alpha}, \zeta_k)$ coincide on Q_{R-R^α} , since they do coincide on $\{T \in \mathcal{T}(\zeta_k) \mid T \cap Q_{R-R^\alpha-4\rho_2} \neq \emptyset\}$, and take the value φ_Λ elsewhere on Q_{R-R^α} . Hence, taking into account Definition 4.1, this yields for all $R > 0$,

$$(55) \quad \tilde{S}^{R^\alpha+4\rho_2}(Q_R, \zeta_k) = \tilde{S}^{4\rho_2}(Q_{R-R^\alpha}, \zeta_k) + R^{d-1}O(R^\alpha).$$

Using (51), (50) and (55) in succession then gives us for $r = 4\rho_2$ that

$$\begin{aligned} \tilde{S}^r(Q_{R_k}, \zeta_k) &= R_k^d \tilde{S}^{\varepsilon_k, R_k^\alpha+r}(Q, \zeta_k) + o(R_k^d) \\ &= \tilde{S}^{R_k^\alpha+r}(Q_{R_k}, \zeta_k) + o(R_k^d) \\ &= \tilde{S}^r(Q_{R_k-R_k^\alpha}, \zeta_k) + o(R_k^d) \end{aligned}$$

and therefore (49) holds with S replaced by $\tilde{S}^{4\rho_2}$ along the subsequence, and hence along the original sequence too. To demonstrate (49) for S itself, note that for any $R > 4\rho_2$ and $\zeta \in \mathcal{A}_{\rho_1, \rho_2}$, if $u \in \mathfrak{S}_1^{Q_{R-2\rho_2}, \Lambda, 2\rho_2}(\zeta)$ then extending u by φ_Λ to $Q_R \setminus Q_{R-2\rho_2}$ gives a function (also denoted u) in $\mathfrak{S}_1^{Q_R, \Lambda, 4\rho_2}(\zeta)$, while conversely if $u \in \mathfrak{S}^{Q_R, \Lambda, 4\rho_2}(\zeta)$ then $u|_{Q_{R-2\rho_2}} \in \mathfrak{S}^{Q_{R-2\rho_2}, \Lambda, 2\rho_2}(\zeta)$. Hence there is a constant $C > 0$ such that in both cases we have

$$0 \leq F_1^{Q_R}(\zeta, u) - F_1^{Q_{R-2\rho_2}}(\zeta, u) \leq CR^{d-1},$$

and hence $|S(Q_R, \zeta) - \tilde{S}^{4\rho_2}(Q_R, \zeta)| \leq CR^{d-1}$. Then it is clear that (49) for S follows from (49) for $\tilde{S}^{4\rho_2}$.

We are thus in position to apply Theorem 2.3, which concludes the proof of Theorem 4.4. \square

4.3. Rubber elasticity and random parking in a bounded set. This subsection is devoted to a second version of the second question of the introduction, whereby we consider a more general domain D than the cubes considered in Section 4.2. Let $D \in \mathcal{O}(\mathbb{R}^d)$ be some fixed open domain. We do not focus here on the approximation of W_{hom} , but rather on the Γ -convergence result itself. We define the discrete model as follows. For all $\varepsilon > 0$, we let $\xi^{1/\varepsilon} := \xi^{D_{1/\varepsilon}}$ be the random parking measure with parameter $\rho_0 > 0$ on $D_{1/\varepsilon} := \{\varepsilon^{-1}x, x \in D\}$, let V_ε be the associated Voronoi tessellation of \mathbb{R}^d , and let \mathcal{T}_ε be the associated Delaunay tessellation of the convex hull of $\xi^{1/\varepsilon}$. Although $\xi^{1/\varepsilon}$ is $(\rho_0/2, 2\rho_0)$ -admissible, some edges of \mathcal{T}_ε may be arbitrarily large. In order to avoid such difficulties (which are physically irrelevant anyway), we proceed as in the previous section and focus on $D_{1,4\varepsilon\rho_0} = \{x \in D \mid d_2(x, \partial D) \geq 4\varepsilon\rho_0\}$. (Note that in the previous section $D = Q$ so that $D_{1,4\varepsilon\rho_0} = Q_{1,4\varepsilon\rho_0} = Q_{1-4\varepsilon\rho_0}$.) By the geometric argument of the proof of Theorem 4.4, $D_{1,4\varepsilon\rho_0}$ has the property that every simplex T of the Delaunay tessellation \mathcal{T}_ε is such that if $\varepsilon T \cap D_{1,4\varepsilon\rho_0} \neq \emptyset$ then $\varepsilon T \subset D$, and therefore all the simplices of \mathcal{T}_ε contained in $D_{\varepsilon^{-1}, 4\rho_0}$ have uniformly bounded edge lengths.

