Universitat de Barcelona
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GRAU DE MATEMÀTIQUES

Treball final de grau

Cohomology Theories of Spacetimes

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Barcelona, June 20, 2021

Abstract

Recently, non-standard de Rham cohomologies with causally restricted supports have proven to be useful in ongoing theoretical physics research. In this work, we present them starting from the most elementary notions of cohomology in manifolfds and spacetime, thereby offering an overview of de Rham cohomology and an introduction to Lorentzian manifolds and their causal structure. We also characterize these cohomologies in globally hyperbolic spacetimes, exhibiting isomorphims with the standard de Rham cohomologies and providing examples of computation for some well-known physical models of the spacetime.

Resum

Recentment, les cohomologies de de Rham no estàndards amb suports causalment restringits han provat ser útils en la recerca en física teòrica. En aquest treball, les presentem partint de les nocions més elementals de cohomologia en varietats i de espaitemps, oferint d'aquesta manera un compendi de cohomologia de de Rham i una introducció a les varietats de Lorentz i a la seva estructura causal. També caracteritzem aquestes cohomologies en espaitemps globalment hiperbòlics, exhibint isomorfismes amb la cohomologia de de Rham estàndard i proporcionant exemples de càlcul per alguns models físics coneguts del espaitemps.

2021 Mathematics Subject Classification. 53C50, 58A12, 58A05

Acknowledgements

First and foremost, I want to express my gratitude to my advisor, Dr. Joana Cirici, for her valuable guidance and her helpful suggestions. Next, I wish to thank my friends, who have been accompanying me since we started together this challenging path, making it easier and gentler. I am also profoundly thankful to my parents, for supporting me no matter what throughout all these years without expecting anything in return. Last but by no means least, I would like to thank my high school mathematics teacher, Susana, for opening my eyes to this exciting world, and all those teachers who are tirelessly striving to their best, day after day, to transmit to their students not only their knowledge but also the passion for learning.

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0 Introduction

The abstract understanding of the physical space and its properties, has always been a human concern. It responds to our inner impulse of modeling the most immediate physical world that we can perceive with our senses. The recognition of geometrical patterns, forms and magnitudes is so deeply embedded in our nature that is even thought to be present in animal cognition. It is no coincidence then that geometry emerged as one of the oldest branches of mathematics and therefore of the human knowledge. Since its origins as such in the ancient Mesopotamia, it has evolved branching out in many subfields, creating new whole areas of mathematics.

One of these areas, topology, which can be traced back to the celebrated L. Euler solution to the problem of the bridges of Königsberg in 1736, emerged from shifting the focus to the structural properties of geometric objects, leaving the metric properties that had dominated classical geometry as secondary. In topology, the exact shape of objects is no longer contemplated; instead its cornerstone are the properties that are preserved under continuous deformations. This gave way to a new class of equivalencies, the homeomorphism and the homotopy equivalence, which allow to identify objects with the same topological properties.

The study of these equivalences and topological properties would not be the same without algebraic invariants. Despite the fact that some invariants such as the Euler characteristic or Betti's numbers had already been discovered, the birth of algebraic topology is often set in the revolutionary papers published by H. Poincaré between 1895 and 1904, which also constitute one of the firsts systematic treatments of topology that consolidate it as a field. Poincaré tackled the until then arduous problem of distinguishing between non-homeomorphic topological spaces by using algebraic structures, namely the fundamental group and simplicial homology. This achievement, reduces the complexity of the problem to a computation of these algebraic entites which are preserved under homeomorphism and homotopy equivalence. Algebraic topology was then further developed by E. Noether, W. Mayer and L. Vietoris and spread out, given rise to the homotopy groups and the wide variety of homology constructions we know nowadays.

In parallel, from the XVII century contributions of Galileo and Newton, physicists had been conceiving the physical space as given by the classical Euclidean axiomatization of geometry, and understood time as an absolute entity totally independent of space. This conception suffered a catastrophic failure in the early XX century with the development of the special and general theories of relativity by A. Einstein.

In order to account for experimental results such as the Michelson-Morley experiment, in his exceptional paper published in 1905 Einstein postulated the invariance of the speed of light under the change of inertial frames of reference, which, together with the principle of special relativity that claims that all laws of physics are the same in every inertial frame of reference, led to the Lorentz transformations as the characterization of the kinematic description relative to different inertial frames. From them follows that time is not absolute but dynamic, as its perception depends on the observer, and it is, furthermore, deeply entangled with the notion of space. Consequently, special relativity ended up being more elegantly accommodated in a model of space and time that merged them together in a unique object, the *spacetime*. More precisely, it was the model of spacetime introduced by H. Minkowski in 1908 that seemed to fit perfectly, a 4-dimensional affine space with a inner product of index 1. Instead of considering physical positions and instants of time separately, points of the affine space represent physical *events*, i.e., ideal physical occurrences without spatial extension or time duration, such as an instant in the trajectory of a point-like particle.

However, the special relativity framework and the Minkowski model of spacetime have a major drawback: they do not account for gravity. It was Einstein himself who in 1915 brilliantly generalized his own theory proposing the theory of general relativity. He developed it from his famous equivalence principle, which states that the outcome of any local experiment in a freely falling laboratory is independent of its velocity, its location in the spacetime and the gravitational field; and the principle of general relativity, that extents the principle of special relativity to all kind of reference frames. In this new formalism, the Minkowski spacetime is just taken to be valid as local approximation of the spacetime, and its affine space description is replaced by a equivalent description in terms of a geometrical object known as a flat *Lorentzian manifold*. General spacetimes are modeled by a wider class of Lorentzian manifolds which allow to describe gravity through the curvature of spacetime caused by matter and energy according to the famous Einstein's field equations. Since in the general relativity framework different configurations of matter and energy give rise to different models of the spacetime, we will refer to them as spacetimes in plural. The study of Lorentzian manifolds, a particular case of semi-Riemannian manifolds, lies in the branch of differential geometry known as semi-Riemannian geometry, which explores the notions derived of endowing smooth manifolds with a so-called semi-Riemannian metric.

At this point, given this more sophisticated geometrical model of spacetimes which leaves behind the trivial flat classical Euclidean geometry, one may ask the question of what does algebraic topology have to say about them, and what applications may it have in physics theories that involve general spacetimes. To present a proper simple answer to this question we first have to narrow it down, constraining the algebraic tools chosen among all the available. Although homotopy groups are more intuitive and can exhibit a vastly richer structure, the computation of homology is far more simpler, specially in high dimensions, and is enough for many applications, remaining as the primary method for classifying topological spaces. Nonetheless, it is *cohomology*, the algebraic dualization of the concept of homology, which has proven to be stronger, as it is suitable for an extra ring structure.

In topological spaces the most paradigmatic example of homology is the *singular homology*, a powerful invariant that encodes plenty of topological information. Therefore, as Lorentzian manifolds are themselves smooth manifolds and thus topological spaces, we can consider the singular homology of spacetimes and its dual cohomology. In addition, the smooth structure with which smooth manifolds are furnished, defines a wide variety of objects linked to differential calculus. One of this objects, differential forms, allows to naturally construct an alternative notion of cohomology in smooth manifolds, *de Rham cohomology*. Astonishingly, this construction which in principle lies entirely in the realm of differential geometry, is deeply linked to bare topology, and is in fact equivalent to the singular cohomology as asserted by the acclaimed *de Rham's theorem*.

Unfortunately, the cohomology of physical spacetimes which are found in physics literature is, by itself, quite simple and uninteresting, as usually they are nothing but elementary manifolds that can be trivially studied with the tools provided by any algebraic topology book. Nevertheless, some results of de Rham cohomology such as *Poincaré duality* have found applications in classical and quantum electromagnetic theory on curved spacetimes, namely in the separability of field configurations for the Faraday tensor (see

[Ben16]). However, it is a slightly modified version of de Rham cohomology motivated by open problems in gauge field theories, the so-called *causally restricted cohomologies*, that is finding many applications in these field theories which are a current area of research (see for instance [Ben16] and [Kha16]). More precisely, they play an important role in understanding the (pre)symplectic and Poisson structure of these field theories.

These cohomologies exploit the fact that spacetimes are not just smooth manifolds, but are endowed with a Lorentzian metric, that allows to consider an extra feature of them, the *causal structure*. Under the classical conception of absolute space and time, the notions of past and future, cause and effect, are trivial. Time flows in one direction, defined by the so-called arrow of time, establishing the causal relations by ordering the physical events. However, when the distinction between space and time fades away until almost becoming a mere mirage, and the speed of light limit constrains the physical acceptable causal relations, some extra considerations have to be made. The reason why Lorentzian manifolds are suitable for model spacetimes is precisely the fact that they provide the natural causal structure we should expect of a spacetime. From the characterization of this causal structure, some sets naturally arise as appropriate to constitute the support of physical fields, the sets that are indeed used to construct the already mentioned cohomologies. This construction can be generalized beyond the de Rham complex to other complexes which have important applications in other field theories, such as the Calabi complex in linearized gravity on constant curvature backgrounds (see [Kha16]).

With the ultimate goal set in understanding and depicting these cohomologies, in this work we will encompass a description of de Rham cohomology in smooth manifolds, together with an outline of Lorentzian manifolds from a physical perspective. For this purpose, we have organized this work in the following way. Section 1 is devoted to lay the foundations of differential geometry needed to carry out our study. In Section 2, we introduce the basic notions of cohomology on smooth manifolds, namely singular and de Rham cohomology, presenting the main results regarding them. Finally, Section 3 is dedicated to the definition and characterization of de Rham cohomologies with non-standard support in causally well-behaved spacetimes. It contains a preliminary introduction to semi-Riemannian manifolds and the causal structure of Lorentzian manifolds, followed by a brief presentation of some tools of the Hodge theory. These are then used to introduce and study the desired cohomology theories.

Overall, we hope that our presentation of these cutting-edge de Rham cohomologies with non-standard support, offers also an interesting introductory dive into the concepts of de Rham cohomoloy and Lorentzian manifolds with a physical perspective, without requiring further previous knowledge than undergraduate general topology, algebra, and calculus.

1 Differential geometry of smooth manifolds

The basic mathematical object studied in differential geometry is the *smooth manifold.* In the same way that topological spaces provide the minimal support necessary to naturally extend the notion of continuity in $\mathbb{R}$ to the most possible general sense, smooth manifolds constitute the minimal mathematical structure that allow us to generalize the essentials of differential calculus. Intuitively, they are topological spaces locally similar to Euclidean spaces in an enough fine way to permit the definition of differentiation. In this section we will introduce the basic notions regarding smooth manifolds mainly following [O'N83], presenting the most relevant related objects and describing their most remarkable properties. These includes the concepts of *tangent vector spaces* and *tensor fields*, which will allow, later on, to present the modern model of spacetimes, Lorentzian manifolds, and to construct de Rham cohomology.

1.1 Smooth manifolds

We denote by $\mathbb{R}^n$ the set of n-tuples of real numbers with the usual algebraic structure and the Euclidean topology. We will often use the Einstein summation convention to simplify the notation, specially when working with coordinates, by which a repeated index on two adjacent terms, one superscript and one subscript, denotes the summation over the range of values of the index, whenever it makes sense and if it is not otherwise specified. For example $a^i b_i$ denotes the sum $\sum_{i\in I} a^i b_i$. However, we will still use the usual Σ notation if we want to specify the range of the indices or to avoid confusion. Let X be a topological space.

Definition 1.1. A *n-dimensional local chart* on X is pair (U, φ) where $U \subseteq X$ is an open set and $\varphi : U \longrightarrow \varphi(U) \subseteq \mathbb{R}^n$ is a homeomorphism. The functions $x^i = u^i \circ \varphi : U \longrightarrow \mathbb{R}$, where $u^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ are the canonical coordinate functions on $\mathbb{R}^n$ $u^i(a^1, \ldots, a^n) = a^i$, are called *coordinate functions* of φ and satisfy that $\varphi = (x^1, \ldots, x^n)$. We say that $\{x^1, \ldots, x^n\}$ is a *coordinate system* on U for which each $p \in U$ is said to have *coordinates* $(x^1(p), \ldots, x^n(p))$.

Remark 1.2. Given two local charts (U, φ), (V, ψ) of X such that $U \cap V \neq \emptyset$ we have the following commutative diagram:

$$\begin{array}{ccc} & U \cap V & \\ \swarrow^{\varphi} & & \searrow^{\psi} \\ \varphi(U \cap V) & \xrightarrow{\psi \circ \varphi^{-1}} & \psi(U \cap V) \end{array}$$

The real multivariable function given by the composition $\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$ which maps the coordinates of ψ into the coordinates of φ is called the *transition map.* Since the restricted applications $\varphi|_{U\cap V}$ and $\psi|_{U\cap V}$ are homeomorphism the transition map is also a homeomorphism.

Definition 1.3. An *atlas* $\mathcal{A}$ on X is a a family of local charts of X, $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$, such that $X = \bigcup_{i\in I} U_i$. If all the charts are of dimension n, we say that $\mathcal{A}$ is a *n-dimensional atlas* on X.

Definition 1.4. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ on X is *smooth* (or $\mathcal{C}^\infty$) if for all $i, j \in I$ such that $U_i \cap U_j \neq \emptyset$ the transition map $\varphi_j \circ \varphi_i^{-1}$ is a function of class $\mathcal{C}^\infty$. The coordinate systems induced by the local charts are said to overlap smoothly.

Definition 1.5. Given $\mathcal{A}_1$ and $\mathcal{A}_1$ two smooth atlases on X, we say that they are *compatible* if $\mathcal{A}_1 \cup \mathcal{A}_2$ is a smooth atlas on X, and we denote $\mathcal{A}_1 \sim \mathcal{A}_2$.

The compatibility of atlases $\sim$ is an equivalence relation. Two compatible atlases will endow a space with equivalent properties, so instead of working with atlases we will consider equivalence classes.

Definition 1.6. A *smooth structure* $[\mathcal{A}]$ on X is an equivalence class of atlases on X.

Definition 1.7. A *topological manifold* M is a second-countable Hausdorff space which is *locally Euclidean*, i.e, there is a positive integer n such that for all $p \in M$ there is a neighbourhood of p homeomorphic to $\mathbb{R}^n$. We say that M is *n-dimensional topological manifold.*

Remark 1.8. The dimension n of a nonempty topological manifold is unique thanks to the the theorem of the topological invariance of the domain, which implies that no nonempty open subset of $\mathbb{R}^n$ is homeomorphic to an open subset of $\mathbb{R}^m$ if $m \neq n$.

Remark 1.9. The Euclidean locality property is equivalent to the existence of an atlas $\mathcal{A}$ on M of a certain dimension n. Moreover, by the previous remark, a n-dimensional topological manifold only admits n-dimensional atlases.

Definition 1.10. A *n-dimensional smooth manifold* is a pair $(M, [\mathcal{A}])$, where M is a n-dimensional topological manifold and $[\mathcal{A}]$ is a (n-dimensional) smooth structure on M.

If there is no need of specify the smooth structure we will often denote a smooth manifold $(M, [\mathcal{A}])$ simply by M. A chart of M will refer to a local chart of one of the atlases of the implicit smooth structure. From now on, we may refer to a smooth manifold simply as a manifold or n-manifold, and M will denote a n-manifold if it is not otherwise specified.

Examples 1.11. Here are some basic examples of smooth manifolds:

1. For all $n \geq 1$, $\mathbb{R}^n$ with the standard smooth structure $[\{(\mathbb{R}^n, id_{\mathbb{R}^n})\}]$ is a n-dimensional smooth manifold. We will refer to it simply as $\mathbb{R}^n$.

2. For all $n \geq 1$, the n-dimensional sphere

$$\mathbb{S}^n = \{(a^1, \dots, a^{n+1}) \in \mathbb{R}^{n+1} : (a^1)^2 + \dots (a^{n+1})^2 = 1\}$$

with smooth structure given by the atlas $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ is a n-dimensional smooth manifold, where

$$U_N = \mathbb{S}^n \backslash \{p_N = (1, 0, \dots, 0)\}, \quad U_S = \mathbb{S}^n \backslash \{p_S = (0, , \dots, 0, 1)\}$$

and $\varphi_N : U_N \longrightarrow \mathbb{R}^n$ and $\varphi_S : U_S \longrightarrow \mathbb{R}^n$ are the stereographic projections from p_N and p_S respectively.

3. Any n-dimensional real vector space V is a n-dimensional smooth manifold with the initial topology for any isomorphism $\phi : V \longrightarrow \mathbb{R}^n$ (which does not depend on the choice of isomorphism) and smooth structure given by $[\{(V, \phi)\}]$. Then, any isomorphism $\psi : V \longrightarrow \mathbb{R}^n$ is a global chart on V.

4. If $(M, [\{(U_i, \varphi_i)\}_{i\in I}])$ and $(N, [\{(V_j, \psi_j)\}_{j\in J}])$ are smooth manifolds of dimension m and n then $M \times N$ is a smooth manifold of dimension $n + m$ with smooth structure represented by the atlas $\{(U_i \times V_j, \varphi_i \times \psi_j)\}_{i\in I, j\in J}$.

Given a smooth manifold M, its smooth structure can naturally induce a smooth structure on some subsets $P \subseteq M$. We call them submanifolds of M.

Definition 1.12. A subset $P \subseteq M$ is said to be a *k-dimensional smooth manifold* if for every $p \in P$ there is a chart (U, φ) of M such that $p \in U$ and

$$\varphi(U \cap P) = \varphi(U) \cap \{\mathbb{R}^k \times \{0\} \times \overset{(n-k)}{\cdots} \times \{0\}\}$$

Remark 1.13. A k-submanifold is a subset of M for which every point is covered by a chart such that they only have the first k coordinates respect to the chart non-zero. Consider the maps

$$\begin{aligned} \pi : \mathbb{R}^n &\longrightarrow \mathbb{R}^k \\ (a^1, \ldots, a^n) &\longmapsto (a^1, \ldots, a^k) \end{aligned} \qquad\qquad \begin{aligned} j : \mathbb{R}^k &\longrightarrow \mathbb{R}^n \\ (a^1, \ldots, a^k) &\longmapsto (a^1, \ldots, a^k, 0, \ldots, 0) \end{aligned}$$

If $\{(U_i, \varphi_i)\}_{i\in I}$ is an atlas that represents the smooth structure on M, a submanifold $P \subseteq M$ is a smooth manifold with the topology of subspace (which ensures that P is Hausdorff and second-countable) and with the smooth structure given by the atlas $\{(U_i \cap P, \psi_i)\}_{i\in I}$ where $\psi_i = \pi \circ \varphi_i|_{U\cap P}$. It is clear that the atlas has dimension k and covers P. Then, by definition of submanifold $\psi_i^{-1} = \varphi_i^{-1} \circ j$ so the transition maps $\psi_k \circ \psi_j^{-1} = \pi \circ \varphi_k \circ \varphi_i^{-1} \circ j$ are $\mathcal{C}^\infty$.

Examples 1.14. Bellow are some prominent examples of submanifolds.

1. Any open subset $U \subseteq M$ is a smooth submanifold of the same dimension that M, since for all $p \in U$ there exists a chart (V, φ) in M such that $p \in V$, so by compatibility $(U \cap V, \phi|_{U\cap V})$ will be another chart of M covering p and satisfying the property of Definition 1.12.

2. Any k-dimensional subspace F of a real vector space V is a k-submanifold of V.

As we previously advanced, the introduction of smooth structures allow us to naturally extend the $\mathcal{C}^\infty$-differentiability (or smoothness) of the usual real multivariable functions in Euclidean spaces $f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to maps between general manifolds.

Definition 1.15. Let M and N be smooth manifolds and $p \in M$. A map $F : M \longrightarrow N$ is said to be *smooth* at p if for all charts (U, φ) of M, (V, ψ) of N such that $p \in U$ and $F(p) \in V$, the *coordinate expression* of F in such charts, which is the real multivariable map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \longrightarrow \psi(V),$$

is of class $\mathcal{C}^\infty$. We say that F is smooth if it is smooth for all $p \in M$.

$$\begin{array}{ccc} U \cap F^{-1}(V) & \xrightarrow{\quad F \quad} & V \\ \varphi \downarrow & & \downarrow \psi \\ \varphi(U \cap F^{-1}(V)) & \xrightarrow{\psi \circ F \circ \varphi^{-1}} & \psi(V) \end{array}$$

Remark 1.16. It is straightforward to see that:

1. It suffices to the check the smoothness for a pair of charts (U, φ) of M and (V, ψ) of N such that $p \in U$ and $F(p) \in V$ thanks to the smooth overlap of charts.
2. Every smooth map is also continuous.
3. The identity on a smooth manifold is a smooth map.
4. The composition of smooth maps is smooth.

Definition 1.17. A map $F : M \longrightarrow N$ such that it is smooth, bijective and its inverse F^{-1} is smooth, is called *diffeomorphism*. If there exists a diffeomorphism between M and N, they are said to be *diffeomorphic* and we denote $M \cong N$.

Remark 1.18. Every diffeomorphism is also a homeomorphism, but not every smooth homeomorphism is also a diffeomorphism. For instance $F : \mathbb{R} \longrightarrow \mathbb{R}$ given by $F(a) = a^3$ is smooth and hence continuous but its inverse, although continuous is not smooth.

There are two particular cases of smooth maps that are of special interest:

Definition 1.19. Let $I \subseteq \mathbb{R}$ be an open real interval. A *smooth curve* or simply a *curve* on M is a smooth map $\gamma : I \longrightarrow M$, where I has the smooth structure given by the inclusion in $\mathbb{R}$ as a global chart.

