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Generating functions for computing the Shapley value in games restricted by a cooperation index

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Abstract

L'objectiu d'aquest treball és descriure un mètode per calcular el valor de Shapley en jocs amb cooperació restringida per un índex de cooperació mitjançant una funció generatriu. Un cop descrit el mètode, estudiem la complexitat computacional del càlcul de la funció generatriu en qüestió i presentem una implementació del mètode i un exemple.

Abstract

Our goal in this paper is to describe a generating function method to compute the Shapley value in games restricted by a cooperation index. Once we have described the method, we study the computational complexity for computing the required generating function and show an implementation of the method and an example.

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Introduction and preliminaries

Game theory is a branch of mathematics that studies the decision-making of some set of agents in situations where the outcome is affected by their choices. This discipline is of interest due to the applications these “situations”, or games, have in a wise range of topics, from economics and political science to computer science.

Broadly speaking, game theory is divided in the study of two different types of games: non-cooperative and cooperative. This paper focuses on cooperative game theory, in which the different agents, or players, reach binding agreements with each other, which determine the game. Although it is not the only class of cooperative games, this study specifically deals with the class of TU-games, short for *transferable utility games*.

A TU-game is a pair (N, v) where N is what we call the set of players of the game, and v is called the characteristic function of the game, which is a function $v : 2^N \to \mathbb{R}$, where $2^N = \{S : S \subseteq N\}$, such that $v(\varnothing) = 0$. As long as no confusion arises, we will usually identify such a game with its characteristic function. Being a cooperative game, in a TU-game we call the sets $S \subseteq N$ *coalitions* of players, and the value $v(S)$ of a coalition is its utility. Moreover, since this value is not related to that of any other coalition, one can interpret $v(S)$ as the benefit this group of players can generate for and by themselves. This benefit can be thought as a certain amount of a common currency (utility) that is to be divided among the members of the coalition.

This division is not trivial, as different players can be of different importance to a coalition. In terms of the characteristic function of a game, given a coalition $S \subseteq N$, the value $v(S \cup \{i\}) - v(S)$ can vary for each $i \in N$. This term $v(S \cup \{i\}) - v(S)$ is called the *marginal contribution* of i to S, and serves as a starting point to discuss how to “fairly” distribute utility among the players of the game.

In this context, given a set of players $N = \{1, 2, \dots, n\}$, we will refer to an n-dimensional real vector as an *allocation*. For our purposes, the i-th component of an allocation is the utility it allots to player i. As an extension of this idea, one can consider mapping each characteristic function $v : 2^N \to \mathbb{R}$ to a specific allocation. More formally, given G^N, the class of characteristic functions of TU-games with player set N, it is of interest to study maps $\varphi : G^N \to \mathbb{R}^n$. We call such maps *allocation rules*.

One well-known allocation rule is the Shapley value, which we will denote by Φ. For each player $i \in N$, the utility this player is allotted, analogously called the Shapley value of i and denoted by $\Phi_i[v]$, is defined as follows:

$$\Phi_i[v] = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \gamma_s \cdot (v(S \cup \{i\}) - v(S))$$

where for each S, $s = |S|$ and $\gamma_s = s!(n - s - 1)!$.

In other words, the Shapley value of a player in v is the weighted sum of their marginal contribution to all possible coalitions[2], the weight being $\frac{\gamma_s}{n!}$, which is dependent on the size of each coalition. Although we will not discuss this here, the Shapley value can also be derived from a set of axioms, with the properties required forcing the choice of these weights[3]. Our focus is instead on studying how to compute $\Phi_i[v]$ for each player of a TU-game v.

For TU-games in general, one could enumerate all possible coalitions of players and add up the weighted marginal contributions of each player. Other useful methods are not as general, as they require the characteristic function of the game to have some structure. As an example that will motivate the contents of this paper, we will show a method to compute the Shapley value in so-called weighted majority games. These games are part of the larger subclass of the simple games, so we will first define what these are.

A *simple* game is a TU-game (N, v) such that $v(S) \in \{0, 1\}$ for all $S \subseteq N$ and, given two coalitions $S, T \subseteq N$ such that $S \supseteq T$, then $v(S) \geqslant v(T)$. We call this latter property *monotonicity*. Simple games are usually characterized by the set of coalitions $S \subseteq N$ for which $v(S) = 1$, called the *set of winning coalitions* of v, and usually denoted by $\mathcal{W}(v)$ (or $\mathcal{W}$, as long as there is no ambiguity regarding the game). The set of coalitions $S \subseteq N$ for which $v(S) = 0$ is called set of losing coalitions.

[2] Note that if $i \in S$, then i's marginal contribution to S is zero.

[3] A priori, other choices are possible. For instance, the sum of the marginal contributions of a player weighted by 2^{1-n} is called the (normalized) Banzhaf power index, which can also be derived from a set of axioms. In any case, the scope of this study is limited to the Shapley value.

A *weighted majority* game is a simple game (N, v) for which there exists a list of weights $w_1, \ldots, w_n \in \mathbb{N}$ for the players and a quota $q > 0$ such that $S \subseteq N$ is a winning coalition of v if and only if $w(S) = \sum_{i \in S} w_i \geqslant q$. In general, we take

$$q \geqslant \frac{1}{2} \sum_{i \in N} w_i + 1$$

Thus, there cannot be two disjoint coalitions S, $T \subseteq N$ such that $w(S)$ and $w(T)$ are simultaneously greater than q. Equivalently, in this type of games there cannot be two disjoint winning coalitions. We say a simple game with this property is *proper*.

The reason why a method limited to the class of weighted majority games is of interest is that several real-life situations can be modeled using these games. As a relevant example of a weighted majority game, we can take a parliament where perfect party discipline is assumed, with the different parties acting as the players of the game and the number of representatives affiliated to each party as their weights. The quota is usually set to be the number of representatives required for a simple majority, assuming only votes in favor or against some proposal are allowed. In this overall scenario, the Shapley value of each player conveys, in a number from 0 to 1[4], the power each party holds in parliament.

Now, in order to compute the Shapley value in these games, note that since a simple game v must be monotonic, in particular, $v(S \cup \{i\}) \geqslant v(S)$ for all S. Furthermore, since for each S, $v(S)$ is either 0 or 1, a player i's marginal contribution to S, $v(S \cup \{i\}) - v(S)$, is also 0 or 1. We can disregard those $S \subseteq N$ for which $v(S \cup \{i\}) - v(S) = 0$, as they will have no effect in the calculation. On the other hand, those coalitions S with $v(S \cup \{i\}) - v(S) = 1$ are the losing coalitions of v for which $S \cup \{i\}$ is a winning coalition. We say a coalition with this property is a *swing* for player i.

[4]One of the axioms that defines the Shapley value, called *efficiency*, states that the sum of the Shapley value of all players of a game amounts to the value of the grand coalition, N, in this game, i.e. $\sum_{i \in N} \Phi_i[v] = v(N)$. Thus, in a weighted majority game, $\sum_{i \in N} \Phi_i[v] = 1$. Moreover, in such game, the marginal contributions are always non-negative, so the Shapley value of each player must also be non-negative.

It follows that, if v is a simple game, we can compute the Shapley value of each player $i \in N$ as

$$\Phi_i[v] = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \gamma_s (v(S \cup \{i\}) - v(S)) = \frac{1}{n!} \sum_{s=0}^{n-1} \gamma_s \cdot d_s^i$$

where d_s^i is the number of swings for i that have size s. Furthermore, if v is a weighted majority game, then S is a swing for i if and only if

$$q - w_i \leqslant w(S) \leqslant q - 1$$

where $w(S) = \sum_{j \in S} w_j$. Indeed, this ensures $w(S) < q$, and thus that S is a losing coalition, while $w(S \cup \{i\}) = w(S) + w_i \geqslant q$, and so $S \cup \{i\}$ is a winning coalition. Hence, the number of swings for player i that have size s is

$$d_s^i = \sum_{k=q-w_i}^{q-1} A_s^i(k)$$

where for each s and k, $A_s^i(k)$ is the number of coalitions $S \subseteq N \setminus \{i\}$ of size s such that $w(S) = \sum_{j \in S} w_j = k$. In short, if v is a weighted majority game, the calculation of the Shapley value of each $i \in N$ is reduced to that of the numbers $A_s^i(k)$. In order to compute these terms, we introduce the concept of a generating function.

A *generating function* is the representation of an m-variable sequence

$$\{s(n_1, \ldots, n_m) : \, n_j \geqslant 0, \, 1 \leqslant j \leqslant m\}$$

as a power series

$$f(x_1, \ldots, x_m) = \sum_{n_1 \geqslant 0} \cdots \sum_{n_m \geqslant 0} s(n_1, \ldots, n_m) \cdot x^{n_1} \cdot x^{n_2} \cdots x^{n_m}$$

This representation is useful as it provides the whole sequence merely as a multivariate polynomial.

For the sequence formed by the numbers $A_s^i(k)$ in a weighted majority game,

$$\{A_s^i(k):\ 0 \leqslant s \leqslant n-1,\, 0 \leqslant k \leqslant \sum_{j\neq i} w_j = w(N) - w_i\}$$

its generating function can be computed as

$$\begin{aligned}\prod_{j\neq i}(1 + zx^{w_j}) = 1 + \sum_{\varnothing\neq S\subseteq N\smallsetminus\{i\}} z^{|S|}\prod_{j\in S} x^{w_j} = \\ = \sum_{S\subseteq N} z^{|S|}x^{w(S)} = \\ = \sum_{s=0}^{n-1}\sum_{k=0}^{w(N)-w_i} A_s^i(k)\cdot z^s \cdot x^k\end{aligned}$$

Computing these polynomials is less demanding than listing all possible coalitions[5], making the *generating function method* a powerful tool to compute the Shapley value in weighted majority games. Moreover, note that the Shapley value of each player is directly computed from the list of weights and the quota for a majority: the explicit calculation of the winning coalitions of the game is not needed.

Unfortunately, despite these improvements on its calculation, the Shapley value as an allocation rule has some limitations. Recall that the Shapley value of a player is the weighted sum of their marginal contribution to all coalitions S, where the weights only depend on the size of the coalition. Thus, besides their size, this sum gives equal weight to each coalition, effectively assuming they are all equally likely to form. Applied to parliament, this would suggest that all agreements between parties are possible and equally likely, which is far from the case in general.

In reality, the relations between players can be complex. In an attempt to capture this complexity, we use *restricted cooperation* models. Some such models use a graph with the players of the games as vertices to convey some information on the relations. For instance, such a graph can be used to indicate pairs of players that will never cooperate. These graphs are called *incompatibility* graphs, and we say a pair of players connected by an edge in such a graph are *incompatible*.

[5] More details on this method and its computational complexity can be found in [4].

Given a restricted cooperation model, we can define its associated *restricted game*, which will be a new TU-game the characteristic function of which will account for the restrictions to cooperation imposed by the model. In particular, given a TU-game (N, v) and an incompatibility graph g, the restricted game is (N, v^g), with $v^g : 2^N \to \mathbb{R}$ defined by

$$v^g(S) = \max_{\mathcal{P} \in P(S,\, g)} \sum_{T \in \mathcal{P}} v(T)$$

where $P(S, g)$ is the set of all partitions of S into subsets that do not contain incompatible players[6], that is, that contain no pair of players connected in g.

Thus, the Shapley value in the restricted game results in an allocation that is arguably more realistic, as it takes into account the restrictions to cooperation given by the incompatibility graph g. As for computing it, although using enumeration is still possible, this would require computing the restricted game first, which is not trivial. Fortunately, if the original game v is a weighted majority game, a generating function method also exists for the Shapley value in a game with cooperation restricted by an incompatibility graph. Similar to the general case, the procedure, fully described in [1], does not require the explicit computation of the restricted game, only the list of weights, the quota for a majority and the incompatibility graph.

This brief introduction on how to compute the Shapley value under a specific restricted cooperation model poses the following question: is it possible to use a similar method for other models? At the same time, one might seek a more general model. Albeit useful, an incompatibility graph is arguably too restrictive, as it completely rejects cooperation between some players, thus eliminating much of the nuances in relations between players in real-life situations. This latter issue is part of the focus of [2], in which its authors introduce the *cooperation index* model, designed to generalize other restricted cooperation models.

Thus, our goal in this paper is to develop a generating function method to compute the Shapley value in TU-games restricted by a cooperation index. The approach is similar to that of [1], in that the method does not require the explicit computation of the restricted game. Furthermore, while the two generating function methods for the Shapley value we have discussed apply only to weighted majority games, the method we will describe here is more general.

[6] This definition is taken directly from [3]. See this paper for a detailed study on games restricted by incompatibility graphs and allocation rules in such games.

Next, Section 1 introduces the cooperation index model and the restricted game it yields and describes the method in full detail. Section 2 discusses the computational complexity of the method and the prior calculations its application might require. Finally, Section 3 provides an implementation of the method using `Mathematica` and a small example, and in Section 4 we give some final comments on the overall process and related questions to be explored.

1 Games restricted by a cooperation index

Let $(N,\, v)$ be a TU-game. In order to develop a method for the calculation that concerns us, we will first introduce the cooperation index concept and the restriction of v it yields, as stated in [2]:

Definition 1. A cooperation index on N is a function $p : 2^N \to [0,\, 1]$ such that $p(\{i\}) = 1 \;\forall i \in N$.

