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Treball final de grau

GRAU DE MATEMÀTIQUES

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PAIRING OF ZEROS AND CRITICAL POINTS FOR RANDOM POLYNOMIALS

Autor: Guillem de la Calle Vicente

Director: Dr. Xavier Massaneda
Realitzat a: Departament de Matemàtiques i Informàtica

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Abstract

In this project we deal with random holomorphic polynomials p_N. Specifically, we study the relationship between zeros and critical points of p_N considering two different probabilistic models. The first one is based on chosing independently and with uniform probability N random points that will be the zeros of our polynomial p_N. The second model is that of the so-called parabolic Gaussian Analytic Function. In this second model, the distribution of points is more rigid, and the striking phenomenon continues to be observed: zeros and critical points appear, with high probability, in pairs.

Resum

En aquest treball tractem polinomis holomorfs aleatoris p_N. En concret, estudiem la relació entre els zeros i els punts crítics de p_N considerant dos models probabilístics diferents. El primer es basa en escollir independentment i amb probabilitat uniforme N punts aleatoris que seran els zeros del nostre polinomi p_N. El segon model és el de l'anomenada Funció Analítica Gaussiana parabòlica. En aquest segon model, la distribució de punts és més rígida, i se segueix observant el sorprenent fenomen: els zeros i els punts crítics apareixen, amb alta probabilitat, aparellats.

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Introduction

In this project we study the relationship between zeros and critical points of certain random holomorphic polynomials $p_N(z)$. This problem is more easily understood if we think of p_N as a meromorphic function defined in the Riemann sphere $\mathbb{S}^2 \simeq \mathbb{CP}^1$.

The main probabilistic model we consider is the following. Given $N \in \mathbb{N}$, let $\eta_1, \dots, \eta_N \in \mathbb{S}^2$ be N random points chosen independently and with uniform probability. Let $\xi_j = \pi(\eta_j) \in \mathbb{C}$, where $\pi\colon S^2 \to \mathbb{C}$ is the stereographic projection, and define a polynomial that has ξ_j as zeros, that is

$$p_N(z) = \prod_{j=1}^{N} (z - \xi_j).$$

Since the projection (push-forward) of the normalized Lebesgue measure in $\mathbb{S}^2$ is $d\nu(z) = \dfrac{dm(z)}{\pi(1+|z|^2)^2}$, one can equivalently think that the zeros in $p_N(z)$ are chosen independently in $\mathbb{C}$ according to the probability measure $d\nu$.

The second model we consider is that of the so-called parabolic Gaussian Analytic function. Consider the monomials

$$e_j(z) = \sqrt{\binom{N}{j}}\, z^j, \qquad j = 0, \dots, N.$$

These are just the usual monomials, but normalised so that

$$||e_j||_N^2 := (N+1) \int_{\mathbb{C}} \frac{|e_j(z)|^2}{(1+|z|^2)^N}\, d\nu(z) = 1.$$

This is a natural norm for polynomials of degree at most N.

Consider then the random polynomial

$$p_N(z) = \sum_{j=0}^{N} a_j e_j(z),$$

where $a_j \in \mathbb{C}$ are independent and identically distributed (i.i.d.) standard Gaussians $N_{\mathbb{C}}(0,1)$.

It is known that the zeros of p_N, i.e. the set

$$\mathcal{Z}(p_N) := \{z \in \mathbb{C} \,:\, p_N(z) = 0\} = \{\xi_1, \dots, \xi_N\}$$

is distributed, in average, uniformly on the sphere (i.e. the number of points $\pi^{-1}(\xi_j) = \eta_j \in \mathbb{S}^2$ lying in a region $U \subseteq S^2$ is expected to be N times the area of U).

However, in this second model, the distribution of points is more rigid, in the sense that the fluctuations around the expected value are smaller than in the first model. In particular there is a local repulsion phenomenon: it is less likely to find two close zeros.

The main goal of this work is to study the following striking phenomenon: zeros and critical points appear, with high probability, in pairs.

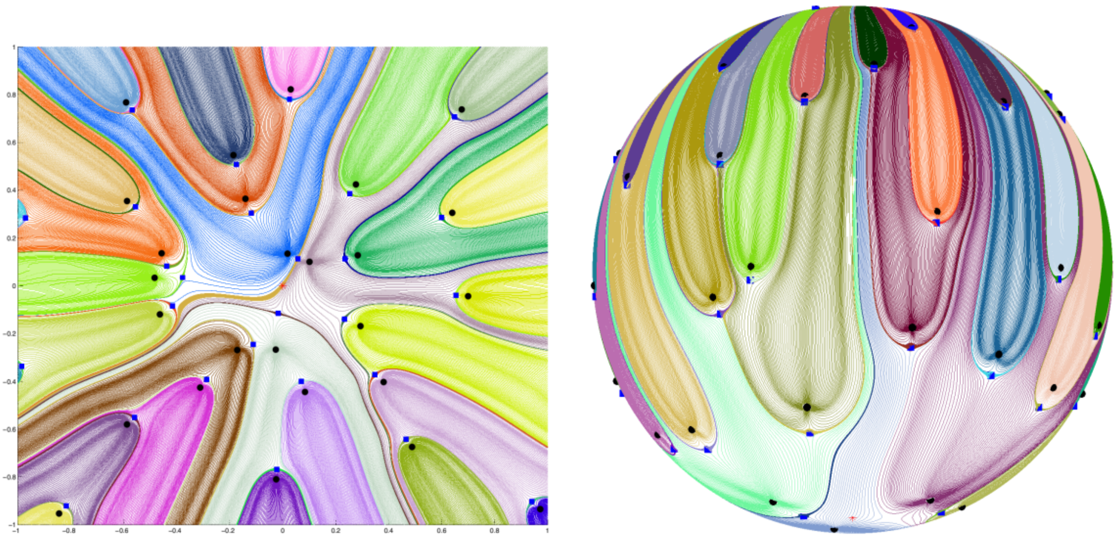

Figure 1: Zeros (black disks) and critical points (blue squares) for a degree 50 parabolic Gaussian Analytic polynomial. (Boris Hanin)

We wonder why this pairing occurs. If we choose a zero deterministically, with what probability can we be sure that there is a critical point around it and how far away will it be? Why does this pairing break down in some places?

Observe that the expected separation between N points chosen randomly on $\mathbb{S}^2$ would be of order $1/\sqrt{N}$. However, it turns out that, for both models, the probability of finding a critical point near a given zero is of order $1/N$ with very high probability. Moreover, this critical point near a given zero is aligned with the zero. We will see this in the Theorems 2.3.1 (Chapter 2) and 3.7.1 (Chapter 3).

The main part of the work is devoted to state and proof the theorem for the first model. For the GAF model, we explain its set-up, its main properties and state the theorem. The proof of this result is out of our scope.

Probably the best-known result relating zeros and critical points of holomorphic polynomials is the Gauss-Lucas theorem: the critical points of a given holomorphic polynomial $p_N(z)$ lie inside the convex hull of its zeros.

The phenomenon we study here is of different nature, and it can be understood with the help of the following electrostatic interpretation.

Write, as before,

$$p_N(z) = \prod_{j=1}^{N} (z - \xi_j).$$

where the zeros ξ_j are chosen with uniform distribution in $\mathbb{S}^2$, i.e., if $A \subseteq \mathbb{S}^2$ and $\eta_j = \pi^{-1}(\xi_j) \in \mathbb{S}^2$, then $P(\eta_j \in A) = \sigma(A)$, where σ is the area measure on $\mathbb{S}^2$ normalized with $\sigma(\mathbb{S}^2) = 1$.

Consider the function of one complex variable (random electrostatic potential)

$$g(w) = \log |p_N(w)|^2 = \sum_{j=1}^{N} \left[\log(w - \xi_j) + \log(\overline{w - \xi_j})\right].$$

Then, we have, for $w \notin \{\xi_j\}_{j=1}^{N}$

$$\partial g(w) = \frac{\overline{p_N(w)} \cdot \partial p_N(w)}{|p_N(w)|^2} = \sum_{j=1}^{N} \partial \log(w - \xi_j) = \sum_{j=1}^{N} \frac{1}{w - \xi_j}.$$

Given $p_N(w)$, the critical points ($\partial p_N(w) = \frac{\partial}{\partial w} p_N(w) = 0$) are therefore solutions of the equation

$$E_N(w) := \sum_{j=1}^{N} \frac{1}{w - \xi_j} = 0. \tag{1}$$

We note that $E_N(w)$ is the electric field at w given by positive charges of value $+1$ located at each point ξ_j. Thus, the critical points can be viewed equilibrium points of this electric field.

We interpret this on $\mathbb{S}^2$. We think of p_N as a meromorphic function (with pole of order N at infinity) on $\mathbb{S}^2$. Then

$$\Delta g = \Delta \log |p_N|^2 = -N\delta_\infty + \sum_{\xi \,:\, p_N(\xi)=0} \delta_{\xi_j},$$

where the Laplacian Δ is understood in the distributional sense and δ_∞, δ_{ξ_j} are the Dirac delta measures. This is interpreted as an electrostatic field on the Riemann sphere generated by N positive charges located at the zeros of the random polynomial and a negative charge of weight N at the north pole, which corresponds to the point at infinity.

This expression of Δg comes out of writing $w = 1/z$ (that is, taking the chart at ∞) and writing

$$p_N(z) = p_N\left(\frac{1}{w}\right) = c \prod_{j=1}^{N} \left(\frac{1}{w} - \xi_j\right) = \frac{c}{w^N} \prod_{j=1}^{N} \xi_j \left(\frac{1}{\xi_j} - w\right).$$

Let now ξ be a fixed zero and write

$$p_{N,\xi}(w) = \frac{1}{w^N}(w-\xi)\prod_{j=1}^{N-1}(w-\xi_j),$$

with the remaining zeros ξ_j, $j = 1, \ldots, N-1$, i.i.d. uniformly distributed on the Riemann sphere. Strictly speaking, we would have ξ_i to be the inverse of zeros of p_N uniformly distributed on $\mathbb{S}^2$, but the distribution of ξ_j and $1/\xi_j$ is equally uniform at $\mathbb{S}^2$.

With this factorization, the electrostatic field becomes

$$E_N(w) = -\frac{N}{w} + \frac{1}{w-\xi} + \sum_{j=1}^{N-1}\frac{1}{w-\xi_j}. \tag{2}$$

A critical point of the polynomial corresponds to a point where the gradient of the electrostatic potential cancels out, i.e., to a point of equilibrium of this electrostatic field. At a point of equilibrium, three types of forces act and must be compensated:

(i) The force of the negative charge of the infinity point, which is proportional to the degree of the polynomial (number of zeros) N.

(ii) The force of the positive charge of the nearest random particle. It is proportional to $1/d$, where d is the distance to the particle. If we choose w with $|w-\xi| \approx 1/N$, then $\frac{1}{w-\xi}$ is of order N.

(iii) The force of the other charges. These are uniformly distributed around the equilibrium point, and the central limit theorem allows us to see that this force is proportional to $\sqrt{N}$ with high probability, i.e. negligible with respect to the force of the charge coming from the point of infinity.

By the uniform distribution of each ξ_j, we have that $\mathbb{E}\left(\frac{1}{w-\xi_j}\right) = 0$, and thus

$$\mathbb{E}\left(\sum_{j=1}^{N-1}\frac{1}{w-\xi_j}\right) = \sum_{j=1}^{N-1}\mathbb{E}\left(\frac{1}{w-\xi_j}\right) = 0.$$

Therefore, generically

$$E_N(w) \approx -\frac{N}{w} + \frac{1}{w-\xi} = \mathbb{E}[E_N(w)].$$

If $\xi \notin \{0,\infty\}$ (i.e. north and south poles of the sphere), we have $\mathbb{E}[E_N(w_{N,\xi})] = 0$ in precisely the point

$$w_{N,\xi} = \xi\left(1-\frac{1}{N}\right)^{-1}.$$

Therefore, there is a point $w_{N,\xi}$ at distance of order $1/N$ from ξ where the mean electrostatic field cancels out.

This means is that to compensate all the charges, it is necessary that the positive charge closest to the critical point is at a distance $1/N$ and, approximately, in the segment that joins the critical point and the origin.

Near the origin (at the South pole on the Riemann sphere) this reasoning does not work, since the force exerted by the North pole charge is zero, by isotropy. As a consequence, the pairing of zeros and critical points breaks down near the origin.

A final word on the structure of the memoir. In the first chapter we introduce (or recall) the necessary elements to write the statements and proofs of the following chapters. Chapter 2 is devoted to study in detail the model of N points chosen independently and uniformly in $\mathbb{S}^2$. Finally, in Chapter 3 we discuss the GAF model.

Chapter 1

Preliminaries

1.1 The Riemann sphere, topology and chordal metric

Stereographic projection and properties

It's useful to consider that rational functions take values (and are defined on) $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$, the extended complex plane or Riemann sphere. Topologically it's just $\mathbb{S}^2$ (unit sphere in $\mathbb{R}^3$).

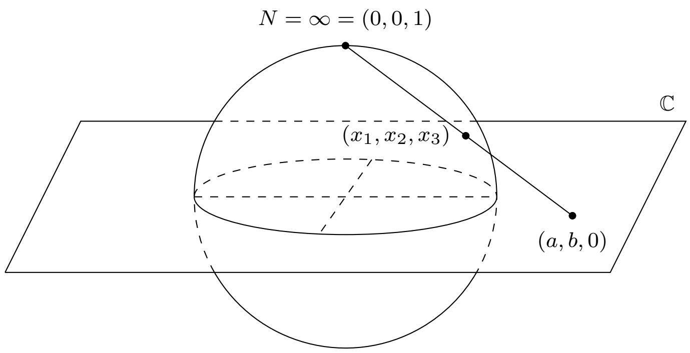

Figure 1.1: The stereographic projection

We consider in $\widehat{\mathbb{C}}$ a topology that induces in $\mathbb{C}$ the usual topology. This is defined by fixing a basis of open sets of each point of $\widehat{\mathbb{C}}$:

(i) If $a \in \mathbb{C}$, the basis is $\{D(a,r)\}_{r>0}$.

(ii) The basis for ∞ is $\widehat{\mathbb{C}} \setminus \{\overline{D(0,r)}\}_{r>0}$.

We recall this well-known fact.

Proposition 1. $\widehat{\mathbb{C}}$ *with this topology is homeomorphic to the unit sphere of* $\mathbb{R}^3$,

$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, x_1^2 + x_2^2 + x_3^2 = 1\},$$

with the topology induced by $\mathbb{R}^3$.

Proof. Set $N = (0, 0, 1) \in \mathbb{S}^2$ the north pole. Identify $\mathbb{C}$ with the equatorial plane $x_3 = 0$, i.e., $\mathbb{R}^2 \times \{0\} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \, : \, x_3 = 0\}$.

Consider the stereographic projection $\pi \colon \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ given in the following way: if $x \in \mathbb{S}^2 \setminus \{N\}$, $\pi(x)$ is the point of intersection with the plane $x_3 = 0$ of the line determined by N and x.

If $x = (x_1, x_2, x_3) \in \mathbb{S}^2 \setminus \{N\}$, $x_3 \neq 1$, then, $\pi(x) = (0, 0, 1) + \lambda(x_1, x_2, x_3 - 1) = (a, b, 0)$ and $\lambda = \frac{-1}{x_3 - 1} = \frac{1}{1 - x_3}$. Therefore, $a = \frac{x_1}{1 - x_3}$, $b = \frac{x_2}{1 - x_3}$ and

$$\pi(x) = \pi(x_1, x_2, x_3) = \frac{x_1}{1 - x_3} + i \frac{x_2}{1 - x_3},$$

an expression that gives us the continuity of π.