Definition 1.20. A *smooth function* or simply a *function* on M is a smooth map of the type $f : M \longrightarrow \mathbb{R}$. The set of all functions on M, denoted by $\mathcal{F}(M)$, has a $\mathbb{R}$-algebra structure with the operations defined by

$$(f+g)(p) := f(p) + g(p), \quad (fg)(p) := f(p) \cdot g(p), \quad (\lambda f)(p) := \lambda \cdot f(p)$$

for all $p \in M$.

1.2 Tangent vector space

Given a smooth manifold M, the *tangent vector* space at a point formalizes the concept of tangency and tangent directions, which is, roughly speaking, the intuitive notion of all the possible different directions in which we can pass through the point within the manifold. In Euclidean spaces, tangent spaces are usually defined through an ambient space in which the manifold is embedded into. However, when making a general treatment, it is more convenient to present a definition intrinsic to the manifold without depending on the existence of a given embedding. The following definition is based upon an abstract generalization of the concept of directional derivative of functions which axiomatize its key properties.

Definition 1.21. Let M be a smooth manifold and $p \in M$. A *vector tangent* to M at p is a map $v : \mathcal{F}(M) \longrightarrow \mathbb{R}$ such that:

1. $v(\lambda f + \mu g) = \lambda \cdot v(f) + \mu \cdot v(g)$ ($\mathbb{R}$-linearity)
2. $v(fg) = v(f) \cdot g(p) + f(p) \cdot v(g)$ (Leibniz rule)

for all $f, g \in \mathcal{F}(M)$ $\lambda, \mu \in \mathbb{R}$. The set of all vectors tangent to M at p, denoted by T_pM, is called the *tangent space* of M at p. It is a real vector space with the operations defined by

$$(v+w)(f) := v(f) + w(f), \quad (\lambda v)(f) := \lambda \cdot v(f)$$

for all $f \in \mathcal{F}(M)$.

This definition is somehow natural because from elementary differential calculus in $\mathbb{R}$ to curves and surfaces on Euclidean spaces, tangent vectors and tangent spaces are formalized through derivatives. In fact, exploiting this idea, the tangent space at p can be defined more intuitively as the quotient set of all smooth curves $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ on M such that $\gamma(0) = p$ by the equivalence relation of "having the same derivative" at 0, in the sense that γ and ξ are said to be equivalent if there is a chart (U, φ) such that $(\varphi \circ \gamma)'(0) = (\varphi \circ \xi)'(0)$ as a curves on $\mathbb{R}^n$. Equivalence classes are often denoted by $\gamma'(0)$. However, the two definitions are canonically isomorphic by the map given by associating to any smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\gamma(0) = p$ the tangent vector at p, $v_\gamma : \mathcal{F}(M) \longrightarrow \mathbb{R}$, defined by $v_\gamma(f) = (f \circ \gamma)'(0)$.

Remark 1.22. Tangent vectors are local objects. This is expressed by the fact that given $v \in T_pM$, it follows from the definition that if if $f \in \mathcal{F}(M)$ is constant in a neighborhood of p then $v(f) = 0$. The linearity then implies that if f and g are equal on a neighborhood of p then $v(f) = v(g)$.

A set of tangent vectors of particular interest are the ones given by partial differentiation, which is locally defined using a chart to work with a coordinate expression of the function in an Euclidean space.

Definition 1.23. Let (U, φ) be a chart on M such that $p \in U$ and $\varphi = (x^1, \ldots, x^n)$. For each $1 \leq i \leq n$ the function $\frac{\partial}{\partial x^i}\big|_p$: $\mathcal{F}(M) \longrightarrow \mathbb{R}$ defined by

$$\left.\frac{\partial}{\partial x^i}\right|_p (f) = \frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p))$$

where $u^1, \ldots, u^n$ are the canonical coordinate functions of $\mathbb{R}^n$ and $\frac{\partial}{\partial u^i}$ denote the partial derivative respect of them, is a vector tangent to M at p, which is said to be the *coordinate vector tangent to M at p in the x_i direction.* If there is no need to specify the coordinate system we will simply denote it by $\partial_i|_p$.

The following result provides a useful description of tangent spaces by linking tangent vectors and coordinates (see for example Section 1 of [O'N83], Theorem 12).

Theorem 1.24. *Let M be a smooth manifold, $p \in M$ and (U, φ) a chart on M with coordinate system $\{x^1, \ldots, x^n\}$ such that $p \in U$. Then, $\{\partial_i|_p\}_{i \in \{1,\ldots,n\}}$ is a basis of T_pM in terms of which every $v \in T_pM$ can be expressed as*

$$v = \sum_{i=1}^{n} v(x^i) \partial_i|_p$$

Corollary 1.25. *The vector space T_pM has the same dimension than M.*

The real numbers $v(x^i)$ are the coordinates of $v \in T_pM$ in the basis $\{\partial_i|_p\}_{i \in \{1,\ldots,n\}}$, and will often be denoted by v^i, so $v = v^i \partial_i|_p$.

Remark 1.26. From Corollary 1.25, if $P \subseteq M$ is a k-dimensional submanifold of M, then for every $p \in P$ the tangent space T_pP is a k-dimensional real vector space that can be regarded as a subspace of the n-dimensional tangent space T_pM by the isomorphism that identifies each $v \in T_pP$ with $\tilde{v} \in T_pM$ defined by $\tilde{v}(f) = v(f|_P)$. In particular, for all $U \subseteq M$ open, $T_pU \cong T_pM$, which express once again the locality of the tangent vectors.

Remark 1.27. In the particular case of M being a real vector space, there exist a natural isomorphism between T_pM and M for all $p \in M$. For any basis $\{e_i\}_{i\in I}$, the isomorphism is given by mapping each $v_p = v^i \partial_i|_p \in T_pM$ onto $v = v^i e_i \in M$, where $\partial_i|_p$ are the coordinate vectors for the global chart that maps $\{e_i\}_{i\in I}$ onto the canonical basis in $\mathbb{R}^n$ (see Page 25 in [O'N83] for instance).

Definition 1.28. Given a manifold M and $p \in M$ the *cotangent space of* M *at* p is the dual space $T_p^*(M)$ of $T_p(M)$. Its elements $\alpha \in T_p^\star(M)$ are called *linear forms* or *covectors*.

Tangent spaces somehow recover the idea always present in differential calculus of locally approximating smooth objects by linear objects. Following this approach, roughly speaking, the so-called differential map induced by a smooth mapping between manifolds will approximate it around each point by a linear transformation of tangent spaces. It formalizes the non-rigorous notion of infinitesimal functional variations used in physics, as the linear approximation resembles more the function the closer we are to the point.

Definition 1.29. Let $F : M \longrightarrow N$ be a smooth map. For each $p \in M$ the *differential map* d_pF of F at p is the linear map defined by

$$\begin{aligned} d_pF\colon T_pM &\longrightarrow T_{F(p)}N \\ v &\longmapsto d_pF(v) : \mathcal{F}(N) \longrightarrow \mathbb{R} \\ &\qquad\qquad\qquad g \longmapsto d_pF(v)(g) = v(g \circ F) \end{aligned}$$

An alternative definition in terms of equivalence classes of curves, which is perhaps more intuitive, is given by mapping each curve on M to the curve in N given by the composition with F:

$$\begin{aligned} d_pF\colon T_pM &\longrightarrow T_{F(p)}N \\ \alpha'(0) &\longmapsto d_pF(\alpha'(0)) = (F \circ \alpha)'(0) \end{aligned}$$

which does not depend on the choice of class representatives.

Remark 1.30. If $F : M \longrightarrow N$ and $G : N \longrightarrow P$ are smooth mappings, then for each $p \in M$ $d_p(G \circ F) = d_{F(p)}G \circ d_pF$. This property is the generalization of the chain rule, in particular, when combined with the next result.

Proposition 1.31. Let $F : M \longrightarrow N$ a smooth mapping and $p \in M$. If (U, φ) is chart of M such that $p \in U$ with coordinate system $\{x^1, \dots, x^n\}$ and (V, ψ) is a chart of N such that $F(p) \in V$ with coordinate system $\{y^1, \dots, y^m\}$, then for all $1 \le j \le n$

$$d_pF\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{i=1}^{m} \frac{\partial F^i}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_{F(p)}$$

where $F^i := y^i \circ F \in \mathcal{F}(M)$.

Proof. We denote $w = d_pF\left(\frac{\partial}{\partial x^j}\big|_p\right) \in T_{F(p)}N$. By Theorem 1.24 we can express w as

$$w = \sum_{i=1}^{m} w(y^i)\ \frac{\partial}{\partial y^i}\bigg|_{F(p)}$$

And by the definition of differential map and vector tangent in the direction of a coordinate we have

$$w(y^i) = d_pF\left(\frac{\partial}{\partial x^j}\right)(y_i) = \frac{\partial}{\partial x^j}\bigg|_p (y^i \circ F) = \frac{\partial F^i}{\partial x^j}(p)$$

□

Definition 1.32. Let $F : M \longrightarrow N$ be a smooth map and $p \in M$. The matrix J_pF associated to $d_pF : T_pM \longrightarrow T_{F(p)}N$ in basis $\{\partial/\partial x^j|_p\}_{j\in\{1,\dots,n\}}$ and $\{\partial/\partial y^i|_{F(p)}\}_{i\in\{1,\dots,m\}}$, is called the *Jacobian matrix* of F at p relative to (U, φ) and (V, ψ), and as a consequence of the previous result is expressed as

$$\left(\frac{\partial F^i}{\partial x^j}(p)\right)_{1\le i\le m,\, 1\le j\le n}$$

Example 1.33. Any linear map $\phi : V_1 \longrightarrow V_2$ between real vector spaces is a smooth map. Using the previous result the differential can be expressed as $d_p\phi(v_p) = (\phi(v))_{\phi(p)}$, with the notation of the isomorphism on Remark 1.27, as we should expect for a linear approximation of a map which is already linear.

Definition 1.34. Given a function $f \in \mathcal{F}(M)$ and $p \in M$ the *differential of f at p* is the linear form $\mathrm{dif}_pf \in T_p^*M$ defined by $\mathrm{dif}_pf(v) = v(f)$.

Since the differential can be naturally associated to the differential map d_pf, by thinking of $d_pf(v)$ as the function $v(\cdot \circ f)$ (or more precisely, by expressing dif_pf as the composition of d_pf with the natural isomorphism between $T_p(\mathbb{R})$ and $\mathbb{R}$ given by Remark 1.27), the differential dif_pf is simply denoted by d_pf.

Remark 1.35. Given a manifold M, a point $p \in M$ and chart (U, φ) such that $p \in U$ with coordinate system $\{x^1, \dots, x^n\}$, for all $1 \le i \le n$ we have that $x^i \in \mathcal{F}(U)$ and the linear form $d_px^i \in T_p^*U \cong T_p^*M$ satisfies that

$$d_px^i(\partial_j|_p) = \frac{\partial x^i}{\partial x^j}(p) = \delta^i_j$$

where δ is the Kronecker delta. Therefore, $\{d_px^i\}_{i\in\{1,\dots,n\}}$ is the dual basis of $\{\partial_i|_p\}_{i\in\{1,\dots,n\}}$.

The following result is the generalization of the inverse function theorem in manifolds, and follows from applying the classical theorem to a coordinate expression for a map F around a point p.

Theorem 1.36. *Let $F : M \longrightarrow N$ be a smooth mapping and $p \in M$. The differential map d_pF is an isomorphism if and only if there is a neighborhood $U \subseteq M$ of p such that $F|_U : U \longrightarrow F(U) \subseteq N$ is a diffeomorphism.*

Note that if d_pF is an isomorphism then F defines a local diffeomorphism at p. More in general, we introduce the following terminology.

Definition 1.37. Let $F : M \longrightarrow N$ be a smooth mapping and $p \in M$. F is a *immersion*, a *submersion*, or a *local diffeomorphism* at p if d_pF is respectively injective, surjective or isomporhism. We say that F is a immersion, a submersion or a local diffeomorphism is the properties hold for all $p \in M$. An injective immersion is called an *embedding*.

Remark 1.38. It follows from the generalized inverse function theorem, that a local diffemorphism which is bijective is a diffemorphism. On the other hand, if a inclusion $i : P \hookrightarrow M$ is an embedding then P is a submanifold of M. In fact, this is a more natural definition of submanifold, which is equivalent to the definition presented.

The differential map also provides a proper generalization of the notion of derivative of a curve as a vector tangent to M. Since the paradigmatic example of curves are physical trajectories parametrized in time, it is often referred as the velocity of the curve.

Definition 1.39. Let $\gamma\colon I \longrightarrow M$ be a curve and $u : I \longrightarrow \mathbb{R}$ the only coordinate given by the inclusion as a global chart of I. The *velocity vector* of γ at $t \in I$ is

$$\gamma'(t) := d_t\gamma \left(\left.\frac{d}{du}\right|_t \right) \in T_{\gamma(t)}M,$$

where $\left(\left.\frac{d}{du}\right|_t \right) \in T_tI$ denotes the unit vector tangent to I at t in the u positive direction.

Remark 1.40. The velocity vector $\gamma'(t)$ applied to any $f \in \mathcal{F}(M)$ gives

$$\gamma'(t)(f) = d_t\gamma \left(\left.\frac{d}{du}\right|_t \right)(f) = \left.\frac{d}{du}\right|_t (f \circ \gamma) = \frac{d(f \circ \gamma)}{du}(t) = (f \circ \gamma)'(t),$$

which is consistent with the notation used previously, as the tangent vector $\gamma'(0)$ will have associated the equivalence class of curves $\gamma'(0)$.

Notice also that, according to Proposition 1.31 and considering the chart in I given by the inclusion, the components of $\gamma'(t)$ on the basis $\{\partial_i|_{\alpha(t)}\}_{i\in\{1,\ldots,n\}}$ given by the coordinate system $\{x^1, \ldots, x^n\}$ of a certain chart are $((x^1 \circ \gamma)'(t), \ldots, (x^n \circ \gamma)'(t))$, which is usually denoted simply by $((\gamma^1)'(t), \ldots, (\gamma^n)'(t))$ generalizing naturally the expression of the derivative of a curve in $\mathbb{R}^n$ by components.

1.3 Vector fields and one-forms

A tangent *vector field* on a manifold is a map that assigns to each point a vector tangent at that point. It results more convenient to express vector fields introducing a new manifold that glues together all tangent spaces.

Definition 1.41. The *tangent bundle* TM of a manifold M is the disjoint union of all tangent spaces of M:

$$TM := \bigsqcup_{p\in M} T_pM = \bigcup_{p\in M} \{p\} \times T_pM = \{(p, v) : p \in M,\, v \in T_pM\}$$

The tangent bundle is, indeed, a smooth manifold with topology and smooth structure defined as follows:

Given a chart (U, φ) of M with coordinates $\varphi = (x^1, \ldots, x^n)$, we consider the projection map

$$\begin{aligned} \pi : TM &\longrightarrow M \\ (p, v) &\longmapsto \pi(p, v) = p \end{aligned}$$

and the function

$$\Psi_U : \pi^{-1}(U) \subseteq TM \longrightarrow \mathbb{R}^{2n}$$
$$(p,v) \longmapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^n)$$

where $v = v^i \partial_i|_p$. Then, the topology of TM is generated by the preimages by Ψ_U of all open sets of $\mathbb{R}^{2n}$ and all charts U of M. In addition, if $\{(U_i, \varphi_i)\}_{i\in I}$ represents the smooth structure of M then $\{(\pi^{-1}(U_i), \Psi_{U_i})\}_{i\in I}$ represents a smooth structure in TM.

Definition 1.42. The set $M_p := \pi^{-1}(p)$ is called the *fibre* of TM at p and is canonically identified with T_pM by mapping each (p,v) onto v.

Definition 1.43. A *smooth vector field* or simply a *vector field* on M is a *section* of the tangent bundle, that is to say a smooth mapping $X : M \longrightarrow TM$ such that $\pi \circ X = Id$.

Vectors fields are then defined by $X(p) = (p, X_p)$ with $X_p \in T_pM$,i.e., by mapping each point $p \in M$ onto a point of the fibre of TM at p, which is identified with T_pM. Therefore, as we advanced, they can be thought as assigning to each point of the manifold a vector tangent to the manifold at that point. For all $f \in \mathcal{F}(M)$ we denote by f_X the function $f_X : M \longrightarrow \mathbb{R}$ defined by $f_X(p) = X_p(f)$. We also denote by $\mathcal{X}(M)$ the set of vector field on M.

Remark 1.44. The fact that X is smooth implies that f_X is also smooth, so $f_X \in \mathcal{F}(M)$ for all $f \in \mathcal{F}(M)$. Moreover, it suffices for f_X to be smooth for all $f \in \mathcal{F}(M)$ to ensure that a function $X : M \longrightarrow TM$ such that $\pi \circ X = Id$ is smooth.

Remark 1.45. $\mathcal{X}(M)$ is a real vector space with operations induced by the operations in T_pM defined by:

$$(X+Y)(p) := (p, X_p + Y_p), \quad (\lambda X)(p) := (p, \lambda X_p)$$

for all $p \in M$. It is also a $\mathcal{F}(M)$-module with the action given by

$$(fX)(p) := (p, f(p)X_p)$$

for all $p \in M$.

Remark 1.46. Given a smooth map $F : M \longrightarrow N$, the differential map d_pF induces a smooth map between the corresponding tangent bundles $dF : TM \longrightarrow TN$ defined by $dF(p, v_p) = (F(p), d_pF(v_p))$ and a linear map (abusing of notation) $dF : \mathcal{X}(M) \longrightarrow \mathcal{X}(N)$ given by $dF(X)(p) = dF(p, X_p)$.

Definition 1.47. A *derivation* D on $\mathcal{F}(M)$ is a map $D : \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$ satisfying

1. $D(\lambda f + \mu g) = \lambda D(f) + \mu D(g)$ ($\mathbb{R}$-linearity)
2. $D(fg) = D(f)g + fD(g)$ (Leibniz rule)

for all $\lambda, \mu \in \mathbb{R}$ and for all $f, g \in \mathcal{F}(M)$.

Remark 1.48. Notice the similarity between the definition of derivations and tangent vectors. In fact, derivations and vector fields can be identified since every vector field $X \in \mathcal{X}(M)$ defines a unique derivation D_X on $\mathcal{F}(M)$ by $D_X(f) = f_X$ and conversely, every derivation D on $\mathcal{F}(M)$ defines a unique vector field $X_D \in \mathcal{X}(M)$ by the property $X_{D_p}(f) = D(f)(p)$. This association is indeed a canonical isomorphism, so the notation is abused by referring to a vector field and the corresponding derivation both as X, and thus $X(f) := f_X$.

Definition 1.49. Let (U, φ) be a chart of M with coordinate system $\{x_1, \dots, x_n\}$. For each $1 \leq i \leq n$ the map

$$\begin{aligned} \partial_i \colon U &\longrightarrow TU \\ p &\longmapsto (p, \partial_i|_p) \end{aligned}$$

is a vector field, since the function $\partial_i(f) = \partial f / \partial x^i : M \longrightarrow \mathbb{R}$ given by $\partial_i(f)(p) = \partial_i|_p(f)$ is smooth for all $f \in \mathcal{F}(M)$. It is called the *coordinate vector field of* (U, φ) *in the* x^i *direction.*

Remark 1.50. From Theorem 1.24 any vector field X can be locally expressed on U as

$$X|_U = X^i \partial_i$$

where $X^i := X(x^i) : U \longrightarrow \mathbb{R}$ are called the *local coordinates of* X *on the chart* (U, φ). Note that the smoothness of X implies that $X(x^i) \in \mathcal{F}(U)$. Since M is covered by charts we can always define local coordinates of a vector field in a neighbourhood of any point $p \in M$. However, this does not mean that $\mathcal{X}(M)$ is finite dimensional as a vector space over $\mathbb{R}$. We have that, in general, $\mathcal{F}(M)$ is infinite dimensional, as given a non-zero vector field X, $\{fX : f \in \mathcal{F}(M)\}$ spans a infinite dimensional subspace of $\mathcal{X}(M)$. As a $\mathcal{F}(M)$-module, the maximal number of linear independent elements is finite, but there does not exist any basis in general.

The notion of vector field on a curve can also be introduced as follows.

Definition 1.51. Let $\gamma : I \longrightarrow M$ a curve on M. A *vector field on* γ is a smooth map $V : I \longrightarrow TM$ such that $\pi \circ V = \gamma$.

A vector field V on γ is given by $V(t) = (\gamma(t), V_{\gamma(t)})$, so it smoothly assigns to each $t \in I$ a tangent vector to M at $\gamma(t)$. The set of all vector fields on γ is denoted by $\mathcal{X}(\gamma)$, and has a $\mathcal{F}(I)$-module structure.

Examples 1.52. Let $\gamma : I \longrightarrow M$ a curve.

1. The *velocity vector field* of γ, γ', is the vector field on γ defined for all $t \in I$ by $\gamma'(t) = (\gamma(t), \gamma'(t))$.
2. The restriction of any vector field $X \in \mathcal{X}(M)$ to $\gamma(I)$ defines a vector field X_γ on γ by $X_\gamma(t) = (\gamma(t), X_{\gamma(t)})$.

We can dualize the notion of vector field using the cotangent spaces.

Definition 1.53. Let M be a manifold. The *cotangent bundle* T^*M is the disjoint union of all the cotangent spaces $T^*M := \bigsqcup_{p \in M} T_p^*M$. The cotangent bundle is a manifold with the smooth structure obtained in the same way as that the tangent bundle. Fibres are defined likewise.

Definition 1.54. A *differential one-form* or simply a *one-form* on M is a section of the cotangent bundle, i.e., a smooth map $\omega : M \longrightarrow T^*M$ such that $\pi^* \circ \omega = Id$, where $\pi^* : T^*M \longrightarrow M$ is the projection defined by $\pi^*(p, \lambda) = p$.