The restriction of $(N,\, v)$ by p (henceforth, as long as no confusion arises, the restricted game) is the TU-game $(N,\, v_p)$, where v_p is defined for every $S \subseteq N$ by

$$v_p(S) = \max_{\mathcal{P} \in P^+(S,\, p)} \sum_{T \in \mathcal{P}} v(T)p(T)$$

where $P^+(S,\, p)$ denotes the set of all partitions of S into subsets $T_1,\, \dots,\, T_k$ such that $p(T_j) > 0 \;\forall j \in \{1,\, \dots,\, k\}$[7].

Remark 1. By convenience, given a cooperation index p we will always assume $p(\varnothing) = 1$. Note that this does not affect the value of $v_p(\varnothing)$.

Intuitively, for each $S \subseteq N$, $v_p(S)$ can be seen as the maximum value weighted by p the members of S can collectively obtain, conditional on reaching agreements only with other members of S in ways that ensure all (sub)coalitions formed have positive cooperation index.

In this study we will describe a way to compute

$$\Phi_i[v_p] = \frac{1}{n!} \sum_{S \subseteq N \smallsetminus \{i\}} \gamma_s \left(v_p(S \cup \{i\}) - v_p(S) \right)$$

the Shapley value of each player $i \in N$ in a restricted game[8], where for each S, $s = |S|$ and $\gamma_s = s!(n - s - 1)!$.

[7] Definition 1 is taken directly from [2]. For the purposes of this study, in this definition we could take the maximum over all partitions, including those with some coalition with null cooperation index. In the cited paper, its authors discuss this alternative definition, and show it coincides with the original one should v be a non-negative game, hence why it will ultimately work for us.

[8] In the aforementioned paper by Amer and Carreras, Theorem 4.1. states that given a game v, this is the unique map $\Psi :\; I(N) \to \mathbb{R}^n$, where $I(N)$ is the class of cooperation indices on N. In other words, they provide an axiomatic description of the value, which we will not further discuss here.

Specifically, we will do so for simple proper games; thus, we will henceforth assume that (N, v) is a TU-game such that:

- v is monotonic, that is, given two coalitions $S, T \subseteq N$ such that $S \supseteq T$, we have that $v(S) \geqslant v(T)$.
- $v(S) \in \{0, 1\}$ for all $S \subseteq N$, and so v is a simple game.
- v is proper, that is, there are no two disjoint coalitions in the set of winning coalitions of v, $\mathcal{W} = \{S \subseteq N : v(S) = 1\}$.

Recall that, given $i \in N$,

$$\Phi_i[v_p] = \frac{1}{n!} \sum_{S \subseteq N \setminus \{i\}} \gamma_s \left(v_p(S \cup \{i\}) - v_p(S)\right)$$

where $s = |S|$ for each S and $\gamma_s = s!(n - s - 1)!$ for each $s \in \{0, \ldots, n - 1\}$. Thus, in order to compute $\Phi_i[v_p]$ it is in our interest to determine when $v_p(S \cup \{i\}) - v_p(S)$ is non-zero; we devote the following section to this objective.

1.1 Marginal contributions in the restricted game

Previously, we stated that given a TU-game (N, v), for each $i \in N$ and $S \subseteq N$ we call

$$v(S \cup \{i\}) - v(S)$$

the marginal contribution of player i to S in game v.

We claim that, since v is a non-negative game, i.e. $v(S) \geqslant 0 \ \forall S$, v_p is monotonic. This result appears in [2] as Proposition 3.4. To prove it, let $S, T \subseteq N$ be such that $S \supseteq T$ and let $\mathcal{Q} \in P^+(T, p)$ be a partition such that

$$v_p(T) = \sum_{T' \in \mathcal{P}} v(T')p(T')$$

Now consider the partition $\mathcal{Q}' = \bigcup_{j \in S \setminus T} \{j\} \cup \mathcal{Q}$ of S. By construction, $p(\{j\}) = 1$ for each $j \in S \setminus T$, and so $\mathcal{Q}' \in P^+(S, p)$.

Furthermore, by the non-negativity of v,

$$v_p(T) = \sum_{T' \in \mathcal{Q}} v(T')p(T') \leqslant \sum_{T' \in \mathcal{Q}} v(T')p(T') + \sum_{j \in S \setminus T} v(\{j\}) =$$

$$= \sum_{T' \in \mathcal{Q}'} v(T')p(T') \leqslant \max_{\mathcal{P} \in P^+(S,\, p)} \sum_{T' \in \mathcal{P}} v(T')p(T') = v_p(S)$$

and the monotonicity of v_p follows.

In particular, the marginal contribution of i to S in v_p, $v_p(S \cup \{i\}) - v_p(S)$ is always non-negative. Hence, we are actually studying when this term is positive. Recall that when v is a simple game, $v(S \cup \{i\}) - v(S) > 0$ if and only if a coalition $S \subseteq N \setminus \{i\}$ is what we call a *swing* for player i, that is, S is losing in v while $S \cup \{i\}$ is winning.

In the restricted game, the situation is somewhat more complex. The next result shows how the expression for v_p is reduced if v is a proper simple game, which will simplify the subsequent treatment of the marginal contribution of a player i to a coalition $S \subseteq N$ in v_p; in particular, that of its value when S is a swing for i in v.

Lemma 1. Given $\mathcal{W} = \{S \subseteq N :\ v(S) = 1\}$, the set of winning coalitions of v, for each $S \subseteq N$ we have

$$v_p(S) = \begin{cases} \max\limits_{T \in \mathcal{W} \cap 2^S} p(T) \text{ if } S \in \mathcal{W} \\ 0 \text{ otherwise} \end{cases}$$

Proof. By the monotonicity of v, if $S \notin \mathcal{W}$, then there is no $T \subseteq S$ such that $v(T) > 0$, and therefore it must be the case that $v_p(S) = 0$. If, on the contrary, $S \in \mathcal{W}$, then for each partition $\mathcal{P}$ of S there is at most one $T \subseteq S$ such that $v(T) = 1$, since we are assuming v is a proper game. Indeed, should there be two (or more) different winning coalitions in a partition $\mathcal{P}$ of S, we would have, in particular, a pair of disjoint coalitions in $\mathcal{W}$.

Now, given $S \in \mathcal{W}$, for each $\mathcal{P} \in P^+(S,\, p)$ consider

$$v_p(\mathcal{P},\, S) = \sum_{T \in \mathcal{P}} v(T)p(T)$$

We seek to maximize $v_p(\mathcal{P},\, S)$. It is clear that $v_p(\mathcal{P},\, S) = 0$ if there are no winning coalitions among the subsets in a partition $\mathcal{P}$. Otherwise, $v_p(\mathcal{P},\, S) = p(T) \geqslant 0$, where $T \in \mathcal{W}$ is the only such coalition contained in $\mathcal{P}$. It follows from here that if $S \in \mathcal{W}$, then $v_p(S) = \max_{T \in \mathcal{W} \cap 2^S} p(T)$. $\square$

Remark 2. As the following example shows, v being proper is not a superfluous hypothesis in the previous result.

Example 1. Let v be the 3-person game where all possible non-empty coalitions are winning except for $\{2\}$ and $\{3\}$ and consider the cooperation index p on $N = \{1, 2, 3\}$ defined as

$$p(S) = \begin{cases} 1 \text{ if } |S| = 1 \\ 1/2 \text{ otherwise} \end{cases}$$

Note that v is not proper: $P_1 = \{1\}$ and $P_2 = \{2, 3\}$ are disjoint winning coalitions of this game. In fact, $\mathcal{P}' = \{P_1, P_2\}$ is a partition of N containing only sets with positive cooperation index and therefore, starting from the initial characterization of v_p given in Definition 1,

$$v_p(N) = \max_{\mathcal{P} \in P^+(N,p)} \sum_{T \in \mathcal{P}} v(T)p(T) \geqslant \sum_{T \in \mathcal{P}'} v(T) \cdot p(T) =$$

$$= v(\{1\})p(\{1\}) + v(\{2, 3\})p(\{2, 3\}) = 3/2 > 1 = \max_{T \in \mathcal{W} \cap 2^N} p(T)$$

In particular, N is such that $v_p(N) \neq \max_{T \in \mathcal{W} \cap 2^N} p(T)$.

Remark 3. The previous example also shows that given a simple game v, in general, v_p is not a simple game.

From Lemma 1 follows that if S is a swing for i in v, since, in particular, S is a losing coalition in v, we have $v_p(S) = 0$, and so

$$v_p(S \cup \{i\}) - v_p(S) = v_p(S \cup \{i\}) - 0 = v_p(S \cup \{i\})$$

In other words, a swing for i in the simple game can give rise to a situation for which the marginal contribution of i to a coalition S in v_p is positive, as long as $v_p(S \cup \{i\})$ is positive in the first place.

On this latter point, Lemma 1 states a trivial necessary condition for $v_p(S)$ to be positive, namely, that S has to be winning in v. But this is clearly not enough: it could be the case that all winning coalitions contained in some $W \in \mathcal{W}$ have null cooperation index and therefore

$$v_p(W) = \max_{T \in \mathcal{W} \cap 2^W} p(T) = 0$$

Corollary 1. Given $S \subseteq N$, $v_p(S) > 0$ if and only if there is some winning coalition $T \subseteq S$ with $p(T) > 0$.

In general, given $W \in \mathcal{W}$, the value of p on all subsets of W must be checked in order to determine whether $v_p(W)$ is positive or not; the same effort is required to compute the exact value of the coalition.

Swings for player i in the simple game are not the only coalitions to which i can have a positive marginal contribution in the restricted game, though. Below these lines we properly characterize all instances in which i's marginal contribution to a coalition $S \subseteq N$ in the restricted game is positive.

Lemma 2. Let $S \subseteq N$ be a coalition not containing i, i.e. $S \subseteq N \setminus \{i\}$. The marginal contribution of i to S in v_p is positive if and only if $\exists T \subseteq S \cup \{i\}$ such that T is winning in v and $p(T) > 0$, and all coalitions T for which $v_p(S \cup \{i\}) = p(T)$ contain i.

Proof. In order to have $v_p(S \cup \{i\}) - v_p(S) > 0$, it must be the case that $v_p(S \cup \{i\}) > 0$. Otherwise the monotonicity of v_p implies that $v_p(S) = 0$, and so

$$v_p(S \cup \{i\}) - v_p(S) = 0 - 0 = 0$$

As stated by Corollary 1, $v_p(S \cup \{i\}) > 0$ is equivalent to there being some not necessarily unique winning coalition T with positive cooperation index contained in $S \cup \{i\}$. In particular, $S \cup \{i\}$ is winning in v and thus Lemma 1 guarantees $v_p(S \cup \{i\}) = p(T)$ for some of these T, still not ensuring uniqueness. It remains to be checked that all such T contain i.

This is trivial if S is losing, that is, if it is a swing for i in v. Note that a winning coalition $T' \subset S \cup \{i\}$ not containing i is a winning coalition contained in S. Should such T' exist, the monotonicity of v would then imply that S is also winning, which contradicts the assumption that S is a swing.

If S is winning to begin with and we suppose there is some winning coalition $T' \subseteq S$ such that $v_p(S \cup \{i\}) = p(T')$ then we claim that $v_p(S) = p(T')$. Since we are assuming that $S \in \mathcal{W}$, by Lemma 1 we have that $v_p(S) = \max_{T \in \mathcal{W} \cap 2^S} p(T) \geqslant p(T')$. Hence, if the claim does not hold, it would be implied that

$$v_p(S) > p(T') = v_p(S \cup \{i\})$$

which contradicts the monotonicity of v_p. It follows that $v_p(S \cup \{i\}) = v_p(S)$.

Conversely, if every winning coalition $T \subseteq S \cup \{i\}$ with $p(T) > 0$ and such that $v_p(S \cup \{i\}) = p(T)$ contains i, then there is no winning $T' \subseteq S$ with $p(T') > 0$ such that $v_p(S \cup \{i\}) = p(T') > 0$. In particular,

$$v_p(S \cup \{i\}) \neq \max_{T \in \mathcal{W} \cap 2^S} p(T) = v_p(S)$$

and the monotonicity of v_p implies that $v_p(S \cup \{i\}) > v_p(S)$. □

In summary, we have two situations in which the marginal contribution of a player i to a coalition S in v_p can be positive. We first saw that a swing for i in the simple game, that is, a situation in which a coalition S is losing while $S \cup \{i\}$ is winning, is one of these scenarios. Lemma 2 shows that, by the nature of the restricted game, it can also be the case that a player's marginal contribution to a coalition S that is winning in the simple game is positive.

In each situation, for i to have a positive marginal contribution to S, this player is required to be part of all subcoalitions of $S \cup \{i\}$ with cooperation index equal to $v_p(S \cup \{i\})$, this coalition being winning in v. Unfortunately, in practice it can be hard to verify whether a player satisfies this property relative to an arbitrary coalition; for now, we will use the following trivial necessary condition:

Corollary 2. Given a coalition $S \subseteq N \setminus \{i\}$, if $v_p(S \cup \{i\}) - v_p(S) > 0$, then there is *some* winning coalition $T \subseteq S \cup \{i\}$ such that $i \in T$ and $p(T) > 0$. In other words, S contains some set of players such that adding i to it results in a winning subcoalition of $S \cup \{i\}$ whose cooperation index is positive.