We note that π is bijective, because if $\pi(x) = z = a + ib$, with $a = \frac{x_1}{1 - x_3}$ and $b = \frac{x_2}{1 - x_3}$, then

(a) $|z|^2 = a^2 + b^2 = \frac{1 + x_3}{1 - x_3}$ and, in particular, $x_3 = \frac{|z|^2 - 1}{1 + |z|^2}$,

(b) $x_1 = a(1 - x_3) = \frac{2a}{1 + |z|^2} = \frac{z + \overline{z}}{1 + |z|^2}$,

(c) $x_2 = \frac{2b}{1 + |z|^2} = \frac{1}{i} \frac{z - \overline{z}}{1 + |z|^2}$.

These calculations give us, for $z \in \mathbb{C}$,

$$\pi^{-1}(z) = \left(\frac{z + \overline{z}}{1 + |z|^2}, \frac{1}{i} \frac{z - \overline{z}}{1 + |z|^2}, \frac{|z|^2 - 1}{1 + |z|^2} \right). \tag{1.1}$$

Note that $\pi^{-1}(z) \neq N$, so π^{-1} is continuous. We have then that $\pi \colon \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ is a homeomorphism and, in particular, $\mathbb{C}$ is topologically equivalent to $\mathbb{S}^2 \setminus \{N\}$. Since $\lim_{x \to N} |\pi(x)| = +\infty$, if we define $\pi(N) = \infty$, we also have that π is a homeomorphism between $\mathbb{S}^2$ and $\widehat{\mathbb{C}}$. To see that π is continuous at the point N observe that if $|x - N| < \delta$, then $|\pi(x)| > M$, i.e. $\pi(D(N, \delta)) \subset \left(\mathbb{C} \setminus \overline{D(0, M)} \right) \cup \{\infty\}$. □

Some well-known properties that come out from the definition of stereographic projection are:

Observation 1.

- *The southern hemisphere projects on the unit disk.*
- *The stereographic projection is conformal, i.e., it preserves angles and orientations. (see [Shu21])*

- *The push-forward in $\mathbb{C}$ of the Lebesgue measure in $\mathbb{S}^2$ by the stereographic projection π is (see [HKPV09] for more details)*

$$d\nu(z) = \frac{dm(z)}{\pi(1+|z|^2)^2}, \quad z \in \mathbb{C}.$$

Chordal distance

The Euclidian distance in $\mathbb{R}^3$ restricted to the sphere projected to $\mathbb{C}$ by the stereographic projection π induces a natural distance in $\mathbb{C}$.

Definition 1.1.1. *(Chordal distance). The cordal distance between $z, w \in \mathbb{C}$ is*

$$d_C(z,w) := |\pi^{-1}(z) - \pi^{-1}(w)|_{\mathbb{R}^3} = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

We justify that the chordal distance has this form. Let us call $z = x+iy$ and $w = u+iv$. By the stereographic projection, $\pi^{-1}(z) = (x_1, x_2, x_3)$ and $\pi^{-1}(w) = (y_1, y_2, y_3)$. Since $x_1^2+x_2^2+x_3^2 = 1$ and $y_1^2+y_2^2+y_3^2 = 1$ we have, from the definition

$$d_C(z,w)^2 = (x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2 = 2(1 - x_1y_1 - x_2y_2 - x_3y_3)$$

Using Equation (**??**) of π^{-1}:

$$d_C(z,w)^2 = 2\left(1 - \frac{4xu+4yv+(|z|^2-1)(|w|^2-1)}{(|z|^2+1)(|w|^2+1)}\right) = \frac{4(|z|^2+|w|^2-2xu-2yv)}{(|z|^2+1)(|w|^2+1)}$$

Using that $|z-w|^2 = |z|^2+|w|^2-2xu-2yv$,

$$d_C(z,w)^2 = \frac{4|z-w|^2}{(1+|z|^2)(1+|w|^2)} \Rightarrow d_C(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}$$

If $w = \infty$ and $z \in \mathbb{C}$,

$$d_C(z,w)^2 = x_1^2 + x_2^2 + (x_3-1)^2 = 2(1-x_3) = 2\left(1 - \frac{|z|^2-1}{|z|^2+1}\right),$$

then

$$d_C(z,w) = \frac{2}{\sqrt{1+|z|^2}}.$$

The Euclidean distance in $\mathbb{R}^3$ is invariant by rotations of the sphere, and therefore the chordal distance is invariant as well. For the Riemann sphere, it is the one point compactification of the plane. So, $(\widehat{\mathbb{C}}, d_C)$ is a compact metric space.

If $a \in \mathbb{C}$ and $r > 0$, then the chordal disk of center a and radius r will be denoted as

$$D_c(a,r) = \{z \in \mathbb{C} : d_C(z,a) < r\}.$$

Finally, we compute the natural area of the chordal disk of center a and radius r, which will be used often.

Proposition 2. *If $a \in \mathbb{C}$ and $r > 0$, then*

$$\nu(D_c(a,r)) = \frac{r^2}{1+r^2}.$$

Proof. Using polar coordinates, we have that:

$$\begin{aligned}\nu(D_c(a,r)) &= \int_{D_c(a,r)} d\nu(z) = \int_{D_c(a,r)} \frac{dm(z)}{\pi(1+|z|^2)^2} \\ &= \int_0^r \int_0^{2\pi} \frac{\rho}{\pi(1+\rho^2)^2}\, d\theta d\rho = \frac{2\pi}{\pi} \int_0^r \frac{\rho}{(1+\rho^2)^2}\, d\rho \\ &= 2\frac{r^2}{2(1+r^2)} = \frac{r^2}{1+r^2}.\end{aligned}$$

□

Rotations in $\mathbb{S}^2$

The transformations we consider are the rotations of $\mathbb{S}^2$, through π, which are seen as

$$\varphi_{\lambda,\theta}(z) = e^{i\theta}\frac{z-\lambda}{1+\bar{\lambda}z}, \quad \lambda \in \mathbb{C};\ \theta \in [0, 2\pi).$$

We note that the point $\lambda = \pi(\eta)$ goes to $0 = \pi(S)$; i.e. in $\mathbb{S}^2$ that is the rotation that takes η to the S pole.

We shall denote simply by φ_λ the canonical transformation corresponding to $\theta = 0$.

The area in $\mathbb{S}^2$ is invariant by rotations, so the measure $d\nu$ is invariant by these transformations. Let us check for the sake of completeness:

Corollary 1.1.1. *The measure $d\nu$ is invariant by $\varphi_{\lambda,\theta}$, for all $\lambda \in \mathbb{C}$, $\theta \in [0, 2\pi)$.*

Proof. We have to see that for all $z \in \mathbb{C}$, $\theta \in [0, 2\pi)$, $d\nu(\varphi_{\lambda,\theta}(z)) = d\nu(z)$:

$$\begin{aligned}d\nu(\varphi_{\lambda,\theta}(z)) &= \frac{dm(\varphi_{\lambda,\theta}(z))}{\pi(1+|\varphi_{\lambda,\theta}(z)|^2)^2} = \frac{|\varphi'_{\lambda,\theta}(z)|^2}{\pi\frac{(1+|\lambda|^2)^2(1+|z|^2)^2}{|1+\bar{\lambda}z|^4}} dm(z) \\ &= \frac{\frac{(1+|\lambda|^2)^2}{|1+\bar{\lambda}z|^4}}{\pi\frac{(1+|\lambda|^2)^2(1+|z|^2)^2}{|1+\bar{\lambda}z|^4}} dm(z) = \frac{dm(z)}{\pi(1+|z|^2)^2} = d\nu(z).\end{aligned}$$

□

1.2 The Gamma and Beta functions

Further calculations will require these two classical special functions.

Definition 1.2.1. *The Gamma function of parameter* $p > 0$ *is defined as*

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx.$$

Some properties of this function are:

1. $\Gamma(1) = 1$.
2. $\Gamma(p) = (p-1)\Gamma(p-1)$, $p > 1$.
3. If $p \in \mathbb{N}$ and $p \geq 1$, then $\Gamma(p) = (p-1)!$.
4. $\Gamma(1/2) = \sqrt{\pi}$.

Definition 1.2.2. *The Beta function of parameters* $p > 0$ *and* $q > 0$ *is defined as:*

$$\beta(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx.$$

The most relevant properties of this function are:

1. $\beta(p, q) = \beta(q, p)$.
2. $\beta(1, q) = \dfrac{1}{q}$.
3. $\beta(p, q) = \dfrac{q-1}{q}\beta(p+1, q-1)$, for all $p > 0$ and $q > 1$.
4. $\beta(p, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

1.3 Complex Gaussian distribution

This section will be used thoroughly Chapter 3.

Throughout this section, we shall encounter complex Gaussian random variables. As conventions vary, we begin by establishing our terminology. By $\mathcal{N}(\mu, \sigma^2)$, we mean the distribution of the real-valued random variable with probability density $\dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are the mean and variance respectively.

Our goal in this section is define and study the univariate complex normal distribution.

Definition 1.3.1. *A complex random variable X can be written as*

$$X = U + iV,$$

where U and V are the uniquely determined real random variables corresponding to the real and the imaginary parts of X, respectively.

When we consider complex random variables, three operators are important. The complex random variable space (where we will define the operators) is given by

$$\mathcal{L}(\mathbb{C}) = \{X|X \text{ is a complex random variable and } \mathbb{E}(X\overline{X}) < \infty\},$$

where $\mathbb{E}$ denotes the expectation operator of a real random variable.

Observe that $\mathcal{L}(\mathbb{C})$ is the vector space of complex random variables having finite square length: writing $X = U + iV$ we can see that

$$\mathbb{E}(X\overline{X}) = \mathbb{E}(U^2 + V^2) = \mathbb{E}(|X|^2).$$

Definition 1.3.2. *Let $X = U + iV$ be a complex random variable. We define the expectation operator of X, $\mathbb{E} : \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ as*

$$\mathbb{E}(X) = \mathbb{E}(U) + i\mathbb{E}(V).$$

Definition 1.3.3. *Let X and Y be complex random variables. The covariance operator of X an Y, $Cov(X, Y) : \mathcal{L}(\mathbb{C}) \times \mathcal{L}(\mathbb{C}) \to \mathbb{C}$ is defined as*

$$Cov(X, Y) = \mathbb{E}\left((X - \mathbb{E}(X))\overline{(Y - \mathbb{E}(Y))}\right)$$

In the case where $X = Y$ it is called the variance operator. This provides us with the following definition.

Definition 1.3.4. *Let X be a complex random variable. The variance operator of X, $Var(X) : \mathcal{L}(\mathbb{C}) \to \overline{\mathbb{R}}_+$ is defined as*

$$Var(X) = Cov(X, X) = \mathbb{E}\left((X - \mathbb{E}(X))\overline{(X - \mathbb{E}(X))}\right) = \mathbb{E}\left(|X|^2\right) - |\mathbb{E}(X)|^2$$

Observe that the complex conjugate is necessary in the variance operator as we requiere the variance to be a nonnegative real number.

Property 1. *Let $X = U + iV$ be a complex random variable. Then,*

$$Var(X) = Var(U) + Var(V).$$

Proof. This property follows immediately from the definitions above:

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left((X - \mathbb{E}(X))\overline{(X - \mathbb{E}(X))}\right) = \mathbb{E}\left(|X - \mathbb{E}(X)|^2\right) = \\ &= \mathbb{E}\left(|U + iV - \mathbb{E}(U + iV)|^2\right) = \mathbb{E}\left(|U + iV - \mathbb{E}(U) - i\mathbb{E}(V)|^2\right) = \\ &= \mathbb{E}((U - \mathbb{E}(U))^2) + \mathbb{E}((V - \mathbb{E}(V))^2) = \mathrm{Var}(U) + \mathrm{Var}(V).\end{aligned}$$

□

We say that X has a univariate standard complex normal distribution (with mean zero and variance one) if and only if

1. X has a bivariate normal distribution on $\mathbb{R}^2$.
2. X has a complex covariance structure.
3. $\mathbb{E}(X) = 0$ and $\mathrm{Var}(X) = 1$.

Remark 1. *Let us discuss these conditions:*

2. Let $X = U+iV$ be a complex random variable. The 2-dimensional real random vector $X = (U, V)$ is said to have a complex covariance structure if

$$Var(X) = \begin{pmatrix} \Sigma & -\xi \\ \xi & \Sigma \end{pmatrix},$$

where $\Sigma, \xi \in \mathbb{R}$. Since a real variance matrix is symmetric it follows that $Cov(U, V) = 0$ and it implies that U and V are independent with $Var(U) = Var(V)$.

3. This condition standarizes the mean and the variance of the real and imaginary part of the complex random variable: using $\mathbb{E}(X) = \mathbb{E}(U) + i\mathbb{E}(V)$, $Var(X) = Var(U) + Var(V)$ and $Var(U) = Var(V)$ we get

$$\mathbb{E}(U) = \mathbb{E}(V) = 0; \qquad Var(U) = Var(V) = \frac{1}{2}.$$

The three conditions lead to the following definition.

Definition 1.3.5. *(The standard complex Gaussian distribution) A complex random variable X has a (univariate) complex Gaussian distribution with mean zero and variance one if*

$$\mathcal{L}(X) = \mathcal{N}\left(0, \frac{1}{2}Id_2\right).$$

This is denoted by $\mathcal{L}(X) = \mathcal{N}_{\mathbb{C}}(0, 1)$.

Proposition 3. *Let X be a complex random variable with $\mathcal{L}(X) = \mathcal{N}_{\mathbb{C}}(0, 1)$. The density function of X with respect to Lebesgue measure on $\mathbb{C}$ is given as*

$$f_X(z) = \frac{1}{\pi} e^{-|z|^2}, \quad z \in \mathbb{C}. \tag{1.2}$$

Proof. From the bivariate real normal distribution we know that, when $\mathcal{L}(X) = \mathcal{N}\left(0, \frac{1}{2}\mathrm{Id}_2\right)$, then the density of X with respect to Lebesgue measure on $\mathbb{R}^2$ is given as

$$\begin{aligned} f_X([x]) &= (2\pi)^{-1} \det\left(\frac{1}{2}\mathrm{Id}_2\right)^{-\frac{1}{2}} e^{-\frac{1}{2}[x]^T\left(\frac{1}{2}\mathrm{Id}_2\right)^{-1}[x]} \\ &= \frac{1}{\pi} e^{-[x]^T[x]}, \quad [x] \in \mathbb{R}^2. \end{aligned}$$

Using the one-to-one correspondence between the univariate complex standard Gaussian distribution and this bivariate normal distribution established by the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$, the density function of X with respect to Lebesgue measure on the complex plane $\mathbb{C}$ is identical to the density function of X with respect to Lebesgue measure on $\mathbb{R}^2$. Then,

$$f_X(z) = \frac{1}{\pi}e^{-\bar{z}z} = \frac{1}{\pi}e^{-|z|^2}, \quad z \in \mathbb{C}.$$

□

We have proved that it's equivalent to define the standard complex Gaussian with the probability density (1.2) and with $X = U + iV$, where U and V are i.i.d. $\mathcal{N}\left(0, \frac{1}{2}\right)$ random variables.