Similarly to vector fields, one-forms are defined by $\omega(p) = (p, \omega_p)$ with $\omega_p \in T_p^*M$, so they assign to each point a covector at that point. Analogously to the derivation approach to vector fields, for each $X \in \mathcal{X}(M)$, we can also define maps $\omega(X) : M \longrightarrow \mathbb{R}$ by $\omega(X)(p) = \omega_p(X_p)$, in terms of which the smoothness can be equivalently defined, and identify each one-form with its corresponding map $\omega : \mathcal{X}(M) \longrightarrow \mathcal{F}(M)$. We denote $\mathcal{X}^*(M)$ the set of all one-forms of M, which is a real vector space and a $\mathcal{F}(M)$-module with operations defined by

$$(\omega + \eta)(p) = (p, \omega_p + \eta_p), \quad (\lambda\omega)(p) = (p, \lambda\omega_p), \quad (f\omega)(p) = (p) = (p, f(p)\omega_p)$$

for all $p \in M$. Notice that, indeed, $\mathcal{X}^*(M)$ can be regarded as the dual space of $\mathcal{X}(M)$ thinking of one-forms as maps $\mathcal{X}(M) \longrightarrow \mathcal{F}(M)$.

Definition 1.55. The *differential* of a function $f \in \mathcal{F}(M)$ is the one-form $df \in \mathcal{X}^*(M)$ defined by $df(p) = (p, d_pf)$. It assigns to each point the differential of f at that point.

Remark 1.56. Given a chart (U, φ) of M with coordinate system $\{x^1, \dots, x^n\}$ we can consider the *coordinate one-forms* $dx^1, \dots, dx^n$ on U which by Remark 1.35 satisfy that $dx^i(\partial_i) = \delta^i_j$. It follows from applying to both sides of the following equation a vector field expressed in terms the coordinate vector fields, that any one-form $\omega \in \mathcal{X}^*(M)$ can be expressed in U as

$$\omega|_U = \omega_i dx^i$$

where $\omega_i := \omega(\partial_i)$ are smooth. In particular, if $f \in \mathcal{F}(M)$, since $df(\partial_i) = \partial f/\partial x_i$ on U:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i$$

1.4 Tensor fields

Functions, vector fields and one-forms can be regarded as a particular case of a more general entity called *tensor field*. Tensor fields provide also the generalization on manifolds of inner product on which semi-Riemannian geometry is based, and lead to the definition of the building block of de Rham cohomology, differential forms. Lets first introduce the concept of a tensor over a module. Let V be a module over a ring R and V^* its dual module. For any integers $r, s \geq 0$, we can consider the R-modules

$$(V^*)^r = V^* \times \overset{(r)}{\cdots} \times V^*, \quad V^s = V \times \overset{(s)}{\cdots} \times V$$

with the usual component-wise operations.

Definition 1.57. Let $r, s \geq 0$. A *tensor of type* (r, s) or (r, s)*-tensor over* V or is an R-multilinear map

$$A : (V^*)^r \times V^s \longrightarrow R$$

i.e., A is R-linear in each slot. For $r = 0, s \neq 0$ we understand A as $A : V^s \longrightarrow R$, for $r \neq 0, s = 0$, $A : (V^*)^r \longrightarrow R$, and for $r = s = 0$, $A \in R$. We denote by $\mathcal{T}^r_s(V)$ the set of all tensors of type (r, s) over V.

Remark 1.58. $\mathcal{T}^r_s(V)$ is an R-module with the usual definitions of functional addition and action by an element of R. A $(0, 0)$-tensor is an element of $\lambda \in R$. A $(0, 1)$-tensor is a linear form $\alpha \in V^*$. A $(1, 0)$-tensor is a linear form on V^*, that is to say an element of the double dual module $\tilde{v} \in V^{**}$, which can be regarded as an element of $v \in V$ by the canonical identification of V and V^{**}, which maps the double dual element defined by $\tilde{v}(\alpha) = \alpha(v)$ into v. A $(0, 2)$-tensor is a bilinear form on V.

Definition 1.59. Let $A \in \mathcal{T}^r_s(V)$ and $B \in \mathcal{T}^{r'}_{s'}(V)$. The *tensor product* of A and B is the tensor $A \otimes B \in \mathcal{T}^{r+r'}_{s+s'}(V)$ defined by

$$(A \otimes B)(\alpha^1, \ldots, \alpha^{r+r'}, v_1, \ldots, v_{s+s'}) =$$
$$= A(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_s)B(\alpha^{r+1}, \ldots, \alpha^{r+r'}, v_{s+1}, \ldots, v_{s+s'})$$

Remark 1.60. The tensor product defines an operation on the set $\mathcal{T}(V) := \bigoplus_{r,s\in\mathbb{N}} \mathcal{T}^r_s(V)$ which is associative, and compatible with the other operations induced by the operations on each $\mathcal{T}^r_s(V)$, endowing it with a (graded) R-algebra structure. It is called the tensor algebra. However, in general, the tensor product it is not commutative.

Proposition 1.61. Let V be a vector space, $\{e_i\}_{i\in I}$ a basis and $\{e^{*j}\}_{j\in I}$ its dual basis. The set $\{e_{i_1} \otimes \cdots e_{i_r} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_s}\}_{i_k,j_l\in I}$ (which is well-defined due to the tensor product associativity) is a basis of $\mathcal{T}^r_s(V)$ and all tensors can be expressed as:

$$A = A^{i_1\ldots i_r}_{j_1\ldots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*j_1} \otimes \cdots \otimes e^{*j_s}$$

where $A^{i_1\ldots i_r}_{j_1\ldots j_s} := A(e^{*i_1}, \ldots, e^{*i_r}, e_{j_1}, \ldots, e_{j_s})$ are called the components of A relative to the basis.

The result follows form a straightforward check of the linear independence of the given set, and the use of multilinearity and the expression of coordinates relative to the basis of V and V^* on the arguments of A to show that $A(\alpha^1, \ldots, v_s) = A^{i_1\ldots i_r}_{j_1\ldots j_s} e_{i_1}(\alpha^1) \ldots e^{*j_s}(v_s)$.

Remark 1.62. From the definition of components of a tensor relative to a basis it follows that:

1. If $A \in \mathcal{T}^r_s$ has components $A^{i_1\ldots i_r}_{j_1\ldots j_s}$ and $B \in \mathcal{T}^{r'}_{s'}$ has components $B^{i_1\ldots k_{r'}}_{j_1\ldots j_{s'}}$ on a certain basis, $C = A \otimes B$ has components

$$C^{i_1\ldots i_{r+r'}}_{j_1\ldots j_{s+s'}} = A^{i_1\ldots i_r}_{j_1\ldots j_s} B^{i_{r+1}\ldots i_{r+r'}}_{j_{s+1}\ldots j_{s+s'}}$$

2. Given two basis $\{e_j\}_{j\in I}$ and $\{e'_i\}_{i\in I}$. We consider the change of basis matrix $Q = (Q^i_j)_{i,j\in I}$ defined by $e_j = Q^i_j e'_i$ and its inverse matrix Q'. If $A^{i_1\ldots i_r}_{j_1\ldots j_s}$ are the components of a tensor A relative to the $\{e_j\}_{j\in I}$ basis, then

$$A'^{i_1\ldots i_r}_{j_1\ldots j_s} = Q^{i_1}_{k_1} \cdots Q^{i_r}_{k_r} Q'^{l_1}_{j_1} \cdots Q'^{l_s}_{j_s} A^{k_1\ldots k_r}_{l_1\ldots l_s}$$

are the components of A relative to the basis $\{e'_i\}_{i\in I}$. The upper indices are said to be contravariant (change according to Q) and the lower indices covariant (change according to Q'). Thus, $(r, 0)$-tensors are called contravariant tensors and $(0, s)$-tensors are called covariant tensors

Definition 1.63. Let A be a covariant or contravariant tensor of type at least two. A is said to be *symmetric* if transposing any two of its arguments leave its image on R unchanged. A is said to be *skew-symmetric* if each such reversal produces a sign change. Tensor fields of type $(1, 0)$, $(0, 1)$ and $(0, 0)$ are taken to be both symmetric and skew-symmetric.

The concept of tensor field in manifolds is the natural way of introducing tensors through the modules of vector fields and one-forms. More precisely, we define them as follows.

Definition 1.64. For any integers $r, s \geq 0$, a *tensor field of type (r,s) on M* is a tensor of type (r, s) over the $\mathcal{F}(M)$-module $\mathcal{X}(M)$, i.e., a $\mathcal{F}(M)$-multilinear map

$$A : (\mathcal{X}^*(M))^r \times (\mathcal{X}(M))^s \longrightarrow \mathcal{F}(M)$$

We denote by $\mathcal{T}_s^r(M)$ the $\mathcal{F}(M)$-module of tensor fields of type (r, s) over M.

Examples 1.65. Smooth functions are $(0, 0)$-tensor fields, vector fields are $(1, 0)$-tensor fields and one-forms are $(0, 1)$-tensor fields.

Remark 1.66. A tensor over the tangent space T_pM at a given point $p \in M$ is called a *tensor at p*. Any tensor field A of type (r, s) on M can be considered as a field on M that assigns smoothly to each point $p \in M$ a tensor at it $A_p : (T_p^*M)^r \times (T_pM)^s \longrightarrow \mathbb{R}$. Since $A(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s)(p)$ depends only on $\omega_p^1, \ldots \omega_p^r, X_{1p}, \ldots X_{sp}$ and not on the entirety of each one-form and vector field evaluated (see for instance Section 2 of [O'N83], Proposition 2), A_p can be defined by

$$A_p(\alpha^1, \ldots, \alpha^r, v_1, \ldots, v_s) = A(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s)(p)$$

where for all $1 \leq i \leq s$, $1 \leq j \leq r$, ω^i is any one-form on M such that $\omega_p^i = \alpha^i \in T_p^*M$ and X_j is any vector field on M such that $X_{ip} = v_i \in T_pM$. In fact, the association between M and the tensor bundle (defined analogously) is smooth. Conversely, a choice of A_p determines a unique tensor field A. In the same way that with vector fields and one-forms, the smoothness can be equivalently defined in terms of the smoothness of the maps $A(\omega^1, \ldots, \omega^s, X_1, \ldots X_s) : M \longrightarrow \mathbb{R}$ for all $\omega^1, \ldots \omega^r \in \mathcal{X}^*(M)$, for all $X_1 \ldots X_s \in \mathcal{X}(M)$, which can be regarded as

$$A(\omega^1, \ldots, \omega^r, X_1, \ldots X_s)(p) = A_p(\omega_p^1, \ldots, \omega_p^r, X_{1p}, \ldots X_{sp})$$

The following result generalizes the coordinate expressions obtained for vector fields and one-forms.

Proposition 1.67. Let (U, φ) be a chart on M with coordinate system $\{x^1, \ldots, x^n\}$. Any tensor field $A \in \mathcal{T}_s^r(M)$ can be expressed on U as:

$$A = A_{j_1 \ldots j_s}^{i_1 \ldots i_r} \partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$$

where $i_k, j_k \in \{1, \ldots, n\}$ for all $k \in \{1, \ldots, s\}$, $l \in \{1, \ldots, r\}$ and the smooth functions $A_{j_1 \ldots j_s}^{i_1 \ldots i_r} := A(dx^{i_1}, \ldots, dx^{i_r}, \partial_{j_1}, \ldots, \partial_{j_s}) \in \mathcal{F}(U)$ are called the components of A relative to (U, φ).

Proof. It is a direct consequence of the fact that according to Proposition 1.61 the set $\{\partial_{i_1}|_p \otimes \cdots \otimes \partial_{i_r}|_p \otimes d_px^{j_1} \otimes \cdots \otimes d_px^{j_s}\}_{i_k, j_k \in \{1, \ldots, n\}}$ is a basis of the real vector space of tensors at p, $\mathcal{T}_s^r(T_pM)$, and the local expressions we have for vector fields and one forms. □

Remark 1.68. Given another chart (V, ψ) with coordinate system $\{y^1, \ldots, y^n\}$ such that $U \cap V \neq \emptyset$. In $U \cap V$ we have that

$$\frac{\partial}{\partial x^j} = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

Thus, the components of A relative to (V, ψ) on $U \cap V$ can be expressed in terms of the components relative to (U, φ) as in Remark 1.62(2) with Q_j^i being the Jacobian matrix

$$Q_j^i = \frac{\partial y^i}{\partial x^j} = \frac{\partial(u^i \circ \psi \circ \varphi^{-1})}{\partial u^j} = J(\psi \circ \varphi^{-1})_j^i$$

Definition 1.69. Let $F : M \longrightarrow N$ a diffeomorphism and $A \in \mathcal{T}^r_s(M)$, $B \in \mathcal{T}^r_s(N)$ with s and r not both 0. The *pushforward of A by F* is the tensor field $F_*(A) \in \mathcal{T}^r_s(N)$ defined by

$$(F_*(A))(\eta^1, \dots, \eta^r, Y_1, \dots, Y_s) = A(\eta^{1*}, \dots, \eta^{r*}, Y_1^*, \dots, Y_s^*)$$

where $\eta^{i*} \in \mathcal{X}^*(M)$ is defined by $\eta^{i*}(X) = \eta^i(dF(X))$ and $Y_j^* \in \mathcal{X}(M)$ is $Y_j^* = dF^{-1}(Y_j)$ for all $i \in \{1, \dots, r\}$ $j \in \{1, \dots, s\}$. The pushforward of a $(0,0)$-tensor is defined as the function given by $F_*(f)(p) = F(f(p))$, i.e. $F_*(g) = F \circ g$.

Analogously, the *pullback of B by F* is the tensor field $F^*(B) \in \mathcal{T}^s_r(M)$ defined by

$$(F^*(B))(\omega^1, \dots, \omega^r, X_1, \dots, X_s) = B(\omega^1_*, \dots, \omega^r_*, X_{1*}, \dots, X_{s*})$$

where $\omega^i_* \in \mathcal{X}^*(N)$ is defined by $\omega^i_*(Y) = \omega^i(dF^{-1}(Y))$ and $X_{j*} \in \mathcal{X}(N)$ is $X_{j*} = dF(X_j)$ for all $i \in \{1, \dots, r\}$ $j \in \{1, \dots, s\}$. The pullback of a (0,0)-tensor is defined as the function given by $F^*(g)(p) = g(F(p))$, i.e. $F^*(g) = g \circ F$.

Remark 1.70. Pushforwards and pullbacks are $\mathcal{F}(M)$-linear and compatible with the tensor product, i.e., $F_*(A \otimes A') = F_*(A) \otimes F_*(A')$ (and idem for the pullback). Note that for smooth maps which are not diffeomorphisms, pullbacks and pushforwards are well-defined for covariant or contravariant tensors respectively.

2 Cohomology of smooth manifolds

Once the preliminaries of differential geometry have been set, the goal of this section is to present an introduction to *de Rham cohomology* that will be the starting point for the study of non-standard cohomologies in Section 3. We will start by providing the general abstract notions of homology and cohomology, to then define the *singular cohomology* of topological spaces, for which we will present the main results, such as invariance under homeomorphism and homotopy equivalence and *Poincaré duality*. Then, we will go on by constructing from *differntial forms* de Rham cohomology and its counterpart with compact support, that will also play a relevant role in Section 3. Finally, we will briefly present the theory of integration of differential forms that will provide the necessary tools, in particular the generalized *Stokes' theorem*, to give an insight into the major result regarding de Rham cohomology and singular cohomology, *de Rham's theorem*. For this section we have mainly followed [Hat02], [War83], and also [Kri99], [Nak03] and [FOT08].

2.1 Homology and cohomology

As we introduced, algebraic topology is the study of topological spaces using algebraic tools, in particular through the association of algebraic entities to the topological spaces. *Homology*, one of the most remarkable among these tools, is based on an algebraic structure known as chain complex. Let R be a commutative unitary ring.

Definition 2.1. A *chain complex* (over R) is a pair $(C_\bullet, d_\bullet)$ formed by a sequence of R-modules $C_\bullet = \{C_k\}_{k\in\mathbb{N}}$ and a sequence of morphisims $d_\bullet = \{d_k : C_k \longrightarrow C_{k-1}\}_{k\in\mathbb{N}\setminus\{0\}}$ such that $d_k \circ d_{k+1} = 0$ for all $k \geq 1$. The elements of C_k are called *k-chains* and the morphisms are called *boundary operators* or *differentials*.

$$\cdots \longleftarrow C_{k-1} \xleftarrow{d_k} C_k \xleftarrow{d_{k+1}} C_{k+1} \longleftarrow \cdots$$

Remark 2.2. For $R = \mathbb{Z}$ the notion of $\mathbb{Z}$-module is equivalent to the notion of abelian group, so a chain complex over $\mathbb{Z}$ can be regarded as sequence of abelian groups and the corresponding differential morphisms.

Definition 2.3. Let $(C_\bullet, d_\bullet)$ be a chain complex. For $k \geq 1$ we define

1. The R-module of *k-cycles* or *closed* elements $Z_k(C_\bullet, d_\bullet) = ker\, d_k \subseteq C_k$.
2. The R-module of *k-boundaries* or *exact* elements $B_k(C_\bullet, d_\bullet) = im\, d_{k+1} \subseteq C_k$.
3. The *k-th homology* R-module $H_k(C_\bullet, d_\bullet) = Z_k/B_k$, which is well-defined since $im\, d_{k+1} \subseteq ker\, d_k$. We may vaguely refer to the sequence of homology R-modules of a given complex as its homology.

A chain complex is said to be *exact* if and only if for all $k \geq 1$, $\ker d_k = \mathrm{im}\ d_{k+1}$, i.e, if $H^k(C_\bullet, d_\bullet) = \{0\}$.

Definition 2.4. A *chain map* $f_\bullet : (C_\bullet, d_\bullet) \longrightarrow (C'_\bullet, d'_\bullet)$ between chain complexes is a sequence of morphisms $f_\bullet = \{f_k : C_k \longrightarrow C'_k\}_{k\in\mathbb{N}}$ such that the diagram

$$\begin{array}{ccccccccc} \cdots \longleftarrow & C_{k-1} & \xleftarrow{d_k} & C_k & \xleftarrow{d_{k+1}} & C_{k+1} & \longleftarrow \cdots \\ & \downarrow{\scriptstyle f_{k-1}} & & \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k+1}} & \\ \cdots \longleftarrow & C'_{k-1} & \xleftarrow{d'_k} & C'_k & \xleftarrow{d'_{k+1}} & C'_{k+1} & \longleftarrow \cdots \end{array}$$

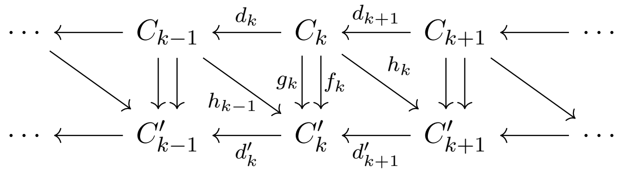
is commutative, i.e., $f_k \circ d_{k+1} = d'_{k+1} \circ f_{k+1}$, for all $k \geq 0$.

Remark 2.5. A straightforward computation shows that a chain map $f_\bullet$ induces a morphism on the homologies $H_k(f_\bullet) : H_k \longrightarrow H'_k$ for all $k \geq 0$ by $H_k(f_\bullet)([z]) = [f_k(z)]$. If $f_\bullet$ is an chain isomorphism, i.e. f_k are isomorphisms for all k, then $H_k(f_\bullet)$ are also isomorphism for all k.

Definition 2.6. Let $f_\bullet, g_\bullet : (C_\bullet, d_\bullet) \longrightarrow (C'_\bullet, d'_\bullet)$ be two chain maps. We say that $f_\bullet$ and $g_\bullet$ are *chain homotopic* and denote $f_\bullet \sim g_\bullet$ if there exists a sequence of R-linear morphisms $h_\bullet = \{h_k : C_k \longrightarrow C'_{k+1}\}_{k\in\mathbb{N}}$ such that for all $k \geq 0$

$$f_k - g_k = d'_{k+1} \circ h_k + h_{k-1} \circ d_k$$

with the convention that $h_{-1} = 0$. In that case we say that $h_\bullet$ is a *chain homotopy* between $f_\bullet$ and $g_\bullet$.



Remark 2.7. If $f_\bullet \sim f_\bullet$ then $H_k(f_\bullet) = H_k(g_\bullet)$ for all $k \geq 0$, because if $[c] \in H_k(C_\bullet, d_\bullet)$ then $d_k c = 0$ and $f_k(c) - g_k(z) = d'_{k+1}(h_k(c)) \Rightarrow f_k(c) - g_k(z) \in \text{im } d'_{k+1} \Rightarrow [f_k(c)] = [g_k(z)]$.

Definition 2.8. A *cochain complex* (over R) is a pair $(C^\bullet, d^\bullet)$ formed by a sequence of R-modules $C^\bullet = \{C^k\}_{k\in\mathbb{N}}$ and a sequence of morphisims $d^\bullet = \{d^k : C^k \longrightarrow C^{k+1}\}_{k\in\mathbb{N}}$ such that $d^{k+1} \circ d^k = 0$ for all $k \geq 0$. The elements of C^k are called *k-cochains* and the morphisms are called *coboundary operators* or *differentials.*

$$\cdots \longrightarrow C^{k-1} \xrightarrow{d^{k-1}} C^k \xrightarrow{d^k} C^{k+1} \longrightarrow \cdots$$

Analogously, for each $k \geq 1$ we can define the R-module of *k-cocycles*(or closed elements) $Z^k(C^\bullet, d^\bullet) = ker\, d^k \subseteq C^k$, the R-module of *k-coboundaries* (or exact elements) $B^k(C^\bullet, d^\bullet) = im\, d^{k-1} \subseteq C^k$ and the *k-th cohomology* R-module $H^k(C^\bullet, d^\bullet) = Z^k/B^k$. *Cochain maps* and *cochain homotopy* are defined likewise, and the results that involve them are straightforward dualized.