Aided by these results, we can write $\Phi_i[v_p]$ as

$$\begin{aligned}
\Phi_i[v_p] &= \frac{1}{n!}\sum_{S\subseteq N\setminus\{i\}} \gamma_s \cdot (v_p(S\cup\{i\} - v_p(S)) = \\
&= \frac{1}{n!}\left(\sum_{S\notin\mathcal{W}\colon S\cup\{i\}\in\mathcal{W}} \gamma_s v_p(S\cup\{i\}) + \sum_{S\in\mathcal{W}\colon i\notin S} \gamma_s(v_p(S\cup\{i\}) - v_p(S))\right) = \\
&= \frac{1}{n!}\left(\sum_{\substack{S\notin\mathcal{W}\colon \exists R\cup\{i\}\in\mathcal{W}\\ R\subseteq S,\, p(R\cup\{i\})>0}} \gamma_s v_p(S\cup\{i\}) + \sum_{\substack{S\in\mathcal{W}\colon \exists R\cup\{i\}\in\mathcal{W}\\ i\notin S,\, R\subseteq S,\, p(R\cup\{i\})>0}} \gamma_s(v_p(S\cup\{i\}) - v_p(S))\right) = \\
&= \frac{1}{n!}\left(\sum_{\substack{S\subseteq N\setminus\{i\}\colon \exists R\cup\{i\}\in\mathcal{W}\\ R\subseteq S,\, p(R\cup\{i\})>0}} \gamma_s v_p(S\cup\{i\}) - \sum_{\substack{S\in\mathcal{W}\colon \exists R\cup\{i\}\in\mathcal{W}\\ i\notin S,\, R\subseteq S,\, p(R\cup\{i\})>0}} \gamma_s v_p(S)\right)
\end{aligned}$$

In the preceding equation, starting from the definition of player i's Shapley value in the restricted game, we have first restricted our work to those sets of coalitions in which it is possible to have $v_p(S\cup\{i\}) - v_p(S) > 0$. As previously discussed, these correspond to cases where either (a) S is a losing coalition of v that becomes winning when i is added to it, for which we know this player's marginal contribution to S in the restricted game is $v_p(S\cup\{i\})$, or (b) S is a winning coalition of v not involving player i.

The next step further prunes both of these sets using Corollary 2. Namely, we are restricting the summations to those S satisfying the necessary condition for $v_p(S\cup\{i\}) - v_p(S) > 0$ to occur given by this result. Hence, in the first summation, accounting for coalitions satisfying (a), we are only adding over the set of losing coalitions $S \subseteq N \setminus \{i\}$ for which there is some $R \subseteq S$ such that $R\cup\{i\}$ is winning and $p(R\cup\{i\}) > 0$. Analogously, in the second summation, accounting for coalitions satisfying (b), we add over the set of winning coalitions $S \subseteq N\setminus\{i\}$ for which there is some $R \subseteq S$ with the same properties as before.

Finally, we can merge the first summation with the positive part of the second one, as they are collectively adding term $\gamma_s v_p(S\cup\{i\})$ over all $S \subseteq N\setminus\{i\}$ for which there is some subcoalition $R \subseteq S$ such that adding i to R results in a winning coalition with positive cooperation index.

Definition 2. For each $i \in N$, let

$$\Phi_i^+[v_p] = \sum_{\substack{S \subseteq N \setminus \{i\} \colon \exists R \cup \{i\} \in \mathcal{W} \\ R \subseteq S,\, p(R \cup \{i\}) > 0}} \gamma_s v_p(S \cup \{i\})$$

$$\Phi_i^-[v_p] = \sum_{\substack{S \in \mathcal{W} \colon \exists R \cup \{i\} \in \mathcal{W} \\ i \notin S,\, R \subseteq S,\, p(R \cup \{i\}) > 0}} \gamma_s v_p(S)$$

We will call these summations the positive and negative parts of $\Phi_i[v_p]$, respectively.

To sum up, we can now compactly write $\Phi_i[v_p]$ as

$$\Phi_i[v_p] = \frac{1}{n!}\left(\Phi_i^+[v_p] - \Phi_i^-[v_p]\right)$$

1.2 The set of essential coalitions

Note that, in essence, the positive and negative parts of $\Phi_i[v_p]$ are sums of $v_p(S)$ (weighted by γ_s) for S in a certain subset of $\mathcal{W}$. Recall that by Lemma 1, for each such S, we have $v_p(S) = p(T)$ for some winning $T \subseteq S$; hence, we are adding up $p(T)$ for T in a certain subset of $\mathcal{W}$.

As the latter subset, although it would be reasonable to use the smallest collection of winning coalitions T for which there is some $S \subseteq N$ such that $v_p(S) = p(T)$, that is, those winning coalitions with cooperation index equal to the value of some coalition in the restricted game, this is not the collection we will use for this purpose. For now, we introduce the collection we will actually use and its key property.

Definition 3. Let $\mathscr{E}(v, p)$ be the set of winning coalitions S of v with positive cooperation index such that every proper winning subcoalition of S has a lower cooperation index than S, that is,

$$\mathscr{E}(v, p) = \{S \in \mathcal{W}_+(p) :\ p(S) > p(R)\ \forall R \in \mathcal{W} \cap 2^S,\ R \neq S\}$$

where $\mathcal{W}_+(p) = \{W \in \mathcal{W} :\ p(W) > 0\}$.

We will call this collection set of essential coalitions of game v associated with cooperation index p. As long as no confusion arises, we will simply call it the set of essential coalitions and denote it by $\mathscr{E}$.

Lemma 3. For each $T \subseteq N$ such that $v_p(T) > 0$ there is some essential coalition S such that $S \subseteq T$ and $v_p(T) = p(S)$.

Proof. Given a coalition $T \subseteq N$ such that $v_p(T) > 0$, let $S \subseteq T$ be a winning coalition such that $v_p(T) = p(S)$ and $|S| \leqslant |S'|$ for all other $S' \in \mathcal{W} \cap 2^T$ for which $v_p(T) = p(S')$. We claim that S is an essential coalition, that is, its cooperation index is greater than that of any winning coalitions of v strictly contained in it.

Should this claim not hold, then there would be some winning $R \subsetneq S$ with $p(R) \geqslant p(S) = v_p(T)$. In fact, since, in particular, $R \subseteq T$, and by Lemma 1,

$$p(S) = v_p(T) = \max_{R' \in \mathcal{W} \cap 2^T} p(R') \geqslant p(R)$$

it follows that $v_p(T) = p(R)$. But, since $|R| < |S|$, this contradicts the assumption that there is no coalition S' smaller than S such that $v_p(T) = p(S')$; we conclude that $S \in \mathscr{E}$. □

Observe that given $\mathcal{C} \subseteq \mathcal{W}$ such that for all $T \subseteq N$ with $v_p(T) > 0$ there is some $C \in \mathcal{C}$ for which $C \subseteq T$ and $v_p(T) = p(C)$, this collection $\mathcal{C}$ must contain all essential coalitions, that is, $\mathscr{E} \subseteq \mathcal{C}$. This is directly implied by the definition of $\mathscr{E}$: any essential coalition S is, by construction, a winning coalition of v with positive cooperation index, and so, by Lemma 1,

$$v_p(S) = \max_{R \in \mathcal{W} \cap 2^S} p(R) \geqslant p(S) > 0$$

Again by construction of $\mathscr{E}$, the only winning coalition $R \subseteq S$ for which $v_p(S) = p(R)$ is S itself, and so $S \in \mathcal{C}$. In summary, we can use the property of $\mathscr{E}$ stated by Lemma 3 as an alternative definition of the set of essential coalitions.

However, while Lemma 3 guarantees that if $v_p(T) > 0$ then there is some essential coalition $E \in \mathscr{E}$ contained in T for which $v_p(T) = p(E)$, such E need not be unique. This is a consequence of Lemma 1 stating that for each winning coalition $T \subseteq N$ there is some winning coalition $S \subseteq T$ for which $v_p(T) = p(S)$, not ensuring uniqueness either; as the following example shows, it can be the case that more than one such S is an essential coalition.

Example 2. Let $N = \{1, 2, 3, 4\}$ and v be the four player simple game with winning coalitions

$$\mathcal{W} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\} \cup \{S \subseteq N : |S| \geqslant 3\}$$

Consider the cooperation index p defined by

$$p(S) = \begin{cases} 1 \text{ if } |S| = 1 \\ 1/3 \text{ if } |S| = 2 \\ 1/4 \text{ if } |S| = 3 \\ 1/2 \text{ if } S = N \end{cases}$$

It is easy to check that $\mathscr{E} = \mathscr{E}(v, p) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, N\}$. Recall that

$$\mathscr{E} = \mathscr{E}(v, p) = \{S \in \mathcal{W}_+(p) : \, p(S) > p(R) \; \forall R \in \mathcal{W} \cap 2^S, \, R \neq S\}$$

The grand coalition of this game, N, has the highest cooperation index among the winning coalitions, and no other matches its value, so $N \in \mathscr{E}$. Coalitions of size 3 cannot be essential in this game, as they all contain at least one winning coalition of size 2, which have greater cooperation index. Finally, each winning coalition of size 2 trivially has strictly larger cooperation index than all winning coalitions it contains. These two player coalitions are the minimal winning coalitions of v, that is, those that contain no further winning coalitions. Having said this, we have that for $T = \{2, 3, 4\}$ both $S_1 = \{2, 3\}$ and $S_2 = \{3, 4\}$ are essential coalitions of this game contained in T and

$$v_p(T) = p(S_1) = p(S_2) = 1/3$$

Thus, $\mathscr{E}$ is not necessarily the smallest collection of winning coalitions W of v for which there is some $S \subseteq N$ with $v_p(S) = p(W)$. In other words, the previous example shows that $\mathscr{E}$ may contain two coalitions with the same cooperation index. However, in spite of $\mathscr{E}$ being a larger collection, we will favor its use over that of a collection containing only one coalition $W(x) \in \mathscr{E}$ for each possible value $x \in [0, 1]$ the cooperation index p takes in $\mathscr{E}$; we can interpret the previous example as showing that, given $\mathscr{E}$, there may not be a unique way to construct such a collection.

In general, given the set of winning coalitions $\mathcal{W}$ of a game v and a cooperation index p, computing the corresponding set of essential coalitions is hard. This is partly caused by p being a rather general function on 2^N: we are only requiring it to return outputs in interval $[0, 1]$ and for it to return exactly 1 for sets of one element. Imposing more structure on p can ease the construction of $\mathscr{E}$. This issue will be fully addressed later on in section 2. In the method below we will suppose the desired set of essential coalitions is given. Not only that, we will require the essential coalitions to be in a certain order.

Namely, given the set of essential coalitions $\mathscr{E}$, let $\Theta(\mathscr{E})$ be a permutation of the elements of this set such that, as a list,

$$\Theta(\mathscr{E}) = (E_1,\, E_2,\, \ldots,\, E_k)$$

is ordered in non-increasing order of p, that is, if $l < j$, then $p(E_l) \geqslant p(E_j)$. We will call $\Theta(\mathscr{E})$ a *sorted list of essential coalitions*, and denote it by Θ as long as no confusion arises regarding the particular set of essential coalitions $\mathscr{E}$ being considered.

Note that the ordering of $\mathscr{E}$ given by Θ above is not necessarily unique. In practice, for our purposes here and later in this section, since the upcoming results hold for any sorted list of coalitions, we are indifferent as to which one is chosen. Even so, for the sake of clarity and consistency, from now on let Θ be the sorted list of coalitions in which, in case some coalitions have the same cooperation index, these are sorted in non-decreasing order of size. Should the tie persist among some of them, let these coalitions be in lexicographical order. To sum up, we are henceforth assuming that

$$\Theta = (E_1, E_2,\, \ldots,\, E_k)$$

is a list of essential coalitions such that given two such coalitions $E_l = \{i_1,\, \ldots,\, i_{|E_l|}\}$ and $E_t = \{j_1,\, \ldots,\, j_{|E_t|}\}$ with $l < t$ and

$$\begin{aligned} i_1 < i_2 < \cdots < i_{|E_l|} \\ j_1 < j_2 < \cdots < j_{|E_t|} \end{aligned}$$

then, either

a) $p(E_l) > p(E_t)$,

b) $p(E_l) = p(E_t)$ and $|E_l| < |E_t|$, or

c) $p(E_l) = p(E_t)$, $|E_l| = |E_t|$ and there is some $m \in \{1,\, \ldots,\, |E_l|\}$ such that $i_r = j_r$ for $r < m$ and $i_m < j_m$.

We will still call Θ a sorted list of essential coalitions.

For the remainder of this section we will also consider a map μ defined for each $S \subseteq N$ with $v_p(S) > 0$ by $\mu(S) = \min\{l \in K : E_l \subseteq S\}$[9]. The lemma below shows a key property of this map; from this result, a necessary condition for a player's marginal contribution to a coalition $S \subseteq N \setminus \{i\}$ to be positive in terms of Θ follows.

Lemma 4. Let v be a simple proper game with player N and p a cooperation index on N. Given a sorted list of essential coalitions

$$\Theta = \Theta\left(\mathscr{E}(v,\, p)\right) = (E_1,\, \ldots,\, E_k)$$

For all $S \subseteq N$ such that $v_p(S) > 0$ we have $v_p(S) = p(E_{\mu(S)})$.