For future use we compute the moments of the standard complex Gaussian.

Property 2. *Let X be a complex random variable with $\mathcal{L}(X) = \mathcal{N}_{\mathbb{C}}(0,1)$ and $n \geq 1$. Then,*

a) $\mathbb{E}(X^n) = 0$.

b) $\mathbb{E}(|X|^n) = \Gamma\left(\frac{n}{2}+1\right)$.

Proof. a) By definition and the density (1.2),

$$\mathbb{E}(X^n) = \int_{\mathbb{C}} z^n f_X(z)\, dm(z) = \int_{\mathbb{C}} z^n \frac{1}{\pi} e^{-|z|^2}\, dm(z)$$

Integrating in polar coordinates

$$\mathbb{E}(X^n) = \int_0^{2\pi}\int_0^{\infty} \rho^n e^{in\theta}\frac{1}{\pi}e^{-\rho^2}\rho\, d\rho\, d\theta = \frac{1}{\pi}\int_0^{2\pi} e^{in\theta}\, d\theta \int_0^{\infty} \rho^{n+1}e^{-\rho^2}\, d\rho = 0$$

because, for $n \geq 1$,

$$\int_0^{2\pi} e^{in\theta}\, d\theta = \frac{1}{in}e^{in\theta}\Big|_{\theta=0}^{\theta=2\pi} = \frac{1}{in}(e^{i2\pi n} - 1) = 0$$

b) Again, using polar coordinates and Definition 1.2.1 (with $p = \frac{n}{2} + 1 > 0$):

$$\begin{aligned}
\mathbb{E}(|X|^n) &= \int_{\mathbb{C}} |z|^n \frac{1}{\pi}e^{-|z|^2}\, dm(z) = \int_0^{2\pi}\int_0^{\infty} \rho^n \frac{1}{\pi}e^{-\rho^2}\rho\, d\rho\, d\theta \\
&= \frac{1}{\pi}\underbrace{\int_0^{2\pi} d\theta}_{2\pi} \int_0^{\infty} \rho^{n+1}e^{-\rho^2}\, d\rho = 2\int_0^{\infty} \rho^{n+1}e^{-\rho^2}\, d\rho \\
&= \int_0^{\infty} t^{\frac{n}{2}}e^{-t}\, dt = \Gamma\left(\frac{n}{2}+1\right)
\end{aligned}$$

□

In Chapter 3 we shall study holomorphic functions whose Taylor series coefficients are Gaussians (or multiples of Gaussians). The following result is important when studying the convergence of such series.

Proposition 4. *Let a_n be complex random independent variables with $\mathcal{L}(a_n) = \mathcal{N}_{\mathbb{C}}(0,1)$. Then $|a_n|^{1/n}$ converges almost surely towards* 1

$$\limsup_{n\to\infty} |a_n|^{1/n} = 1 \qquad a.s.$$

Proof. We are going to use Borel−Cantelli lemma. First note that $|a_n|^2$ is an exponential variable with parameter 1, because

$$P\left(|a_n|^2 \leq x\right) = \int_{\{z\in\mathbb{C}:|z|^2\leq x\}} \frac{1}{\pi} e^{-|z|^2}\, dz = \int_0^{2\pi}\int_0^{\sqrt{x}} \frac{1}{\pi} e^{-r^2} r\, dr\, d\theta = 1 - e^{-x}. \tag{1.3}$$

Seeing that $|a_n|^{1/n}$ converges almost surely towards 1 is equivalent to prove that

$$P\left(\limsup_{n\to\infty} |a_n|^{1/n} < 1\right) = P\left(\limsup_{n\to\infty} |a_n|^{1/n} > 1\right) = 0.$$

By definition:

$$\begin{aligned}\limsup_{n\to\infty} |a_n|^{1/n} < 1 &\iff \exists\varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ |a_n|^{1/n} < 1-\varepsilon \\ &\iff \exists\varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0\ |a_n| < (1-\varepsilon)^n.\end{aligned}$$

Let's consider

$$A = \limsup_{n\to\infty} A_n = \bigcap_{n\geq 1}\bigcup_{k\geq n} A_k,$$

where

$$A_k = \{w\ :\ |a_n(w)|^2 < (1-\varepsilon)^{2k}\} \quad k = 0,1,2,\ldots$$

We observe that $P(A_k) = 1 - e^{-(1-\varepsilon)^{2k}}$ (by Equation 1.3), and using the Taylor's approximation $1 - e^{-t} \approx t$ for $t \approx 0$, we obtain

$$\sum_k P(A_k) = \sum_k \left[1 - e^{-(1-\varepsilon)^{2k}}\right] \approx \sum_k (1-\varepsilon)^{2k} < +\infty.$$

Applying Borel-Cantelli's lemma, we conclude that

$$P\left(\limsup_{n\to\infty} |a_n|^{1/n} < 1\right) = 0.$$

□

Finally, we write a property of a sequence of independent Gaussians that we will use when computing the covariance kernel of a GAF (Section 3.4):

Proposition 5. *If* $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ *i.i.d., then*

$$\mathbb{E}(a_n\overline{a_m}) = \begin{cases} 1 & \textit{if} \quad n = m \\ 0 & \textit{if} \quad n \neq m \end{cases} .$$

Proof. If $n = m$, then, by Property 2(b) with $n = 2$ (or because $|a_n|^2$ is an exponential variable with parameter 1)

$$\mathbb{E}(a_n\overline{a_n}) = \mathbb{E}(|a_n|^2) = \Gamma\left(\frac{2}{2} + 1\right) = \Gamma(2) = 1.$$

Otherwise, if $n \neq m$, then, by the independence of $(a_k)_k$, we have that

$$\mathbb{E}(a_n\overline{a_m}) = \mathbb{E}(a_n)\mathbb{E}(\overline{a_m}) = 0.$$

□

Chapter 2

Probabilistic model: roots of the polynomial

2.1 Polynomials on the sphere

Let $\mathbb{P}_N[\mathbb{C}]$ be the space of polynomials of degree at most $N \in \mathbb{N}$. By the Fundamental Theorem of Algebra, $p_N \in \mathbb{P}_N[\mathbb{C}]$ has a factorization

$$p_N(z) = c(z - \xi_1) \cdots (z - \xi_n) = c \prod_{i=1}^{n} (z - \xi_i)$$

where each root ξ_i appears as many times as its multiplicity.

One of the nice reasons for studying polynomials on the Riemann sphere is that we can naturally see them as meromorphic functions $p_N : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ in the extended complex plane $\widehat{\mathbb{C}}$ (specifically, they have a single pole of order N and at ∞). To study p_N at ∞ we have to study $f(z) = p_N(1/z)$ at $z = 0$. A more general result is the following: (for all the details, see [Tai16])

Proposition 6. *A function is meromorphic on the Riemann sphere if and only if it is a rational function.*

There is a relationship between the location of the zeros of p_N and its critical points (the zeros of p'_N). This is precisely what we want to explore, in the case that the polynomials are random.

By Rolle's theorem, if a real polynomial p_N has all its roots real, then all the roots of its derivative are in the smallest closed interval containing all the roots of p_N. Let's see a beautiful extension of this result to $\mathbb{C}$, the classical Gauss-Lucas theorem.

Theorem 2.1.1. *(Gauss-Lucas) The critical points of a polynomial in one complex variable lie inside the convex hull of its zeros, that is, in the smallest convex polygon containing the zeros.*

Proof. Let $\xi_1, \xi_2, \dots, \xi_n \in \mathbb{C}$ be the zeros of the complex polynomial $p_N \in \mathbb{P}_N[\mathbb{C}]$.

Let β be a zero of p_N'. If β is a zero of p_N, then β is automatically in the convex hull of the points $\xi_1, \dots, \xi_n$. So, we suppose that β is not a zero of p_N.

By equation (1) in the introduction, we know that the critical points that are not zeros are the points where

$$0 = \sum_{i=1}^{n} \frac{1}{\beta - \xi_i}.$$

This is equivalent to

$$0 = \sum_{i=1}^{n} \frac{\overline{\beta} - \overline{\xi_i}}{|\beta - \xi_i|^2}$$

which is

$$\sum_{i=1}^{n} \frac{\overline{\xi_i}}{|\beta - \xi_i|^2} = \overline{\beta} \sum_{i=1}^{n} \frac{1}{|z - \xi_i|^2}.$$

Letting $\zeta = \sum_{i=1}^{n} \frac{1}{|z - \xi_i|^2}$, we get

$$\sum_{i=1}^{n} \frac{\frac{1}{|\beta - \xi_i|^2}}{\zeta} \xi_i = \beta.$$

So β is a linear combination of the roots ξ_i by non-negative coefficients adding up to 1. This proves that β is a convex combination of the ξ_i. □

A nice consequence of this theorem is the following: if the polynomial has real zeros, then the derivative must have real roots.

2.2 Electrostatic Interpretation of Zeros and Critical Points

From the introduction, we consider the polynomial

$$p_{N,\xi}(z) = \frac{1}{w^N}(w - \xi) \prod_{j=1}^{N-1} (w - \xi_j),$$

where ξ is a fixed zero and ξ_j, $j = 1, \dots, N-1$ are the remaining zeros i.i.d. uniformly distributed on the Riemann sphere.

The polynomial $p_{N,\xi}$ generates the electric field given by equation (2):

$$E_N(w) = -\frac{N}{w} + \frac{1}{w - \xi} + \sum_{j=1}^{N-1} \frac{1}{w - \xi_j}.$$

The first term is the contribution from the $-N$ charges at ∞ and is of order N. The second term is the contribution from the $+1$ charge at ξ, which is also of order N if $|w-\xi| \approx N^{-1}$. Since the zeros $\xi_1, \ldots, \xi_{N-1}$ by assumption are distributed uniformly, for large N, the third term has generically order $\sqrt{N}$, by the central limit theorem. Let's see this:

Proposition 7. *The random term* $\displaystyle\sum_{j=1}^{N-1} \frac{1}{w-\xi_j}$ *has order* $\sqrt{N}$ *with high probability.*

Proof. We note that $\mathbb{E}\left(\dfrac{1}{w-\xi_j}\right) = 0$ for all $j = 1, \ldots, N-1$, by the spherical symmetry. Thus, $\mathbb{E}\left(\displaystyle\sum_{j=1}^{N-1} \frac{1}{w-\xi_j}\right) = 0$.

Letting $X_j = \dfrac{1}{w-\xi_j}$, the central limit theorem yields

$$X_1 + \cdots + X_{N-1} = \frac{\frac{X_1+\cdots+X_{N-1}}{N-1} - 0}{\frac{\sigma}{\sqrt{N-1}}} \cdot \frac{N-1}{\sqrt{N}}\sigma \sim N_{\mathbb{R}}(0,1)\sqrt{N}\sigma$$

and therefore $X_1 + \cdots + X_{N-1}$ has order $\sqrt{N}$ with very high probability. □

In the formula (2), we have that $\mathbb{E}\left(\displaystyle\sum_{j=1}^{N-1} \frac{1}{w-\xi_j}\right) = 0$, and therefore the expected critical point $w_{N,\xi}$ is the one that results from solving the equation

$$-\frac{N}{w} + \frac{1}{w-\xi} = 0 \Longrightarrow w_{N,\xi} = \xi\left(1 - \frac{1}{N}\right)^{-1}. \tag{2.1}$$

Then the expected value of $E_N(w)$ is

$$\mathbb{E}[E_N(w)] = -\frac{N}{w} + \frac{1}{w-\xi}.$$

We want to use Rouché's Theorem and the concentration of these variables around the mean to justify the existence of a true critical point (a zero of E_N) near the zero of the expectation.

If we find a curve $\Gamma_{N,\xi}$ of index 1 around $w_{N,\xi}$ with:

i) $|E_N(w) - \mathbb{E}[E_N(w)]| < |\mathbb{E}[E_N(w)]| \quad w \in \Gamma_{N,\xi}$,

ii) $d_C(w, w_{N,\xi}) \lesssim 1/N \quad w \in \Gamma_{N,\xi}$

we will have, by Rouché's theorem,

$$\#\{\text{zeros of } E_N(w) - \mathbb{E}[E_N(w)] \text{ closed by } \Gamma_{N,\xi}\} = \\ = \#\{\text{zeros of } \mathbb{E}[E_N(w)] \text{ closed by } \Gamma_{N,\xi}\} = 1$$

That is, there will be a single critical point χ_ξ of p_N in the region bounded by $\Gamma_{N,\xi}$. By the second condition, we will have

$$d_C(\xi, \chi_\xi) \leq d(\xi, w_{N,\xi}) + d(w_{N,\xi}, \chi_\xi) \lesssim 1/N$$

In summary, we will have that χ_ξ is a small perturbation of $w_{N,\xi}$, with very high probability.

2.3 Statement and proof of main result

To write this section I followed [Han16].

Theorem 2.3.1. *(Pairing of a Zero and a Critical Point). Let $\eta \in \mathbb{S}^2 \setminus \{0, \infty\}$ be fixed and let $\xi = \pi(\eta) \in \mathbb{C}$, i.e. ξ is the projection of point $\eta \in \mathbb{S}^2$. Let $p_{N,\xi}$ be a random polynomial as described above: the zero ξ fixed and the remaining $N-1$ zeros $\xi_j = \pi(\eta_j)$ are chosen in $\mathbb{S}^2$ with uniform probability. Fix $r > 0$ and define*

$$\Gamma_N = \left\{ w \in \mathbb{C} \,:\, d_C(w, w_{N,\xi}) = \frac{r}{N} \right\},$$

where $w_{N,\xi}$ is defined in the equation (2.1). Suppose that $\xi \notin \Gamma_N$ for all $N \geq 1$. Then, for any $\delta \in (0,1)$, there exists $C = C(r,\delta) > 0$ such that for all $N \geq 1$

$$P\left(\exists!\, w \in D_c\left(w_{N,\xi}, \frac{r}{N}\right) \text{ with } \frac{dp_{N,\xi}}{dw}(w) = 0 \right) \geq 1 - \frac{C}{N^\delta}.$$

Observation 2. *The $\xi \notin \{0, \infty\}$ condition is not a drawback of this method. At $\xi = 0$ (south pole) the contribution of the N charges to the opposite pole is not felt, because of isotropy. Thus, near 0 in the field E_N dominates the part of the zeros, which has statistical fluctuations that cannot be neglected now.*

Remark 2. *The theorem holds equally well if we replace $\Gamma_{N,\xi}$ by a curve with winding number 1 around $w_{\xi,N}$ that does not pass through ξ and satisfies:*

(i) There exists $c_1 > 0$ such that

$$\inf_{w \in \Gamma_{N,\xi}} \left| \mathbb{E}\left[\partial \log |p_{N,\xi}(w)|^2 \right] \right| \geq c_1 N.$$

This means that $\inf_{w \in \Gamma_{N,\xi}} |\mathbb{E}[E_N(w)]| \geq c_1 N$, and this is the condition that allows us to obtain

$$|E_N(w) - \mathbb{E}[E_N(w)]| < |\mathbb{E}[E_N(w)]| \quad w \in \Gamma_{N,\xi}$$

and then apply Rouché's Theorem.

(ii) There exists $c_2 > 0$ such that for all N

$$\sup_{w \in \Gamma_{N,\xi}} d_C(w, \xi) \leq \frac{c_2}{N}.$$

This is the condition that allows us to conclude that the critical point (zero of $E_N(w) - \mathbb{E}[E_N(w)]$) is at distance $\lesssim 1/N$ from $w_{N,\xi}$ (zero of $\mathbb{E}[E_N(w)]$).