Definition 2.9. Let $(C_\bullet, d_\bullet)$ be a chain complex (over R). Its *dual* complex $(C^\bullet, d^\bullet)$ is the cochain complex defined by the dual R-modules $C^k = C_k^*$ and the dual morphisms $d^k = d_{k+1}^* : C_k^* \longrightarrow C_{k+1}^*$ given by $d_k^*(\varphi) = \varphi \circ d_{k+1}$.

Remark 2.10. From this construction, one may expect to have a natural isomorphism $H^k(C^\bullet, d^\bullet) \cong (H_k(C_\bullet, d_\bullet))^*$ but this is not always true. Nonetheless, the universal coefficient theorem for principal ideal domains (see [Hat02] for instance) ensures that this is the case for chain complexes over a field F. Thus, in this case, we will treat them indistinctly. In addition, this implies that for complexes over fields, $H^k(C^\bullet, d^\bullet)$ is (non-canonically) isomorphic to $H_k(C_\bullet, d_\bullet)$.

The notion of exactness can be generalized to any sequence of linear maps by the property that the images of the preceding maps are included in the kernels of the succeeding ones. With this consideration in mind, we end this section presenting the following well-known result (see for instance [BT82], Page 17) that will be useful to prove the main result on Subsection 3.4. We have also the completely analogous result for homology.

Proposition 2.11. Given a exact sequence of cochain maps

$$\{0\} \longrightarrow A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \longrightarrow \{0\}$$

known as *short sequence*, there exist linear maps $\tau^k : H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet)$ induced by the differentials of the complexes such that the so-called *long sequence* in cohomologies

$$\cdots H^k(A^\bullet) \xrightarrow{H^k(f^\bullet)} H^k(B^\bullet) \xrightarrow{H^k(g^\bullet)} H^k(C^\bullet) \xrightarrow{\tau^k} H^{k+1}(A^\bullet) \xrightarrow{H^{k+1}(f^\bullet)} H^{k+1}(B^\bullet) \cdots$$

is also exact.

2.2 Singular cohomology

As we advanced, in the realm of topological spaces the most prominent construction of homology is the *singular homology.*

Definition 2.12. (Singular homology) Let X be a topological space, R be a commutative unitary ring and

$$\Delta^k = \left\{ (a^1, \dots, a^{k+1}) \in \mathbb{R}^{k+1} : a^i \geq 0 \, \forall i \text{ and } \sum_{i=1}^{k+1} a^i = 1 \right\}$$

be the *k-simplex standard.* A *singular k-simplex on X* is a continuous map $\sigma : \Delta^k \longrightarrow X$. We define the *singular k-chain R-module of X* $S_k(X, R)$ as the free R-module generated by the singular k-simplices, i.e

$$S_k(X, R) = \bigoplus_{\sigma : \Delta^k \longrightarrow X \, cont.} R(\sigma)$$

where $R(\sigma)$ is the free R-module generated by σ. The elements of $S_k(X, R)$ are called *k-chains over R of X*. For each $k \geq 1$ we consider the maps

$$\delta_i^k : \Delta^{k-1} \longrightarrow \Delta^k$$
$$(a^1, \dots, a^{k+1}) \longmapsto (a^i, \dots, a^{i-1}, 0, a^i, \dots a^{k+1})$$

for all $i \leq k$, and we define the *singular differential morphism* or *boundary operator* ∂_k as

$$\partial_k : S_k(X, R) \longrightarrow S_{k-1}(X, R)$$
$$\sigma \longmapsto \sum_{i=0}^{k} (-1)^i \sigma \circ \delta_i^k$$

and $\partial_0 = 0$. They satisfy that $\partial_{k-1} \circ \partial_k = 0$ for all $k > 0$, so we can define the *singular chain complex over R of X* as $(S_\bullet(X, R), \partial_\bullet)$, and the singular k-th homology R-modules of X, $H_k(X, R) := H_k(S_\bullet(X, R), \partial_\bullet)$. In particular, for $R = \mathbb{Z}$ we obtain the *singular group chain complex of X* $(S_\bullet(X), \partial_\bullet)$ and the k-th homology groups of X, $H_k(X)$.

A close look into the definition of k-chains and the differential morphism shows that k-cycles of the singular chain complex can be regarded as k-dimensional loops on X whereas k-boundaries can be regarded as the boundaries of $(k+1)$-dimensional chains on X (with an orientation given by ∂). Intuitively, the existence of a k-cycle which is not the

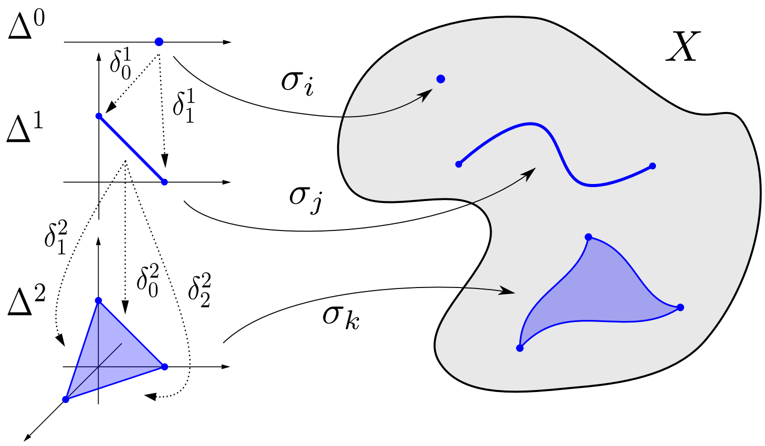


Figure 1: Visualization of the construction of the singular chain complex.

boundary of any $(k+1)$-chain, indicates the presence of a k-dimensional hole in X (see Chapter 2 in [Hat02]). The homology spaces are simply the quotient spaces of k-cycles obtained by factoring out the k-boundaries, that are not linked to holes. More precisely, given a topological space X, the dimension of the singular k-th homology vector spaces (or the dimension of the singular k-th homology group or modules if they are free) can be thought as formalizing the number of k-dimensional holes of X.

A continuous map between topological spaces $f : X \longrightarrow Y$ induces a chain map between their singular chain complexes $S_\bullet(f)$ by considering $S_k(f) : S_k(X,R) \longrightarrow S_k(Y,R)$ defined by $S_k(f)(\sigma) = f \circ \sigma$, and therefore f induces a morphism on their homologies. In particular, if f is a homeomorphism, the induced morphism on the homologies is an isomorphism. Furthermore, the singular homology R-modules are aslo homotopically invariant, i.e., if X is homotopic-equivalent to Y, $X \sim Y$, then $H_k(X,R) \cong H_k(Y,R)$.

Despite he fact that Δ^k is not a submanifold of $\mathbb{R}^{k+1}$, and therefore given a smooth manifold M we can not, in principle, discuss whether a k-simplex on M, $\sigma : \Delta^k \longrightarrow M$ is smooth or not, a notion of smoothness can be introduced on simplices as follows: we say that a singular k-simplex σ on M is smooth at a given $p \in \Delta^k$ if there exists a submanifold $P \subset \mathbb{R}^{k+1}$ with $p \in P$ and a smooth function $F : P \longrightarrow M$ such that $f|_{P\cap\Delta^k} = \sigma$. Furthermore, σ is said to be smooth if it is smooth for every $p \in \Delta^k$. With these considerations, given a smooth manifold M, we can also consider the *smooth singular chain complex over R of M* $(S_{\infty,\bullet}(M,R), \partial_\bullet)$ defined by the R-submodules generated by the smooth k-simplices, $S_{\infty,k}(M,R) = \langle\{\sigma : \Delta^k \longrightarrow M : \sigma \text{ is smooth}\}\rangle \subseteq S_k(M,R)$ and the restriction of the singular differential morphisms ∂_k. However, the inclusion $i : S_{\infty,k}(M,R) \hookrightarrow S_k(M,R)$ defines a chain map and the induced morphism on homologies is, in fact, a isomorphism, giving $H_k(M,R) \cong H_{\infty,k}(M,R)$.

Examples 2.13. Some well-known results of the singular homology (see for example [Hat02] Section 2.1) are:

1. If X is a non-empty path-connected space $H_0(X,R) \cong R$.

2. If $\{X_i\}_{i\in I}$ are the path-connected components of a non-empty space, then, for all $k \geq 0$, $H_k(X,R) = \bigoplus_{i\in I} H_k(X_i,R)$.

3. If $X = \{p\}$ then $H_k(X,R) = \{0\}$ for all $k > 0$

4. For all $n \geq 1$,

$$H_k(\mathbb{S}^n, R) \cong \begin{cases} R & \text{if } k = 0, n \\ \{0\} & \text{otherwise} \end{cases}$$

Given a topological space X, in addition to the singular chain complex we can consider, we can also consider its dual complex, the *singular cochain complex over* R, $(S^\bullet(X,R), \partial^\bullet)$ and the *singular cohomology* R-modules of X, $H^k(X,R)$. The main properties of the singular homology, namely its invariance, are directly transfer to singular cohomology. This dualization allows to define a natural product in the singular cochain complex which does not have a well-behaved analogous in the chain complex. The *cup product* of singular cochains is defined by its action on simplices by

$$\begin{aligned} \smile\colon S^k(X,R) \times S^l(X,R) &\longrightarrow S^{k+l}(X,R) \\ (\varphi,\psi) &\longmapsto \varphi \smile \psi : \ S_k(X,R) \longrightarrow R \\ &\qquad\qquad\qquad \sigma \longmapsto \varphi\left(\sigma|_{[0,\dots,k]}\right) \psi\left(\sigma|_{[k,\dots,k+l]}\right) \end{aligned}$$

where $[0,\dots,k] = \{(a^1, \cdots, a^{k+l+1}) \in \Delta^{k+l} : a^i = 0 \, \forall i > k\}$ and $[k,\dots,k+l]$ is defined likewise. This product endows the R-module $S(M,R) = \bigoplus_{k\in\mathbb{N}} S^k(M,R)$ with a (graded) R-algebra structure, and induces a product on cohomology

$$\smile\colon H^k(X,R) \times H^l(X,R) \longrightarrow H^{k+l}(X,R)$$

which also endows the R-module $H(M,R) = \bigoplus_{k\in\mathbb{N}} H^k(M,R)$ with a (graded) R-algebra structure (see for instance [Hat02], Section 3.2).

We can also define the *cap product* for each $k \geq l$ by

$$\begin{aligned} \frown\colon S_k(X,R) \times S^l(X,R) &\longrightarrow S_{k-l}(X,R) \\ (\sigma,\varphi) &\longmapsto \sigma \frown \varphi = \varphi\left(\sigma|_{[0,\dots,l]}\right) \sigma|_{[l,\dots,k]} \end{aligned}$$

which, again, descends to a product on homologies and cohomologies

$$\frown\colon H_k(X,R) \times H^l(X,R) \longrightarrow H_{k-l}(X,R)$$

Then, this map is used to construct one of the main results in the singular cohomology theory the so-called Poincaré duality, that asserts that in a n-dimensional compact oriented smooth manifold we have natural isomorphisms $H^k(M,R) \cong H_{n-k}(M,R)$ for all $k \leq n$ (see for instance [Hat02], Section 3.3). Although the natural isomorphism is given between the cohomology and the homology, if R is a field, thanks to Remark 2.10, in practice we can use Poincaré duality with all indices up or down. This will hold, in particular, for Poincaré duality of de Rham cohomology.

2.3 Differential forms and de Rham cohomology

As we already mentioned, *differential forms* compose the core of a natural construction of cohomology on smooth manifolds, *de Rham cohomology*. They are a particular type of tensor field that provide a generalization of the integration on $\mathbb{R}$, curves and surfaces to general manifolds under a unified approach. Intuitively, interpreting dx^i as the formalization of an infinitesimal variation or increment in the x^i coordinate direction, covariant tensors of the form $dx^i \otimes dx^j$ may be interpreted as representing infinitesimal 2-dimensional variations or surfaces which locally approximate the $\partial_i - \partial_j$ plane. The same intuition extends to k dimensions. However, the fact that the tensor product of covariant tensors is commutative implies that there is no distinction in the order of the coordinates, as for instance $dx^i \otimes dx^j = dx^j \otimes dx^i$. Then, k-forms are defined as covariant skew-symmetric tensors so that they naturally carry with them a notion of orientation, which, as we will later discuss, is crucial to define a consistent integration.

Definition 2.14. Given an integer $k \geq 0$, a *differential k-form* or simply a *k-form* on M is a skew-symmetric tensor field of type $(0, k)$. The set of k-forms on M is denoted by $\Omega^k(M)$.

Remark 2.15. Notice that $\Omega^0(M) = \mathcal{F}(M)$ and $\Omega^1(M) = \mathcal{X}^*(M)$. In general, the set of k-forms $\Omega^k(M) \subseteq \mathcal{T}_k^0(M)$ is a $\mathcal{F}(M)$-submodule of $\mathcal{T}_k^0(M)$.

Although the tensor product of differential forms is not in general a differential form because is not skew-symmetric, we can define a product of forms by skew-symmetrizing the tensor product. This new operation, denoted by $\wedge$, allows to consider, for example, differential forms of the type $dx^i \wedge dx^j$ which can be thought as representing infinitesimal oriented surfaces, as the order of the coordinates changes their sign $dx^i \wedge dx^j = -dx^j \wedge dx^i$. As we will see, these coordinate forms play a fundamental role, because every k-form can be expressed locally as a $\mathcal{F}(M)$-linear combination of k-coordinates forms.

Definition 2.16. Given $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ the *exterior product* $\omega \wedge \eta \in \Omega^{k+l}(M)$ is defined by

$$\omega \wedge \eta = \frac{1}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \varepsilon(\sigma)\sigma(\omega \otimes \eta)$$

where $\mathcal{S}_n$ is the symmetric group, $\varepsilon(\sigma)$ is the sign of the permutation σ, and $\sigma(\omega \otimes \eta)$ is the tensor defined by

$$\sigma(\omega \otimes \eta)(X_1, \dots, X_{k+l}) = (\omega \otimes \eta)(X_{\sigma(1)}, \dots X_{\sigma(k+l)})$$

Remark 2.17. The exterior product is associative and skew-commutative, i.e.,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

which implies that $\omega \wedge \omega = 0$. It defines a binary operation on the $\mathcal{F}(M)$-module $\Omega(M) := \bigoplus_{k \geq 0} \Omega^k(M)$ that is compatible with the other operations yielding a structure of a (graded) $\mathcal{F}(M)$-algebra. It is called the *differential forms algebra* or the *de Rham algebra*.

In particular, for k 1-forms $\alpha^1, \dots, \alpha^k \in \Omega^1(M)$ we have that

$$\alpha^1 \wedge \cdots \wedge \alpha^k = \sum_{\sigma \in \mathcal{S}_k} \varepsilon(\sigma) \alpha^{\sigma(1)} \otimes \cdots \otimes \alpha^{\sigma(k)}$$

Given a local chart (U, φ) of M with coordinate system $\{x_1, \dots, x_n\}$ any k-form ω, which is in particular a tensor, can be locally expressed as

$$\omega|_U = \omega_{j_1 \dots j_k} dx^{j_1} \otimes \cdots \otimes dx^{j_k}$$

where $\omega_{j_1 \dots j_k} = \omega(\partial_{j_1}, \dots, \partial_{j_k})$, and since the skew-symmetry implies that if there is a repeated index the coefficient is zero because $\omega_{j_{\sigma(1)} \cdots j_{\sigma(k)}} = \varepsilon(\sigma)\omega_{j_1 \dots j_k}$, we can rewrite the expression as

$$\omega|_U = \sum_{j_1 < \cdots < j_k} \sum_{\sigma \in \mathcal{S}_k} \omega_{j_{\sigma(1)} \cdots j_{\sigma(k)}} dx^{j_{\sigma(1)}} \otimes \cdots \otimes dx^{j_{\sigma(k)}} = \sum_{j_1 < \cdots < j_k} \omega_{j_1 \dots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

or equivalently using Einstein's summation convention, as

$$\omega|_U = \frac{1}{k!}\omega_{j_1\dots j_k} dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

In particular, for $k = n$ we have that the summation for $j_1 < \cdots < j_n$ just have the term $j_1 = 1, \dots, j_n = n$ so

$$\omega|_U = \omega_{1\cdots n} dx^1 \wedge \cdots \wedge dx^n$$

Note that if $k > n$ there must be a repeated index on each coordinate so $\Omega^k(M) = \{0\}$. For $k \leq n$, if we consider the real vector spaces of k-forms as tensors at a point $p \in M$, we can construct basis from the local coordinate expressions. Since there are $\binom{n}{k}$ different choices for k indices over a set of n indices, the spaces will be generated by $\binom{n}{k}$ linear independent elements, and therefore their dimension will be $\binom{n}{k}$.

The following result proves the existence of the differential map needed to construct the sought cochain complex of differential forms.

Proposition 2.18. Let M be a manifold. There exists a unique map $d : \Omega(M) \longrightarrow \Omega(M)$ called *exterior derivative* satisfying:

(i) If $\omega \in \Omega^k(M)$, $d\omega \in \Omega^{k+1}(M)$.

(ii) d is $\mathbb{R}$-linear.

(iii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for all $\omega \in \Omega^k(M)$ for all $\eta \in \Omega^l(M)$.

(iv) If $f \in \Omega^0(M)$ df is the differential of f.

(v) If $f \in \Omega^0(M)$ $d(df) = 0$.

Proof. We first suppose that such a map exists to prove the uniqueness. From the definition of exterior product and Properties (iii) and (v) it follows that if $g_1 \dots, g_k \in \Omega^0(M)$ then $d(dg_1 \wedge \cdots \wedge dg_k) = 0$. Since $f\omega = f \wedge \omega$ for $f \in \Omega^0(M), \omega \in \Omega^k(M)$, this implies that

$$d(f dg_1 \wedge \cdots \wedge dg_k) = df \wedge dg_1 \wedge \cdots \wedge dg_k$$

Therefore, if $\omega_{j_1\dots j_k} \in \mathcal{F}(M)$ are the local components of a k-form ω relative to a chart on U, which are unique, then $d\omega$ is completely determined on U by the expression

$$d\omega = \frac{1}{k!} d(\omega_{j_1\dots j_k}) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} \tag{2.1}$$

which, as M can be covered by charts, proves the global uniqueness.

The existence is then given by defining d on any local chart by its action on coordinates according to Equation 2.1 and check that the properties hold (see Theorem 2.5.1 on [Kri99]). Then, as the properties imply the uniqueness the expressions in different charts must agree on overlaps. Therefore d is well-defined globally proving the existence. □

Note that the exterior derivative is a $\mathbb{R}$-linear map, that can be regarded indeed as a sequence of $\mathbb{R}$-linear maps $\{d^k : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)\}_{k\in\mathbb{N}}$. Hence, we shall consider the $\mathcal{F}(M)$-modules of differential k-forms $\Omega^k(M)$ just as real vector spaces. Furthermore, applying the exterior derivative to Equation 2.1 directly shows that Property (v) in Proposition 2.18 is, in fact, stronger, as for any k-form $d(d\omega) = 0$ in all coordinate system, so $d^2 = d \circ d = 0$. This gives the last ingredient needed to finally define the following cochain complex.

Definition 2.19. Let M be a smooth manifold. The *de Rham complex* on M is the cochain complex $(\Omega^\bullet(M), d^\bullet)$, where $\Omega^\bullet(M)$ is the sequence of the real vector spaces of k-forms and d the exterior derivative. The *de Rham cohomology of* M are the real vector spaces defined by the cohomology of the de Rham complex, $H^k_{dR}(M) = H^k(\Omega^\bullet(M), d^\bullet)$.

$$\cdots \longrightarrow \Omega^{k-1}(M) \xrightarrow{d^{k-1}} \Omega^k(M) \xrightarrow{d^k} \Omega^{k+1}(M) \longrightarrow \cdots$$

Remark 2.20. In the same way that the cup product induces a R-algebra structure in $H(X, R)$, the exterior product induces a product in the cohomology classes endowing $H_{dR}(M) = \bigoplus_{k\in\mathbb{N}} H^k_{dR}(M)$ with a (graded) $\mathbb{R}$-algebra structure.

Whereas the condition $d^2 = 0$ implies that any exact form is closed, the converse is not true in general. Nonetheless, for $f \in \Omega^0(M)$, if $df = 0$, from its coordinate expression follows that the partial derivatives of f are zero, which implies that f is locally constant. For the remaining degrees we have the following extension (see for instance, [Kri99] Theorem 2.5.2).

Theorem 2.21. *(Poincaré lemma) Let M be a manifold and $\omega \in \Omega^k(M)$ with $k \geq 1$. If $d\omega = 0$, then for every $p \in M$ there exists a neighbourhood of p and a $(k-1)$-form $\eta \in \Omega^{k-1}(M)$ such that $\omega|_U = d\eta$.*

Remark 2.22. As we will justify in Section 3.3, Poincaré lemma is nothing but a generalization of the fact that every real function with zero derivative is locally constant and in 3-dimensional Euclidean spaces every curl (divergence)-free vector field has a local scalar(vector) potential. Note that in the same way that, roughly speaking, singular homology measures how cycles are not boundaries, which as we have discussed indicates the presence of holes, de Rham cohomology measures how closed forms are not exact, or in other words the failure of Poincaré lemma globally.