Proof. Suppose the lemma is not true. Then, since $E_{\mu(S)} \subseteq S$ and $v_p(S) > 0$ implies that S is winning, by Lemma 1

$$p(E_{\mu(S)}) \leqslant \max_{T \in \mathcal{W} \cap 2^S} p(T) = v_p(S)$$

and so $v_p(S) > p(E_{\mu(S)})$.

Now, let $m \in \{1,\, \ldots,\, k\}$ be such that the essential coalition E_m is contained in S and $v_p(S) = p(E_m)$. By Lemma 3, m is well defined. Moreover, we have seen that $p(E_m) > p(E_{\mu(S)})$. Thus, since Θ is a sorted list of essential coalitions, it must be the case that $m < \mu(S)$. However, this contradicts the assumption that $E_{\mu(S)}$ is the first coalition in Θ that is contained in S; we conclude that $v_p(S) = p(E_{\mu(S)})$. $\square$

Corollary 3. Let v be a simple proper game with player set N and p a cooperation index on N. Given a coalition $S \subseteq N \setminus \{i\}$ and a sorted list of essential coalitions

$$\Theta = \Theta\left(\mathscr{E}(v,\, p)\right) = (E_1,\, \ldots,\, E_k)$$

If $v_p(S \cup \{i\}) - v_p(S) > 0$, then $\mu(S \cup \{i\}) = \min\{l \in K : E_l \subseteq S \cup \{i\}\}$ is well defined and $i \in E_{\mu(S \cup \{i\})}$.

[9] A map such as μ can be built for each sorted list of essential coalitions, and so a parametric family of maps μ_Θ could be considered. However, we need not use this, as we have already set Θ to be a particular list of essential coalitions.

Proof. As seen in Lemma 2, if $v_p(S \cup \{i\}) - v_p(S) > 0$, then $v_p(S \cup \{i\}) > 0$. Thus, $\mu(S \cup \{i\})$ is indeed well defined, and, by Lemma 4, $v_p(S \cup \{i\}) = p(E_{\mu(S\cup\{i\})})$.

And, since, by construction, $E_{\mu(S\cup\{i\})} \subseteq S \cup \{i\}$, if $i \notin E_m$, then, by Lemma 2, $v_p(S \cup \{i\}) - v_p(S) = 0$, and the corollary follows. □

While still not sufficient, the necessary condition given in Corollary 3 is at least as restrictive as the one given in Corollary 2. Indeed, given $S \subseteq N \smallsetminus \{i\}$ such that $i \in E_{\mu(S\cup\{i\})}$, where $\mu(S \cup \{i\})$ is well defined, then there is some coalition $R \subseteq S$ such that $R \cup \{i\}$ is winning in v and has positive cooperation index, namely, $E_m \smallsetminus \{i\}$. The converse statement is not true however, as the following example shows.

Example 3. Consider again Example 2, that is, the game v with player set $N = \{1, 2, 3, 4\}$ and winning coalitions

$$\mathcal{W} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\} \cup \{S \subseteq N : |S| \geqslant 3\}$$

and the cooperation index p defined by

$$p(S) = \begin{cases} 1 \text{ if } |S| = 1 \\ 1/3 \text{ if } |S| = 2 \\ 1/4 \text{ if } |S| = 3 \\ 1/2 \text{ if } S = N \end{cases}$$

We already saw that these inputs yield

$$\mathscr{E} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, N\}$$

Note that the list of essential coalitions $(\{1, 2\}, \{2, 3\}, \{3, 4\}, N)$ is not in non-increasing order of p, since N is in last position while it has strictly larger cooperation index than all other essential coalitions; in this example,

$$\Theta = (N, \{1, 2\}, \{2, 3\}, \{3, 4\}) = (E_1, E_2, E_3, E_4)$$

is a sorted list of essential coalitions.

Now, consider the coalition $S = \{1, 2\}$. For player 3, subcoalition $R = \{2\}$ is such that $R \cup \{3\} = \{2, 3\}$ is a winning coalition that is contained in $S \cup \{3\} = \{1, 2, 3\}$ and $p(\{2, 3\}) = \frac{1}{3} > 0$ implies that $v_p(\{1, 2, 3\}) > 0$. However,

$$\mu(\{1, 2, 3\}) = \min\{l \in K :\ E_l \subseteq \{1, 2, 3\}\} = 2$$

and $3 \notin E_2 = \{1, 2\}$.

Thus, we obtain

$$\begin{aligned}
\Phi_i^+[v_p] - \Phi_i^-[v_p] &= \sum_{\substack{S \subseteq N \smallsetminus \{i\} \colon R \cup \{i\} \in \mathcal{W} \\ R \subseteq S,\, p(R \cup \{i\}) > 0}} \gamma_s v_p(S \cup \{i\}) - \sum_{\substack{S \in \mathcal{W} \colon R \cup \{i\} \in \mathcal{W} \\ i \notin S,\, R \subseteq S,\, p(R \cup \{i\}) > 0}} \gamma_s v_p(S) \\
&= \sum_{S \subseteq N \smallsetminus \{i\} \colon i \in E_{\mu(S \cup \{i\})}} \gamma_s v_p(S \cup \{i\}) - \sum_{\substack{S \in \mathcal{W} \colon i \in E_{\mu(S \cup \{i\})} \\ i \notin S}} \gamma_s v_p(S)
\end{aligned}$$

where $\mu(S \cup \{i\}) = \min\{l \in K :\ E_l \subseteq S \cup \{i\}\}$[10].

Since the two resulting summations depend on the particular ordering of the sorted list of essential coalitions Θ, we will denote them by

$$\Phi_i^+[v_p,\, \Theta] = \sum_{S \subseteq N \smallsetminus \{i\} \colon i \in E_{\mu(S \cup \{i\})}} \gamma_s v_p(S \cup \{i\})$$

$$\Phi_i^-[v_p,\, \Theta] = \sum_{\substack{S \in \mathcal{W} \colon i \in E_{\mu(S \cup \{i\})} \\ i \notin S}} \gamma_s v_p(S)$$

and call them the positive and negative parts of $\Phi_i[v_p]$, respectively, *associated with* Θ[11]. All in all, we have

$$\Phi_i[v_p] = \frac{1}{n!}\left(\Phi_i^+[v_p,\, \Theta] - \Phi_i^-[v_p,\, \Theta]\right)$$

[10] There is an abuse of notation here, as in the summations above it is implied that $\mu(S \cup \{i\})$ is well defined, i.e. $v_p(S \cup \{i\}) > 0$. Recall that if $v_p(S \cup \{i\}) = 0$, then, by the monotonicity of v_p, $v_p(S) = 0$, so S would not contribute to any of the summations in the equation.

[11] Although we have previously set Θ to be a specific sorted list of essential coalitions, this notation conveys that, while $\Phi_i^+[v_p]$ and $\Phi_i^-[v_p]$ are inherent to the game, $\Phi_i^+[v_p,\, \Theta]$ and $\Phi_i^-[v_p,\, \Theta]$ depend on the actual choice of Θ.

Without further ado now, we proceed to discuss how to add up the cooperation indices of essential coalitions to obtain the positive and negative parts of $\Phi_i[v_p]$ associated with Θ. We will first focus on the positive part,

$$\Phi_i^+[v_p,\, \Theta] = \sum_{S\subseteq N\setminus\{i\}:\, i\in E_{\mu(S\cup\{i\})}} \gamma_s v_p(S\cup\{i\})$$

This is a sum over coalitions of size between 0 and $n-1$, and each of its terms $s \in \{0,\, \dots,\, n-1\}$ is itself a sum adding up $v_p(S\cup\{i\})$ for those S of size s such that $i \in E_{\mu(S\cup\{i\})}$, weighted by $\gamma_s = s!(n-s-1)!$, that is,

$$\Phi_i^+[v_p,\, \Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{S\subseteq N\setminus\{i\}:\, |S|=s,\, i\in E_{\mu(S\cup\{i\})}} v_p(S\cup\{i\})$$

Thus, our calculation has been reduced to properly adding the value of some coalitions in the restricted game. Recall that Lemma 4 shows that if $v_p(S\cup\{i\}) > 0$, then $v_p(S\cup\{i\}) = p(E_{\mu(S\cup\{i\})})$, and so

$$\sum_{S\subseteq N\setminus\{i\}:\, |S|=s,\, i\in E_{\mu(S\cup\{i\})}} v_p(S\cup\{i\}) = \sum_{S\subseteq N\setminus\{i\}:\, |S|=s,\, i\in E_{\mu(S\cup\{i\})}} p(E_{\mu(S\cup\{i\})})$$

Note that, in practice, we are adding the cooperation index of essential coalitions that contain i, and so

$$\sum_{S\subseteq N\setminus\{i\}:\, |S|=s,\, i\in E_{\mu(S\cup\{i\})}} p(E_{\mu(S\cup\{i\})}) = \sum_{\substack{l\in K\\ i\in E_l}} p(E_l)\cdot|\{S \subseteq N\setminus\{i\} :\ \mu(S\cup\{i\}) = l \text{ and } |S| = s\}|$$

which proves the following result.

Lemma 5. Let v be a simple proper game with player set N and p a cooperation index on N. Given a sorted list of essential coalitions

$$\Theta = \Theta\left(\mathscr{E}(v,\, p)\right) = (E_1,\, \ldots,\, E_k)$$

for each player $i \in N$, the positive part of their Shapley value in the restricted game associated with Θ satisfies

$$\Phi_i^+[v_p,\, \Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{l \in K \\ i \in E_l}} p(E_l) \cdot \left| D_s^i(l) \right|$$

where $\gamma_s = s!(n-s-1)!$, $K = \{1,\, 2,\, \ldots,\, k\}$ and, for each $l \in K$,

$$D_s^i(l) = \{S \subseteq N \setminus \{i\} :\; \mu(S \cup \{i\}) = l \text{ and } |S| = s\}$$

where for each $T \subseteq N$ such that $v_p(T) > 0$, $\mu(T) = \min\{l \in K :\; E_l \subseteq T\}$.

Thus, $D_s^i(l)$ represents the set of coalitions of size s that do not contain player i for which $E_l \subseteq S \cup \{i\}$ while none of the essential coalitions $E_1,\, \ldots,\, E_{l-1}$ that precede it in the sorted list Θ are subcoalitions of $S \cup \{i\}$.

The procedure we will use to compute the negative part,

$$\Phi_i^-[v_p,\, \Theta] = \sum_{\substack{S \in \mathcal{W} :\, i \in E_{\mu(S \cup \{i\})} \\ i \notin S}} \gamma_s v_p(S)$$

is quite similar. Once again, we are performing a sum over subsets of $N \setminus \{i\}$, which have cardinality $s \in \{0,\, \ldots,\, n-1\}$. Now, for each term s of the sum, the weight $\gamma_s = s!(n-s-1)!$ is to be multiplied by the sum of $v_p(S)$ over those winning coalitions $S \subseteq N \setminus \{i\}$ of size s such that $i \in E_{\mu(S \cup \{i\})}$, that is,

$$\Phi_i^-[v_p,\, \Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{S \in \mathcal{W} :\, |S| = s,\, i \in E_{\mu(S \cup \{i\})} \\ i \notin S}} v_p(S)$$

We can safely disregard those winning coalitions for which $v_p(S) = 0$, as they do not contribute to the sum above. For those with $v_p(S) > 0$, Lemma 4 ensures that

$v_p(S) = p(E_{\mu(S)})$, where $\mu(S) = \min\{l \in K : E_l \subseteq S\}$. Note that as $i \notin S$ for these S, it must also be the case that $i \notin E_{\mu(S)}$, so, in essence, we are merely adding the cooperation index of essential coalitions that do not contain i, and so

$$\sum_{\substack{S \in \mathcal{W}\colon |S|=s,\, i \in E_{\mu(S\cup\{i\})} \\ i \notin S}} v_p(S) = \sum_{l \in K\colon i \notin E_l} \sum_{\substack{S \subseteq N \setminus \{i\}\colon i \in E_{\mu(S\cup\{i\})} \\ \text{and } \mu(S)=l}} p(E_l)$$

Finally, for each $l \in K$ such that $i \notin E_l$, the coalitions $S \subseteq N \setminus \{i\}$ for which $i \in E_{\mu(S\cup\{i\})}$ while $\mu(S) = l$ are precisely those for which $\mu(S \cup \{i\}) < \mu(S)$. It suffices to observe that, by construction, $\mu(S \cup \{i\}) \leqslant \mu(S)$ in any case, since $E_{\mu(S)} \subseteq S \subseteq S \cup \{i\}$, and equality holds if and only if $i \notin E_{\mu(S\cup\{i\})}$, as this is equivalent to $E_{\mu(S\cup\{i\})} \subseteq S$. All in all, we have proven the following result.

Lemma 6. *Let v be a simple proper game with player set N and p a cooperation index on N. Given a sorted list of essential coalitions*

$$\Theta = \Theta\left(\mathscr{E}(v,\, p)\right) = (E_1,\, \ldots,\, E_k)$$

for each player $i \in N$, the negative part of their Shapley value in the restricted game associated to Θ satisfies

$$\Phi_i^-[v_p,\, \Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{l \in K \\ i \notin E_l}} p(E_l) \cdot \left|\{S \in \overline{D_s^i}(l) : \mu(S \cup \{i\}) < l\}\right|$$

where $\gamma_s = s!(n-s-1)!$, $K = \{1,\, 2,\, \ldots,\, k\}$, for each $l \in K$,

$$\overline{D_s^i}(l) = \{S \subseteq N \setminus \{i\} : \mu(S) = l \text{ and } |S| = s\}$$

and $\mu(S) = \min\{l \in K : E_l \subseteq S\}$.