Observation 3. *The order of growth $1 - C \cdot N^{-\delta}$ is optimal, in the sense that, due to the uniformity and independence of the zeros*

$$P\left(\exists \xi_j \in D_c\left(\xi, \frac{1}{N}\right)\right) \approx \frac{1}{N} \tag{2.2}$$

and if such a zero exists it will distort the presence of the critical point.

Now, we justify equation (2.2): we define the variables

$$X_r := \#\{\xi_j \,:\, \xi_j \in D_c(a, r)\}$$

for $r > 0$. Then

(i) X_r does not depend on a (center of the disk).

(ii) X_r follows a binomial distribution with parameters $N-1$ and $p = \nu\left(D_c(a, r)\right) = \dfrac{r^2}{1+r^2}$, by Proposition 2.

We consider $X_{1/N}$ to be a binomial with parameters as described above: $N-1$ and $p = \nu\left(D_c\left(\xi, \frac{1}{N}\right)\right) \simeq \frac{1}{N^2}$. Then,

$$\begin{aligned}
P\left(\exists \xi_j \in D_c\left(\xi, \frac{1}{N}\right)\right) &= P(X_{1/N} > 0) = 1 - P(X_{1/N} = 0) \\
&= 1 - \binom{N}{0} p^0 (1-p)^N = \left[1 - \left(1 - \frac{1}{N^2}\right)^N\right] \\
&= \left[1 - \left(1 - \frac{1}{N^2}\right)\right] \times \\
&\times \left[1^{N-1} + 1^{N-2}\left(1 - \frac{1}{N^2}\right)^1 + \cdots + 1^0\left(1 - \frac{1}{N^2}\right)^{N-1}\right] \\
&\simeq \frac{1}{N^2}\left[1 + 1 + \cdots + 1\right] = \frac{1}{N}.
\end{aligned}$$

Proof. (of Theorem 2.3.1)

We work in coordinates centered at ∞ and we fix $\xi \in S^2 \setminus \{0, \infty\}$. Let $p_{N,\xi}$ and

$w_{\xi,N} =: w_\xi$ be as described in Equation (2.1). Let also be the equation obtained in (2)

$$E_N(w) = -\frac{N}{w} + \frac{1}{w-\xi} + \sum_{j=1}^{N-1} \frac{1}{w-\xi_j}.$$

Recall that a critical point is a zero of $E_N(w)$, and that

$$\mathbb{E}[E_N(w_\xi)] = -\frac{N}{w_\xi} + \frac{1}{w_\xi - \xi} = 0.$$

The curve Γ_N satisfies

(i) There exists $c_2 = c_2(r, \xi) > 0$ such that

$$\sup_{w \in \Gamma_N} |w - \xi| \leq \frac{c_2}{N}.$$

This follows directly from the triangular inequality

$$|w - \xi| \leq |w - w_\xi| + |w_\xi - w| = |w - w_\xi| + \frac{1}{N-1}$$

and the condition of Γ_N. In the chordal metric, since w and ξ are nearby

$$d_C(w, \xi) = \frac{|w-\xi|}{\sqrt{1+|w|^2}\sqrt{1+|\xi|^2}} \approx \frac{|w-\xi|}{1+|\xi|^2} \lesssim \frac{1/N}{1+|\xi|^2}.$$

(ii) There exists $c_1 = c_1(r, \xi) > 0$ such that

$$\inf_{w \in \Gamma_N} \left| -\frac{N}{w} + \frac{1}{w-\xi} \right| \geq c_1 N.$$

This follows from the fact that, by (i), there exists c such that

$$\left| -\frac{N}{w} + \frac{1}{w-\xi} \right| = N \left| -\frac{1}{w} + \frac{1/N}{w-\xi} \right| \geq N \left| \frac{1}{|w|} - \frac{1/N}{|w-\xi|} \right| \geq N \left| \frac{1}{|w|} - cr \right|$$

It is now sufficient to choose r so that

$$\left| \frac{1}{|w|} - cr \right| \geq c_3(r, \xi) > 0 \qquad \text{for } w \in \Gamma_N$$

We note that this is equivalent to $|1 - cr|w|| \geq c_3|w|$, i.e.

$$1 \geq (c_3 + cr)|w| \quad ; \quad |w| \leq \frac{1}{c_3 + cr}$$

or

$$(cr - c_3)|w| \geq 1 \quad ; \quad |w| \geq \frac{1}{cr - c_3}$$

We can always choose c_3 and r that fulfills this.

By the way, these two conditions (i) and (ii) are precise those in Remark 2.

Let

$$\widetilde{E}_N(w) := E_N(w) - \mathbb{E}[E_N(w)] = \sum_{j=1}^{N-1} \frac{1}{w-\xi_j}$$

be the random part of $E_N(w)$. It will be enough to see the following.

Lemma 1. *Fix $\delta > 0$. There exists $\gamma = \gamma(c_1, c_2, \delta) > 0$ and $C_3 = C_3(c_1, c_2, \delta)$ such that*

$$P\left(\sup_{w\in\Gamma_N} \left|\widetilde{E}_N(w)\right| \le N^{1-\gamma}\right) \ge 1 - \frac{C_3}{N^\gamma}.$$

As soon as we prove this, in the event that $\sup_{w\in\Gamma_N} \left|\widetilde{E}_N(w)\right| \le N^{1-\gamma}$ we have, for $w \in \Gamma_N$ (for (ii) and $N \ge N_0$)

$$\left|\widetilde{E}_N(w)\right| \le N^{1-\gamma} < c_1 N \le \left|\mathbb{E}[E_N(w)]\right|,$$

and by Rouché's Theorem

$$\#\mathcal{Z}(\widetilde{E}_N) \cap \Gamma_N^\circ = \#\mathcal{Z}(\mathbb{E}[E_N]) \cap \Gamma_N^\circ = \#\{w_\xi\} = 1,$$

where $\Gamma_N^\circ = D_c\left(w_{N,\xi}, \frac{r}{N}\right)$ denotes the region closed by Γ_N.

That is, there is a (unique) critical point of $p_{N,\xi}$ inside Γ_N with probability at least $1 - C_3 \cdot N^{-\gamma}$, as desired.

The proof of Lemma 1 is not sophisticated, but it is a bit technical. Before giving the details we give a brief outline.

1. $\sup_{w\in\Gamma_N} |\widetilde{E}_N(w)|$ is not simple to estimate directly, since it fluctuates a lot. We will estimate the fluctuations of $\widetilde{E}_N(w) - \widetilde{E}_N(w_\xi)$ and $\widetilde{E}_N(w_\xi)$ separately considering that

$$\widetilde{E}_N(w) = \widetilde{E}_N(w) - \widetilde{E}_N(w_\xi) + \widetilde{E}_N(w_\xi).$$

2. To estimate $\widetilde{E}_N(w) - \widetilde{E}_N(w_\xi)$ we will use that $|w - w_\xi| \approx 1/N$ to discard the contribution coming from the zeros that are far from ξ (and therefore from w

and from w_ξ, since $w \in \Gamma_N$). To see this, estimate

$$\begin{aligned}
\sup_{w\in\Gamma_N}\left|\sum_{|\xi_j-\xi|>\frac{1}{N^{1/2-\delta/2}}}\left(\frac{1}{w-\xi_j}-\frac{1}{w_\xi-\xi_j}\right)\right| &\leq \sup_{w\in\Gamma_N}\sum_{|\xi_j-\xi|>\frac{1}{N^{1/2-\delta/2}}}\frac{|w_\xi-w|}{|w-\xi_j|\,|w_\xi-\xi_j|}\\
&\approx \sup_{w\in\Gamma_N}\sum_{|\xi_j-\xi|>\frac{1}{N^{1/2-\delta/2}}}\frac{1/N}{N^{-(1/2-\delta/2)}N^{-(1/2-\delta/2)}}\\
&\approx \frac{N^{1-\delta}}{N}\#\left\{\xi_j\ :\ |\xi_j-\xi|>\frac{1}{N^{1/2-\delta/2}}\right\}\\
&\leq \frac{N^{1-\delta}}{N}(N-1)\\
&\leq N^{1-\delta}.
\end{aligned}$$

This shows that the contribution of the ξ_j with $|\xi_j-\xi|>N^{-\frac{1-\delta}{2}}$ (zeros that are far from ξ) is negligible. That is, fixed $\delta>0$, the contribution of these is at most of order $N^{1-\delta}$.

3. It remains to estimate the sum corresponding to the zeros ξ_j near ξ. Let

$$\widetilde{E}_N(w,\delta) := \sum_{|\xi-\xi_j|\leq\frac{1}{N^{1/2-\delta/2}}}\frac{1}{w-\xi_j}.$$

To prove Lemma 1 it is enough to see that there exist constants $\gamma=\gamma(c_1,c_2,\delta)>0$, $K_2=K_2(c_1,c_2,\delta)>0$ and $K_3=K_3(c_1,c_2,\delta)>0$ such that

$$P\left(\sup_{w\in\Gamma_N}\left|\widetilde{E}_N(w_\xi,\delta)-\widetilde{E}_N(w,\delta)\right|\geq N^{1-\gamma}\right)\leq\frac{K_2}{N^\delta} \tag{2.3}$$

and

$$P\left(\left|\widetilde{E}_N(w_\xi)\right|\geq N^{1-\gamma}\right)\leq\frac{K_3}{N^\gamma}. \tag{2.4}$$

With this, since

$$\widetilde{E}_N(w)=\widetilde{E}_N(w)-\widetilde{E}_N(w_\xi)+\widetilde{E}_N(w_\xi)$$

we see that

$$\begin{aligned}
\left\{\sup_{w\in\Gamma_N}\left|\widetilde{E}_N(w)\right|\leq 2N^{1-\gamma}\right\} \supseteq \left\{\sup_{w\in\Gamma_N}\left|\widetilde{E}_N(w)-\widetilde{E}_N(w_\xi)\right|\leq N^{1-\gamma}\right\}\cap\\
\cap\left\{\left|\widetilde{E}_N(w_\xi)\right|\leq N^{1-\gamma}\right\}
\end{aligned}$$

and therefore

$$\begin{aligned}
P\left(\sup_{w\in\Gamma_N}\left|\widetilde{E}_N(w)\right|\leq 2N^{1-\gamma}\right) &\geq P\left(\sup_{w\in\Gamma_N}\left|\widetilde{E}_N(w)-\widetilde{E}_N(w_\xi)\right|\leq N^{1-\gamma}\right)\cdot\\
&\cdot P\left(\left|\widetilde{E}_N(w_\xi)\right|\leq N^{1-\gamma}\right)\\
&\geq\left(1-\frac{K_2}{N^\delta}\right)\left(1-\frac{K_3}{N^\delta}\right)\geq 1-\frac{(K_2+K_3)}{N^\delta},
\end{aligned}$$

as desired.

There are two reasons to believe that (2.3) and (2.4) hold. First, if $|\xi - \xi_j| \approx 1/N$ for some j, then both $\widetilde{E}_N(w)$ and $\widetilde{E}_N(w_\xi)$ will be generically of order N, because of the $\frac{1}{w-\xi_j}$ (or $\frac{1}{w_\xi-\xi_j}$) term. But this occurs with very small probability. Therefore, we will not have large $\widetilde{E}_N(w_\xi,\delta)$ or $\widetilde{E}_N(w,\delta) - \widetilde{E}_N(w_\xi,\delta)$ because there is a large summand (except in cases of small probability).

The other way to make these terms large is for there to be more than $N^{1/2+\delta/2}$ zeros ξ_j with $|\xi_j - \xi| \leq \frac{1}{N^{1/2-\delta/2}}$; then each $\frac{1}{w-\xi_j}$ term in $\widetilde{E}_N(w_\xi,\delta)$ will be large, of size minus $N^{1/2-\delta/2}$, and all added together can produce an $\widetilde{E}_N(w,\delta)$ of size $N^{1/2+\delta/2}N^{1/2-\delta/2} = N$. But this grouping of zeros has also very small probability.

In order to quantify all this and prove the steps outlined above, we consider the random variable explained in Observation 3

$$\mathcal{N}(\xi, R) := X_R = \#\{j \,|\, d_C(\xi_j,\xi) \leq R\}.$$

Observe that $\mathcal{N}(\xi,R)$ is a binomial variable with parameters $N-1$ and $p = \nu(D_c(\xi_j,R))$ and the distribution of $\mathcal{N}(\xi,R)$ does not depend on ξ. Proposition 2 yields

$$p = \frac{R^2}{1+R^2}.$$

Next lemma follows from the properties of the binomial random variable and it will be crucial in our estimates.

Lemma 2. *Fix $\eta \in \left(0,\frac{1}{2}\right)$ and $\kappa > 0$. There exist $K = K(\eta) > 0$ and $K' = K'(\kappa,\delta) > 0$ so that*

(i) $P\left(\mathcal{N}\left(w_\xi, \frac{1}{N^{1-\eta}}\right) \geq 1\right) \leq \frac{K}{N^{1-2\eta}}$

(ii) $P\left(\mathcal{N}\left(w_\xi, \frac{1}{N^{1/2-\delta/2}}\right) \geq N^{\delta+\kappa}\right) \leq \frac{K'}{N^{\delta+2\kappa}}.$

For the estimates, we will want to use Chebyshev's inequality and so we want to estimate the variance.

As ξ_j are independent and $\mathbb{E}\left(\frac{1}{w-\xi}\right) = 0$ for all ξ, we have

$$\mathbb{E}\left[\left|\widetilde{E}_N(w)\right|^2\right] = \operatorname{Var}\left[\widetilde{E}_N(w)\right] = \sum_{j=1}^{N-1} \operatorname{Var}\left(\frac{1}{w-\xi_j}\right) = \sum_{j=1}^{N-1} \mathbb{E}\left[\frac{1}{|w-\xi_j|^2}\right].$$

Each $\mathbb{E}\left(\frac{1}{|w-\xi|^2}\right)$ term is infinite, since the singularity of order 2 is not integrable with respect to the area.

But the conditional variance given $\mathcal{N}\left(w, \frac{1}{N^{1-\eta}}\right) = 0$ (no zeros ξ in $D_c\left(w, \frac{1}{N^{1-\eta}}\right)$) is quite small and allows us to get a good estimate on the tail probability in (2.4). This is the content of the following lemma.

Lemma 3. *Fix $\eta \in \left(0, \frac{1}{2}\right)$ and write A for the event*

$$A = \left\{ \omega \,:\, \sum_{j=1}^{N-1} \frac{1}{|w_\xi - \xi_j|^2} > N^{2(1-\eta)} \right\}.$$

There exists $K = K(\eta)$ such that

$$P(A) \leq K \frac{\log N}{N^{1-2\eta}}.$$

Observation 4. *This implies that for any $\varepsilon > 0$ with probability at least $1 - \frac{K}{N^{1-(2\eta+\varepsilon)}}$ we have $\sum_{j=1}^{N-1} \frac{1}{|w_\eta - \xi_j|^2} < N^{2(1-\eta)}$. In this situation we will be able to use Markov and Chebyshev inequalities.*

Now, we are going to give all the details. We start with Lemma 2.

Proof. (of Lemma 2)

(i) Recall that if X follows a binomial of parameter (M, p), then

$$P(X = k) = \binom{M}{k} p^k (1-p)^{M-k} \qquad k = 0, \ldots, M.$$

Here we have variable $X = \mathcal{N}(w, R)$ with parameters $M = N - 1$ and $p = \dfrac{R^2}{1 + R^2}$.