However, surprisingly, this failure can also be linked to holes. For example, we can define in $\mathbb{R} \setminus \{0\}$ a function by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x > 0$ which is differentiable in the whole domain with zero derivative but is non-constant. The existence of such functions can be interpreted as an indication of a 0-dimensional hole in the domain at $x = 0$. Similarly, in a cylinder which has a 1-dimensional hole we can define a curl-free vector field which has not a scalar potential, and the same happens for divergence-free vector fields and 3-dimensional holes. This fact seems to hide a deep connection between the two homologies that appear to be two sides of the same coin encoding the same information. The next section will introduce the necessary tools to present the result that makes this relation explicit, de Rham's theorem.

Before, to end this section, we present a variant of de Rham cohomology that will by pivotal for the discussion carried out in the last section. Recall that given some module V, a topological space X and a map $f : X \longrightarrow V$, the *support of* f is defined as $\text{supp}(f) = \overline{\{p \in X : f(p) \neq 0\}}$. Then, in particular, we can consider the subsets of differential k-forms $\Omega^k_c(M) = \{\omega \in \Omega^k(M) : \text{supp}(\omega) \text{ is compact}\}$. Note that for any $\lambda \in \mathbb{R} \setminus \{0\}$ and for any $\omega \in \Omega^k_c(M)$ $\text{supp}(\lambda\omega) = \text{supp}(\omega)$ and therefore is compact. On the other hand for any $\omega, \eta \in \Omega^k(M)$

$$\text{supp}(\omega + \eta) \subseteq \overline{\{p \in M : \omega_p \neq 0\} \cup \{p \in M : \eta_p \neq 0\}} = \text{supp}(\omega) \cup \text{supp}(\eta)$$

which is compact because is a finite union of compact sets. Consequently, $\text{supp}(\omega + \eta)$ is a closed subset of a compact set and thus compact. These two observations together with the fact $0 \in \Omega^k(M)$ is trivially compactly supported, show that $\Omega_c^k(M)$ are real vector subspaces of $\Omega^k(M)$.

Furthermore, if a k-form ω has compact support then $d\omega$ is also compactly supported, i.e., d restricts to $\Omega_c^k(M)$. Therefore, the following cochain complex is well-defined.

Definition 2.23. Let M be a smooth manifold, the *compactly supported de Rham complex* is the cochain complex given by $(\Omega_c^\bullet(M), d^\bullet)$. The corresponding cohomology is called *de Rham cohomology with compact support* and is denoted by $H_{c,dR}^k(M)$.

Remark 2.24. If M is compact then for any $\omega \in \Omega^k(M)$ $\operatorname{supp}(\omega)$ is a closed set of a compact set and therefore compact. Consequently, if M is compact $\Omega^k(M) = \Omega_c^k(M)$ and $H_{dR}^k(M) = H_{c,dR}^k(M)$.

The exterior product also endows $H_{c,dR}(M)$ with $\mathbb{R}$-algebra structure. However, the compactly supported de Rham cohomology exhibits, in general, a different behaviour compared to the standard de Rham cohomology. In particular, regarding the cohomology of degree $k = 0$, we have the following result.

Proposition 2.25. Let M be a smooth manifold with m compact connected components. Then $H_{c,dR}^0(M) \cong \mathbb{R}^m$. In particular, if M is non-compact and connected then $H_{c,dR}^0(M) \cong \{0\}$.

Proof. By definition,

$$H_{c,dR}^0(M) = Z_c^0(M) = \{f \in \Omega^0(M) : df = 0 \text{ and } \operatorname{supp}(f) \text{ is compact}\}$$

Then, if $f \in H_{c,dR}^0(M)$, $df = 0$ implies that f is locally constant, and the smoothness implies that it has to be constant on each connected component. In addition, since f has compact support, it has to be zero on all non-compact connected components. Consequently, the statemet follows from the fact that every element of $H_{c,dR}^0(M)$ is determined by the value of the function on each connected compact component, i.e., a m-tuple of real numbers where m is the number of connected compact components. □

From this result follows directly that the compactly supported de Rham cohomology is not homotopy invariant. As a counterexample, $\mathbb{R}^n$ is homotopy equivalent to a point $\{p\}$, but $H_{c,dR}^0(\mathbb{R}^n) \cong \{0\}$ for being non-compact and connected whereas $H_{c,dR}^0(\{p\}) \cong \mathbb{R}$ for being compact and connected. In contrast, for the standard de Rham cohomology, the homotopy invariance will be given directly by the homotopy invariance of the singular cohomology thanks to de Rham's theorem. Nevertheless, there is a closed relation between the standard de Rham cohomology and its compactly supported counterpart in oriented manifolds, Poincaré duality, that will also be presented in the next section.

2.4 Integration of forms and de Rham's theorem

De Rham's theorem is perhaps the most remarkable result regarding de Rham cohomology. As we introduced, it beautifully connects the worlds of differential geometry and topology. To get an overall idea of how it operates, we need to understand the integration of forms and its main result, *Stokes' Theorem.* However, prior to that, we have to introduce the concept of *oriented manifold* which is essential provide a consistent notion of integration.

In $\mathbb{R}^3$, the orientation of a curve (as a 1-submanifold) is a formalization of the notion of a direction on it. It can be defined as a smooth unitary tangent vector field on the curve, or as an equivalence class on the smooth tangent vector fields on the curve by the relation $V_\gamma \sim U_\gamma \iff$ for all $t \in I$ $\exists \lambda_t > 0$ such that $V_\gamma(t) = \lambda_t U_\gamma(t)$, for which unit vector fields can be chose as representative. On the other hand, regrading surfaces of $\mathbb{R}^3$ as 2-submanifolds, orientation formalizes the notion of consistently identifying an inner and an outer side by defining a smooth unitary normal vector field (or an equivalence class on the set of normal vector fields). Equivalently, orientation of surfaces can be thought as a consistent choice of a positive or clockwise direction of the loops on the surface. The general formulation, which encompasses these two cases and can be naturally generalized to n-dimensional manifolds, is to define orientation through the concept vector space orientation, which is a equivalence class of basis by the equivalence relation defined by

$$\{e_i\}_{i\in I} \sim \{v_i\}_{i\in I} \iff |Q| > 0$$

where $|Q|$ is the determinant of the matrix of basis change $e_j = Q^i_j v_i$. Then, on curves, a consistent choice of orientation on its tangent vector spaces uniquely determines a orientation on the curve and vice versa. In the same way, on surfaces, an orientation in its tangent spaces uniquely determines a vector product on it, so a consistent choice of orientations uniquely determines a orientation on the surface. However, in general, such consistent choices may not exist. We define the concept of general manifold orientability as follows.

Definition 2.26. Two charts (U, φ), (V, ψ) of a manifold M such that $U \cap V \neq \emptyset$ are said to be *positively compatible* if their coordinates basis at T_pM are equivalent in the sense specified above for all $p \in U \cap V$, i.e, if $|J_p(\varphi \circ \psi^{-1})| > 0$ for all $p \in U \cap V \neq \emptyset$. A manifold M is said to be *orientable* if there exists an atlas formed by positively compatible charts.

Although all manifolds are locally orientable, i.e., each point has a neighbourhood than can covered by positively compatible charts (choosing only one chart for instance), not all of them are orientable, the paradigmatic counterexample being the Möbius strip (see Example 2.5.1 in [Kri99]).

Definition 2.27. Let M be an orientable manifold and $\mathcal{P}$ be the set of all atlases of the smooth structure of M formed by positive compatible charts. Two atlases of $\mathcal{P}$ are said to be positively compatible if all of their overlapping charts are positively compatible. This defines an equivalence relation, the classes determined by which are called *orientations of* M. An orientable manifold is said to be *oriented* if an orientation is fixed, or, equivalently, an oriented manifold is considered to be a pair $(M, \mathcal{O})$ where M is a an orientable manifold and $\mathcal{O}$ a orientation in M.

The orientation $\mathcal{O}$ will often be omitted to simplify the notation if it does not need to be specified. Orientability can be alternative characterize using differential n-forms.

Definition 2.28. A *volume form* ω on a n-dimensional manifold M is a nowhere vanishing n-form on M, i.e., a n-form ω such that for all vector fields $X_1, \dots, X_n \in \mathcal{X}(M)$ the functions $\omega(X_1, \dots, X_n)(p)$ are non-zero for all $p \in M$, or equivalently for all $p \in M$ $\omega_p(v_1, \dots, v_n) \in \mathbb{R}$ is non-zero for all $v_1, \dots, v_n \in T_pM$.

Given a volume form ω, the collection of charts that pullback ω to a positive multiple of the volume form $du^1 \wedge \cdots \wedge du^n$ of $\mathbb{R}^n$ form an atlas of positively compatible charts.

Conversely, an atlas formed by positively compatible charts allows to define a nowhere vanishing n-form. More precisely, we have the following result (see for instance [Kri99] Proposition 2.5.2).

Proposition 2.29. A manifold is orientable if and only if there exists a volume form.

Given a volume form ω on M a ordered basis of tangent vectors to M at p, $(v_1, \ldots, v_n)$ is said to be *positively (negatively) oriented* or *right (left)-handed* by ω if $\omega_p(v_1, \ldots, v_n) > 0$ (< 0). Then, orientations of M can be equivalently defined as the equivalence classes in the set of volume forms by the equivalence relation of defining the same set of positively oriented vectors.

Examples 2.30. Below are some examples regarding orientations.

1. The orientation defined by volume form $\omega = du^1 \wedge \cdots \wedge du^n$ in $\mathbb{R}^n$ is called the *standard orientation.*
2. A connected orientable manifold with volume form ω admits just two orientations, namely $\mathcal{O}_1 = \{f\omega : f \in \mathcal{F}(M) \text{ and } f(p) > 0 \, \forall p \in M\}$ and $\mathcal{O}_2 = -\mathcal{O}_1$.

With these considerations in mind, we can begin to introduce the notion of integration of forms.

Definition 2.31. Let ω be a n-form of $\mathbb{R}^n$ and $\omega_{1\cdots n}$ the unique function that defines the coordinate expression in the canonical basis $\omega = \omega_{1\cdots n} du^1 \wedge \cdots \wedge du^n$. Let $A \subseteq \mathbb{R}^n$ be a measurable set. The *the integral of ω on A* with the standard orientation is defined (if it exists) as

$$\int_A \omega = \int_A \omega_{1\cdots n} du^1 \wedge \cdots \wedge du^n := \int_A \omega_{1\cdots n} du^1 \cdots du^n$$

where the last expression denotes the usual integration on $\mathbb{R}^n$.

Note that fixing an orientation is required for the integral to be well-defined because forms are skew-symmetric but the integral is independent of the order of variables. Forms do not necessarily have to be smooth or to be defined on the entire manifold but just on a neighbourhood of the domain for the integration to be well-defined. Moreover, we just need to be able to define an orientation on the domain of integration, so we can weaken the previous and the following definitions. On the other hand having a compact support is a sufficient condition for a form in order to ensure that its integral exist. Although it is not a necessary condition it will sometimes be assumed.

Definition 2.32. Let M be an oriented manifold and $A \subseteq M$ such that there exists a chart (U, φ) positively compatible with the orientation of M with $A \subseteq U$ and $\varphi(A)$ measurable. Let ω be a n-form. The *integral of ω on A* is defined (if it exists) as the integral of the pullback $(\varphi^{-1})^*\omega \in \Omega^n(\mathbb{R}^n)$ on $\varphi(A)$, i.e.:

$$\int_A \omega := \int_{\varphi(A)} (\varphi^{-1})^*(\omega)$$

Remark 2.33. In terms of the coordinate expressions on the given chart (U, φ), if the coordinate of ω is $\omega_{1\cdots n} = \omega(\partial_1, \ldots, \partial_n)$ then the coordinate of its pullback in the canonical basis is

$$((\varphi^{-1})^*\omega)_{1\cdots n} = (\varphi^{-1})^*\omega(\tilde{\partial}_1, \ldots, \tilde{\partial}_n) = \omega(d\varphi^{-1}(\tilde{\partial}_1), \ldots, d\varphi^{-1}(\tilde{\partial}_n)) = \omega_{1\cdots n} \circ \varphi^{-1}$$

where $\tilde{\partial}_j = \partial/\partial u^j$ and we have used that by definition of coordinate tangent vectors

$$d_p\varphi^{-1}(\tilde{\partial}_j|_p)(f) = \tilde{\partial}_j|_p(f \circ \varphi^{-1}) = \partial_j|_{\varphi^{-1}(p)}(f)$$

Thus,

$$\int_A \omega_{1\cdots n} dx^1 \wedge \cdots \wedge dx^n = \int_{\varphi(A)} (\omega_{1\cdots n} \circ \varphi^{-1}) du^1 \cdots du^n$$

The definition is independent of the coordinates chosen for cover A as long as they are positively oriented, because for another chart (V, ψ) positively compatible with (U, φ) with coordinate system $\{y^1, \ldots, y^n\}$, using the tensor change of coordinates of Remark 1.68 if $\omega'_{1\cdots n}$ is the local coordinate of ω on (V, ψ), then

$$\omega_{1\cdots n} = \frac{\partial y^{l_1}}{\partial x^1} \cdots \frac{\partial y^{l_n}}{\partial x^n} \omega'_{l_1 \cdots l_n} = \sum_{l_1 < \cdots < l_n} \sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) \frac{\partial y^{l_{\sigma(1)}}}{\partial x^1} \cdots \frac{\partial y^{l_{\sigma(n)}}}{\partial x^n} \omega'_{l_1 \cdots l_n} =$$

$$= \sum_{\sigma \in \mathcal{S}_n} \varepsilon(\sigma) \frac{\partial y^{\sigma(1)}}{\partial x^1} \cdots \frac{\partial y^{\sigma(n)}}{\partial x^n} \omega'_{1\cdots n} = \det J(\psi \circ \varphi^{-1}) \omega'_{1\cdots n}$$

Consequently, since $|\det J(\psi \circ \varphi^{-1})| = \det J(\psi \circ \varphi^{-1})$, using the change of coordinates formula on $\mathbb{R}^n$, we obtain

$$\int_{\psi(A)} \omega'_{1\cdots n} \circ \psi^{-1} = \int_{\varphi(A)} \omega'_{1\cdots n} \circ \varphi^{-1} |\det J(\psi \circ \varphi^{-1})| = \int_{\varphi(A)} \omega_{1\cdots n} \circ \varphi^{-1}$$

For a chart which is not positively compatible the integral will change the sign.

Remark 2.34. We can define the integration of a k-forms ω in subsets $A \subseteq P$ where P is a k-submanifold of M. Given the inclusion $i : P \hookrightarrow M$ which is an embedding, we consider the k-form $i^*(\omega)$ on P (which is simply $\omega|_P$) and define the integral as

$$\int_A \omega := \int_A i^*(\omega)$$

where the last expressions refers to the integral with the smooth structure of P. In particular the integral of a function f as a 0-form over a point p is given by

$$\int_p f = f(p)$$

We can extend the definition of integral to smooth k-chains over $\mathbb{R}$ by imposing linearity and using the fact that smooth simplices can be extended to smooth maps so their pullback is well-defined (point by point, and it does not depend on the extensions chosen because the exterior derivative is local).

Definition 2.35. Let M be a manifold and $c = \sum_{i=1}^k \lambda_i \sigma_i \in S_{\infty,k}(M, \mathbb{R})$ a real smooth k-chain and ω a k-form. The *integral of* ω *over* c is defined by

$$\int_c \omega := \sum_{i=1}^k \lambda_i \int_{\Delta^k} \sigma_i^*(\omega)$$

Notice that there is no need in orienting the manifold for defining the integration along chains, because the pullback is directly defined by the simplices. Observe that, in terms of chains and forms, the fundamental theorem of calculus can be expressed as

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a) = \int_b f - \int_a f \Rightarrow \int_{[a,b]} df = \int_{\partial[a,b]} f$$

In manifolds, this result turns out to be just a particular case of a more general theorem (see [War83], Theorem 4.7).

Theorem 2.36. *(Stokes' theorem for smooth chains) Let M be a n-dimensional manifold, ω a $(k-1)$-form, and c a real smooth k-chain ($k > 0$). Then,*

$$\int_c d\omega = \int_{\partial c} \omega$$

Stokes' theorem not only generalizes the fundamental theorem of calculus to manifolds, but also, as we will justify, generalizes and simplifies the classic vector calculus theorems, such as Gauss's divergence theorem and Kelvin-Stokes theorem in $\mathbb{R}^3$, Green's theorem in $\mathbb{R}^2$, and the fundamental theorem of calculus on curves. It synthesizes the notion that integrating an object along the boundary of a domain is equal to integrating a certain differential operation acting on it over the whole domain. In other words, roughly speaking, it formalizes the intuitive idea of that the total change in the boundary equals to the total sum of infinitesimal changes within the domain.

Stokes' theorem plays also crucial role in de Rham's theorem. As we advanced, it states that de Rham cohomology and singular cohomology, although having been constructed in very different ways with objects of different nature that do not seem to be related, are the same entity describing the same idea.

Theorem 2.37. *(De Rham's theorem) Let M be a manifold. Then $H^k_{dR}(M) \cong H^k_\infty(M,\mathbb{R})$ with a natural isomorphism.*

The proof is non-trivial and we shall refer to [War83] for a detailed discussion. Bellow we offer just an sketch. The result is based on the fact that the linearity of the integration allows to define bilinear form for every $k \geq 0$

$$\begin{aligned} \langle \cdot , \cdot \rangle : \Omega^k(M) \times S_{\infty,k}(M,\mathbb{R}) &\longrightarrow \mathbb{R} \\ (\omega, c) &\longmapsto \langle \omega, c \rangle = \int_c \omega \end{aligned}$$

which induces $\mathbb{R}$-linear maps

$$\begin{aligned} \Lambda^k : \Omega^k(M) &\longrightarrow S^k_\infty(M,\mathbb{R}) \\ \omega &\longmapsto \Lambda^k(\omega) : S_{\infty,k}(M,\mathbb{R}) \longrightarrow \mathbb{R} \\ & \qquad c \longmapsto \Lambda^k(\omega)(c) = \langle \omega, c \rangle \end{aligned}$$

Stokes' theorem ensures that they are compatible with the differential morphisms in the sense that $\langle d\omega, c \rangle = \langle \omega, \partial c \rangle$. Therefore, the following diagram is commutative

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega^k(M) & \xrightarrow{d^k} & \Omega^{k+1}(M) & \longrightarrow & \cdots \\ & & \downarrow{\Lambda^k} & & \downarrow{\Lambda^{k+1}} & & \\ \cdots & \longrightarrow & S^k_\infty(M,\mathbb{R}) & \xrightarrow{\partial^k} & S^{k+1}_\infty(M,\mathbb{R}) & \longrightarrow & \cdots \end{array}$$

since for all $\omega \in \Omega^k(M)$, $c \in S^{k+1}_\infty(M,\mathbb{R})$

$$\begin{aligned} (\partial^k \circ \Lambda^k)(\omega)(c) &= \Lambda^k(\omega)(\partial c) = \langle \omega, \partial c \rangle \\ (\Lambda^{k+1} \circ d^k)(\omega)(c) &= \Lambda^{k+1}(d\omega)(c) = \langle d\omega, c \rangle \end{aligned}$$

Consequently they define a cochain map between $(\Omega^\bullet(M), d^\bullet)$ and $(S^\bullet_\infty(M,\mathbb{R}), \partial^\bullet)$ which by Remark 2.5 induces $\mathbb{R}$-linear maps on cohomology by

$$\begin{aligned} H^k(\Lambda) : H^k_{dR}(M) &\longrightarrow H^k_\infty(M,\mathbb{R}) \\ [\omega] &\longmapsto H^k(\Lambda)([\omega]) = [\Lambda^k(\omega)] \end{aligned}$$

which are indeed isomorphisms.

Remark 2.38. Note that since $H^k(M,\mathbb{R}) \cong H^k_\infty(M,\mathbb{R})$ de Rham's theorem gives indeed an isomorphism between de Rham cohomology spaces and the singular cohomology spaces. In fact the isomorphism is an isomorphism of $\mathbb{R}$-algebras between $H_{dR}(M)$ and $H(M,\mathbb{R})$. Having shown the isomorphism between cohomologies, from now on we will denote de Rham cohomology symply by $H^k(M)$ if no additional specification is needed.

To end this section, we consider a further extension of the integration that will lead to a version of Poincaré duality for de Rham cohomology. By using partitions of unity (see [War83] for a detailed discussion) we can define the integration of compactly supported forms over regular domains on oriented manifolds. A regular domain is a subset $D \subseteq M$ such that for every $p \in M$ one of the following holds:

(i) There is an open neighbourhood of p contained in $M \setminus D$.

(ii) There is an open neighbourhood of p contained in D.

(iii) There is a chart (U, φ) with $p \in U$ such that $\varphi(U \cap D) = \varphi(U) \cap \mathbb{H}^n$ where $\mathbb{H}^n$ is the half-space of $\mathbb{R}^n$, $\mathbb{H}^n = \{(a^1, \dots, a^n) \in \mathbb{R}^n : a^n \geq 0\}$.

Points of type (iii) define the boundary of D, ∂D which is an oriented $(n-1)$-dimensional submanifold with orientation induced by the orientation on M. In particular, any manifold is itself a regular domain with no boundary. Stokes' theorem is then expressed as follows (see Theorem 4.9 in [War83]).

Theorem 2.39. *(Stokes' theorem on regular domains) Let D be a regular domain in an oriented n-dimensional manifold M, and let ω be a smooth $(n-1)$-form with compact support (on D). Then*

$$\int_D d\omega = \int_{\partial D} \omega$$

In addition, for any oriented n-dimensional smooth manifolds we can define the following bilinear form

$$\begin{aligned} (\,\cdot\,|\,\cdot\,) : \Omega^k(M) \times \Omega^{n-k}_c(M) &\longrightarrow \mathbb{R} \\ (\omega, \eta) &\longmapsto (\omega|\eta) = \int_M \omega \wedge \eta \end{aligned}$$

Notice that we can consider $(\omega|\eta)$ and for forms with not necessarily compact support as long as $\mathrm{supp}(\omega) \cap \mathrm{supp}(\eta)$ is compact so that the exterior product also has compact support and the integral on M is well-defined. In particular if $\mathrm{supp}(\eta)$ is compact, then $\mathrm{supp}(\omega) \cap \mathrm{supp}(\eta)$ is a closed subset of a compact set and therefore compact. Finally, as discussed in [BT82], Page 44, this bilinear form is non-degenerate and descends to a non-degenerate bilinear form in cohomology, which gives the following result.