Thus, $\overline{D_s^i}(l)$ represents the set of coalitions of size s that do not contain player i for which $E_l \subseteq S$ while none of the essential coalitions $E_1,\, \ldots,\, E_{l-1}$ that precede it in the sorted list Θ are subcoalitions of S.

1.3 Generating functions for the Shapley value of the restricted game

After the work done in the previous subsection, the calculation of each player's Shapley value in the restricted game has now been reduced to computing the cardinality of some subsets of

$$\{S \subseteq N \setminus \{i\} : \; i \in E_{\mu(S \cup \{i\})} \text{ and } |S| = s\}$$

where $\Theta = (E_1, \ldots, E_k)$ is a sorted list of essential coalitions and μ is the map

$$\mu : \; \{S \subseteq N : \; v_p(S) > 0\} \to K$$

$$S \mapsto \min\{l \in K : \; E_l \subseteq S\}$$

with $K = \{1, 2, \ldots, k\}$. Recall that, by construction,

$$p(E_1) \geqslant p(E_2) \geqslant \cdots \geqslant p(E_k)$$

In this subsection, we will perform the required computation for each $i \in N$ using the generating function

$$\sum_{s_1 \geqslant 0} \sum_{s_2 \geqslant 0} \cdots \sum_{s_k \geqslant 0} \sum_{s \geqslant 0} A_s^i(s_1, \ldots, s_k)$$

where $A_s^i(s_1, \ldots, s_k)$ is the number of coalitions $S \subseteq N \setminus \{i\}$ of size s such that $|S \cap E_l| = s_l$ for each $l \in K$.

Thus, it is in our interest to study the value of $|S \cap E_l|$ for each $l \in K$ and S in the sets that concern us.

For the positive part of the Shapley value $\Phi_i[v_p]$, Lemma 5 states that

$$\Phi_i^+[v_p, \Theta] = \sum_{S \subseteq N \setminus \{i\} : \, i \in E_{\mu(S \cup \{i\})}} \gamma_s v_p(S \cup \{i\})$$

can be written as

$$\Phi_i^+[v_p, \Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{l \in K \\ i \in E_l}} p(E_l) \cdot \left| D_s^i(l) \right|$$

where for each l,

$$D_s^i(l) = \{S \subseteq N \setminus \{i\} : \; \mu(S \cup \{i\}) = l \text{ and } |S| = s\}$$

In plain words, given i and s, the elements of $D^i_s(l)$ are the coalitions S of size s such that adding i to S results in a coalition $S \cup \{i\}$ that contains essential coalition E_l, while the essential coalitions $E_1, \ldots, E_{l-1}$, which have cooperation index greater or equal than $p(E_l)$, are not contained in $S \cup \{i\}$. Note that in our computation we are only interested in $D^i_s(l)$ for $l \in K$ such that $i \in E_l$.

Take one such l and consider $S \in D^i_s(l)$. First of all, since it is required that

$$l = \mu(S \cup \{i\}) = \min\{l \in K : \, E_l \subseteq S \cup \{i\}\}$$

in particular, $E_l \subseteq S \cup \{i\}$, and so it must be the case that $S \cap E_l = E_l \setminus \{i\}$, that is, $|S \cap E_l| = |E_l| - 1$. For all other $l' \in K$ we must have $|S \cap E_{l'}| \leqslant |S| = s$; also recall that $0 \leqslant s \leqslant n-1$, as $S \subseteq N \setminus \{i\}$. For $l' < l$, since it is required that $E_{l'} \not\subseteq S \cup \{i\}$, we need $|(S \cup \{i\}) \cap E_{l'}| \leqslant |E_{l'}| - 1$ as well.

Let $m < l$ be such that $i \in E_m$. In this case, we cannot have $|S \cap E_m| = |E_m| - 1$; this would imply $E_m \subseteq S \cup \{i\}$, which, since $m < l$, contradicts the assumption that E_l is the first essential coalition in the sorted list Θ contained in $S \cup \{i\}$. In short, if $m < l$ and $i \in E_m$, there must be at least one player in E_m not belonging to S other than i themselves, that is, we must have $0 \leqslant |S \cap E_m| \leqslant \min\{s, \, |E_m| - 2\}$.

As for those $t < l$ for which $i \notin E_t$, we have $|S \cap E_t| = |(S \cup \{i\}) \cap E_t|$, and so requiring $0 \leqslant |S \cap E_t| \leqslant \min\{s, \, |E_t| - 1\}$ suffices. Finally, for those $j > l$, the construction of $D^i_s(l)$ imposes no conditions on the relation between $S \in D^i_s(l)$ and E_j, regardless of whether i is in E_j or not. Thus, since $i \notin S$ for $j \in K$ such that $j > l$, all that can be said is $0 \leqslant |S \cap E_j| \leqslant \min\{s, \, |E_j|\}$ if $i \notin E_j$ and $0 \leqslant |S \cap E_j| \leqslant \min\{s, \, |E_j| - 1\}$ if $i \in E_j$.

All in all, we have seen that for $l \in K$ such that $i \in E_l$,

$$\left|D^i_s(l)\right| = \sum_{\substack{m<l \\ i \in E_m}} \sum_{s_m=0}^{\min\{s, |E_m|-2\}} \sum_{\substack{t<l \\ i \notin E_t}} \sum_{s_t=0}^{\min\{s, |E_t|-1\}} \sum_{\substack{j>l \\ i \notin E_j}} \sum_{s_j=0}^{\min\{s, |E_j|\}} \sum_{\substack{j'>l \\ i \in E_{j'}}} \sum_{s_{j'}=0}^{\min\{s, |E_{j'}|-1\}} A^i_s(s_1, \ldots, s_k)\Big|_{s_l = |E_l|-1}$$

where $A^i_s(s_1, \ldots, s_k)$ is the number of coalitions $S \subseteq N \setminus \{i\}$ of size s such that $|S \cap E_{l'}| = s_{l'}$ for each $l' \in K$. Note that, given s, by construction, if there is some $l' \in K$ such that $s_{l'} > s$, then $A^i_s(s_1, \ldots, s_{l'}, \ldots, s_k) = 0$; otherwise it would be implied that $|S \cap E_{l'}| = s_{l'} > s = |S|$. Therefore, the overall calculation is unchanged if we replace the minima from the upper bounds of the summations above with the terms different than s. We can also merge the summations over $j > l$, since if $i \in E_{j'}$ for $j' > l$, there is no coalition not containing i for which $|S \cap E_{j'}| = |E_{j'}|$, that is, $A^i_s(s_1, \ldots, |E_{j'}|, \ldots, s_k) = 0$ as well in this scenario.

Thus, following from Lemma 5, we have proven that the proposition below characterizes the computation of $\Phi^+_i[v_p, \Theta]$.

Proposition 1. Let v be a simple proper game with player set N and p a cooperation index on N. Given a sorted list of essential coalitions

$$\Theta = \Theta\left(\mathscr{E}(v, p)\right) = (E_1, \ldots, E_k)$$

for each player $i \in N$, the positive part of their Shapley value in the restricted game associated with Θ can be computed as

$$\Phi^+_i[v_p, \Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{l \in K \\ i \in E_l}} p(E_l) \cdot d^i_s(l)$$

where $\gamma_s = s!(n-s-1)!$, $K = \{1, 2, \ldots, k\}$ and for each $l \in K$ for which $i \in E_l$,

$$d^i_s(l) = \sum_{\substack{m<l \\ i \in E_m}} \sum_{s_m=0}^{|E_m|-2} \sum_{\substack{t<l \\ i \notin E_t}} \sum_{s_t=0}^{|E_t|-1} \sum_{j>l} \sum_{s_j=0}^{|E_j|} A^i_s(s_1, \ldots, s_k)\big|_{s_l=|E_l|-1}$$

and $A^i_s(s_1, \ldots, s_k)$ is the number of coalitions $S \subseteq N \setminus \{i\}$ of size s such that $|S \cap E_{l'}| = s_{l'}$ for each $l' \in K$.

At the end of this section we will describe how to compute the terms $A^i_s(s_1, \ldots, s_k)$. Before that, we will show a result analogous to Proposition 1 for the negative part of the Shapley value associated with Θ.

Recall that by Lemma 6,

$$\Phi_i^-[v_p,\,\Theta] = \sum_{\substack{S\in\mathcal{W}:\, i\in E_{\mu(S\cup\{i\})}\\ i\notin S}} \gamma_s v_p(S)$$

can be written as

$$\Phi_i^-[v_p,\,\Theta] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{l\in K\\ i\notin E_l}} p(E_l)\cdot \left|\{S\in \overline{D_s^i}(l)\,:\, \mu(S\cup\{i\})<l\}\right|$$

where for each l,

$$\overline{D_s^i}(l) = \{S\subseteq N\setminus\{i\}\,:\, \mu(S)=l \text{ and } |S|=s\}$$

and for each $T\subseteq N$ such that $v_p(T)>0$, $\mu(T)=\min\{l\in K\,:\, E_l\subseteq T\}$.

The elements of $\overline{D_s^i}(l)$ are the coalitions S of size s that contain the essential coalition E_l, while the essential coalitions $E_1,\,\ldots,\,E_{l-1}$, which have cooperation index greater or equal than $p(E_l)$, are not contained in S. Note that in our computation we are only interested in $\overline{D_s^i}(l)$ for $l\in K$ such that $i\notin E_l$. For these l, our goal is to count how many coalitions $S\in\overline{D_s^i}(l)$ are such that the first coalition in the list Θ that is a subcoalition of $S\cup\{i\}$ is positioned before E_l. In other words, we seek to compute the number

$$\widetilde{d_s^i} = \left|\{S\in\overline{D_s^i}(l)\,:\, \mu(S\cup\{i\})<l\}\right|$$

that appears in the previous expression for $\Phi_i^-[v_p,\,\Theta]$.

To do so, we will first study what values $|S\cap E_{l'}|$ can take for each $l'\in K$ where $S\in\overline{D_s^i}(l)$ for some $l\in K$ such that $i\notin E_l$. The procedure will be similar to the one used when discussing the positive part. Now it is required that

$$l=\mu(S)=\min\{l\in K\,:\, E_l\subseteq S\}$$

In particular, $E_l\subseteq S$, so it must be the case that $|S\cap E_l|=|E_l|$. Again, for all other $l'\in K$, we must have $|S\cap E_{l'}|\leqslant|S|=s$ and $0\leqslant s\leqslant n-1$, and if $l'<l$, we have $|S\cap E_{l'}|\leqslant|E_{l'}|-1$ as well, since it is required that $E_{l'}\nsubseteq S$.

Thus, for $t < l$ such that $i \notin E_t$, $0 \leqslant |S \cap E_t| \leqslant \min\{s,\, |E_t| - 1\}$. For $j > l$, the construction of $\overline{D^i_s}(l)$ imposes no conditions on the relation between $S \in \overline{D^i_s}(l)$ and E_j in any case. Hence, once more, all we can say is that $0 \leqslant |S \cap E_j| \leqslant \min\{s,\, |E_j|\}$ if $i \notin E_j$ and $0 \leqslant |S \cap E_j| \leqslant \min\{s,\, |E_j| - 1\}$ if $i \in E_j$. As, in particular, we are assuming $S \subseteq N \smallsetminus \{i\}$, if $l' \in K$ is such that $i \in E_{l'}$ it can never be the case that $E_{l'} \subseteq S$. Hence, by merely requiring $S \in \overline{D^i_s}(l)$, no non-trivial restrictions on the value $|S \cap E_{l'}|$ arise either.

This changes when we impose $S \in \overline{D^i_s}(l)$ to be such that

$$l > \mu(S \cup \{i\}) = \min\{l' \in K :\ E_{l'} \subseteq S \cup \{i\}\}$$

We will now discuss what values $|S \cap E_{l'}|$ can take for $l' < l$ such that $i \in E_{l'}$ conditional on $\mu(S \cup \{i\}) < l$. In such case we would have $\left|S \cap E_{\mu(S\cup\{i\})}\right| = \left|E_{\mu(S\cup\{i\})}\right| - 1$, as player i is, by construction, the only player in $E_{\mu(S\cup\{i\})}$ not in S.

It is also implied that for each $u < \mu(S \cup \{i\})$ for which $i \in E_u$ there must be at least one more player besides i that is in E_u but not in S, that is, we need $0 \leqslant |S \cap E_u| \leqslant \min\{s,\, |E_u| - 2\}$. As for those $r \in K$ for which $\mu(S \cup \{i\}) < r < l$ and $i \in E_r$, no further restrictions on $|S \cap E_r|$ have been introduced, and so, as $i \notin S$, the best bounds we can give for $|S \cap E_r|$ are $0 \leqslant |S \cap E_r| \leqslant \min\{s,\, |E_r| - 1\}$.

However, we do not know beforehand what $\mu(S \cup \{i\})$ will be for an arbitrary $S \in \overline{D^i_s}(l)$. Given such a coalition S for which $\mu(S \cup \{i\}) < l$, we have already discussed the bounds for $|S \cap E_{m'}|$ for all other $m' < l$. Thus, in order to compute

$$\widetilde{d^i_s}(l) = \left|\{S \in \overline{D^i_s}(l) :\ \mu(S \cup \{i\}) < l\}\right|$$

as desired, we will add over all $m < l$ such that $i \in E_m$ and, for each such m, count how many coalitions S are such that $\mu(S \cup \{i\}) = m$.