Thus,

$$\begin{aligned}
P\left(\mathcal{N}(w, N^{-1+\eta}) \geq 1\right) &= 1 - P\left(\mathcal{N}(w, N^{-1+\eta}) < 1\right) \\
&= 1 - \nu(D_c(0, N^{-1+\eta})) \\
&= 1 - (1-p)^{N-1} \\
&= p\left[(1-p)^{N-2} + \cdots + 1\right] \\
&\leq p(N-1) \approx \frac{1}{N^{2-2\eta}} N = \frac{1}{N^{1-2\eta}}
\end{aligned}$$

since for R small (here $R = 1/N^{1-\eta}$), $\nu(D_c(\xi, R^2)) \approx \pi R^2$ by Proposition 2.

(ii) Since for a binomial X of parameters (M, p), $\operatorname{Var} X = Mp(1-p)$, we have

$$\begin{aligned}
\operatorname{Var}\left(\mathcal{N}\left(w, N^{-\frac{1}{2}+\frac{\delta}{2}}\right)\right) &\approx (N-1)\left(N^{-\frac{1}{2}+\frac{\delta}{2}}\right)^2 \left(1 - \left(N^{-\frac{1}{2}+\frac{\delta}{2}}\right)^2\right) \\
&\approx N \cdot N^{-1} N^{\delta} \left(1 - \frac{1}{N^{1-\delta}}\right) \leq N^{\delta}.
\end{aligned}$$

Then by Chebyshev's inequality,

$$P\left(\mathcal{N}\left(w, N^{-\frac{1}{2}+\frac{\delta}{2}}\right) \geq N^{\delta+\kappa}\right) \leq \frac{\operatorname{Var}\left(\mathcal{N}\left(w, N^{-\frac{1}{2}+\frac{\delta}{2}}\right)\right)}{\left(N^{\delta+\kappa}\right)^2} \leq \frac{N^{\delta}}{N^{2\delta}N^{2\kappa}}.$$

□

Proof. (of Lemma 3) Define the event

$$B = \{\omega : \mathcal{N}(w_\xi, N^{-1+\eta}) = 0\},$$

which has probability

$$P(B) = \binom{N-1}{0} q^0 (1-q)^{N-1} \sim \left(1 - \frac{c}{N^{2-2\eta}}\right)^{N-1},$$

because

$$q = \frac{\frac{1}{N^{(1-\eta)2}}}{1 + \frac{1}{N^{(1-\eta)2}}} \sim \frac{1}{N^{(1-\eta)2}}.$$

Observe that:

$$-\log P(B) = -(N-1)\log\left(1 - \frac{c}{N^{2-2\eta}}\right) \approx (N-1)\frac{c}{N^{2-2\eta}} \approx \frac{1}{N^{1-2\eta}}.$$

So, with a suitable constant c,

$$P(B) \geq e^{-\frac{c}{N^{1-2\eta}}} \geq 1 - \frac{c}{N^{1-2\eta}}.$$

We now study the variance on $\widetilde{E}_N(w_\xi)$ under this event. We have

$$\mathbb{E}\left[\sum_{j=1}^{N-1} \frac{1}{|w_\xi - \xi_j|^2} \Big| B\right] = \sum_{j=1}^{N-1} \mathbb{E}\left[\frac{1}{|w_\xi - \xi_j|^2}\Big| B\right]$$

Rotating the sphere so that $w_\xi = 0$ (south pole) and computing in polar coordinates, we have

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{|w_\xi - \xi_j|^2}\Big| B\right] &= \mathbb{E}\left[\frac{1}{|\xi_j|^2}\Big|\mathcal{N}(0, N^{-1+\eta}) = 0\right] \\
&= \int_{|z|>N^{-1+\eta}} \frac{1}{|z|^2}\frac{dm(z)}{\pi(1+|z|^2)^2} \\
&= \int_{N^{-1+\eta}}^{\infty} \frac{2r}{r^2(1+r^2)^2}\, dr \\
&= \int_{N^{-2+2\eta}}^{\infty} \frac{dt}{t(1+t)^2} \\
&= \int_{N^{-2+2\eta}}^{1} \frac{dt}{t(1+t)^2} + \int_1^{\infty} \frac{dt}{t(1+t)^2} \\
&\leq \int_{N^{-2+2\eta}}^{1} \frac{dt}{t} + \int_1^{\infty} \frac{dt}{t^3} \\
&= \log\frac{1}{N^{-2+2\eta}} + \frac{1}{2} = (2-2\eta)\log N + \frac{1}{2}.
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\sum_{j=1}^{N-1} \frac{1}{|w_\xi - \xi_j|^2} \Big| B\right] \leq (N-1)\left[2(1-\eta)\log N + \frac{1}{2}\right]$$
$$\leq c(1-\eta)\cdot N\log N. \tag{2.5}$$

With this and by Lemma (2), we deduce that

$$\begin{aligned}
|P(A) - P(A|B)| &= \left|P(A\cap B) + P(A\cap B^c) - \frac{P(A\cap B)}{P(B)}\right| \\
&\leq P(A\cap B)\left|1 - \frac{1}{P(B)}\right] + P(A\cap B^c) \\
&\leq \frac{1-P(B)}{P(B)} + P(B^c) \\
&\leq \frac{cN^{-1+2\eta}}{1-cN^{-1+2\eta}} + cN^{-1+2\eta} \\
&\leq 3cN^{-1+2\eta}.
\end{aligned}$$

By Markov's inequality and by the definition of A

$$\begin{aligned}
P(A|B) &= P\left[\sum_{j=1}^{N-1} \frac{1}{|w_\xi - \xi_j|^2} > N^{2(1-\eta)} \;\Big|\; B\right] \\
&\leq \frac{\mathbb{E}\left[\sum_{j=1}^{N-1} \frac{1}{|w_\eta - \xi_j|^2} \;\Big|\; B\right]}{N^{2(1-\eta)}} \\
&\leq c\frac{(1-\eta)N\log N}{N^{2-2\eta}} = c\frac{(1-\eta)\log N}{N^{1-2\eta}}.
\end{aligned}$$

Then, finally

$$\begin{aligned}
P(A) &\leq P(A|B) + |P(A) - P(A|B)| \leq c(1-\eta)\frac{\log N}{N^{1-2\eta}} + \frac{3c}{N^{1-2\eta}} \\
&\leq K\cdot\frac{\log N}{N^{1-2\eta}},
\end{aligned}$$

as desired. □

Proof. (of estimate 2.3)

Recall that

$$\widetilde{E}_N(w,\delta) := \sum_{|\xi-\xi_j|\leq \frac{1}{N^{1/2-\delta/2}}} \frac{1}{w-\xi_j}.$$

By definition, and by the condition $\sup_{w\in\Gamma_N} |w-\xi| \leq c_2/N$, for $w \in \Gamma_N$

$$\left|\widetilde{E}_N(w_\xi,\delta) - \widetilde{E}_N(w,\delta)\right| = \left| \sum_{|\xi-\xi_j|\leq \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{w-w_\xi}{(w_\xi-\xi_j)(w-\xi_j)} \right|$$
$$\leq \frac{c_2}{N} \left| \sum_{|\xi-\xi_j|\leq \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{1}{(w_\xi-\xi_j)(w-\xi_j)} \right|$$

We split

$$\begin{aligned}\frac{1}{(w_\xi-\xi_j)(w-\xi_j)} &= \frac{1}{(w_\xi-\xi_j)(w-\xi_j)} - \frac{1}{(w_\xi-\xi_j)^2} + \frac{1}{(w_\xi-\xi_j)^2} \\ &= \frac{1}{w_\xi-\xi_j}\left[\frac{1}{w-\xi_j} - \frac{1}{w_\xi-\xi_j}\right] + \frac{1}{(w_\xi-\xi_j)^2} \\ &= \frac{1}{w_\xi-\xi_j}\frac{w_\xi-w}{(w_\xi-\xi_j)(w-\xi_j)} + \frac{1}{(w_\xi-\xi_j)^2}\end{aligned}$$

As $|w_\xi - w| \leq c_2/N$ and $(w_\xi - w)$ does not depend on the summation index

$$\left|\widetilde{E}_N(w_\xi,\delta) - \widetilde{E}_N(w,\delta)\right| \leq \frac{c_2^2}{N^2} \left| \sum_{|\xi-\xi_j|\leq \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{1}{(w_\xi-\xi_j)^2(w-\xi_j)} \right| +$$
$$+ \frac{c_2}{N} \left| \sum_{|\xi-\xi_j|\leq \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{1}{(w_\xi-\xi_j)^2} \right|$$

Iterating this procedure, for every $l \geq 1$, adding and subtracting $(w_\xi - \xi_j)^{-l}$, we get

$$\left|\widetilde{E}_N(w_\xi,\delta) - \widetilde{E}_N(w,\delta)\right| \leq \frac{c_2^l}{N^l} \left| \sum_{|\xi-\xi_j|\leq \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{1}{(w_\xi-\xi_j)^l(w-\xi_j)} \right| \tag{2.6}$$
$$+ \sum_{k=1}^{l-1} \frac{c_2^k}{N^k} \left| \sum_{|\xi-\xi_j|\leq \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{1}{(w_\xi-\xi_j)^{k+1}} \right| \tag{2.7}$$

We note that w appears only in the first summand; the second summand only w_ξ comes out. We choose l large enough so that

$$\delta < \frac{l+1}{l+5}$$

in order to be able to achieve a bound of (2.6).

By Lemma 2, there exists $C = C(\delta) > 0$ such that, except perhaps for an event of probability at most C/N^δ, we have

$$\mathcal{N}\left(w_\xi, \frac{1}{N^{\frac{1+\delta}{2}}}\right) = 0 \quad \text{and} \quad \mathcal{N}\left(w_\xi, \frac{1}{N^{\frac{1-\delta}{2}}}\right) \le N^{2\delta}.$$

Then, since under this event (by assumption (ii))

$$\begin{aligned} |w - \xi_j| \ge |w_j - \xi_j| - |w_\xi - w| &\ge \frac{1}{N^{1/2+\delta/2}} - \frac{1}{N} \\ &= \frac{1}{N^{1/2+\delta/2}}\left(1 - \frac{1}{N^{1/2-\delta/2}}\right) \ge \frac{C}{N^{1/2+\delta/2}}, \end{aligned}$$

we can bound the terms of the first summand (2.6):

$$\begin{aligned} N^{-l} \sum_{|w_\xi - \xi_j| \le \frac{1}{N^{1/2-\delta/2}}} \frac{1}{|w_\xi - \xi_j|^l |w - \xi_j|} &\le N^{-l} \sum_{|w_\xi - \xi_j| \le \frac{1}{N^{1/2-\delta/2}}} \frac{1}{N^{-l(\frac{1+\delta}{2})} N^{\frac{1+\delta}{2}}} \\ &\le N^{-l} N^{2\delta} N^{l\frac{1+\delta}{2}} N^{\frac{1+\delta}{2}} \\ &= N^{-l\frac{1-\delta}{2}} N^{\frac{1+5\delta}{2}} = N^{1-\gamma}, \end{aligned}$$

with

$$\gamma = 1 + l\frac{1-\delta}{2} - \frac{1+5\delta}{2} = \frac{1}{2}(1-5\delta) + l\frac{1-\delta}{2} \ge \frac{1-5\delta}{2}$$

because $\delta < \dfrac{l+1}{l+5}$.

This occurs with probability at least $1 - \dfrac{C''}{N^\delta}$, and therefore with this high probability we have the first summand (7) bounded by $CN^{1-\gamma}$.

Now we bound the other summand (term (8)).

By Lemma 3 with $\eta \in \left(\dfrac{1-\delta}{2}, \dfrac{1}{2}\right)$ we have $\displaystyle\sum_{j=1}^{N-1} \frac{1}{|w_\xi - \xi_j|^2} \le N^{2(1-\eta)}$ with probability $1 - K\dfrac{\log N}{N^{1-\eta}}$ or higher.

We also use this well-known inequality for ℓ^p-norms: if $\{x_k\}_k$ is a sequence in $\mathbb{C}$ and $1 \le p \le q < \infty$, then

$$\left(\sum_{k=1}^{\infty} |x_k|^q\right)^{\frac{1}{q}} \le \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}}.$$

We use this with $p = 2$ and $q = k+1 \ge 2$. Then, with probability at least

$1 - K\dfrac{\log N}{N^{1-2\eta}}$

$$\sum_{k=1}^{l-1} c_2^k N^{-k} \left| \sum_{|w_\xi - \xi_j| \le N^{-1/2+\delta/2}} \frac{1}{(w_\xi - \xi_j)^{k+1}} \right| \le \sum_{k=1}^{l-1} c_2^k N^{-k} \left(\sum_{|w_\xi - \xi_j| \le \frac{1}{N^{\frac{1-\delta}{2}}}} \frac{1}{|w_\xi - \xi_j|^2} \right)^{\frac{k+1}{2}}$$
$$\le C' \sum_{k=1}^{l-1} N^{-k} N^{(2-2\eta)\frac{k+1}{2}} = C' N^{1-\eta} \sum_{k=1}^{l-1} N^{-k+(1-\eta)k}$$
$$\approx N^{1-\eta} \sum_{k=1}^{l-1} \left(\frac{1}{N^\eta}\right)^k \le N^{1-\eta} \sum_{k=1}^{\infty} \left(\frac{1}{N^\eta}\right)^k$$
$$= N^{1-\eta} \frac{\frac{1}{N^\eta}}{1 - \frac{1}{N^\eta}} \approx N^{1-2\eta}.$$

As $2\eta < 1$, there is a $\delta > 0$ with $N^{1-2\eta} \lesssim N^\delta$. □

Proof. (of estimate (2.4)) We set $\eta = \dfrac{1}{2}(1-\delta)$ and consider, as before, the event

$$B = \{\omega : \mathcal{N}(w_\xi, N^{-1+\eta}) = 0\} = \{\omega : \mathcal{N}(w_\xi, N^{-\frac{1+\delta}{2}}) = 0\}.$$

Recall that $\mathcal{N}$ (see Observation 3) is a Bernoulli with parameter $p = \nu(D_c(w_\xi, N^{-\frac{1+\delta}{2}})) = \dfrac{N^{-(1+\delta)}}{1+N^{-(1+\delta)}} \approx N^{-(1+\delta)}$ by Proposition 2. Therefore

$$P(B) = (1-p)^{N-1} \ge 1 - \frac{C}{N^\delta}, \tag{2.8}$$

because

$$-\log P(B) = -(N-1)\log(1-p) \le Np \approx N^{-\delta}$$

that is

$$P(B) \le e^{-\frac{C}{N^\delta}} \approx 1 - \frac{C}{N^\delta}.$$

Then, by the independence of the $\dfrac{1}{w_\xi - \xi_j}$, $j = 1, \dots, N-1$, and by the inequality (2.5) given on the proof of the lemma 3 ($1-\eta = \frac{1+\delta}{2}$)

$$\mathbb{E}\left[\left| \widetilde{E}_N(w_\xi) \right|^2 \,\middle|\, B \right] = \mathbb{E}\left[\sum_{j,k=1}^{N-1} \frac{1}{w_\xi - \xi_j} \cdot \overline{\frac{1}{w_\xi - \xi_k}} \,\middle|\, B \right]$$
$$= \mathbb{E}\left[\sum_{j=1}^{N-1} \frac{1}{|w_\xi - \xi_j|^2} \,\middle|\, B \right]$$
$$\le C\frac{1+\delta}{2} N \log N.$$

From (2.8), $P(B^c) \leq \dfrac{C}{N^\delta}$, and therefore

$$P\left(|\widetilde{E}_N(w_\xi)| > N^{1-\gamma}\right) \leq P\left(|\widetilde{E}_N(w_\xi)| > N^{1-\gamma} \,|\, B\right) + \frac{C}{N^\delta}.$$

Using Markov's inequality and the above estimate,

$$\begin{aligned} P\left(|\widetilde{E}_N(w_\xi)| > N^{1-\delta}\right) &\leq \frac{\mathbb{E}\left[|\widetilde{E}_N(w_\xi)|^2 \,|\, B\right]}{N^{2(1-\gamma)}} + \frac{C}{N^\delta} \\ &\leq C\frac{N\log N}{N^{2-2\gamma}} + \frac{C}{N^\delta} \\ &\leq C\frac{\log N}{N^{1-2\gamma}} + \frac{C}{N^\delta}. \end{aligned}$$

Taking $\gamma \in \left(0, \dfrac{1-\delta}{2}\right)$ we have $1 - 2\gamma > \delta$ and therefore

$$\lim_{N\to\infty} \frac{\dfrac{\log N}{N^{1-2\gamma}}}{\dfrac{1}{N^\delta}} = 0.$$

Then,

$$P\left(|\widetilde{E}_N(w_\xi)| > N^{1-\delta}\right) \leq \frac{C'}{N^\delta}.$$

□

This completes the proof of the estimate (2.4) and therefore the Theorem. □

Chapter 3

Probabilistic model: coefficients of the polynomial (GAFs)

3.1 Local repulsion of this model

We begin this chapter by explaining informally why it is reasonable to expect local repulsion of the zeros of polynomials with random coefficients. Before going into this, we look at the following figure. All the three samples shown are portions of certain translation invariant point processes in the plane, with the same average number of points per unit area. Nevertheless, they visibly differ from each other qualitatively, in terms of the clustering they exhibit.