Theorem 2.40. *(Poincaré duality for de Rham cohomology) Let M be a n-dimensional oriented smooth manifold. Then $H^k(M) \cong H_{c,n-k}(M)$ for all $k \leq n$.*

3 Cohomology of Lorentzian manifolds

In the previous sections, we have been working with smooth manifolds, defining the elementary concepts regarding them and introducing the most paradigmatic examples of cohomologies. Now it is time to add a new layer of structure, the Lorentzian metric. As we presented, the importance of Lorentzian manifolds, smooth manifolds endowed with a Lorentzian metric, lies in the fact that they are the bedrock of the modern relativistic model of spacetime. A primary understanding of them and their causal structure is required to define the sets that will lead to the cohomologies that we want to study. These sets, which we will refer to as *causally compact sets*, emerge from somehow being compatible with the *causal sets* relative to compact subsets of the spacetime. Then, the compactly supported de Rham cohomology is adapted to them giving rise to the *cohomologies with causally restricted support*.

We will start this section by offering a general introduction to semi-Riemannian manifolds that will supply us with the necessary tools to approach our study of spacetimes. Then, we will continue by describing the causal structure arose in Lorentzian manifolds from their metric, providing a parallelism with the intuitive Minkowski spacetime. We will also discuss and justify from a physical perspective the causality constrains that will be assumed, in particular the *gloablly hyperbolicity.* This part is mainly based on [O'N83], [Kri99] and also [Ben16].

Finally, following [War83], [Bär15] and [Kha16], we will briefly present some tools of the Hodge theory, namely the *Laplace-de Rham operator* and its *Green's operator*, before defining and characterizing the already mentioned cohomologies. More precisely, based on the work carried out in [Kha16], we will show isomorphisms between these causally restricted cohomologies and the standard de Rham cohomology and its counterpart with compact support in globally hyperbolic spacetimes, which are essential for simplify their computation.

3.1 Semi-Riemannian manifolds

Semi-Riemannian geometry generalizes the concept of metric or inner product from Euclidean vector spaces to manifolds, allowing us to introduce the metric notions derived from it. This is done through furnishing manifolds with an object known as a *metric tensor*, which induces a inner product on the tangent spaces. Then Lorentzian manifolds emerge as a particular case. The description of semi-Riemannian manifolds is often accompanied with a presentation of metric related objects, namely the connection and the curvature tensor, but we have not included it here in the sake of brevity and due to the lack of direct applications for our purposes.

First, let us recall the basic definitions of a metric on a real vector space. Let V be a finite dimensional real vector space.

Definition 3.1. A symmetric bilinear form on V $g \in \mathcal{T}^r_s(V)$ is said to be a *inner product on V* if it is non-degenerate, i.e., if $g(v, w) = 0$ for all $w \in V$ then $v = 0$.

Definition 3.2. An inner product g on V is *positive(negative)-definite* if $g(v, v) > 0$ $(g(v, v) < 0))$ for all $v \neq 0$.

The positive-definiteness is sometimes included in the definition of inner product. We will adopt the convention of naming positive-definite inner products as *scalar products.*

Definition 3.3. The *index* ν of an inner product g on V is the higher dimension of a subspace $F \subseteq V$ for which $g|_F$ is negative-definite.

Definition 3.4. Two vectors $v, w \in V$ are said to be *orthogonal* for an inner product g if $g(v, w) = 0$. A vector $v \in V$ is called a *unit vector* for g if $g(v, v) = \pm 1$. A basis $\{e_i\}_{i \in I}$ of V consisting of mutually orthogonal unit vectors, that is that $g(e_i, e_j) = 0$ if $i \neq j$ and $g(e_i, e_i) = \pm 1$ for all $i, j \in I$, is called a n *orthonormal basis* for g.

Some well-known results of linear geometry, based on the generalization of the Gram-Schmidt process are summarized on the following theorem.

Theorem 3.5. *Let g be a inner product on a n-dimensional vector space V. There exists an orthonormal basis $\{e_1, \ldots, e_n\}$ for g. The number of basis vectors e_i for which $g(e_i, e_i) = -1$ is the same for any such basis and equals the index of g.*

Remark 3.6. Notice that an orthonormal basis diagonalizes the matrix defined by the components of g, as $g_{ij} = g(e_i, e_j) = \delta_{ij}\varepsilon_j$, where $\varepsilon_j = g(e_j, e_j) = \pm 1$. We will consider all orthonormal basis ordered so that the n-tuple $(\varepsilon_1, \ldots, \epsilon_n)$, called *signature* of g satisfy that $\varepsilon_j = -1$ if $j \leq \nu$ and $\varepsilon_j = 1$ if $j \geq \nu$, which is often denoted by $(\nu, n - \nu)$. If $v = v^i e_i$ and $w = w^i e_i$ then

$$g(v, w) = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j w^j$$

The inner product notion is introduced on smooth manifolds in the following way:

Definition 3.7. A *metric tensor* g or simply a *metric* on a manifold M is a symmetric non-degenerate $(0, 2)$-tensor field on M of constant index ν.

The non-degeneracy and the constant index can be understood in terms of regarding $g \in \mathcal{T}_2^0(M)$ as smoothly assigning to each point $p \in M$ an inner product g_p on T_pM, with the index of g_p being the same for all p.

Definition 3.8. A *pseudo-Riemannian* or *semi-Riemannian manifold* is a pair (M, g) where M is a smooth manifold and g a metric tensor on M. We say that (M, g) is a

1. *Riemannian manifold* if $\nu = 0$. g is called a *Riemannian metric*
2. *Lorentzian manifold* if $dim\, M \geq 2$ and $\nu = 1$. g is called a *Lorentzian metric*

We will usually denote semi-Riemannian manifolds simply by M if the metric does not need to be specified.

Remark 3.9. Any smooth manifold admits a Riemannian metric but not all manifolds admit a non-Riemannian metric (see [Kri99], Theorem 4.1.1 and Corollary 4.1.3). However, under certain conditions, the existence of Lorentzian metrics, is guaranteed (see for instance [Kri99], Theorem 4.1.2). A Riemanniam metric defines on each tangent space a scalar product, allowing the definition of a distance induced by the metric.

Remark 3.10. Given a local chart (U, φ) of M a metric tensor can be locally expressed on U as

$$g = g_{ij} dx^i \otimes dx^j$$

with $g_{ij} = g(\partial_i, \partial_j)$ the local components relative to the chart. The symmetry implies that $g_{ij} = g_{ji}$, and the non-degeneracy implies that $(g_{ij})_{i,j}$ is a regular matrix. If X and Y are vector fields such that locally $X = X^i \partial_i$ and $Y = Y^j \partial_j$ then

$$g(X, Y) = g_{ij} X^i Y^j$$

Remark 3.11. Given a semi-Riemannian manifold M, $g : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{F}(M)$ defines a natural $\mathcal{F}(M)$-linear isomorphism (see Section 3 of [O'N83], Proposition 10)

$$\begin{aligned} \flat : \mathcal{X}(M) &\longrightarrow \mathcal{X}^*(M) \\ X &\longmapsto X^\flat : \mathcal{X}(M) \longrightarrow \mathcal{F}(M) \\ & \qquad\qquad Y \longmapsto X^\flat(Y) = g(X, Y) \end{aligned}$$

with inverse map is denoted by

$$\begin{aligned} \sharp : \mathcal{X}^*(M) &\longrightarrow \mathcal{X}(M) \\ \omega &\longmapsto \omega^\sharp \end{aligned}$$

which are called *musical isomorphism.* The notation is motivated by the fact that, if in a local coordinate system $X = X^i \partial_i$ then $X^\flat = X_i dx^i$ with $X_i = g_{ij} X^j$ so given $Y = Y^k \partial_k$

$$X^\flat(Y) = g_{ij} X^j dx^i (Y^k \partial_k) = g_{ij} X^j Y^i = g(X, Y)$$

Therefore, the association is said to be *lowering-raising the index by the metric.* The situation is analogous for the real tangent and cotangent spaces at a point p with g_p, for which the ismorphism follows immediately from the non-degeneracy of g_p.

Examples 3.12. Here are some examples of semi-Riemannian manifolds:

1. $\mathbb{R}^n$ with the metric defined for each $p \in \mathbb{R}^n$ by

   $$g_p(v^i \partial_i|_p, w^i \partial_i|_p) = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j w^j$$

   where $\partial_i|_p$ are the canonical coordinate vectors, is a semi-Riemannian manifold denoted by $\mathbb{R}^n_\nu$. This metric can be understood as the metric induced by a inner product h in $\mathbb{R}^n$ defined for any $a = (a^1, \ldots, a^n)$, $b = (b^1, \ldots, b^n) \in \mathbb{R}^n$ by

   $$h(a, b) = -\sum_{i=1}^{\nu} a^i b^i + \sum_{j=\nu+1}^{n} a^j b^j$$

   through the isomorphism given by Remark 1.27, i.e, $g_p(v_p, w_p) = h(v, w)$. The Lorentzian manifold $\mathbb{R}^n_1$ is called the *standard Minkowski n-space*, and the Riemannian manifold $\mathbb{R}^n_0$ is called the *Euclidean n-space* and is denoted simply by $\mathbb{R}^n$. The metric in $\mathbb{R}^n_\nu$ can be expressed globally in terms of the canonical coordinates, using that $g_{ij} = \delta_{ij} \varepsilon_i$, as $g = \varepsilon_i du^i \otimes du^i$ where $(\varepsilon_1, \ldots, \varepsilon_n) = (-1, \overset{(\nu)}{\ldots}, -1, 1 \ldots, 1)$ is the signature of g.

2. More in general, any real vector space E endowed with a inner product h is itself a semi-Riemannian manifold with the metric tensor g defined for each point $p \in M$ by $g_p(v_p, w_p) = h(v, w)$, using the isomorphism of Remark 1.27.

3. If (M, g) and (N, h) are semi-Riemannian manifolds and π and ρ are the projections of $M \times N$ onto M and N respectively, then the tensor field given by the pullbacks $\pi^*(g) + \rho^*(h)$ is a metric tensor on $M \times N$. In fact, $\mathbb{R}^n_\nu$ can be regarded in these terms.

In the same way that with smooth manifolds, under certain circumstances, a subset of a semi-Riemannian manifold inherits its metric structure.

Definition 3.13. Let P be a smooth submanifold of a semi-Riemannian manifold M of metric g, and $j : P \hookrightarrow M$ the inclusion map. If the pullback $j^*(g)$ is a metric tensor on P, we say that $(P, j^*(g))$ is a *semi-Riemannian submanifold* of (M, g).

Remark 3.14. If (M, g) is a Riemannian manifold, any smooth submanifold P of M is itself Riemannian with the restriction $g|_P$. However, in general, this is not true for non-Riemannian metric, because the pullback of the metric tensor may be degenerate or without constant index. Nevertheless if (M, g) is a semi-Riemannian manifold with index ν and $U \subseteq M$ is open, then $(U, g|_U)$ is a semi-Riemannian submanifold with index ν.

Similarly to diffeomorphisms in smooth manifolds, the notion of equivalence for semi-Riemannian manifolds is expressed as follows.

Definition 3.15. Let (M, g) and (N, h) be two semi-Riemannian manifolds. A map $F : N \longrightarrow M$ is an *isometry* if it is a diffeomorphism and it preserves the metric, i.e., $F^*(h) = g$. Equivalently an isometry F is a diffeomorphism for which for all $p \in M$ the differential map d_pF is a linear isometry, that is to say an isomporhism satisfying $g_p(v, w) = h_p(d_pF(v), d_pF(w))$. If there exists a isometry between M and N we say that they are isometric and we denoted by $M \cong N$.

Example 3.16. Every linear isometry between real vector spaces with inner products is an isometry between them as a semi-Riemannian manifolds with the induced metric tensor. In particular, any vector space E with a inner product h of index ν is isometric to $\mathbb{R}^n_\nu$. Consequently every tangent space, which is itself a semi-Riemannian manifold, is isometric to $\mathbb{R}^n_\nu$.

3.2 Causal structure of spacetimes

As we introduced, the postulates of the special theory of relativity led to the modeling of the physical (flat) spacetime as a *Minkowski spacetime*, a pair (V, η) where V is a 4-dimensional real vector space and η a inner product of index 1. Strictly, the affine space structure is more adequate to describe the concept of spacetime, with a set of points that we call *events* and a vector space. However, the bijection between the set of points and the underlying vector space imposed by the axioms of affine spaces allows us to identify them, so, for simplicity, we will abusively consider just the vector space.

A choice of an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on (V, η) in which the inner product has components $\eta(e_i, e_j) = \eta_{ij} = \varepsilon_i \delta_{ij}$ with $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-1, 1, 1, 1)$, allows to identify the spacetime with $(\mathbb{R}^4, \tilde{\eta})$ through the linear isometry given by the coordinates, where $\tilde{\eta}$ is the inner product on $\mathbb{R}^4$ with components in the canonical basis $\tilde{\eta}_{ij} = \varepsilon_i \delta_{ij}$. This formalizes the notion in physics of an inertial frame of reference centered at the origin, although, to be precise, a further condition regarding the time orientation needs to be imposed to obtain physically acceptable frames of references. Then, relative to a fixed frame of reference, physical events are characterized by a quadruple of coordinates $(x^1, x^2, x^3, x^4) \in \mathbb{R}^4$, where $x^1 = ct$ represents the time coordinate, (x^1, x^2, x^3) the spatial Cartesian coordinates, and c the speed of light, introduced to homogenize the physical dimensions. The *worldline* of a physical point-like particle is defined as the set of events associated to it, and is described by a smooth curve $\gamma : I \longrightarrow V$, or more precisely its image $\gamma(I)$.

The well-known fact implied by the special relativity postulates that no information, energy or matter can be transferred faster than c, conditions the admissible particle worldlines and the causal relations in the spacetime. First, note that given a vector v with coordinates relative to the frame of reference (ct, x, y, z) then

$$\eta(v, v) = -c^2t^2 + x^2 + y^2 + z^2$$

Consequently,

$$\eta(v, v) = 0 \iff c|t| = \sqrt{x^2 + y^2 + z^2}$$

which means that v represents an event such that the spatial distance from the origin equals the distance travelled by the light in the time distance to the origin. Therefore vectors with $\eta(v, v) = 0$, called *lightlike* (if non-zero), represent events associated to light rays emitted from the origin. Then, the set set of events experienced by all possible such light rays $\{v \in V : \eta(v, v) = 0\}$ can be written in coordinates as

$$\{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : -x^1 + x^2 + x^3 + x^4 = 0\}$$

which is a cone in $\mathbb{R}^4$ known as the *light cone* at the origin. The speed limit of c implies that an event can be causally related to the origin if and only if the spatial separation to it is smaller or equal than the distance that light travels in the temporal distance to it, i.e., if it lies inside the light cone, $\eta(v, v) \leq 0 \iff c|t| \leq \sqrt{x^2 + y^2 + z^2}$. Such vectors are said to be *causal* (if they are non-zero). Furthermore, the speed of a mass particle must be strictly smaller than c, which imposes that, parametrizing its worldline by $\gamma(t) = (ct, x(t), y(t), z(t))$ then $\gamma'(t) = (c, x'(t), y'(t), z'(t))$ must satisfy $\eta(\gamma'(t), \gamma'(t)) < 0$ for all t to ensure that

$$c > \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = v(t)$$

Note that given a reparametrization $\lambda(t)$, since $(\gamma(\lambda(t)))' = \gamma'(\lambda)\lambda'(t)$, we have that $\eta(\gamma'(\lambda), \gamma'(\lambda)) = \lambda'(t)^2\eta(\gamma'(t), \gamma'(t))$ so the condition $v < c$ for a mass particle is equivalent to having a worldline described by a curve with velocity vector satisfying $\eta(\gamma', \gamma') < 0$ regardless of the parametrization. This implies that, the events that a particle may experience (or may have experienced) must lie in the interior of the region limited by its light cone. A vector satisfying $\eta(v, v) < 0$ is said to be *timelike* whereas if it satisfies $\eta(v, v) > 0$ we say it is *spacelike*.

In order to account for the fact that our experience of time distinguishes between past and future and the events that causally affect and are causally affected by another event, we have to introduce an arrow of time by fixing the so-called *time-orientation*. It is defined as class in the set of timelike vectors by the relation

$$v \sim u \iff \eta(v, u) < 0$$

which is an equivalence relation with only two classes $[v]$ and $[-v]$, for a timelike vector v. The vectors of the time orientation chosen are said to point to the future whereas the vectors of the remaining class are said to point to the past, which allows us to distinguish between the future light cone and the past light cone.

Since any vector space with a inner product is itself a semi-Riemannian manifold, the (flat) spacetime can also be formalized as a Lorentzian manifold (M, η) isometric to the standard Minkowski 4-space $\mathbb{R}^4_1$, with the events regarded as points $p \in M$ and the wordlines described by curves $\gamma : I \longrightarrow M$. This approach seems to unnecessary increase the conceptual complexity as introduces different tangent vector spaces instead of a unique vector space (although they can be naturally identified). Nevertheless, as we already motivated, it has proven to be the more adequate to be extended to the general theory of relativity, in which spacetimes are modeled by a wider class of Lorentzian manifolds (M, g). The causality structure in these general spacetimes emerges from the local causality induced by the tangent vector spaces (T_pM, g_p), which are themselves vector spaces with inner product of index 1, as the Minkowski spacetime. More precisely, we have the following definitions.

Definition 3.17. Let (M, g) be a Lorentzian manifold. Given a $p \in M$, a tangent vector $v \in T_pM$ is said to be

1. *Timelike* if $g_p(v, v) < 0$
2. *Spacelike* if $g_p(v, v) > 0$
3. *Lightlike* or *null* if $g_p(v, v) = 0$ and $v \neq 0$

A tangent vector $v \in T_pM$ is said to be *causal* if it is a non-zero, non-spacelike vector.

Similarly, a vector field $X \in \mathcal{X}(M)$ is said to be *timelike* (respectively, *spacelike*, *lightlike* or *null*, *causal*) if for all $p \in M$ X_p is timelike (respectively, spacelike, lighlike or null, or causal). A curve $\gamma : I \longrightarrow M$ is called *timelike* (respectively, *spacelike*, *lightlike* or *null*, *causal*) if its velocity vector field γ' is timelike (respectively, spacelike, lightlike or null, or casual). The class into which a vector, a vector field or a curve falls (if it does) is called its *causal character*.

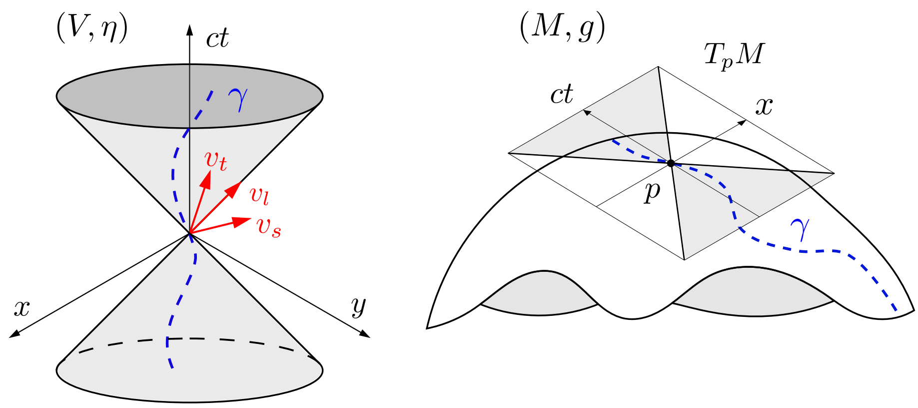


Figure 2: Left: Minkowski (3-dimensional) spacetime. The lightcone, a timelike curve (γ) and a timelike (v_t), a lightlike (v_l) and a spacelike (v_s) vectors are represented. Right: Lorentzian (2-dimensional) manifold. A tangent space with its corresponding light cone and a timelike curve are represented.

Definition 3.18. Let (M, g) be a Lorentzian manifold. A smooth submanifold $S \subseteq M$ is said to be

1. *Timelike* if $g_p|_{T_pS}$ is non-degenerate of index 1 for all $p \in S$.
2. *Spacelike* if $g_p|_{T_pS}$ is non-degenerate positive definite for all $p \in S$.
3. *Lightlike* if $g|_p|_{T_pS}$ is degenerate in for all $P \in S$.

Remark 3.19. It follows immediately that a submanifold is timelike if all of its tangent spaces contains a timelike vector, is spacelike if all non-zero tangent vectors are spacelike, and is lighlike if it contains a null vector but not a timelike vector (see Section 5 of [O'N83], Lemma 28,29). Note that an arbitrary smooth submanifold may not have a definite causal character, and semi-Riemannian submanifolds of a Lorentzian manifold are either timelike or spacelike

Definition 3.20. A Lorentzian manifold (M, g) is said to be *time-orientable* if there exists a timelike vector field on M. A *time-orientation* is an equivalence class in the set of timelike vector fields by the relation

$$X \sim Y \iff g(X, Y) < 0$$

A *time-oriented* manifold is a pair $(M, \mathcal{T})$ where M is a Lorentzian manifold and $\mathcal{T}$ a time-orientation on M. We will denote it simply as M if the time-orientation does not need to be specified.