Finally, as we will once again use the terms $A^i_s(s_1, \ldots, s_k)$, which count how many coalitions $S \subseteq N \smallsetminus \{i\}$ of size s are such that $s_{l'} = |S \cap E_{l'}|$ for each $l' \in K$, the same arguments used when discussing the positive part allow us to safely discard the minima we used in the current discussion and replace them with the terms that do not depend on s. Merging the conditions for $j > l$ to $0 \leqslant |S \cap E_j| \leqslant E_j$ as well, following from Lemma 6 we obtain the following result.

Proposition 2. Let v be a simple proper game with player set N and p a cooperation index on N. Given a sorted list of essential coalitions

$$\Theta = \Theta\left(\mathscr{E}(v,\, p)\right) = (E_1,\, \dots,\, E_k)$$

for each player $i \in N$, the negative part of their Shapley value in the restricted game associated with Θ can be computed as

$$\Phi_i^-[v_p] = \sum_{s=0}^{n-1} \gamma_s \sum_{\substack{l\in K\\ i\notin E_l}} p(E_l) \cdot \widetilde{d}_s^i(l)$$

where $\gamma_s = s!(n-s-1)!$, $K = \{1,\, 2,\, \dots,\, k\}$ and for each $l \in K$ for which $i \notin E_l$,

$$\widetilde{d}_s^i(l) = \sum_{\substack{m<l\\ i\in E_m}} \sum_{\substack{u<m\\ i\in E_u}} \sum_{s_u=0}^{|E_u|-2} \sum_{\substack{m<r<l\\ i\in E_r}} \sum_{s_r=0}^{|E_r|-1} \sum_{\substack{t<l\\ i\notin E_t}} \sum_{s_t=0}^{|E_t|-1} \sum_{j>l} \sum_{s_j=0}^{|E_j|} A_s^i(s_1,\, \dots,\, s_k)\Big|_{s_l=|E_l|,\, s_m=|E_m|-1}$$

and $A_s^i(s_1,\, \dots,\, s_k)$ is the number of coalitions $S \subseteq N \smallsetminus \{i\}$ of size s such that $|S \cap E_{l'}| = s_{l'}$ for each $l' \in K$.

In subsection 1.2 we showed that the Shapley value of player i the restricted game, $\Phi_i[v_p]$, can be computed as

$$\Phi_i[v_p] = \frac{1}{n!}\left(\Phi_i^+[v_p,\, \Theta] - \Phi_i^-[v_p,\, \Theta]\right)$$

In this subsection we have solved the computation of $\Phi_i^+[v_p,\, \Theta]$ and $\Phi_i^-[v_p,\, \Theta]$ in terms of the numbers $A_s^i(s_1,\, \dots,\, s_k)$. To conclude, the following result shows how to compute the generating function of the terms $A_s^i(s_1,\, \dots,\, s_k)$.

Lemma 7. The generating function for the terms $A_s^i(s_1,\, \dots,\, s_k)$ is given by

$$SS_i(x_1,\, \dots,\, x_k,\, z) = \prod_{j\neq i} \left(1 + z \prod_{\substack{l\in K\\ j\in E_l}} x_l\right)$$

Proof. Note that the function

$$
\begin{aligned}
SS(x_1,\,\ldots,\,x_k,\,z) &= \prod_{j=1}^{n}\left(1 + z\prod_{\substack{l\in K\\ j\in E_l}} x_l\right) = \\
&= 1 + \sum_{\varnothing\neq S\subseteq N} z^{|S|}\prod_{j\in S}\prod_{\substack{l\in K\\ j\in E_l}} x_l = \sum_{S\subseteq N} z^{|S|}\prod_{l\in K} x_l^{|S\cap E_l|} = \\
&= \sum_{s_1=0}^{|E_1|}\cdots\sum_{s_k=0}^{|E_k|}\sum_{s=0}^{n} A_s(s_1,\,\ldots,\,s_k) z^s x_1^{s_1}\cdots x_k^{s_k}
\end{aligned}
$$

is the generating function for $A_s(s_1,\,\ldots,\,s_k)$, where each of these terms corresponds to the number of coalitions $S \subseteq N$ of size s such that $|S \cap E_l| = s_l$ for each $l \in K$. The terms $A_s^i(s_1,\,\ldots,\,s_k)$ are obtained by omitting the i-th factor of $SS(x_1,\,\ldots,\,x_k,\,z)$. □

2 Computing the set of essential coalitions

With the method now fully described, before implementing it, we will discuss its input parametres. These are the number of players n, the cooperation index p, and the set of essential coalitions of the original game v associated with p, $\mathscr{E}(v, p)$. Recall that this latter set is defined by

$$\mathscr{E}(v, p) = \{S \in \mathcal{W}_+(p) : \, p(S) > p(R) \ \forall R \in \mathcal{W} \cap 2^S, \, R \neq S\}$$

where $\mathcal{W}_+(p)$ denotes the winning coalitions of v with positive cooperation index.

We will always assume n and p are given, as is the set of winning coalitions of the simple proper game v, $\mathcal{W}$. In the previous section, it was also assumed that $\mathscr{E}(v, p)$ was given. However, this set can be built from p and $\mathcal{W}$. One possible approach for this is to use dynamic programming; for instance, one might follow these steps:

1) Let $\mathcal{C} = (W_1, W_2, \ldots, W_m)$ be the list of winning coalitions with positive cooperation index, ordered so that $p(W_i) \geqslant p(W_j)$ for all $i < j$. In case of a tie, sort the coalitions in non-decreasing order of size; should the tie persist among some coalitions, order them lexicographically.

2) For each $i \in \{1, \ldots, m\}$, remove from $\mathcal{C}$ all W_j with $j > i$ such that $W_i \subsetneq W_j$. By step 1), besides containing W_i, these W_j are winning coalitions with $p(W_j) \leqslant p(W_i)$ and hence cannot be in the set of essential coalitions.

This process ends with $\mathcal{C} = (C_1, \ldots, C_k)$ being a sorted list of essential coalitions as described in subsection 1.2, that is,

$$p(C_1) \geqslant p(C_2) \geqslant \cdots \geqslant p(C_k)$$

While the method presented here is not difficult to implement, it is computationally costly. For a start, even though we are only studying p restricted to $\mathcal{W}$, even in a proper game the number of winning coalitions is in $O(2^n)$. At worst, all winning coalitions have positive cooperation index, and so $|\mathcal{W}_+(p)|$ is in $O(2^n)$ as well. On the other hand, as suggested at the end of subsection 1.2, some complexity arises from the fact that a cooperation index p on N is effectively a 2^n-dimensional vector with $n + 1$ components equal to 1 (those associated with the singletons plus the empty set), while the $2^n - n - 1$ that remain can take any real value ranging from 0 to 1. The number of initially unknown components of p is therefore also in $O(2^n)$.

With such a function, in order to compute $\mathscr{E}(v,\, p)$ we are forced to explore the cooperation indices of the winning coalitions to exhaustion. In the method described above, in step 2) for each W_i in list $\mathcal{C}$ we check whether W_i is a subset of W_j for all $j \in \{i+1,\, \dots,\, m\}$. Hence, we perform $m-i$ such tests for each $i \in \{1,\, \dots,\, m\}$, i.e. a total of $\sum_{i=1}^{m}(m-i)$ in the worst case scenario. The latter sum is equal to that of the first $m-1$ natural numbers, which is in $O(m^2)$. All in all, since $m = |\mathcal{W}_+(p)|$ is in $O(2^n)$, we have that step 2) alone requires checking if one set is a subset of another a number of times in $O(4^n)$.

We conclude our discussion on the computation of $\mathscr{E}(v,\, p)$ with a comment by Amer and Carreras on the structure, or lack thereof, of the cooperation index p. When introducing the restricted cooperation model given by the cooperation index in [2], the authors state that since "no conditions are demanded to [the cooperation index], the scope of the model is maximized", in the sense that it "encompasses" other models of restriction to cooperation. In other words, in order to allow it to generalize other models, the cooperation index is designed to have a limited structure.

Next, we focus on the effect the set of essential coalitions has on the complexity of the generating function method described in section 1. In order to study this, recall that given a sorted list of essential coalitions $\Theta = (E_1,\, \dots,\, E_k)$, for each $i \in N$ computing the Shapley value of i in the restricted game requires the calculation of the function

$$SS_i(x_1,\, \dots,\, x_k,\, z) = \prod_{j \neq i} \left(1 + z \prod_{\substack{l \in K \\ j \in E_l}} x_l \right)$$

where $K = \{1,\, 2,\, \dots,\, k\}$. We showed how to compute these functions in Lemma 7; it is not difficult to determine the time complexity for these computations.

Lemma 8. Let v be a simple proper game with n players and p a cooperation index on $N = \{1,\, \dots,\, n\}$. Let $\Theta = (E_1,\, \dots,\, E_k)$ be a sorted list of essential winning coalitions of v associated with p. Then,

(i) The number c of non-zero terms of

$$SS(x_1,\, \dots,\, x_k,\, z) = \prod_{j \in N} \left(1 + z \prod_{\substack{l \in K \\ j \in E_l}} x_l \right)$$

satisfies

$$n+1 \leqslant c \leqslant \min \left\{ 2^n,\, (n+1) \prod_{l=1}^{k} (|E_l|+1) \right\}$$

Proof. The lower bound for c is obtained when the only essential coalition is the grand coalition of the game, N. Indeed, in such case the generating function we are studying is reduced to $(1 + zx_1)^n$, which has $n + 1$ terms. To study its upper bounds, consider $SS(x_1, \ldots, x_k, z)$ as

$$SS(x_1, \ldots, x_k, z) = \sum_{s_1=0}^{|E_1|} \cdots \sum_{s_k=0}^{|E_k|} \sum_{s=0}^{n} A_s(s_1, \ldots, s_k)$$

where $A_s(s_1, \ldots, s_k)$ is the number of coalitions $S \subseteq N$ of size s such that $|S \cap E_l| = s_l$ for eack $l \in K$. This function is a polynomial of degree $|E_l|$ in x_l for each $l \in K$ and degree n in z; therefore, c is at most $(n+1)\prod_{l=1}^{k}(|E_l|+1)$. On the other hand, at worst each of the terms of $SS(x_1, \ldots, x_k, z)$ is equal to 1, meaning no two coalitions $S, S' \subseteq N$ of the same size are such that $|S \cap E_l| = |S' \cap E_l|$ for all $l \in K$. In this case, $SS(x_1, \ldots, x_k, z)$ has 2^n non-zero terms, one for each possible coalition, and the result follows. $\square$

(ii) For each $i \in N$, to expand the polynomial

$$SS_i(x_1, \ldots, x_k, z) = \prod_{j \neq i} \left(1 + z \prod_{\substack{l \in K \\ j \in E_l}} x_l \right)$$

an $O(nC)$ time is required, where

$$C = \min\{2^n, (n+1)\prod_{l=1}^{k}(|E_l|+1)\}$$

Proof. For $i \in N$, let c_i be the number of non-zero terms of $SS_i(x_1, \ldots, x_k, z)$. In Lemma 7 we argued that

$$SS(x_1, \ldots, x_k, z) = SS_i(x_1, \ldots, x_k, z) \cdot \left(1 + z \prod_{\substack{l \in K \\ i \in E_l}} x_l \right)$$

so c_i is at most c, the number of non-zero terms of $SS(x_1, \ldots, x_k, z)$.

Thus, by the first part of this lemma, $c_i \leqslant C$, where

$$C = \min\{2^n,\ (n+1)\prod_{l=1}^{k}(|E_l|+1)\}$$

Finally, $SS_i(x_1,\,\dots,\,x_k,\,z)$ can be computed as follows.

$SS_i(x_1,\,\dots,\,x_k,\,z) \leftarrow 1$
for $j \in \{1,\,\dots,\,n\},\ j \neq i$ **do**
$\quad SS_i(x_1,\,\dots,\,x_k,\,z) \leftarrow SS_i(x_1,\,\dots,\,x_k,\,z) + SS_i(x_1,\,\dots,\,x_k,\,z)\cdot z \prod_{\substack{l\in K\\ j\in E_l}} x_l$
end for

The line in the loop is computed in $O(C)$ time; at worst, it performs C additions, one for each non-zero term of $SS_i(x_1,\,\dots,\,x_k,\,z)$. Therefore, since this calculation is executed $n-1$ times, the full procedure takes an $O(nC)$ time to be completed. $\square$

In a manner analogous to what Propositions 13 and 14 achieve in [1], we have used the function $SS(x_1,\,\dots,\,x_k,\,z)$ in our discussion on the complexity of computing the generating functions $SS_i(x_1,\,\dots,\,x_k,\,z)$; the number of non-zero terms these functions have are bounded by that of the former function. Broadly speaking, in practice, the lower the number c of non-zero terms $SS(x_1,\,\dots,\,x_k,\,z)$ has in a particular case, the faster the method will run. In this sense, the size of Θ has a direct influence on c. Namely, the upper bound $(n+1)\prod_{l\in K}(|E_l|+1)$ given for c increases with the amount of essential coalitions, as for each additional coalition in Θ, a new variable is added to $SS(x_1,\,\dots,\,x_k,\,z)$.