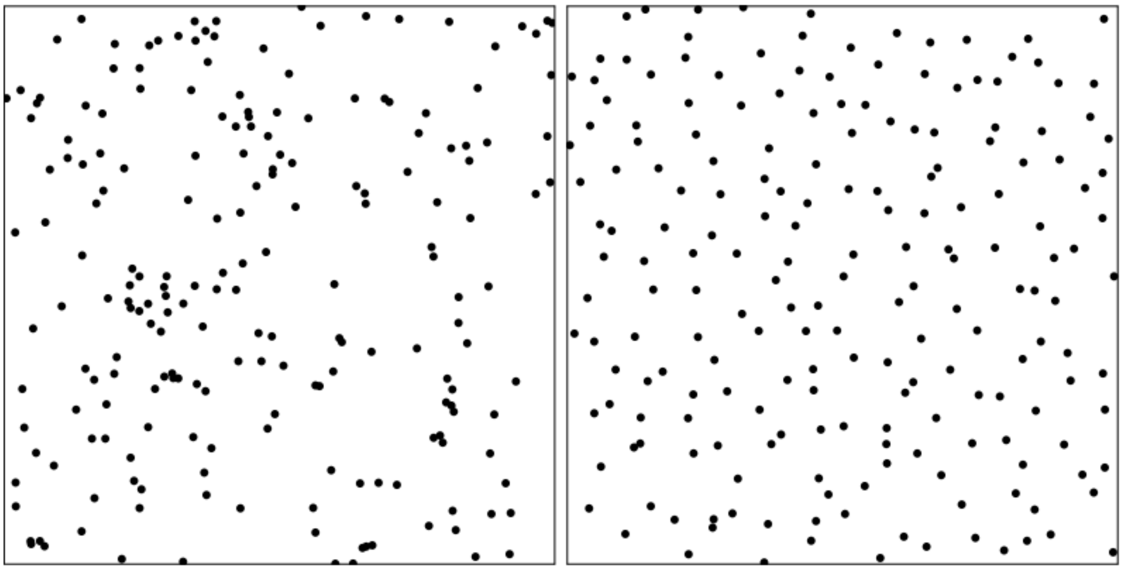

Figure 3.1: The figure on the left corresponds to the model in Chapter 2 and the figure on the right corresponds to the model in Chapter 3. (J. Ben Hough)

We try to understand the phenomenon of local repulsion observed in the central figure above. A heuristic explanation is as follows: consider a polynomial

$$p_N(z) = \sum_{k=0}^{N-1} a_k z^k + z^N,$$

where the coefficients are random variables and let us see how the random roots of p_N are distributed. This is just a matter of looking at the change from coefficients to the roots.

Lemma 4. *Let* $p_N(z) = \prod_{k=1}^{N} (z - \xi_k)$ *have coefficients* a_k, $0 \le k \le N-1$, *i.e.*

$$p(z) = z^N + a_{n-1} z^{N-1} + \cdots + a_1 z + a_0.$$

Then the transformation $T\colon \mathbb{C}^n \to \mathbb{C}^n$ *defined by*

$$T(\xi_1, \ldots, \xi_N) = (a_{N-1}, \ldots, a_0),$$

has real Jacobian determinant $\prod_{i<j} |\xi_i - \xi_j|^2$.

Proof. Let us consider the Jacobian matrix J, constructed by the transformation $T(\xi_1, \xi_2, \xi_3, \ldots, \xi_N)$. It has the following form:

$$J = \begin{pmatrix} \frac{\partial a_{N-1}}{\partial \xi_1} & \frac{\partial a_{N-1}}{\partial \xi_2} & \frac{\partial a_{N-1}}{\partial \xi_3} & \cdots & \frac{\partial a_{N-1}}{\partial \xi_{N-1}} & \frac{\partial a_{N-1}}{\partial \xi_N} \\ \frac{\partial a_{N-2}}{\partial \xi_1} & \frac{\partial a_{N-2}}{\partial \xi_2} & \frac{\partial a_{N-2}}{\partial \xi_3} & \cdots & \frac{\partial a_{N-2}}{\partial \xi_{N-1}} & \frac{\partial a_{N-2}}{\partial \xi_N} \\ \frac{\partial a_{N-3}}{\partial \xi_1} & \frac{\partial a_{N-3}}{partial \xi_2} & \frac{\partial a_{N-3}}{\partial \xi_3} & \cdots & \frac{\partial a_{N-3}}{\partial \xi_{N-1}} & \frac{\partial a_{N-3}}{\partial \xi_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\partial a_1}{\partial \xi_1} & \frac{\partial a_1}{\partial \xi_2} & \frac{\partial a_1}{\partial \xi_3} & \cdots & \frac{\partial a_1}{\partial \xi_{N-1}} & \frac{\partial a_1}{\partial \xi_N} \\ \frac{\partial a_0}{\partial \xi_1} & \frac{\partial a_0}{\partial \xi_2} & \frac{\partial a_0}{\partial \xi_3} & \cdots & \frac{\partial a_0}{\partial \xi_{N-1}} & \frac{\partial a_0}{\partial \xi_N} \end{pmatrix}.$$

To compute the coefficients, we consider the polynomial

$$\begin{aligned} p(z) &= z^N + a_{N-1} z^{N-1} + \cdots + a_0 \\ &= \prod_{k=1}^{N} (z - \xi_k) \\ &= z^N - \sum_{k=1}^{N} \xi_k \cdot z^{N-1} + \sum_{1 \le i,j \le n} \xi_i \xi_j \cdot z^{N-2} + \cdots + (-1)^N \sum_{1 \le i_1, \ldots, i_N \le N} \xi_{i_1} \cdots \xi_{i_N}, \end{aligned}$$

where we used Vieta's formula for computations. This gives an expression of the a_j in terms of the ξ_j.

Compute the derivatives:

$$\begin{aligned}
\frac{\partial a_{N-1}}{\partial \xi_1} &= \frac{\partial a_{N-1}}{\partial \xi_2} = \cdots = \frac{\partial a_{N-1}}{\partial \xi_N} = -1, \\
\frac{\partial a_{N-2}}{\partial \xi_i} &= \sum_{k\neq i} \xi_k, && i = 1, \dots, N \\
&\vdots \\
\frac{\partial a_0}{\partial \xi_i} &= (-1)^N \sum_{\substack{1\leq i_1,\dots,i_{N-1}\leq n \\ i_k\neq i}} \xi_{i_1}\cdots\xi_{i_{N-1}} && i = 1, \dots, N
\end{aligned}$$

As we can see from computations of derivatives, $\det(J)$ is a polynomial in the variables $\xi_1, \dots, \xi_N$. Its degree is $1 + 2 + \cdots + N - 1 = \dfrac{N(N-1)}{2}$.

By the symmetry in ξ_j's it follows that if $\xi_i = \xi_j$ for some $j \neq i$, then the ith and jth column are equal and the determinant $\det(J)$ vanishes. Thus, the determinant is divisible by $\prod_{i<j}(\xi_i - \xi_j)$. Since the degree of the polynomial is equal to $\dfrac{N(N-1)}{2}$, we have:

$$\det(J) = C \prod_{i<j}(\xi_i - \xi_j).$$

We conclude that $C = (-1)^{N(N+1)/2}$. Thus, we get:

$$|\det(J)|^2 = \prod_{i<j} |\xi_i - \xi_j|^2,$$

because we are looking for the real Jacobian determinant $|\det(J)|^2$. □

The Lebesgue measure of the coefficients is pulled back to the measure

$$\left(\prod_{i<j} |\xi_i - \xi_j|^2\right) dm(\xi),$$

which is small near the zeros $\{\xi_j\}$. Thus, given a fixed zero ξ, it is unlikely to find another one nearby. That is, there is negative correlation between the zeros.

This makes the probability of finding multiple zeros zero and the probability of finding zeros very close to each other very small, since the coefficients of the polynomials are random variables and locally the probability measure is uniform in the coefficients.

3.2 Gaussian analytic functions

For this section we have used the references [HKPV09], [Fel13], [NS10] and [Sod04].

A Gaussian analytic function (GAF) is a random element of the space of analytic functions on a certain domain in the complex plane. Given a domain $\Omega \subseteq \mathbb{C}$ let $H(\Omega)$ be the space of analytic functions with the topology of uniform convergence on compact subsets of Ω.

Definition 3.2.1. *(GAF) Let $\mathcal{H} \subseteq H(\Omega)$ be a Hilbert space and let $\{e_k(z)\}_k$ be an orthonormal system in $\mathcal{H}$ with $\sum_k |e_k(z)|^2 < +\infty$ uniformly on compact subsets of $\mathcal{H}$. Then a Gaussian Analytic Function (GAF) is a function of the form:*

$$F(z) = \sum_{k=0}^{\infty} a_k e_k(z)$$

where $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d.

Remark 3. *As $a_k \sim \mathcal{N}_{\mathbb{C}}(0,1)$ i.i.d., for any fixed $z \in \Omega$ the value $F(z)$ is a linear combination of zero mean Gaussians, and therefore $F(z)$ follows a normal distribution with mean 0.*

Example 1. ***The Complex Plane**: Let $\Omega = \mathbb{C}$ and $L > 0$. Consider the Segal-Bargmann space of weight $L > 0$*

$$\mathcal{H}_L = \left\{ f \in H(\mathbb{C}) \,:\, ||f||^2_{\mathcal{H}_L} = \frac{L}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-L|z|^2} \, dm(z) < +\infty \right\}.$$

This is a Hilbert Space with inner product

$$\langle f, g \rangle_{\mathcal{H}_L} = \frac{L}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-L|z|^2} \, dm(z)$$

and the functions $e_k(z) = \sqrt{\frac{L^k}{k!}} z^k$ form an orthonormal basis of $\mathcal{H}_L$. Then if $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ (i.i.d.), a GAF in the complex plane is

$$F(z) = \sum_{k=0}^{\infty} a_k \sqrt{\frac{L^k}{k!}} z^k$$

This is the canonical example; the so-called planar GAF.

We can prove that F is a GAF on $\mathbb{C}$ using Proposition 4. The radius of convergence of the series above is given by

$$r = \frac{1}{\limsup_{n\to\infty} \left| a_n \sqrt{\frac{L^n}{n!}} \right|^{\frac{1}{n}}} = \frac{1}{\sqrt{L} \limsup_{n\to\infty} \underbrace{|a_n|^{\frac{1}{n}}}_{\to 1} \underbrace{\left(\frac{1}{n!}\right)^{\frac{1}{2n}}}_{\to 0}} = \infty \quad a.s.$$

We have that F is analytic on the entire plane. Also, it converges uniformly on compact subsets.

Example 2. ***The Hyperbolic Plane***: *Let* $\Omega = \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *(unit disk) and* $L > 1$*. Consider the weighted Bergman space*

$$\mathbb{B}_L = \Big\{ f \in H(\mathbb{D}) \, : \, ||f||_L^2 = \frac{L-1}{\pi} \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{L-2} \, dm(z) < +\infty \Big\}.$$

This is a Hilbert Space with inner product

$$\langle f, g \rangle_L = \frac{L-1}{\pi} \int_{\mathbb{D}} f(z)\overline{g(z)} (1-|z|^2)^{L-2} \, dm(z)$$

It can be checked that the functions $e_k(z) = \sqrt{\frac{L(L+1)\cdots(L+k-1)}{k!}} z^k = \sqrt{\binom{L+k-1}{k}} z^k$ *form an orthonormal basis of* $\mathbb{B}_L$*.Hence, if* $a_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$ *(i.i.d.), a GAF in the unit disk is*

$$F(z) = \sum_{k=0}^{\infty} a_k \sqrt{\frac{L(L+1)\cdots(L+k-1)}{k!}} z^k.$$

Again, we can prove that F *is a GAF on* $\mathbb{D}$ *using again Proposition 4: the radius of convergence* r *of the serie above is given by*

$$\begin{aligned} \frac{1}{r} &= \limsup_{n\to\infty} \left| a_n \sqrt{\frac{L(L+1)\cdots(L+n-1)}{n!}} \right|^{\frac{1}{n}} \\ &= \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \left(\frac{L(L+1)\cdots(L+n-1)}{n!} \right)^{\frac{1}{2n}} = 1 \quad a.s. \end{aligned}$$

Thus $r = 1$ *a.s.*

We have that F *is analytic on the unit disk.*

3.3 Covariance kernel of a GAF

$F(z)$ is Gaussian of mean 0, and therefore all the probabilistic properties are determined by $\mathrm{Var}[F(z)]$, or more generally, by $\mathrm{Cov}[F(z), F(w)]$.

Definition 3.3.1. *The covariance kernel of a GAF* F *on* Ω *is*

$$\begin{aligned} K(z,w) = Cov[F(z), F(w)] &= \mathbb{E}\left[F(z)\overline{F(w)}\right] - \mathbb{E}[F(z)]\mathbb{E}[F(w)] \\ &= \mathbb{E}\left[F(z)\overline{F(w)}\right], \qquad z, w \in \Omega. \end{aligned}$$

A formula that will be very important for next computations is:

$$\begin{aligned}
K(z,w) &= \mathbb{E}\left[F(z)\overline{F(w)}\right] = \mathbb{E}\left[\left(\sum_{k=0}^{\infty} a_k e_k(z)\right)\left(\sum_{k=0}^{\infty} \overline{a_k e_k(w)}\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\sum_{n=0}^{k} a_n e_n(z)\overline{a_{k-n}e_{k-n}(w)}\right] \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} \mathbb{E}\left[a_n e_n(z)\overline{a_{k-n}e_{k-n}(w)}\right] \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{k} e_n(z)\overline{e_{k-n}(w)}\mathbb{E}[a_n\overline{a_{k-n}}] \\
&= \sum_{k=0}^{\infty} e_k(z)\overline{e_k(w)},. \qquad (3.1)
\end{aligned}$$

by Proposition 5.