Put $\rho_1 = \rho_0/2, \rho_2 = 2\rho_0$, and fix D . For each $\varepsilon > 0$ and $\zeta \in \mathcal{A}_{\rho_1, \rho_2}(D_{1/\varepsilon})$, and open $A \in \mathcal{O}(D)$, we define an energy functional $\overline{F}_\varepsilon^A(\zeta) := \overline{F}_\varepsilon^A(\zeta, \cdot)$ on $L^p(A)$ as:

$$(56) \quad \overline{F}_\varepsilon^A(\zeta, u) := F_\varepsilon^{A \cap D_{1, 2\varepsilon\rho_2}}(\zeta, u),$$

where $F_\varepsilon^A(\zeta, \cdot)$ is defined as in (27). We then have the following result:

Theorem 4.10. *Assume that f_{nn} and W_{vol} are of class \mathcal{U}_p and \mathcal{V}_p for some $p > 1$. Then the functionals $\overline{F}_\varepsilon^D(\xi^{1/\varepsilon})$ Γ -converge as $\varepsilon \rightarrow 0$ to the deterministic integral functional $F_{\text{hom}}^D : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ defined by*

$$F_{\text{hom}}^D(u) = \begin{cases} \int_D W_{\text{hom}}(\nabla u(x)) \, dx & \text{if } u \in W^{1,p}(D, \mathbb{R}^n), \\ +\infty & \text{otherwise,} \end{cases}$$

where W_{hom} coincides with the energy density defined by the asymptotic homogenization formula (32) of Theorem 4.2 when the random point process \mathcal{L} is the random parking measure ξ with parameter ρ_0 in \mathbb{R}^d .

In particular, combining Theorems 4.10 and 4.2 shows that considering the restriction of the random parking measure ξ to $D_{1/\varepsilon}$ or considering the random parking measure $\xi^{1/\varepsilon}$ on $D_{1/\varepsilon}$ yield the same thermodynamic limit for the discrete model of rubber dealt with in this section.

To prove this result, we need the following localized version of Lemma 4.7 for point sets defined on bounded domains, which is proved similarly to Lemma 4.7; taking $D_{1,4\varepsilon\rho_0}$ instead of D does not change the estimates in [ACG] since they are all set on domains compactly included in D .

Lemma 4.11. *Let $D \in \mathcal{O}(\mathbb{R}^d)$ be an open set, and $\{\varepsilon_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers converging to zero. Let $\{\zeta_k\}_{k \in \mathbb{N}}$ be a sequence of (ρ_1, ρ_2) -admissible point sets in D_{1/ε_k} . Then there exist a subsequence (not relabeled) and a Carathéodory function $W : D \times \mathbb{M}^{n \times d} \rightarrow \mathbb{R}^+$ such that for all open sets $A \in \mathcal{O}(D)$ the functionals $\overline{F}_{\varepsilon_k}^A(\zeta_k)$ given by (56) satisfy*

$$(57) \quad \Gamma - \lim_{k \rightarrow \infty} \overline{F}_{\varepsilon_k}^A(\zeta_k) = F^A$$

where the integral functional $F^A : L^p(D, \mathbb{R}^n) \rightarrow [0, +\infty]$ is defined by

$$(58) \quad F^A(u) = \begin{cases} \int_A W(x, \nabla u(x)) \, dx & \text{if } u \in W^{1,p}(A, \mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

In addition, W is quasiconvex in its second variable and satisfies a standard growth condition (30) of order p .