Thus far, in this section we have seen that 1) the limited structure of the cooperation index p makes the computation of the set of essential coalitions a difficult endeavor and 2) the larger the size of the set of essential coalitions is, the more time consuming the calculation of the generating function described in Lemma 7 becomes. In order to address both issues at once, in the remainder of this section we will discuss the lower bounds of the set of essential coalitions, i.e. we will look for collections $L \subseteq \mathcal{W}$ such that $L \subseteq \mathscr{E}(v,\,p)$.

At the cost of some of the generality of the cooperation index model, we will see that one can guarantee equality for certain bounds, which will ease the calculation of $\mathscr{E}(v,\,p)$ with respect to the general case. Moreover, we will see that the conditions we will impose on p are reasonable within a particular interpretation of what this function represents.

For a first rather naive bound, recall Example 2, and note it suggests that every minimal winning coalition of v is an essential coalition. This will actually hold for all minimal winning coalitions with non-zero cooperation index. Recall that

$$\mathscr{E} = \mathscr{E}(v, p) = \{S \in \mathcal{W}_+(p) : \ p(S) > p(R) \ \forall R \in \mathcal{W} \cap 2^S, \ R \neq S\}$$

and

$$\mathcal{M} = \{W \in \mathcal{W} : \ S \notin \mathcal{W} \, \forall S \subsetneq W\}$$

is the set of minimal winning coalitions of v.

No winning coalition of v contained in a minimal winning coalition M can exceed its cooperation index since, by definition, there cannot be any winning coalition contained in M. Therefore, merely requiring $p(M) > 0$ ensures $M \in \mathscr{E}$. In short, we have $\mathcal{M} \cap \mathcal{W}_+(p) \subseteq \mathscr{E}$ indeed.

In general, the bound this collection gives can be improved, namely, by the collection of minimal sets in the collection of winning coalitions with positive cooperation index, which we will denote by

$$\mathcal{M}_+(p) = \{W \in \mathcal{W}_+(p) : \ S \notin \mathcal{W}_+(p) \ \forall S \subsetneq W\}$$

Note that, in particular, the minimal winning coalitions of v contain no coalition from $\mathcal{W}_+(p)$, and therefore all minimal winning coalitions with positive cooperation index are also minimal in $\mathcal{W}_+(p)$, that is, $\mathcal{M} \cap \mathcal{W}_+(p) \subseteq \mathcal{M}_+(p)$. However, equality need not hold, as Example 4 will later show. For now, we still have to show that $\mathcal{M}_+(p)$ provides a lower bound for the set of essential coalitions, $\mathscr{E}$. But indeed, those winning coalitions W with positive cooperation index that contain no other coalitions with these properties in particular satisfy $p(W) > p(R)$ for all winning $R \subsetneq W$, and so $\mathcal{M}_+(p) \subseteq \mathscr{E}$. The following result gives a sufficient condition for equality to hold.

Lemma 9. If p is monotonically non-increasing in $\mathcal{W}_+(p)$, that is, given a pair $S_1, S_2 \in \mathcal{W}_+(p)$, having $S_1 \subseteq S_2$ implies $p(S_1) \geqslant p(S_2)$, then $\mathscr{E} = \mathcal{M}_+(p)$.

Proof. Let p be a cooperation index satisfying the property stated by the lemma. Since $\mathcal{M}_+(p) \subseteq \mathscr{E}$ regardless, it will suffice to show that $\mathscr{E} \subseteq \mathcal{M}_+(p)$ in this case.

Suppose, on the contrary, that there is some $S \in \mathscr{E}$ that is not a minimal set in $\mathcal{W}_+(p)$. Recall that, in particular, $S \in \mathcal{W}_+(p)$, and therefore at least one of its proper subsets is minimal in the collection of winning coalitions with positive cooperation index; let M be one such coalition. Since $S \in \mathscr{E}$, that is, all winning coalitions $T \subsetneq S$ are such that $p(S) > p(T)$. In particular, $p(S) > p(M)$, which contradicts our assumption that p is monotonically non-increasing in $\mathcal{W}_+(p)$. □

Corollary 4. If p is monotonically non-increasing, then $\mathscr{E} = \mathcal{M}_+(p)$.

While finding minimal sets in a large collection of sets is not as computationally demanding a calculation as the construction of $\mathscr{E}$ is in the general case, it can be proven that equality $\mathcal{M}_+(p) = \mathcal{M}$ can happen. This is useful as the latter collection is sometimes given as defining a simple game. We can also characterize when such equality occurs.

Lemma 10. Equality $\mathcal{M} = \mathcal{M}_+(p)$ holds if and only if all minimal winning coalitions of v have positive cooperation index, i.e. $\mathcal{M} \subseteq \mathcal{W}_+(p)$.

Proof. All coalitions in $\mathcal{M}_+(p)$ have positive cooperation index, so if equality holds we have

$$\mathcal{M} \subseteq \mathcal{M}_+(p) \subseteq \mathcal{W}_+(p)$$

Conversely, in order to prove inclusion $\mathcal{M} \subseteq \mathcal{M}_+(p)$, since we already saw prior to Lemma 9 that $\mathcal{M} \cap \mathcal{W}_+(p) \subseteq \mathcal{M}_+(p)$, all minimal winning coalitions having positive cooperation index yields

$$\mathcal{M} = \mathcal{M} \cap \mathcal{W}_+(p) \subseteq \mathcal{M}_+(p)$$

It only remains to be checked that if $\mathcal{M} \subseteq \mathcal{W}_+(p)$, then all minimal sets in $\mathcal{W}_+(p)$ are actually minimal winning coalitions of v, that is, $\mathcal{M}_+(p) \subseteq \mathcal{M}$. Recall that $M \in \mathcal{M}_+(p)$ if and only if M is winning, $p(M) > 0$ and no proper subcoalition of M fulfills both properties at the same time. Our goal is to see that for such M there is no winning coalition $T \subsetneq M$.

On the contrary, suppose such a coalition T exists. Then, there would be some minimal winning coalition R contained in T. But, since we are assuming all minimal winning coalitions have positive cooperation index, we would have a winning $R \subsetneq M$ with $p(R) > 0$. This contradicts the assumption that $M \in \mathcal{M}_+(p)$; we conclude M is indeed a minimal winning coalition of v. □

Corollary 5. If p is positive and monotonically non-increasing, then $\mathscr{E} = \mathcal{M}$.

Proof. If p is monotonically non-increasing, by Corollary 4, we have $\mathscr{E} = \mathcal{M}_+(p)$. By Lemma 10, if p is positive this latter set is precisely $\mathcal{M}$, the set of minimal winning coalitions of v, and the result follows. □

Corollary 6. If $\mathcal{M} \subseteq \mathcal{W}_+(p)$, then $\mathcal{M} \cap \mathcal{W}_+(p) = \mathcal{M}_+(p)$.

Proof. By Lemma 10, should all minimal winning coalitions of v have positive cooperation index, then $\mathcal{M} = \mathcal{M}_+(p)$. Using $\mathcal{M} \subseteq \mathcal{W}_+(p)$ again, we obtain

$$\mathcal{M}_+(p) = \mathcal{M} = \mathcal{M} \cap \mathcal{W}_+(p)$$ □

In general, however, collections $\mathcal{M}$ and $\mathcal{M}_+(p)$ are non-comparable with respect to inclusion, as the following example shows.

Example 4. Let $N = \{1, 2, 3, 4\}$ and v be the simple game with winning coalitions

$$\mathcal{W} = \{S \subseteq N : \ |S| \geqslant 2 \text{ and } 1 \in S\} \cup \{\{2, 3, 4\}\}$$

also known as the 4-player majority game with a strong player. It can be checked that the minimal winning coalitions of this game are those of size 2 that involve player 1, along with $\{2, 3, 4\}$, that is

$$\mathcal{M} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3, 4\}\}$$

Now consider the cooperation index p on N defined by

$$p(S) = \begin{cases} 0 \text{ if } |S| = 2 \text{ and } 4 \notin S \\ |S|^{-1} \text{ otherwise} \end{cases}$$

With this p the winning coalitions of v with positive cooperation index are

$$\mathcal{W}_+(p) = \{\{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, N\}$$

and

$$\mathcal{M} \cap \mathcal{W}_+(p) = \{\{1, 4\}, \{2, 3, 4\}\}$$

However, the minimal sets in $\mathcal{W}_+(p)$ are

$$\mathcal{M}_+(p) = \{\{1,\, 4\},\, \{1,\, 2,\, 3\},\, \{2,\, 3,\, 4\}\}$$

yielding $\mathcal{M} \cap \mathcal{W}_+(p) \subsetneq \mathcal{M}_+(p)$ and that neither $\mathcal{M}_+(p)$ contains all coalitions in $\mathcal{M}$ nor does the opposite happen. Moreover, note that p is trivially monotonically non-increasing in $\mathcal{W}_+(p)$. Hence, by Lemma 9, $\mathscr{E}(v,\, p) = \mathcal{M}_+(p)$ in this example.

All in all, we have seen that

$$\mathcal{M} \cap \mathcal{W}_+(p) \subseteq \mathcal{M}_+(p) \subseteq \mathscr{E}(v,\, p) \subseteq \mathcal{W}_+(p) \subseteq \mathcal{W}$$

Corollary 6 provides a sufficient condition for the first inclusion to be an equality and Example 4 shows a case in which it is strict. As for the second inclusion, we have proven it can be an equality, with Lemma 9 and Corollary 4 now providing sufficient conditions for this to occur. Moreover, as shown in Lemma 10, $\mathcal{M}_+(p)$ and $\mathcal{M}$ are equal under certain circumstances, with Example 4 showing this is not the case in general. It can also be the case that $\mathscr{E}(v,\, p) = \mathcal{W}_+(p)$; in fact, as an anologous counterpart of Lemma 9, the next result provides a sufficient condition for this to hold.

Corollary 7. If p is strictly monotonically non-decreasing in $\mathcal{W}_+(p)$, that is, given a pair $S_1,\, S_2 \in \mathcal{W}_+(p)$, having $S_1 \subseteq S_2$ implies $p(S_1) < p(S_2)$, then $\mathscr{E} = \mathcal{W}_+(p)$. If p is also positive, then $\mathscr{E} = \mathcal{W}$.

Note that in many results in this section, p being non-increasing has been a key requirement in order to achieve the desired bounds. A cooperation index with this property can be interpreted as assessing the likelihood that each member of a coalition individually reaches an agreement with all other members. In this sense, a three player coalition $\{1,\, 2,\, 3\}$ must have at most the same cooperation index as the two players coalitions it contains; for instance, for player 1 to reach an agreement with 2 and 3, this player must cooperate with 2 and 3 separately. Similarly, a four player coalition must have at most the same cooperation index as the three players coalitions it contains, and so on.

Checking whether p is non-increasing is not easy, but a function p with such property can be easily constructed. For instance, this can be achieved by making p dependent solely on the size of each coalition. We already used this technique in Example 4; in the following example we introduce a parametric family of monotonically non-increasing positive cooperation indices.

Example 5. Given an arbitrary n-player simple proper game, v, for each $\alpha > 1$ consider the cooperation index $p_\alpha : 2^N \to [0, 1]$ defined by

$$p_\alpha(S) = \alpha^{1-|S|} \; \forall S \subseteq N = \{1, \ldots, n\}$$

We claim that p_α is well defined and monotonically non-increasing. It is clear that $p_\alpha(\{i\}) = 1 \; \forall i \in N$ and $p_\alpha(S) > 0$ for all $S \subseteq N$; furthermore, given an arbitrary non-empty coalition $T \subseteq N$ contained in S, $|S| \geqslant |T|$ implies that

$$0 < p_\alpha(S) = \alpha^{1-|S|} = \alpha^{1-|T|+|T|-|S|} \leqslant \alpha^{1-|T|} = p_\alpha(T)$$

which proves both the monotonicity of p_α and the fact that $p_\alpha(S) \in [0, 1]$ for all $S \subseteq N$.

All in all, no matter what coalitions are winning in v, by Corollary 5, the set of essential coalitions $\mathscr{E}(v, p_\alpha)$, is guaranteed to coincide with the set of minimal winning coalitions of v.

To conclude this section, we provide two examples of games restricted by a positive cooperation index that is non-monotonic, but in which the set of essential coalitions matches one of the bounds.

Example 6. Let $N = \{1, 2, 3\}$ and consider the simple game v with minimal winning coalitions $\mathcal{M} = \{\{1, 2\}, \{1, 3\}\}$ (equivalently, $\mathcal{W} = \{\{1, 2\}, \{1, 3\}, N\}$) and the cooperation index p defined by

$$p(S) = \begin{cases} 1 \text{ if } |S| = 1 \\ 1/3 \text{ if } |S| = 2 \\ 2/3 \text{ if } S = N \end{cases}$$

Despite N not being a minimal winning coalition, we have $v_p(N) = p(N) = 2/3 > p(W)$ for all other $W \in \mathcal{W}$, and so $N \in \mathscr{E}(v, p)$. Therefore this is an example of a game where $\mathscr{E}(v, p) = \mathcal{W}$.