Observation 5. *If $F(z)$ is a Gaussian (linear combination of Gaussians), we have that $\mathbb{E}[F(z)] = 0$ and $Var[F(z)] = \mathbb{E}[|F(z)|^2] - (\mathbb{E}[F(z)])^2 = \mathbb{E}[F(z)\cdot\overline{F(z)}) = K(z,z)$. Then,*

$$\frac{F(z)-0}{\sqrt{Var[F(z)]}} = \frac{F(z)}{\sqrt{K(z,z)}} \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

Example 3. *Let's compute the covariance kernel of the planar and the hyperbolic GAFs.*

The Complex Plane: *by definition and since $e_k(z) = \sqrt{\frac{L^k}{k!}}\, z^k$,*

$$K(z,w) = \sum_{k=0}^{\infty} e_k(z)\overline{e_k(w)} = \sum_{k=0}^{\infty} \frac{L^k}{k!}(z\overline{w})^k = e^{Lz\overline{w}},$$

using Equation (3.1).

The Hyperbolic Plane: *similarly, the covariance function (or kernel) is*

$$K(z,w) = \sum_{k=0}^{\infty} \underbrace{\frac{L(L+1)\cdots(L+k-1)}{k!}}_{=\binom{L+k-1}{k}}(z\overline{w})^k = (1-z\overline{w})^{-L}.$$

Remark 4. *If F is a GAF on Ω, then for any $z_1,\ldots,z_n \in \Omega$ we have that $(F(z_1),\ldots,F(z_n)) \sim \mathcal{N}_{\mathbb{C}}^n(0,\Sigma)$ where $\Sigma = (\Sigma_{jk})_{j,k}$ and $\Sigma_{jk} =: K(z_j,z_k)$. As such the covariance function (kernel) $K(\cdot,\cdot)$ determines all finite dimensional marginals of F, then $K(\cdot,\cdot)$ on $\Omega\times\Omega$ determines distribution of F. F is Gaussian, and therefore is determined by their mean and covariance.*

3.4 GAFs on the sphere

For any $N \geq 1$ consider the space

$$\mathcal{P}_N = \left\{ p \in \mathbb{P}_N[\mathbb{C}] \,:\, ||p||_N^2 = (N+1) \int_{\mathbb{C}} \frac{|p(z)|^2}{(1+|z|^2)^N} \frac{dm(z)}{\pi(1+|z|^2)^2} < +\infty \right\}.$$

This norm is natural: the measure $\dfrac{dm(z)}{\pi(1+|z|^2)^2} = d\nu(z)$ is area measure in $\mathbb{S}^2$ projected in $\mathbb{C}$, and the term $\dfrac{|p(z)|^2}{(1+|z|^2)^N}$ measures the size of p normalized by the degree.

This is a Hilbert space with inner product given by

$$\begin{aligned} \langle p, q \rangle_{\mathcal{P}_N} &= \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{p(z)\overline{q(z)}}{(1+|z|^2)^N} \frac{dm(z)}{(1+|z|^2)^2} \\ &= (N+1) \int_{\mathbb{C}} \frac{p(z)\overline{q(z)}}{(1+|z|^2)^N} \, d\nu(z), \quad p, q \in \mathcal{P}_N \end{aligned}$$

Lemma 5. *The monomials*

$$e_k(z) = \sqrt{\binom{N}{k}} z^k, \quad k = 0, \dots, N,$$

form an orthonormal basis of $\mathcal{P}_N$.

Proof. It is clear that $\{e_k\}_{k=0}^N$ generate all $\mathcal{P}_N$, because $\mathcal{P}_N$ has dimension $N+1$ and e_k are, up to a constant, the standard monomials.

Let us see that these functions are orthonormal. By definition

$$\begin{aligned} \langle e_n, e_m \rangle_{\mathcal{P}_N} &= \frac{N+1}{\pi} \int_{\mathbb{C}} \sqrt{\binom{N}{n}} \sqrt{\binom{N}{m}} \frac{z^n \overline{z}^m}{(1+|z|^2)^N} \frac{dm(z)}{(1+|z|^2)^2} = \\ &= \frac{N+1}{\pi} \sqrt{\binom{N}{n}\binom{N}{m}} \int_0^{2\pi} \int_0^{+\infty} \frac{r^{m+n+1} e^{i\theta(n-m)}}{(1+r^2)^N (1+r^2)^2} \, dr \, d\theta \end{aligned}$$

Since

$$\int_0^{2\pi} e^{i(n-m)\theta} \, d\theta = \begin{cases} 0 & \text{if} \quad n \neq m \\ 2\pi & \text{if} \quad n = m \end{cases}$$

then

$$\langle e_n, e_m \rangle_{\mathcal{P}_N} = 0, \quad \text{if } n \neq m.$$

Also, for the case $n = m$, substituting $r^2 = u$ and $\frac{1}{1+u} = t$ and using the functions Γ and β seen in the Preliminaries, we get

$$\begin{aligned}
\langle e_n, e_n \rangle_{\mathcal{P}_N} &= 2\pi \frac{N+1}{\pi} \binom{N}{n} \int_0^{+\infty} \frac{r^{2n+1}}{(1+r^2)^N (1+r^2)^2}\, dr \\
&= 2\pi \frac{N+1}{\pi} \binom{N}{n} \frac{1}{2} \int_0^{+\infty} \frac{u^n}{(1+u)^N (1+u)^2}\, du \\
&= (N+1) \binom{N}{n} \int_0^1 (1-t)^n \cdot t^{N-n}\, dt \\
&= (N+1) \binom{N}{n} \beta(N-n+1, n+1) \\
&= (N+1) \binom{N}{n} \frac{\Gamma(n+1)\Gamma(N-n+1)}{\Gamma(N+2)} \\
&= (N+1) \frac{N!}{(N-n)!n!} n! \times \frac{(N-n)!}{(N+1)!} \\
&= \frac{1}{N!} \times N! = 1.
\end{aligned}$$

□

Definition 3.4.1. *The Gaussian Analytic Function (GAF) in the sphere is the random polynomial*

$$p_N(z) = \sum_{k=0}^{N} a_k \sqrt{\binom{N}{k}} z^k, \quad a_k \sim \mathcal{N}_{\mathbb{C}}(0,1) \ i.i.d.$$

An important feature of this GAF is that its zeros are distributed proportionally to the area of the sphere. This is actually the reason for the normalization we have chosen.

In order to understand the probabilistic properties of p_N let us compute the covariance kernel of p_N using Equation (3.1):

$$\begin{aligned}
K_N(z,w) = \mathbb{E}[p_N(z)\overline{p_N(w)}] &= \sum_{k=0}^{N} e_k(z)\overline{e_k(w)} = \\
&= \sum_{k=0}^{N} \sqrt{\binom{N}{k}} z^k \sqrt{\binom{N}{k}} \overline{w}^k = \sum_{k=0}^{N} \binom{N}{k} (z\overline{w})^k \\
&= (1 + z\overline{w})^N.
\end{aligned}$$

Observation 6. *Using Observation 5, we have that*

$$\frac{p_N(z)}{\sqrt{Var(p_N(z))}} = p_N(z)(1+|z|^2)^{-N/2} \sim \mathcal{N}_{\mathbb{C}}(0,1).$$

In the rest of this subsection we will see that the distribution of the zero set of the above defined function on $\mathbb{S}^2$ is invariant under the transformations $\varphi_{\lambda,\theta}$ considered in Section 1.1.

For other results about invariant zero sets see [HKPV09].

Proposition 8. *The zero set of p_N is invariant in distribution under the automorphisms of the form*

$$\varphi_{\lambda,\theta}(z) = e^{i\theta}\frac{z-\lambda}{1+\bar{\lambda}z}, \quad \lambda \in \mathbb{C},\ \theta \in [0, 2\pi).$$

Proof. Let p_N as in Definition 3.4.1 and let $\lambda \in \mathbb{C}$ and $\theta \in [0, 2\pi)$. We define $G = p_N \circ \varphi_{\lambda,\theta}$, which is a GAF over $\mathbb{S}^2$.

We need to prove that

$$\{z \in \mathbb{S}^2 \,:\, p_N(z) = 0\} \overset{(d)}{=} \{z \in \mathbb{S}^2 \,:\, G(z) = 0\}$$

where (d) means that the two sets have the same distribution.

The covariance kernel of G is:

$$\begin{aligned} K_G(z,w) &= \mathbb{E}\left(p_N\left(e^{i\theta}\frac{z-\lambda}{1+\bar{\lambda}z}\right)\overline{p_N\left(e^{i\theta}\frac{w-\lambda}{1+\bar{\lambda}w}\right)}\right) = \left(1+\frac{z-\lambda}{1+\bar{\lambda}z}\frac{\bar{w}-\bar{\lambda}}{1+\lambda\bar{w}}\right)^N \\ &= \left(\frac{(1+|\lambda|^2)(1+z\overline{w})}{(1+\overline{\lambda}z)(1+\lambda\overline{w})}\right)^N. \end{aligned}$$

Therefore, we have that

$$\{z \in \mathbb{S}^2 \,:\, p_N(z) = 0\} \overset{(d)}{=} \left\{z \in \mathbb{S}^2 \,:\, G(z)\left(\frac{1+|\lambda|^2}{(1+\overline{\lambda}z)^2}\right)^{-\frac{N}{2}} = 0\right\}.$$

If we denote by $K_H(z,w)$ the covariance kernel of $H(z) := G(z)\left(\frac{1+|\lambda|^2}{(1+\overline{\lambda}z)^2}\right)^{-\frac{N}{2}}$ we have:

$$\begin{aligned} K_H(z,w) &= K_G(z,w)\left(\frac{1+|\lambda|^2}{(1+\overline{\lambda}z)^2}\right)^{-\frac{N}{2}}\left(\frac{1+|\lambda|^2}{(1+\overline{\lambda}w)^2}\right)^{-\frac{N}{2}} \\ &= \left(\frac{(1+|\lambda|^2)(1+z\overline{w})}{(1+\overline{\lambda}z)(1+\lambda\overline{w})}\right)^N\left(\frac{1+|\lambda|^2}{(1+\overline{\lambda}z)(1+\lambda\overline{w})}\right)^{-N} \\ &= (1+z\overline{w})^N = K_{p_N}(z,w). \end{aligned}$$

Notice that H and G have the same zeros because the factor $\frac{1+|\lambda|^2}{(1+\overline{\lambda}z)^2}$ does not vanish. By Remark 4 we have that $H \overset{(d)}{=} p_N$. The function H differs from G by a multiplication with a nowhere vanishing function, then

$$\{z \in \mathbb{S}^2 \,:\, G(z) = 0\} = \{z \in \mathbb{S}^2 \,:\, H(z) = 0\} \overset{(d)}{=} \{z \in \mathbb{S}^2 \,:\, p_N(z) = 0\}.$$

□

3.5 GAF's zeros. The empirical measure

In this section we want to prove that the zeros of the GAF are distributed, in average, as in the model considered in the previous chapter.

Let $p_N \in \mathcal{P}_N$ be a GAF on the sphere.

Definition 3.5.1. *The empirical measure μ_{p_N} is the discrete measure*

$$\mu_{p_N} = \sum_{i=1}^{N} \delta_{\xi_i}$$

where ξ_i are the zeros of p_N counted as many times as their multiplicity, and δ_{ξ_i} is the Dirac delta on ξ_i.

If A is a set of $\mathbb{C}$, we consider the random variable counting the number of zeros of p_N in A

$$n_{p_N}(A) = \#(\mathcal{Z}_{p_N} \cap A) = \int_A d\mu_{p_N}.$$

This random variable is the analog of the binomial of parameters N and $p = \nu(A)$ considered in the previous chapter.

It is well-known that the empirical measure coincides, in the sense of distributions with the Laplacian of $\log |p_N|$. Let us recall this briefly.

Definition 3.5.2. *Let $\Omega \subseteq \mathbb{C}$ be a region and let μ be a finite measure over $K \subset \Omega$, where K is compact. A function $g \in L^1_{loc}(\Omega)$ is a solution of $\Delta g = \mu$ on Ω in a distributional sense if*

$$\int_\Omega g(z)\Delta\phi(z)\, dm(z) = \int_\Omega \phi(z)\, d\mu(z) \qquad \forall \phi \in \mathcal{C}_c^\infty(\Omega).$$

Proposition 9. *Let p_N be a holomorphic polynomial and let μ_{p_N} be the empirical measure of its zeros. Then*

$$\mu_{p_N} = \frac{1}{2\pi}\Delta \log |p_N|,$$

in the distributional sense.

For more details, see [HKPV09] (chapter 2) and [BC08] (empirical measure).

The average behaviour of $\mathcal{Z}(p_N)$ is described by the following measure which, in a sense, is the average of the empiric measure μ_{p_N}.

Definition 3.5.3. *The first intensity of the GAF p_N is the measure μ defined by*

$$\langle \mu, \phi \rangle = \mathbb{E}\left[\langle \mu_{p_N}, \phi \rangle\right], \quad \phi \in C_c^\infty(\Omega).$$

In terms of integrals,

$$\int_\Omega \phi\, d\mu = \mathbb{E}\left(\int_\Omega \phi\, d\mu_{p_N}\right), \quad \phi \in C_c^\infty(\Omega).$$

The formula gives the measure that counts the average number of zeros of a GAF in a set. In other words, the counting measure that maps a set S to the average number of zeros of a Gaussian analytic function contained in the set S.

Theorem 3.5.1. ***(Edelman-Kostlan formula)*** *(see [HKPV09], p. 23) Let*

$$p_N(z) = \sum_{j=0}^{N} a_j \sqrt{\binom{N}{j}}\, z^j, \qquad a_j \sim \mathcal{N}_{\mathbb{C}}(0,1) \ i.i.d.$$

be the GAF in $\mathbb{S}^2$ *and let* μ_{p_N} *denote its empirical measure.*

Then the first intensity is given by the formula

$$\mu(z) = \frac{1}{4\pi}\Delta \log K_{p_N}(z,z) = N\, d\nu(z), \tag{3.2}$$

where the Laplacian Δ *is understood in distributional sense and* $K_{p_N}(z,z)$ *is the covariance kernel of* p_N.

Observation 7. *By formula (3.2) the expectation of* $\frac{1}{2\pi}\Delta \log |p_N|$ *is* $\frac{1}{2\pi}\Delta \log \mathbb{E}(|p_n|)$, *since* $\mathbb{E}(|p_N|^2) = K_{p_N}(z,z)$.

Before proving Theorem 3.5.1, let us compute the first intensity for a GAF on the sphere.

Then, since $K_{p_N}(z,w) = (1+z\overline{w})^N$ and $\Delta = 4\partial\overline{\partial}$, we have

$$\begin{aligned}\Delta \log\left(K_{p_N}(z,z)\right) &= 4N \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\left[\log(1+|z|^2)\right] \\ &= 4N\frac{\partial}{\partial z}\left(\frac{z}{1+|z|^2}\right) = \frac{4N}{(1+|z|^2)^2},\end{aligned}$$

and the first intensity is

$$\mu(z) = \frac{1}{4\pi}\Delta \log K_{p_N}(z,z) = \frac{N}{\pi}\frac{dm(z)}{(1+|z|^2)^2} = N\, d\nu,$$

where $d\nu(z)$ corresponds to the invariant measure of the sphere.