Definition 3.21. Let $(M, \mathcal{T})$ be a time-oriented manifold and $X \in \mathcal{T}$. Given $p \in M$ a causal vector $v \in T_pM$ is called *future-directed* if $g_p(v, X_p) < 0$ and is called *past-directed* if $g_p(v, X_p) > 0$. A causal vector field Y is said to be *future-directed* (*past-directed*) if Y_p is future-directed (past-directed) for all $p \in M$. A causal curve $\gamma : I \longrightarrow M$ is called *future-directed* (*past-directed*) if its velocity vector field γ' is future-directed (past-directed).

From a physics point of view, the restrictions regarding the speed limit for the transfer of information, energy and matter in general spacetimes are also inherit locally from the tangent vector spaces and extended globally. Thus, analogously to the Minkowski spacetime, the physical acceptable possible worldines on a general spacetime for mass particle are described by timelike curves. Two events may be chronologically related (as part of a particle worldline) if there exists a timelike curve connecting them and may be causally related if there exists a causal curve connecting them. Taking into account the time-orientation, this motivates the following definitions.

Definition 3.22. Let M be a time-oriented manifold and $p, q \in M$. We say that p *chronological precedes* q and denoted by $p \ll q$ if there is a future-directed timelike curve from p to q. We say that p *causally precedes* q and denoted by $p < q$ if there is a future-directed causal curve form p to q. Given a subset $U \subset M$ and $p, q \in U$ we say that $p \ll q$ ($p < q$) in U if there exists a future-directed timelike (causal) curve in U from p to q.

Definition 3.23. Let $(M, \mathcal{T})$ be a time oriented manifold and $A \subseteq U \subseteq M$. The *chronological future of A relative to U* is

$$I^+(A, U) = \{q \in U : \exists p \in A \text{ such that } p \ll q \text{ in } U\}$$

and the *causal future of A relative to U* is

$$J^+(A, U) = \{q \in U : \exists p \in A \text{ such that } p < q \text{ in } U\}$$

If $U = M$ we will simply denote $I^+(A)$ and $J^+(A)$.

Remark 3.24. Note that $p \ll q \implies p < q$ so $I^+(A) \subset J^+(A)$. All these concepts have homologous definitions replacing *future* by *past*, reversing p and q in the inequalities and replacing $+$ by $-$. In particular we can define the *chronological past* and the *causal past*, $I^-(A)$ and $J^-(A)$, of a subset A. We can also define the *chronological set of A*, $I(A) = I^-(A) \cup I^+(A)$ and the *causal set of A*, $J(A) = J^-(A) \cup J^+(A)$. This sets are the generalization of the interior and adherence of the region limited by light cones in the Minkowki spacetimes, and, locally, they have a similar behaviour.

Therefore, as we advanced, time-oriented Lorentzian manifolds are endowed with the rudiments of the causal structure we expect from a model of spacetime, allowing the formal definitions of future, past and the permitted causal relations. However, when considering models of our physical spacetime, we shall disregard not connected manifolds, as it would be (in principle) impossible for us to notice the existence of a hypothetical disconnected component. On paper, we could consider time-oriented non-orientable manifolds as time-orinetability, which depends on the metric, is not related with orientability (there exist orientable and non-orientable manifolds with both time-orientable and non-time-orientable metric). Nonetheless, it does not seem too bold to constrain ourselves to orientable manifolds, as we have no evidence of non-orinetability in our spacetime. This, moreover, will allow us to overcome major technical obstacles in the study of cohomologies regarding the integration of forms and Poincaré duality. These considerations lead to our following definition of spacetime.

Definition 3.25. A *spacetime* is a connected oriented time-oriented Lorentzian manifold.

From a physical point of view, two isometric spacetimes are equivalent, so one can define a physical spacetime as an equivalence class of spacetimes by the relation of being isometric. Nevertheless, the object we have defined is still too general and allow certain causal pathologies that are usually seen as not physically acceptable because they lead to paradoxes. Thus, some additional restrictions are often made in discussions with certain physical perspective. In addition, it goes without saying that, a part from the imposition of further conditions regarding the causal structure that we are going to briefly present, General Relativity confines physical spacetimes to 4-dimensional Lorentzian manifolds. However, we will proceed with a general treatment for an arbitrary dimension n.

Definition 3.26. A spacetime M is said to be *chronological* (*causal*) if there does not exist a closed timelike (causal) curve on M.

The chronological condition prevents the possibility of time-traveling back to a past event of your worldline whereas the causal condition further imposes the impossibility of communicating with a past event of your worldline (without violating the speed of light limit). It is straightforward to check that the chronological condition already rules out compact spacetimes (see Section 14 of [O'N83] Lemma 10). However, timelike or causal curves that return arbitrary close to their origin are still allowed. With the next condition this second type of problematic curves are forbidden.

Definition 3.27. A spacetime M is said to be *strongly causal* if for every $p \in M$ and for every neighbourhood $U \subseteq M$ of p there exists a neighbourhood $V \subseteq U$ of p such that V is crossed at most once by any causal curve on M, i.e., $\gamma^{-1}(V)$ is connected for all causal curve γ.

Note that a strongly causal spacetime is causal and therefore chronological. The strongest causality condition that is often assumed, further imposes that the set of causal curves connecting two given points is compact.

Definition 3.28. A spacetime M is called *diamond-compact* if for all $p, q \in M$ the set $J^+(p) \cap J^-(q)$ is compact. A spacetime M is said to be *globally hyperbolic* if it is strongly causal and *diamond-compact.*

Remark 3.29. The definition of globally hyperbolic spacetime can be weaken to a causal spacetime diamond compact, because, as shown in [BS07] the strong causality is already guaranteed by these two properties.

The most relevant properties of globally hyperbolic spacetimes that justify its assumption when considering physical spacetimes, are regarding the following special type of hypersurfaces.

Definition 3.30. A *inextensible curve* on a smooth manifold M is a curve $\gamma : I \longrightarrow M$ such that for every curve $\xi : J \longrightarrow M$ satisfying that $I \subseteq J$ and $\xi|_I = \gamma$ then $I = J$ and $\xi = \gamma$. A *Cauchy hypersurface* in a spacetime M is a subset $\Sigma \subseteq M$ that is intersected exactly once by any inextensible timelike curve.

Cauchy hypersurfaces are indeed topological $(n-1)$-submanifolds of n-dimensional spacetimes met by any inextensible causal curve in M (see Section 14 of [O'N83], Lemma 29) and any two Cauchy hypersurfaces in M are homeomorphic (see Section 14 of [O'N83], Corollary 32), but, without further assumptions, they are not guaranteed to be smooth.

As it may be found in [Ger70], a spacetime M is globally hyperbolic if and only if it contains a Cauchy hypersurface. In addition, [Ger70] laid the foundations for proving the following characterization of globally hyperbolic spacetimes in terms of Cauchy hypersurfaces.

Theorem 3.31 ([BS03],[BS05][BS06])**.** *Let M be a globally hyperbolic spacetime. Then, it contains a (smooth) spacelike Cauchy hypersurface. Furthermore, for each spacelike Cauchy hypersurface Σ, M is isometric to $M_\Sigma := \mathbb{R} \times \Sigma$ with the natural smooth structure and endowed with the metric $-\beta dt \otimes dt + h_t$ where, $t : \mathbb{R} \times \Sigma \longrightarrow \mathbb{R}$ is the projection onto the first component, $\beta \in \mathcal{F}(\mathbb{R} \times \Sigma)$ is strictly positive and h_t is a Riemannian metric on $\{t\} \times \Sigma$ for each $t \in \mathbb{R}$, which is a spacelike Cauchy hypersurface in M_Σ for all t.*

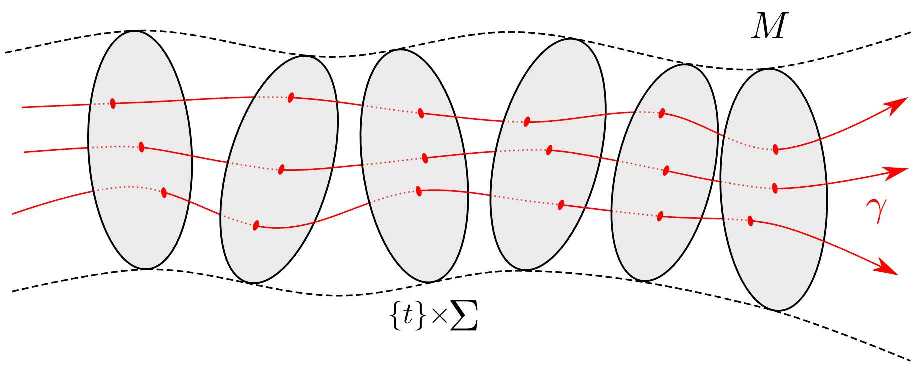


Figure 3: Schematic representation of a 3-dimensional globally hyperbolic spacetime. Cauchy hypersurfaces $\{t\} \times \Sigma$ and inextensible timelike curves γ are represented.

This last theorem allows to understand globally hyperbolic spacetimes in terms of a foliation along a time axis of spacelike hypersurfaces $\{t\} \times \Sigma$ that are interpreted as spatial hypersurfaces of simultaneity for each time t (in which we have a Riemannian metric and hence a distance) and are intersected by each particle wordline exactly once, preventing any kind of time travel or communicating back with the past. Thus, we should naturally expect the globally hyperbolicity condition to be present in any physically reasonable spacetime, as it is the case, for instance, of the Minkowski spacetime and the FLRW universes. Consequently, the characterization of cohomologies that we will present, which is constrained to globally hyperbolic spacetimes, is completely natural from a physical perspective.

Now, we can finally introduce the *causally compact sets*, from which we will define the cohomologies in question.

Definition 3.32. Let M be a spacetime. A closed set $S \subseteq M$ is said to be:

1. *Advanced* (respectively *retarded*) if there is a compact $K \subseteq M$ such that $S \subseteq J^+(K)$ (respectively $S \subseteq J^-(K)$).
2. *Spacelike compact* if there is a compact $K \subseteq M$ such that $S \subseteq J(K)$.
3. *Future compact* (respectively *past compact*) if $S \cap J^+(K)$ (respectively $S \cap J^-(K)$) is compact for all $K \subseteq M$ compact.
4. *Timelike compact* if S is both future compact and past compact.

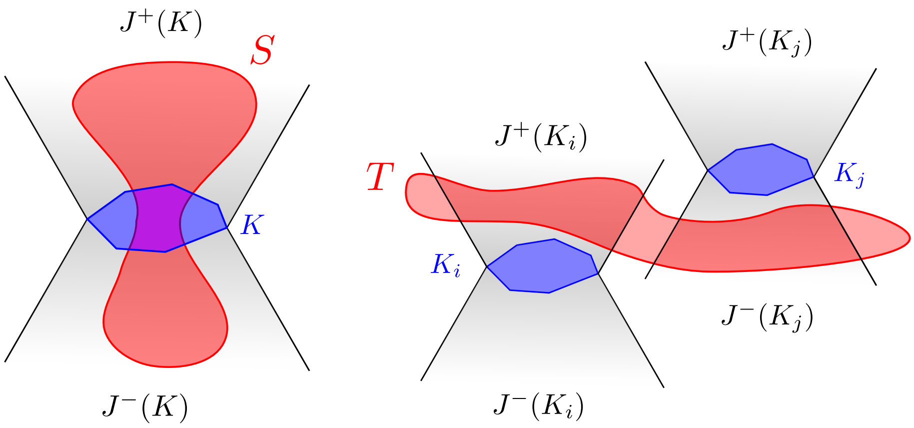


Figure 4: Representation of spacelike compact set S and a timelike compact set T.

Remark 3.33. If S is a compact set, then it follows from the definition of J that S is advanced, retarded and spacelike compact. Moreover, in a globally hyperbolic spacetime M, if a closed set S is advanced then it is also past compact. The reason is that for any compact set $K' \subseteq M$, $S \cap J^-(K') \subseteq J^+(K) \cap J^-(K')$ which is compact as implied by the definition of globally hyperbolicity. The same argument holds for retarded sets in relation to future compact sets. The conversely, however is false (see Example 1.1 in [Bär15]). Thus, advanced and retarded sets are also called *strictly past compact* and *strictly future compact* respectively. As a result, in a globally hyperbolic spacetime, compact set is also future compact, past compact and timelike compact. This is sumarized in the following diagram of implications, where the vertical arrows hold in globally hyperbolic spacetimes.

$$\begin{array}{ccccc} & & \text{compact} & & \\ & \swarrow & & \searrow & \\ \text{advanced} & \Longrightarrow & \text{spacelike compact} & \Longleftarrow & \text{retarded} \\ \Downarrow & & & & \Downarrow \\ \text{past compact} & \Longleftarrow & \text{timelike compact} & \Longrightarrow & \text{future compact} \end{array}$$

The importance of these sets lies on the fact that, for certain differential equations which occur often in physics, a field on a spacetime M which satisfies them with compactly supported initial data will have spacelike compact support. Furthermore, the naturally smooth dual evaluations of spacelike compactly supported fields turn out to have timelike compact support. For a meticulous characterization of these sets in globally hyperbolic spacetimes we shall refer to Section 1 in [Bär15]. In it, an interesting duality is exhibited between them as well as results regarding its behaviour relative Cauchy hypersurfaces. The later are summarized in the following theorem.

Theorem 3.34. *Let M be a globally hyperbolic spacetime and $S \subseteq M$ a closed subset.*

1. *S is future compact (respectively past compact) if and only if there exists a Cauchy hypersurface $\Sigma \subseteq M$ such that $S \subseteq J^+(\Sigma)$ (respectively $S \subseteq J^-(\Sigma)$). Consequently, S is timelike compact if and only if there exist two Cauchy hypersurfaces $\Sigma_1, \Sigma_2 \subseteq M$ such that $S \subseteq J^+(\Sigma_1) \cap J^-(\Sigma_2)$.*

2. *S is advanced (respectively retarded) if and only if, for some compact subset K_Σ of some Cauchy hypersurface Σ, $S \subseteq J^+(K_\Sigma)$ (respectively $S \subseteq J^-(K_\Sigma)$). S is spacelike compact if and only if $S \subseteq J(K_\Sigma)$ for some compact subset K_Σ of any Cauchy hypersurface Σ. Then, for any Cauchy hypersurface Σ, the intersection $S \cap \Sigma$ is compact.*

3.3 Hodge theory for semi-Riemannian manifolds

Our last step before defining the causally restricted cohomologies, is to present another object needed not only for their definition but also for analyse them, the *Laplace-de Rham operator*. With this purpose, we first need to briefly introduce Hodge theory. Although we are just interested in the Lorentzian case, we will proceed for general semi-Riemannian manifolds.

Definition 3.35. Let (M, g) be a n-dimensional semi-Riemannian manifold. A *local frame field* on $U \subseteq M$ is a set $\{E_1, \dots, E_n\}$ of orthonormal vector fields on U, that is that for all $p \in U$ $\{E_1(p), \dots, E_n(p)\}$ is an orthonormal basis on T_pM. Its *local coframe field* is the set of dual one-forms $\{\omega^1, \dots, \omega^n\}$ defined by $\omega^i(E_j) = \delta^i_j$.

Given a $p \in M$, the existence of a local frame field on some neighbourhood of p is guaranteed by the existence of the so-called normal neighbourhoods on each point which allow to extend any orthonormal basis at a pont to a frame field through radial geodesics (see Section 3 of [O'N83], Corollary 46).

Definition 3.36. Let (M, g) be an oriented n-dimensional semi-Riemannian manifold. Given two local coframe fields $\{\omega^1, \dots, \omega^n\}$ and $\{\eta^1, \dots, \eta^n\}$, on U and V respectively, since the corresponding change of basis matrix Q will be orthogonal on each point, the same calculations of Remark 2.33 yield

$$\omega^1 \wedge \dots \wedge \omega^n = \frac{1}{\det(Q)} \eta^1 \wedge \dots \wedge \eta^n = \pm \eta^1 \wedge \dots \wedge \eta^n$$

on $U \cap V$. The choice of a local coframe field on each point compatible with the orientation implies that the corresponding n-forms will agree on overlaps, defining a unique global volume form ω_g called the *metric volume form*.

Remark 3.37. Let (U, φ) be a chart on M with coordinate system $\{x^1, \dots, x^n\}$ such that $\{E_1, \dots, E_n\}$ is a frame field on U with coframe field $\{\omega^1, \dots, \omega^n\}$ and $\omega_g = \omega^1 \wedge \dots \wedge \omega^n$ on U. Let Q be the change of basis matrix $E_i = Q^j_i \partial_j$. We have that

$$\omega_g = \omega^1 \wedge \dots \wedge \omega^n = \frac{1}{\det(Q)} dx^1 \wedge \dots \wedge dx^n$$

Now, since in the $\{E_1, \ldots, E_n\}$ frame $g(E_i, E_j) = \varepsilon_j \delta_{ij}$ we have that

$$\varepsilon_j \delta_{ij} = g(E_i, E_j) = Q_i^k Q_j^l g(\partial_k, \partial_l) = Q_i^k Q_j^l g_{kl} \Longrightarrow \pm 1 = \det(QgQ^t) = \det(Q)^2 \det(g)$$

so for a positively oriented coordinate system

$$\omega_g = \sqrt{|\mathrm{det} g|} dx^1 \wedge \cdots \wedge dx^n$$

Recall that, as discussed in Remark 2.17, given $p \in M$ and considering the spaces of k-forms ($k \leq n$) as tensors at p, their dimension is $\binom{n}{k}$. The equality $\binom{n}{k} = \binom{n}{n-k}$ implies that the space of k-forms at p and the space of $(n-k)$-forms at p have the same dimension. More in general, the following proposition ensures the existence of a canonical isomorphism between $\Omega^k(M)$ and $\Omega^{n-k}(M)$ in semi-Riemannian manifolds (see for instance [Kri99], Proposition 4.2.1).

Proposition 3.38. Let (M, g) be an oriented n-dimensional semi-Riemannian manifold with metric index ν and metric volume form ω_g. There exists a unique $\mathcal{F}(M)$-linear operator $\star : \Omega(M) \longrightarrow \Omega(M)$, called the *Hodge star operator* satisfying:

(i) For all $k \leq n$, if $\omega \in \Omega^k(M)$ then $\star\omega \in \Omega^{n-k}(M)$.

(ii) $\star 1 = \omega_g$ where 1 is the constant 1 function.

(iii) For all $k \leq n$, if $\omega \in \Omega^k(M)$, $\star \star \omega = (-1)^{k(n-k)+\nu}\omega$.

(iv) $\omega \wedge \star\omega = 0$ if and only if $\omega = 0$.

(v) For all ω, $\eta \in \Omega^k(M)$, $\omega \wedge \star\eta = \eta \wedge \star\omega$.

Remark 3.39. Property (ii) implies that, for all $k \leq n$, $(-1)^{k(n-k)+\nu} \star\star = id$, so, as we advanced, $\star : \Omega^k(M) \longrightarrow \Omega^{n-k}(M)$ is an $\mathcal{F}(M)$-isomorphism with inverse defined by $\star^{-1} = (-1)^{k(n-k)+\nu}\star$.

Remark 3.40. Given $P \subseteq M$ a semi-Riemannian k-submanifold and $i : P \hookrightarrow M$ its inclusion, we can consider the induced metric volume for $\omega_g|_P = \omega_{i^*(g)}$ and the corresponding Hodge star operator on P.

Examples 3.41. With these tools we can finally see, as we advanced in Section 2, how Poincaré lemma and Stokes' theorem generalize the corresponding classic statements of calculus in Euclidean spaces.

1. For $f \in \mathcal{F}(M)$ and $X, Y \in \mathcal{X}(M)$, with the isomorphism of Remark 3.10

$$\text{grad } f := (df)^\sharp \quad \text{div } X := \star d \star X^\flat$$

And on a 3-dimensional manifold

$$X \times Y := (\star(X^\flat \wedge Y^\flat))^\sharp \quad \text{curl } X := (\star dX^\flat)^\sharp$$

With these considerations, we have for example:

$$\text{curl } X = (\star dX^\flat)^\sharp = 0 \Longrightarrow dX^\flat = 0 \Longrightarrow \exists f \in \Omega^0(M) \text{ such that } X^\flat|_U = df$$
$$\Longrightarrow X|_U = \text{grad } f$$

2. We can integrate functions as 0-forms over a point, but we can also define the integral of a function $f \in \mathcal{F}(M)$ on a regular domain $D \subseteq M$ of an oriented semi-Riemannian n-dimensional manifold as

$$\int_D f := \int_D \star f = \int_D f\omega_g$$

which generalizes the integration of functions on surfaces of Euclidean spaces. Likewise, the integral of a vector field $X \in \mathcal{X}(M)$ on a regular domain S of a $(n-1)$-submanifold and on a regular domain l of a 1-submanifold, are respectively defined by

$$\int_S X := \int_S \star X^\flat \qquad \int_l X := \int_l X^\flat$$

With these considerations, under the Stokes' theorem hypothesis we have, for instance

$$\int_{\partial D} X = \int_{\partial D} \star X^\flat = \int_D d \star X^\flat = \int_D (-1)^{n(n-n)+\nu} \star (\star d \star X^\flat) = (-1)^\nu \int_D \star \text{div } X$$

which recovers the usual formula for the standard Riemannian metric of Euclidean spaces.

Definition 3.42. Let M be an oriented n-dimensional semi-Riemannian manifold. The *codifferential* is the map $\delta : \Omega(M) \longrightarrow \Omega(M)$ defined by the sequence of maps on each degree $\delta^k : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ given by $\delta^k = (-1)^k \star^{-1} d \star$.