Example 7. Let N, v and p be as in the previous example and consider the cooperation index $\widetilde{p}$ defined as

$$\widetilde{p}(S) = \begin{cases} 3/4 \text{ if } S = \{1,\, 3\} \\ p(S) \text{ otherwise} \end{cases}$$

As was the case with p, $\widetilde{p}(N) > \widetilde{p}(\{1,\, 2\})$, so not only are $\widetilde{p}$ and p not monotonically non-increasing, we also have a minimal winning coalition with positive cooperation index, but (strictly) lower cooperation index than one of its supersets. In particular, their restrictions on $\mathcal{W}_+(p)$ are also not monotonically non-increasing. Despite all this, $\widetilde{p}(\{1,\, 3\}) > \widetilde{p}(N)$ is enough to ensure $N \notin \mathscr{E}(v,\, \widetilde{p})$ and, in turn, $\mathscr{E}(v,\, \widetilde{p}) = \mathcal{M}$.

3 An implementation and an example

Now that we have glossed over the potential prior calculations to be performed before computing the Shapley value of a simple proper game v restricted by a cooperation index p, we proceed to provide an implementation of the method described in section 1. The implementation that follows uses **Mathematica**. We will use a five player game as an example to see what results the code yields and compare them to the Shapley values in the non-restricted game.

Namely, we will let v be the five player weighted majority game where a player has a weight of 3 (henceforth, player 1) while all other players have a weight of 1. We will use the quota for an absolute majority, $q = 4$.

```
In[1]:= w = {3, 1, 1, 1, 1}
        n = Length[w]
        q = 4
```

Recall that in such a game the winning coalitions are those S such that $w(S) = \sum_{i \in S} w_i \geqslant q$. Thus, in this game, the minimal winning coalitions are the two player coalitions that involve player 1, along with the coalition involving all players but player 1; all other winning coalitions contain some superfluous player. Alternatively, we can compute the winning coalitions using the following code:

```
In[2]:= W = Select[Subsets[Range[n]], Plus @@ w[[#]] ≥ q &]
Out[2]= {{1, 2}, {1, 3}, {1, 4}, {1, 5}, {1, 2, 3}, {1, 2, 4},
        {1, 2, 5}, {1, 3, 4}, {1, 3, 5}, {1, 4, 5}, {1, 2, 3, 4},
        {1, 2, 3, 5}, {1, 2, 4, 5}, {1, 3, 4, 5}, {2, 3, 4, 5},
        {1, 2, 3, 4, 5}}
```

Consider now a cooperation index p defined on the resulting set of winning coalitions `W`. In order to let p be as general as possible, we will set it to a vector with `Length[W]` components, and set each `p[[i]]` to a random value between 0 and 1, to be interpreted as the cooperation index of `W[[i]]`.

```
In[3]:= p = Table[RandomReal[],{i, Length[W]}]
Out[3]= {0.930139, 0.642749, 0.82492, 0.438921, 0.917439, 0.376106,
        0.0983529, 0.399605, 0.768252, 0.434036, 0.907991, 0.196418,
        0.363942, 0.539378, 0.639526, 0.670222}
```

Now, given the set of winning coalitions `W` and the cooperation index `p` defined on `W`, the function `EssentialWp[W, p]` computes the set of essential winning coalitions of the original game associated with cooperation index `p`, using the algorithm described at the start of section 2.

```
In[4]:= EssentialWp[W_List, p_List] :=
         Module[{M, E, pE},
          M = SortBy[Select[Range[Length[W]], p[[#]] > 0 &],
          {1 - p[[#]] &, Length[W[[#]]] &}];
          E = Table[W[[i]], {i, M}]; pE = Table[p[[i]], {i, M}];
          M = Length[M];
          For[l = 1, l ≤ M, l++,
           For[j = l + 1, j ≤ M, j++,
            If[SubsetQ[E[[j]], E[[l]]], pE = Delete[pE, j];
             E = Delete[E, j--]; M--]]]; {E, pE}]
```

We will use variables **Theta** and **pTheta** to store the resulting sorted list of essential coalitions $\Theta(\mathscr{E}(v, p))$ and the vector containing the value of the cooperation index in its elements, respectively.

```
In[5]:= Timing[{Theta, pTheta} = EssentialWp[W, p]]

Out[5]= {0.000639, {{{1, 2}, {1, 4}, {1, 3, 5}, {1, 3}, {2, 3, 4, 5},
        {1, 5}},   {0.930139, 0.82492, 0.768252, 0.642749, 0.639526,
        0.438921}}}
```

Recall that in section 2 we showed that this procedure takes an $O(4^n)$ time. However, in this specific example of a five player game, it was completed in less than 10^{-3} seconds. This suggests that, while complex, the set of essential coalitions can be computed in a reasonable amount of time for games with a larger amount of players.

Without further ado, the following function computes the Shapley value in the restricted game.

```
In[6]:= ShValInd[n_Integer, E_List, p_List] :=
         Module[{g, sl, sum, in, Kact, Klow, Kactlow, coefi, slaux, sm,
           vN = p[[1]] * n!, k = Length[E], K}, K = Range[k];
          Kact = Table[Select[K, MemberQ[E[[#]], i] &], {i, n}];
          Table[g = Product[1 + z * Product[Indexed[x, l],
            {l, Kact[[j]]}], {j, Complement[Range[n], {i}]}];
          Klow = {}; Kactlow = {}; sum = 0;
```

```
Do[in = MemberQ[Kact[[i]], l];
 sl = Coefficient[g, Indexed[x, l],
   Length[E[[l]]] - Boole[in]];
 If[in,
  Do[coefi = CoefficientList[sl, Indexed[x, m]];
   sl = Apply[Plus, coefi[[Range[Min[Length[E[[m]]] - 1,
     Length[coefi]]]]]], {m, Kactlow}];
  Kactlow = Union[Kactlow, {l}], slaux = sl; sl = 0;
  Do[sm = Coefficient[slaux, Indexed[x, m],
    Length[E[[m]]] - 1];
   Do[coefi = CoefficientList[sm, Indexed[x, u]];
    sm = Apply[Plus, coefi[[Range[Min[Length[E[[u]]]
      - Boole[u < m], Length[coefi]]]]]],
      {u, DeleteCases[Kactlow, m]}];
   sl = sl + sm, {m, Kactlow}]];
 Do[coefi = CoefficientList[sl, Indexed[x, t]];
  sl = Apply[Plus, coefi[[Range[Min[Length[E[[t]]],
    Length[coefi]]]]]], {t, Complement[Klow, Kactlow]}];
 Do[sl = Apply[Plus, CoefficientList[sl, Indexed[x, j]]],
  {j, l + 1, k}];
  Klow = Union[Klow, {l}];
  sum = sum - Power[-1, Boole[in]] * sl * p[[l]], {l, k}];
N[Sum[Coefficient[sum, z, s] s! * (n - s - 1)!,
  {s, 0, n - 1}] / vN, 10], {i, n}]]
```

Note that the resulting calculation is actually the *normalized* Shapley value of the restricted game. One of the axioms that define the Shapley value states that for an n-player game with characteristic function v,

$$\sum_{i=1}^{n} \Phi_i[v] = v(N)$$

This property is called the efficiency axiom.

In the restricted game we are studying,

$$\sum_{i=1}^{n} \Phi_i[v_p] = v_p(N) = \max_{T \in \mathcal{W} \cap 2^N} p(T)$$

Since we are given a sorted list of essential coalitions, Θ, we know the winning coalition of v with greatest cooperation index is E_1, and so $v_p(N) = p(E_1)$. Albeit $p(E_1)$ can be any real positive value smaller than 1, regardless of the game v, the cooperation index p and the sorted list of essential coalitions Θ,

$$\widetilde{\Phi}[v_p] = \left(\widetilde{\Phi}_1[v_p], \ldots, \widetilde{\Phi}_n[v_p]\right) = \frac{1}{p(E_1)} \left(\Phi_1[v_p], \ldots, \Phi_n[v_p]\right)$$

is a vector with components adding up to 1. Using this normalization of the Shapley value will allow us to directly compare the results yielded by the function for different cooperation indices.

Finally, we compute $\widetilde{\Phi}[v_p]$:

```
In[7]:= Timing[SSIndRd = ShValInd[n, Theta, pTheta]]

Out[7]= {0.007377, {0.595979, 0.15869, 0.0807314, 0.10213, 0.0624699}}
```

For comparison purposes, we also compute the Shapley value of the original game using the function **ShPowerPlus** described in [4], also displayed below.

```
In[8]:= ShG[w_List] := Times @@ (1 + z * x^w)
        ShPowerPlus[w_List, q_Integer] :=
         Module[{n = Length[w], delw, sw, g, coefi, gg},
          Table[delw = Delete[w, i]; sw = Apply[Plus, delw] + 1;
           g = ShG[delw]; coefi = CoefficientList[g, x];
           gg = Apply[Plus, coefi[[Range[Max[1, q - w[[i]] + 1],
             Min[q, sw]]]]];
           N[Sum[Coefficient[gg, z, s] s! * (n - s - 1)!,
             {s, 0, n - 1}], 10], {i, n}] / n!]
        Timing[SS = ShPowerPlus[w, q]]

Out[8]= {0.000578, {0.6000000000, 0.1000000000, 0.1000000000, 0.1000000000,
          0.1000000000}}
```

The results are summarized in the following table:

Player	$\Phi[v]$	$\widetilde{\Phi}[v_p]$
1	0.6000	0.5960
2	0.1000	0.1587
3	0.1000	0.0807
4	0.1000	0.1021
5	0.1000	0.0625

First of all, note that the computation of the Shapley value in the restricted game was completed in $O(10^{-2})$ seconds, while that of the value in the original game took $O(10^{-3})$ seconds.

As for the values themselves, in the original game players 2 through 5 have the same Shapley value, while that of player 1 is larger, since this player appears in all winning coalitions except $\{2, 3, 4, 5\}$. Compared to these, players 1, 3 and 5 all have lower values in the restricted game. This can be explained by the sorted list of essential coalitions in this case being

$$\Theta = (\{1, 2\}, \{1, 4\}, \{1, 3, 5\}, \{1, 3\}, \{2, 3, 4, 5\}, \{1, 5\})$$

and their respective cooperation indices forming the list

$$p_\Theta = (0.930139, 0.82492, 0.768252, 0.642749, 0.639526, 0.438921)$$

All the coalitions in Θ are minimal winning coalitions of the original game except for $\{1, 3, 5\}$. Furthermore, $\{1, 3\}$ and $\{1, 5\}$ have lower cooperation index than $\{1, 2\}$ and $\{1, 4\}$. In fact,

$$p(\{1, 2\}) > p(\{1, 4\}) > p(\{1, 3\}) > p(\{1, 5\})$$

and we end up having

$$\widetilde{\Phi}_2[v_p] > \widetilde{\Phi}_4[v_p] > \widetilde{\Phi}_3[v_p] > \widetilde{\Phi}_5[v_p]$$

As for player 1, the cooperation indices of the winning coalitions they appear in are such that $\widetilde{\Phi}_1[v_p]$ is also lower than the Shapley value of this player in the original game.

4 Concluding remarks

We conclude this study with a few comments on what has been achieved. To begin with, it must be noted that the method we developed in Section 1 applies to simple proper games. This is relevant as other generating function methods used to compute the Shapley value, such as those described in [1] and [4], only apply to weighted majority games. Moreover, the scope could be extended to all simple games by slightly modifying Definition 1.

There, given a TU-game v and a cooperation index p, we defined the characteristic function v_p of v restricted by p by

$$v_p(S) = \max_{\mathcal{P} \in P^+(S,\, p)} \sum_{T \in \mathcal{P}} v(T)p(T)$$

where $P^+(S,\, p)$ is the set of all partitions of S into subsets with positive cooperation index. In Lemma 1 we showed that this definition is reduced to

$$v_p(S) = \begin{cases} \max\limits_{T \in \mathcal{W} \cap 2^S} p(T) \text{ if } S \in \mathcal{W} \\ 0 \text{ otherwise} \end{cases}$$

provided that v is a simple proper game. Should we take this lemma as the definition of v_p for all simple games, our method would apply to this class. Equivalently, replacing the sum in the original definition of v_p by a maximum effectively makes Lemma 1 valid for all simple games.

A similar comment appears in [1] regarding the definition of a game restricted by an incompatibility graph. There, given a TU-game $(N,\, v)$ and an incompatibility graph g on N, the restricted game is defined as in [3], that is, a TU-game $(N,\, v^g)$ with

$$v^g(S) = \max_{\mathcal{P} \in P(S,\, g)} \sum_{T \in \mathcal{P}} v(T)$$

where $P(S,\, g)$ is the set of all partitions of S into subsets that contain no pair of players connected in g. In [1], its authors argue that this definition is "appropriate for general cooperative games, but possibly is not the best choice for simple games". They then propose an alternative definition analogous to the one discussed above these lines.

On the other hand, some aspects related to the set of essential coalitions, $\mathscr{E}(v, p)$, could be studied in more detail. For instance, continuing from Section 2, some other structure could be imposed on p in order to study the resulting set of essential coalitions for cases in which p is not merely a random function. One could also study the maximum amount of players, n, for which the calculation of $\mathscr{E}(v, p)$ as implemented in Section 3 is completed in a reasonable time. We saw that the method described there is relatively fast for a game with a small number of players, but further examples of larger games could be tested.

Finally, for the program to work as intended the list of essential coalitions must be sorted in non-increasing order of p. As such an order is not unique, in Subsection 1.2 we set on a particular order, which is used in the code in Section 3. However, it is unclear if the program would run faster with another valid order, or whether such an order depends on the specific set of essential coalitions being considered.

References

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