Proof of Theorem 4.10. By Lemma 4.11, there exists almost surely an energy density $W : D \times \mathbb{M}^{n \times d} \rightarrow \mathbb{R}^+$ and a subsequence such that for all open sets $A \in \mathcal{O}(D)$ the functionals $\overline{F}_\varepsilon^A(\xi^{1/\varepsilon})$ Γ -converge along the subsequence to $F^A : L^p(A, \mathbb{R}^n) \rightarrow [0, +\infty]$ defined by

$$F^A(u) = \begin{cases} \int_A W(x, \nabla u(x)) \, dx & \text{if } u \in W^{1,p}(A, \mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

As in [ACG, Theorem 2] we then appeal to the characterization of non-homogeneous quasiconvex functions by their minima: for almost every $x \in D$ and for all $\Lambda \in \mathbb{M}^{n \times d}$,

$$W(x, \Lambda) = \lim_{\rho \rightarrow 0} \frac{1}{|Q_\rho|} \inf_{v \in W_0^{1,p}(x+Q_\rho, \mathbb{R}^n)} \left\{ \int_{x+Q_\rho} W(y, \Lambda + \nabla v(y)) \right\}.$$

By Lemma 4.9, for all $\rho > 0$ small enough so that $x + \overline{Q}_{2\rho} \subset D$, we have

$$\begin{aligned} & \inf_{v \in W_0^{1,p}(x+Q_\rho, \mathbb{R}^n)} \left\{ \int_{x+Q_\rho} W(y, \Lambda + \nabla v(y)) \right\} \\ &= \lim_{\varepsilon \rightarrow 0} \inf \left\{ F_\varepsilon^{x+Q_\rho}(\xi^{1/\varepsilon}, v) \mid v \in \mathfrak{S}_\varepsilon^{x+Q_\rho}(\xi^{1/\varepsilon}), v \equiv \varphi_\Lambda \text{ on } \varepsilon \xi^{1/\varepsilon} \cap (x + (Q_\rho \setminus Q_{\rho-4\varepsilon\rho_0})) \right\}. \end{aligned}$$

On the one hand, as proved in Lemma 2.5, for all $0 < \alpha < 1$ there exists an almost surely finite random variable R_0 such that the measures ξ and $\xi^{1/\varepsilon}$ coincide on D_{R, R^α} for all $R \geq R_0$. Since for all ε small enough, $\{\varepsilon^{-1}y \mid y \in x + Q_\rho\} \subset D_{\varepsilon^{-1}, \varepsilon^{-\alpha}}$, there exists ε_0 such that $\varepsilon\xi$ and $\varepsilon\xi^{1/\varepsilon}$ coincide on $x + Q_\rho$ for all $\varepsilon < \varepsilon_0$. Therefore, for all $\varepsilon < \varepsilon_0$,

$$\begin{aligned} & \inf \left\{ F_\varepsilon^{x+Q_\rho}(\xi^{1/\varepsilon}, v) \mid v \in \mathfrak{S}_\varepsilon^{x+Q_\rho}(\xi^{1/\varepsilon}), v \equiv \varphi_\Lambda \text{ on } \varepsilon \xi^{1/\varepsilon} \cap (x + (Q_\rho \setminus Q_{\rho-4\varepsilon\rho_0})) \right\} \\ &= \inf \left\{ F_\varepsilon^{x+Q_\rho}(\xi, v) \mid v \in \mathfrak{S}_\varepsilon^{x+Q_\rho}(\xi), v \equiv \varphi_\Lambda \text{ on } \varepsilon \xi \cap (x + (Q_\rho \setminus Q_{\rho-4\varepsilon\rho_0})) \right\}. \end{aligned}$$

On the other hand, as proved in Step 1 of the proof of [ACG, Theorem 2], by a suitable application of the subadditive ergodic theorem,

$$\lim_{\varepsilon \rightarrow 0} \inf \left\{ F_\varepsilon^{x+Q_\rho}(\xi, v) \mid v \in \mathfrak{S}_\varepsilon^{x+Q_\rho}(\xi), v \equiv \varphi_\Lambda \text{ on } \varepsilon \xi \cap (x + (Q_\rho \setminus Q_{\rho-4\varepsilon\rho_0})) \right\} = W_{\text{hom}}(\Lambda)$$

almost surely. Hence, $W(x, \cdot)$ coincides with W_{hom} for almost every $x \in D$. In particular, by uniqueness of the limit, the entire sequence converges almost surely. \square

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