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{(-1)^k\delta^k} & \Omega^{k-1}(M) \\ \downarrow \star & & \downarrow \star \\ \Omega^{n-k}(M) & \xrightarrow{d^{n-k}} & \Omega^{n-k+1}(M) \end{array}$$

Remark 3.43. For a semi-Riemannian manifold M we can define another bilinear form from $(\,\cdot\,|\,\cdot\,)$ as follows

$$\begin{aligned} \langle\,\cdot\,|\,\cdot\,\rangle : \Omega^k(M) \times \Omega^k_c(M) &\longrightarrow \mathbb{R} \\ (\omega, \eta) &\longmapsto \langle \omega|\eta\rangle = \int_M \omega \wedge \star\eta = (\omega| \star \eta) \end{aligned}$$

Following the same reasoning as for $(\,\cdot\,|\,\cdot\,)$, as long as $\text{supp}(\omega) \cap \text{supp}(\eta)$ is compact, we can always consider $\langle\omega|\eta\rangle$. Moreover, in that case, $\langle\,\cdot\,|\,\cdot\,\rangle$ is symmetric. Since M is a regular domain without boundary, Stokes' theorem implies that $\int_M d\omega = 0$ for all $\omega \in \Omega^{n-1}_c(M)$. Therefore, using that

$$d(\omega \wedge \star\eta) = d\omega \wedge \star\eta + (-1)^{k-1}\omega \wedge d \star \eta$$

we have that for all $\omega \in \Omega^k(M)$, $\eta \in \Omega^{k+1}(M)$ such that $\text{supp}(\omega) \cap \text{supp}(\eta)$ is compact,

$$\begin{aligned} \langle d\omega|\eta\rangle = \int_M d\omega \wedge \star\eta = \int_M d(\omega \wedge \star\eta) - (-1)^{k-1}\omega \wedge d \star \eta = (-1)^k \int_M \omega \wedge d \star \eta = \\ = (-1)^k \int_M \omega \wedge (\star \star^{-1} d \star \eta) = \int_M \omega \wedge \star\delta\eta = \langle\omega|\delta\eta\rangle \end{aligned}$$

which justifies the prefactor $(-1)^k$ in the definition of codifferential.

Definition 3.44. Let M be a semi-Riemannian manifold, the *Laplace-de Rham operator* is the map $\Delta : \Omega(M) \longrightarrow \Omega(M)$ defined by $\Delta = d\delta + \delta d$.

Remark 3.45. If $\omega \in \Omega^k(M)$ then $\Delta\omega \in \Omega^k(M)$, so we can consider the family of maps $\Delta^k : \Omega^k(M) \longrightarrow \Omega^k(M)$. In fact, the Laplace-de Rham operator is a cochain map from the de Rham complex to itself because $d\Delta = d\delta d = \Delta d$. Furthermore, by definition, it is cochain homotopic to zero.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d^{k-1}} & \Omega^k(M) & \xrightarrow{d^k} & \Omega^{k+1}(M) & \longrightarrow & \cdots \\
 & \swarrow & \downarrow{\scriptstyle \Delta^{k-1}} & \overset{\delta^k}{\swarrow} & \downarrow{\scriptstyle \Delta^k} & \overset{\delta^{k+1}}{\swarrow} & \downarrow{\scriptstyle \Delta^{k+1}} & \swarrow & \\
\cdots & \longrightarrow & \Omega^{k-1}(M) & \xrightarrow{d^{k-1}} & \Omega^k(M) & \xrightarrow{d^k} & \Omega^{k+1}(M) & \longrightarrow & \cdots
\end{array}$$

In addition, similarly to d, Δ restricts to $\Omega^k_c(M)$, defining a cochain map from the compactly supported de Rham complex to itself cochain homotopic to zero.

Remark 3.46. We define the so-called *solutions spaces* as $\Omega^k_\Delta(M) = \ker\, \Delta^k$. The exterior derivative d restricts to these spaces, because if $\omega \in \ker\, \Delta$ then

$$(d\delta + \delta d)\omega = 0 \Longrightarrow d\delta\omega = -\delta d\omega$$

and therefore

$$\Delta d\omega = (d\delta + \delta d)d\omega = d\delta d\omega = -\delta dd\omega = 0$$

Hence, they define a subcomplex of the de Rham complex $(\Omega^\bullet_\Delta(M), d^\bullet)$ for which we can consider the cohomology $H^k_\Delta(M)$. In addition, $\Omega^k_{\Delta,c}(M) = \ker \Delta \cap \Omega^k_c(M)$ is a vector subspace in which d also restricts, so we can consider the corresponding complex and the corresponding cohomology $H^k_{\Delta,c}(M)$.

In the particular case of Riemannian geometry, $\Omega^k_\Delta(M)$ are called the spaces of *harmonic forms* $\mathcal{H}^k$ and play a fundamental role in the Hodge decomposition theorem (see Section 6 in [War83]) which asserts that in a compact oriented Riemannian manifold there is an orthogonal direct sum decomposition

$$\Omega^k(M) = \mathcal{H}^k \oplus \text{im}\ \Delta = \mathcal{H}^k \oplus \text{im}\ d \oplus \text{im}\ \delta$$

In this case, the cohomology $H^k_\Delta(M)$ is simply $\mathcal{H}^k$ because $\mathcal{H}^k \cap \text{im}\ d = \{0\}$, and furthermore, is isomorphic to $H^k_{dR}(M)$ (see [FOT08], Theorem A.12).

3.4 Cohomology with causally restricted supports

At this point, we can finally present the cohomologies that we wanted to introduce. After having described the causally compact sets and introduced the solution spaces for the Laplace-de Rham operator all we have left to do is to adapt the compactly supported de Rham cohomology to these sets.

Definition 3.47. Let M be a spacetime. Since supports are closed by definition we can consider the sets of k-forms $\Omega^k_x(M) = \{\omega \in \Omega^k(M) : \text{supp}(\omega) \text{ is } \mathcal{X}\}$, where $\mathcal{X}$ = retarded, advanced, spacelike compact, future compact, past compact, timelike compact and respectively $x = -, +, sc, fc, pc, tc$. Similarly to the compact supported spaces, $\Omega^k_x(M)$ are indeed vector subspaces and the exterior derivative restricts to them. Therefore we can consider the corresponding complexes and cohomologies $H^k_x(M)$. In addition, in the same way as in Remark 3.46, we can also consider the subspaces with causally restricted support on the solution space, denoted by $\Omega^k_{\Delta,x}(M)$, and the corresponding complex and cohomologies $H^k_{\Delta,x}(M)$. All these cohomologies are known as *causally restricted de Rham cohomologies* or *cohomologies with causally restricted supports.*

To provide a proper characterization of these cohomologies in globally hyperbolic spacetimes we have to introduce some operators related to Δ. Nonetheless, we would like to remark that [Ben16] presents an alternative approach that avoids them. However, the one presented here, based on [Kha16], is simpler once the required tools are introduced and, moreover, it can be generalized to other complexes.

Definition 3.48. Let M be a globally hyperbolic spacetime and Δ its Laplace-de Rham operator. An *advanced (retarded) Green's operator* of Δ is a $\mathbb{R}$-linear map of the type $G_\pm : \Omega_c(M) \longrightarrow \Omega(M)$ such that for all $\omega \in \Omega^k_c(M)$

(i) $G_\pm\omega \in \Omega^k(M)$.

(ii) $G_\pm\Delta(\omega) = \Delta G_\pm(\omega) = \omega$.

(iii) $\mathrm{supp}(G_\pm\omega) \subseteq J^\pm(\mathrm{supp}\ \omega)$.

Although we have just considered the particular case of the Laplace-de Rham operator, one can define Green's operators for any differential linear operator, and generalize the following discussion. The Laplace-de Rham operator is indeed one of the most prominent examples of *Green-hyperbolic operators*, that is to say that there exist advanced and retarded Green's operators for it (it follows from Corollary 3.4.3 in [BGP07]). Furthermore, as it may be found in [Bär15], Corollary 3.12, Green's operators of a Green-hyperbolic operator are unique.

Remark 3.49. Note that condition (iii) implies that $G_\pm\omega$ has advanced (retarded) support, so indeed $G_\pm : \Omega_c(M) \longrightarrow \Omega_\pm(M)$. Green's operators admit unique linear extensions (see [Bär15], Theorem 3.8 and Corollary 3.10) $G_+ : \Omega_x(M) \longrightarrow \Omega_x(M)$ for $x = +, pc$, and $G_- : \Omega_x(M) \longrightarrow \Omega_x(M)$ for $x = -, fc$. Then, it follows from the definition that they are isomorphisms with inverse Δ (which restricts to $\Omega^k_x(M)$ as in Remark 3.45). Moreover $G_\pm$ commutes not only with Δ, but also with d, defining the corresponding cochain maps.

With this observation, we can already show that, in globally hyperbolic spacetimes, the spacelike and the timelike compactly supported cohomologies are the only relevant cohomologies.

Theorem 3.50. *Let M be a globally hyperbolic spacetime. For all $k \geq 0$ the cohomologies $H^k_x(M)$ and $H^k_{\Delta,x}(M)$ are trivial for $x = +, -, fc, pc$.*

Proof. We consider, first, $H^k_x(M)$. As noted in Remark 3.45 the Laplace-de Rham operator Δ is a cochain map of the de Rham complex into itself cochain homotopic to zero and therefore induces the zero map in the cohomology, which also holds for $(\Omega^\bullet_x(M), d^\bullet)$. Moreover, on a globally hyperbolic spacetime the restriction of Δ to $\Omega^k_x(M)$ for $x = +, -, fc, pc$ is invertible with inverse given by G_+, when $x = +, pc$ or G_- when $x = -, fc$. Therefore the map in cohomology is both zero and bijective so the cohomology must be trivial.

For $H^k_{\Delta,x}(M)$ note that using again the fact that Δ is bijective for $x = +, -, fc, pc$, $\Omega^k_{\Delta,x}(M) = \ker \Delta|_{\Omega^k_x(M)} = \{0\}$ which implies that the cohomologies are also trivial. $\square$

Definition 3.51. The *causal propagator* of Δ is the map

$$G := G_+ - G_- : \Omega_c(M) \longrightarrow \Omega(M)$$

Remark 3.52. Note that $\text{supp}(G\omega) \subseteq J(\text{ supp } \omega)$ so, in fact, $G : \Omega_c(M) \longrightarrow \Omega_{sc}(M)$.

The properties of the causal propagator in which we have the most interest are given by the following result, stated in [Kha12], Proposition 4.

Theorem 3.53. *Let M be a globally hyperbolic spacetime. For each $k \geq 0$ the sequences of maps*

$$\{0\} \longrightarrow \Omega^k_c(M) \xrightarrow{\Delta} \Omega^k_c(M) \xrightarrow{G} \Omega^k_{sc}(M) \xrightarrow{\Delta} \Omega^k_{sc}(M) \longrightarrow \{0\}$$

$$\{0\} \longrightarrow \Omega^k_{tc}(M) \xrightarrow{\Delta} \Omega^k_{tc}(M) \xrightarrow{G} \Omega^k(M) \xrightarrow{\Delta} \Omega^k(M) \longrightarrow \{0\}$$

are exact.

The result for the first sequence is proven in [BGP07], Theorem 3.4.7, with the last surjection covered by the proof in [BGC+09], Chapter 3, Corollary 5. Then for the second sequence the result follows from the duality exhibited in Section 1 of [Bär15]. These sequence allow to prove the following result that gives isomorphisms between the casually restricted cohomologies and the usual de Rham cohomologies.

Theorem 3.54. *Let M be a globally hyperbolic spacetime. Then, for all $k \geq 0$*

$$H^k_{sc}(M) \cong H^{k+1}_c(M), \qquad H^k_{\Delta,sc} \cong H^k_c(M) \oplus H^{k+1}_c(M),$$
$$H^k_{tc}(M) \cong H^{k-1}(M), \qquad H^k_{\Delta}(M) \cong H^k(M) \oplus H^{k-1}(M)$$

with the convention that all cohomologies $H^k_x(M)$ and $H^k_{\Delta,x}(M)$ vanish whenever $k < 0$.

Proof. Let us start with the spacelike compact support case. Recall again that both Δ and G commute with d ans therefore define cochain maps between the de Rham complexes with appropriate supports, inducing the corresponding maps in cohomology. The exactness of the first sequence of Theorem 3.53 implies that im $G = \ker \Delta = \Omega^k_{\Delta,sc}(M)$, so for each $k \geq 0$, we can break the sequence into two exact sequences of cochain maps

$$\begin{gathered} \{0\} \longrightarrow \Omega^k_c(M) \xrightarrow{\Delta} \Omega^k_c(M) \xrightarrow{G} \Omega^k_{\Delta,sc}(M) \longrightarrow \{0\} \\ \{0\} \longrightarrow \Omega^k_{\Delta,sc}(M) \xhookrightarrow{i} \Omega^k_{sc}(M) \xrightarrow{\Delta} \Omega^k_{sc}(M) \longrightarrow \{0\} \end{gathered} \tag{3.1}$$

where i is the inclusion. Therefore, since Δ is cochain homotopic to zero and consequently induces a zero map in cohomology, the long exact sequences associated to Equation 3.1 (see Proposition 2.11) can be broken up for each k into the following exact sequences

$$\{0\} \longrightarrow H^k_c(M) \xrightarrow{H^k(G)} H^k_{\Delta,sc}(M) \xrightarrow{\tau^k} H^{k+1}_c(M) \longrightarrow \{0\}$$

$$\{0\} \longrightarrow H^{k-1}_{sc}(M) \xrightarrow{\tau^{k-1}} H^k_{\Delta,sc}(M) \xrightarrow{H^k(i)} H^k_{sc}(M) \longrightarrow \{0\}$$

with the convention that any $H^k_x(M)$ and $H^k_{\Delta,x}(M)$ vanishes for $k<0$. The exactness of the sequences yields isomorphisms

$$H^k_c(M)\oplus H^{k+1}_c(M)\cong H^k_{\Delta,sc}(M)\cong H^{k-1}_{sc}(M)\oplus H^k_{sc}(M)$$

which already gives one of the sought isomorphisms. Given that M is connected and non-compact and thus $H^0_c(M)=\{0\}$ (see Proposition 2.25), plugging $k=0$ into the above isomorphism yields $H^0_{sc}(M)\cong H^1_c(M)$. Then, induction on k shows that, as claimed, $H^k_{sc}(M)\cong H^{k+1}_c(M)$ for all k.

The exact same argument to the second exact sequence of Theorem 3.53 gives the last two isomorphisms. □

This theorem together with the characterization of globally hyperbolic spacetimes as a foliation of Cauchy Hypersurfaces, directly implies the following result that allows to compute cohomologies with causally restricted supports as ordinary cohomologies on a Cauchy hypersurface.

Corollary 3.55. *Let M be a globally hyperbolic spacetime and Σ a spacelike Cauchy hypersurface. Then, for al $k\geq 0$*

$$H^k_{sc}(M)\cong H^k_c(\Sigma),\qquad H^K_{\Delta,sc}(M)\cong H^k_c(\Sigma)\oplus H^{k-1}_c(\Sigma),$$
$$H^k_{tc}(M)\cong H^{k-1}(\Sigma),\qquad H^k_{\Delta}(M)\cong H^k(\Sigma)\oplus H^{k-1}(\Sigma)$$

with the convention that all cohomologies vanish for $k<0$.

Proof. By Theorem 3.31, $M\cong\mathbb{R}\times\Sigma$ which implies that M is homotopy equivalent to Σ and thus $H^k(M)\cong H^k(\Sigma)$. On the other hand, using Poincaré duality and the fact that, over fields, cohomology is the dual space of the homology (see Remark 2.10) and therefore they are (non-canonically) isomorphic, so we can use them indistinctly, we get

$$H^k_c(M)\cong H^{n-k}(M)\cong H^{n-k}(\Sigma)=H^{(n-1)-(k-1)}(\Sigma)\cong H^{k-1}_c(\Sigma)$$

The sought isomorphisms follow directly from applying this identity to Theorem 3.54. □

Remark 3.56. If M is a n-dimensional manifold we have that $H^n_{sc}(M)$ is trivial because Σ is a $(n-1)$-dimensional manifold so there are no non-trivial n-forms. On the other hand, directly from the previous corollary we also see $H^0_{tc}(M)$ is always trivial. In addition, if Σ is compact then we have that $H^k_c(\Sigma)=H^k(\Sigma)$ and consequently we get isomorphisms $H^k_{\Delta,sc}(M)\cong H^k_{sc}(M)\oplus H^k_{tc}(M)\cong H^k_{\Delta}(M)$.

Finally, we would like to point out, that Theorem 3.54, also gives an analogous result to Poincaré duality for causally restricted cohomologies.

Corollary 3.57. *Let M be a n-dimensional globally hyperbolic spacetime. Then, for all $k\geq 0$, $H^k_{sc}(M)\cong H_{tc,n-k}(M)$ and $H^k_{\Delta,sc}(M)\cong H_{\Delta,n-k}(M)$.*

Proof. As a direct consequence of Theorem 3.54 we have that $H^k_{sc}(M)\cong H^{k+1}_c(M)$, and $H^{n-k}_{tc}(M)\cong H^{n-(k+1)}(M)$. Thus, Poincaré duality implies that $H^k_{sc}(M)\cong H_{tc,n-k}(M)$. On the other hand, again from Theorem 3.54, $H^k_{\Delta,sc}(M)\cong H^k_c(M)\oplus H^{k+1}_c(M)$ and $H^{n-k}_{\Delta}(M)\cong H^{n-k}\oplus H^{n-(k+1)}(M)$ which also give the corresponding isomorphism using Poincaré duality. □

Examples 3.58. We end the section by reviewing the cohomology with causally restricted support of two well-known physical spacetimes solution to Einsten's field equations. We use again the convention that H^k vanishes for $k < 0$.

1. *Einstein's static universe*, $\mathcal{E}$, was the first relativistic cosmological model in history, proposed by A. Einstein shortly after completing the general theory of relativity. Assuming a homogeneous universe static in time he was led to a finite universe of spherical spatial curvature, defined on $\mathbb{R} \times \mathbb{S}^3$. In particular, any spacelike Cauchy hypersurface is diffeomorphic to $\mathbb{S}^3$. Therefore, using Corollary 3.55 and the fact that $\mathbb{S}^3$ is compact, $H^k_{sc}(\mathcal{E}) \cong H^k_c(\mathbb{S}^3) \cong H^k(\mathbb{S}^3)$. Thus, the singular homology of spheres (Example 2.13.4), isomorphic to its de Rham counterpart by de Rham's theorem, implies that
$$H^k_{sc}(\mathcal{E}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, 3 \\ \{0\} & \text{otherwise} \end{cases}$$
On the other hand, $H^k_{tc}(\mathcal{E}) \cong H^{k-1}(\mathbb{S}^3)$ and therefore
$$H^k_{tc}(\mathcal{E}) \cong \begin{cases} \mathbb{R} & \text{if } k = 1, 4 \\ \{0\} & \text{otherwise} \end{cases}$$
Moreover, $H^k_{\Delta,sc}(\mathcal{E}) \cong H^k(\mathbb{S}^3) \oplus H^{k-1}(\mathbb{S}^3) \cong H^k_{\Delta}(\mathcal{E})$ so
$$H^k_{\Delta,sc}(\mathcal{E}) \cong H^k_{\Delta}(\mathcal{E}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, 1, 3, 4 \\ \{0\} & \text{otherwise} \end{cases}$$

2. The *Schwarzschild spacetime*, $\mathcal{S}$, is a solution that describes the gravitational field created by a spherical mass with electric charge and angular momentum null. The resulting space contains a *singularity* at the center of the mass distribution, i.e., a point in which the curvature of the metric is not well-defined, as it becomes infinite. Although it was initially interpreted as a non-physical solution, now we regard it as describing a *black hole*, a region of spacetime where gravity is so strong than no particles or even light can escape from it. This first solution, defined on $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\}) \cong \mathbb{R} \times (0, \infty) \times \mathbb{S}^2$, can be maximally extended to a full spacetime $\mathcal{S} \cong \mathbb{R}^2 \times \mathbb{S}^2$, where an hypothetical region completely opposite to the black hole emerges, a *white hole*. In this spacetime, any spacelike Cauchy hypersurface is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$ which is homotopy equivalent to $\mathbb{S}^2$. Consequently, using again Corollary 3.55 and Poincaré duality, $H^k_{sc}(\mathcal{S}) \cong H^k_c(\mathbb{R} \times \mathbb{S}^2) \cong H^{3-k}(\mathbb{R} \times \mathbb{S}^2) \cong H^{3-k}(\mathbb{S}^2)$ which yields
$$H^k_{sc}(\mathcal{S}) \cong \begin{cases} \mathbb{R} & \text{if } k = 1, 3 \\ \{0\} & \text{otherwise} \end{cases}$$
On the other hand, $H^k_{tc}(\mathcal{S}) \cong H^{k-1}(\mathbb{R} \times \mathbb{S}^2) \cong H^{k-1}(\mathbb{S}^2)$ which implies that
$$H^k_{tc}(\mathcal{S}) \cong \begin{cases} \mathbb{R} & \text{if } k = 1, 3 \\ \{0\} & \text{otherwise} \end{cases}$$
In addition, $H^k_{\Delta,sc}(\mathbb{S}) \cong H^{3-k}(\mathbb{S}^2) \oplus H^{3-k-1}(\mathbb{S}^2)$ and $H^k_{\Delta}(\mathcal{S}) \cong H^k(\mathbb{S}^2) \oplus H^{k-1}(\mathbb{S}^2)$ which gives
$$H^k_{\Delta,sc}(\mathcal{S}) \cong H^k_{\Delta}(\mathcal{S}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, 1, 2, 3 \\ \{0\} & \text{otherwise} \end{cases}$$

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