This shows that the average number of zeros of the GAF in a region of the sphere is proportional to the area of the region in $\mathbb{S}^2$.

Proof. By Proposition 9, for any smooth function ϕ compactly supported in Ω

$$\int_\Omega \phi(z)\, d\mu_{p_N}(z) = \int_\Omega \Delta\phi(z) \frac{1}{2\pi} \log|p_N(z)|\, dm(z).$$

Taking expectations, we get

$$\mathbb{E}\left[\int_\Omega \phi(z)\, d\mu_{p_N}(z)\right] = \mathbb{E}\left[\int_\Omega \Delta\phi(z)\frac{1}{2\pi}\log|p_N(z)|\, dm(z)\right]$$
$$= \int_\Omega \Delta\phi(z)\frac{1}{2\pi}\mathbb{E}[\log|p_N(z)|]\, dm(z) \tag{3.3}$$

by Fubini's theorem. Now we are going to justify the application of this theorem seeing that

$$\mathbb{E}\left[\int_\Omega \left|\Delta\phi(z)\frac{1}{2\pi}\log|p_N(z)|\, dm(z)\right|\right] < +\infty.$$

We note that

$$\mathbb{E}\left[\int_\Omega |\Delta\phi(z)|\frac{1}{2\pi}|\log|p_N(z)||\, dm(z)\right] = \int_\Omega |\Delta\phi(z)\frac{1}{2\pi}\mathbb{E}\left[|\log|p_N(z)||\right]\, dm(z).$$

For a fixed $z \in \Omega$, $p_N(z)$ is a complex Gaussian random variable with mean zero and variance $K_{p_N}(z,z)$. Therefore, by observation 6 if a is a standard complex Gaussian variable

$$\begin{aligned}
\mathbb{E}\left[|\log|p_N(z)||\right] &= \mathbb{E}\left[\left|\log\left|\frac{p_N(z)}{\sqrt{K_{p_N}(z,z)}}\right|\right|\right] + \log\left|\sqrt{K_{p_N}(z,z)}\right| \\
&= \mathbb{E}[|\log|a||] + \log\left|\sqrt{K_{p_N}(z,z)}\right| \\
&= \int_{\mathbb{C}} |\log|z||\frac{e^{-|z|^2}}{\pi}\, dm(z) + \frac{1}{2}\log|K_{p_N}(z,z)| \\
&= \int_0^\infty 2\rho|\log(\rho)|e^{-\rho^2}\, d\rho + \frac{1}{2}\log|K_{p_N}(z,z)| \\
&= \int_0^\infty |\log(r)|e^{-r}\, dr + \frac{1}{2}\log|K_{p_N}(z,z)| \\
&= C + \frac{1}{2}\log|K_{p_N}(z,z)|,
\end{aligned}$$

for a finite constant C.

Then we can apply Fubini's theorem to obtain Equation (3.3).

If we denote by a a standard complex Gaussian, we deduce that

$$\begin{aligned}
\mathbb{E}[\log|p_N(z)|] &= \mathbb{E}\left[\log\left|\frac{p_N(z)}{\sqrt{K_{p_N}(z,z)}}\right|\right] + \log\sqrt{K_{p_N}(z,z)} \\
&= \mathbb{E}[\log|a|] + \frac{1}{2}\log K_{p_N}(z,z) \\
&= \gamma + \frac{1}{2}\log K_{p_N}(z,z)
\end{aligned}$$

where γ is a constant independent of z. Thus:

$$\mathbb{E}\left[\int_\Omega \phi(z)\,d\mu_{p_N}(z)\right] = \int_\Omega \frac{1}{4\pi}\Delta \log K_{p_N}(z,z)\phi(z)\,dm(z),$$

and by the definition of the first intensity we have

$$\mu(z) = \frac{1}{4\pi}\Delta \log K_{p_N}(z,z).$$

□

We have just seen that the mean distribution is like the area of the sphere, and is therefore invariant by rotations of $\mathbb{S}^2$ (or the transformations $\varphi_{\lambda,\theta}$ in $\mathbb{C}$). In Proposition 8 we have seen that not only the expectation is invariant.

Now, we present the result of Sodin that two GAFs having the same first intensity are essentially equal. In particular we get the remarkable conclusion that the distribution of the zero set is completely determined by its first intensity! We first name a standard fact from complex analysis that will be used in the proof of the Theorem.

Lemma 6. *Let $K(z,w)$ be analytic in z and anti-analytic in w (i.e., analytic in $\overline{w}$) for $z \times w \in \Omega \times \Omega$. If $K(z,z) = 0$ for all $z \in \Omega$, then $K(z,w) = 0$ for all z and w from Ω.*

Sodin discovered the following result and related it to Calabi's rigidity theorem in complex geometry (see [HKPV09], Chapter 2).

Theorem 3.5.2. *(Calabi's rigidity). Suppose f and g are two GAFs in a region Ω such that the first intensity measures of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ are equal.*
Then there exists a nonrandom analytic function ϕ on Ω that does not vanish anywhere, such that, with probability 1,

$$f = \phi g.$$

In particular,

$$\mathcal{Z}(f) = \mathcal{Z}(g) \quad a.s.$$

Corollary 3.5.1. *The random power series of the GAF on $\mathbb{S}^2$ described above is, the only GAFs [up to multiplication by deterministic nowhere vanishing analytic functions] whose zeros are isometry-invariant under the automorphism φ_λ, $\lambda \in \mathbb{C}$.*

3.6 Fluctuations of the zero set of an $\mathbb{S}^2$ GAF

This section is based in the references [Buc15] and [Arr19].

Here we are going to measure the rigidity of this point process by evaluating the variance of some counting random variables.

Let $A \subseteq \mathbb{C}$ and consider the counting random variable

$$n_F(A) = \#(\mathcal{Z}(F) \cap A) = \int_A d\mu_F.$$

To see how fluctuates $n_F(A)$ we compute its variance using this general result:

Theorem 3.6.1. *(see [Arr19], p. 33) Let F be a GAF on a Hilbert space $\mathcal{H} \subseteq H(\Omega)$ and let A be a subset of Ω with $\mathcal{C}^1$ boundary. Let $n_F(A) = \#(\mathcal{Z}(F) \cap A)$. Then*

$$Var[n_F(A)] = -\frac{1}{4\pi^2} \int_{\partial A} \int_{\partial A} \frac{1}{1 - I(z,w)} \frac{\partial}{\partial \overline{z}} \left(\frac{K_F(w,z)}{K_F(z,z)} \right) \frac{\partial}{\partial \overline{w}} \left(\frac{K_F(z,w)}{K_F(w,w)} \right) d\overline{z} d\overline{w},$$

where $K_F(z,w)$ denotes the GAF covariance kernel and

$$I(z,w) = \frac{|K_F(z,w)|^2}{K_F(z,z)K_F(w,w)}.$$

Our goal is to compute the general formula for the $\mathbb{S}^2$ GAF. For simplicity, we consider $A = D_c(a,r)$ to be a chordal disk of radius $r > 0$ and center $a \in \mathbb{C}$. Due to the invariance under rotations on S^2, the distribution of $n_F(D_c(a,r))$ is independent of a, so we can just take $n_F(D_c(0,r))$.

Proposition 10. *Let F be an $\mathbb{S}^2$-GAF. For a chordal disk $D_c(a,r)$, $a \in \mathbb{C}$ and $r > 0$, we have:*

$$Var[n_F(D_c(a,r))] = \frac{N^2}{2\pi} \frac{r}{1+r^2} \int_0^{\frac{4r^2}{(1+r^2)^2}} \frac{(1-s)^N}{1-(1-s)^N} \frac{\sqrt{s}}{1-s} \frac{1}{\sqrt{1 - \frac{(1+r^2)^2}{4r^2} s}} ds.$$

We comment a little this formula (without doing the computations): when $N \to \infty$, we have that

$$\mathrm{Var}[n_F(D_c(a,r))] = \left(\frac{\sqrt{N}}{4\sqrt{\pi}} \zeta\left(\frac{3}{2}\right) \frac{r}{2} \sqrt{1 - \left(\frac{r}{2}\right)^2} \right) (1 + o(1)), \tag{3.4}$$

where ζ is the Riemann zeta function, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1,$$

and $o(1)$ is a term tending to 0 as $N \to +\infty$. The formula (3.4) comes out of looking at the dominant term in the integral.

Fixed $r > 0$, $\mathrm{Var}[n_F(D_c(a,r))] \sim \sqrt{N}$, and that this is much less than in the other case, which is what we will see now.

Let's compute the variance of $n_F(D_c(a,r))$ in the first model: fixed an $r > 0$, we have that the random variable $n_F(D_c(a,r))$ follows a binomial distribution with parameters N and $p = \nu(D_c(a,r)) = \dfrac{r^2}{1+r^2}$. Therefore, its variance is:

$$\begin{aligned}\operatorname{Var}[n_F(D_c(a,r))] &= Np(1-p) \\ &= N\frac{r^2}{(1+r^2)^2}. \end{aligned} \tag{3.5}$$

Comparing Equations (3.4) and (3.5), we see that the GAF case gives a variance of the order of $\sqrt{N}$, which is much smaller than the variance in the first model which is of the order of N. The GAF zeros, despite having the same average as the uniform model, have more rigidity.

Now we prove Proposition 10.

Proof. Since F is an $\mathbb{S}^2$ GAF, we have that $K_F(z,w) = (1+z\overline{w})^N$.

Then

$$I(z,w) = \frac{|K_F(z,w)|^2}{K_F(z,z)K_F(w,w)} = \frac{|1+z\overline{w}|^{2N}}{(1+z\overline{z})^N(1+w\overline{w})^N} = \frac{|1+z\overline{w}|^{2N}}{(1+|z|^2)^N(1+|w|^2)^N}.$$

Computing the derivatives that appear in the formula of Theorem 3.6.1:

$$\frac{\partial}{\partial\overline{z}}\left(\frac{K_F(w,z)}{K_F(z,z)}\right) = \frac{\partial}{\partial\overline{z}}\left(\frac{(1+w\overline{z})^N}{(1+|z|^2)^N}\right) = -N(z-w)\frac{(1+w\overline{z})^{N-1}}{(1+|z|^2)^{N+1}},$$

$$\frac{\partial}{\partial\overline{w}}\left(\frac{K_F(z,w)}{K_F(w,w)}\right) = \frac{\partial}{\partial\overline{w}}\left(\frac{(1+z\overline{w})^N}{(1+|w|^2)^N}\right) = N(z-w)\frac{(1+z\overline{w})^{N-1}}{(1+|w|^2)^{N+1}}.$$

Also,

$$\frac{1}{1-I(z,w)} = \frac{(1+|z|^2)^N(1+|w|^2)^N}{(1+|z|^2)^N(1+|w|^2)^N - |1+z\overline{w}|^{2N}}.$$

Taking polar coordinates, for a given $r > 0$ and denoting $z = re^{i\theta}$ and $w = re^{i\phi}$, for all $\theta, \phi \in (0, 2\pi)$:

$$\begin{aligned}&\operatorname{Var}[n_F(D_c(a,r))] = \\ &= \frac{N^2}{4\pi^2}\int_{\partial D}\int_{\partial D}\frac{|1+z\overline{w}|^{2N-2}(z-w)^2}{(1+|z|^2)^{N+1}(1+|w|^2)^{N+1} - |1+z\overline{w}|^{2N}(1+|z|^2)(1+|w|^2)}\,d\overline{z}d\overline{w} \\ &= \frac{N^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{|1+r^2e^{i(\theta-\phi)}|^{2N-2}(re^{i\theta}-re^{i\phi})^2|D(\theta,\phi)|}{(1+r^2)^{2N+2} - |1+r^2e^{i(\theta-\phi)}|^{2N}(1+r^2)^2}\,d\theta d\phi\end{aligned}$$

where $|D(\theta,\phi)|$ denotes the determinant of the Jacobian matrix:

$$|D(\theta,\phi)| = \left|\begin{vmatrix}-ire^{-i\theta} & 0 \\ 0 & -ire^{-i\phi}\end{vmatrix}\right| = \left|-r^2e^{-i(\theta+\phi)}\right| = r^2e^{-i(\theta+\phi)}.$$

Using the change of variables $t = \theta - \phi$, we obtain:

$$\begin{aligned}
&\operatorname{Var}[n_F(D_c(a,r))] = \\
&= \frac{N^2 r^4}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{|1+r^2 e^{i(\theta-\phi)}|^{2N-2}(e^{2i\theta} - 2e^{i(\theta+\phi)} + e^{2i\phi})e^{-i(\theta+\phi)}}{(1+r^2)^{2N+2} - |1+r^2 e^{i(\theta-\phi)}|^{2N}(1+r^2)^2} \, d\theta d\phi \\
&= \frac{N^2 r^4}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{|1+r^2 e^{i(\theta-\phi)}|^{2N-2}(e^{i(\theta-\phi)} - 2 + e^{i(\phi-\theta)})}{(1+r^2)^{2N+2} - |1+r^2 e^{i(\theta-\phi)}|^{2N}(1+r^2)^2} \, d\theta d\phi \\
&= \frac{N^2 r^4}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{|1+r^2 e^{i(\theta-\phi)}|^{2N-2}|1 - e^{i(\theta-\phi)}|^2}{(1+r^2)^{2N+2} - |1+r^2 e^{i(\theta-\phi)}|^{2N}(1+r^2)^2} \, d\theta d\phi \\
&= \frac{N^2 r^4}{4\pi^2} \int_0^{2\pi} \frac{|1+r^2 e^{it}|^{2N-2}|1-e^{it}|^2}{(1+r^2)^2 \left[(1+r^2)^{2N} - |1+r^2 e^{it}|^{2N}\right]} \, dt \\
&= \frac{N^2}{2\pi} \frac{r^4}{(1+r^2)^2} \int_0^{2\pi} \frac{|1+r^2 e^{it}|^{2N}}{(1+r^2)^{2N} - |1+r^2 e^{it}|^{2N}} \frac{|1-e^{it}|^2}{|1+r^2 e^{it}|^2} \, dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
|1+r^2 e^{it}|^2 &= (1+r^2\cos t)^2 + (r^2 \sin t)^2 = 1 + 2r^2 \cos t + r^4 \\
&= (1+r^2)^2 - 2r^2(1-\cos t) = (1+r^2)^2 \left[1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)\right] \\
|1-e^{it}|^2 &= (1-\cos t)^2 + (\sin t)^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2(1-\cos t).
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Var}[n_F(D_c(a,r))] &= \frac{N^2}{\pi} \left(\frac{r}{1+r^2}\right)^4 \int_0^{2\pi} \frac{\left[1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)\right]^N}{1 - \left[1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)\right]^N} \times \\
&\quad \times \frac{1-\cos t}{1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)} \, dt \\
&= \frac{2N^2}{\pi} \left(\frac{r}{1+r^2}\right)^4 \int_0^{\pi} \frac{\left[1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)\right]^N}{1 - \left[1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)\right]^N} \times \\
&\quad \times \frac{1-\cos t}{1 - \frac{2r^2}{(1+r^2)^2}(1-\cos t)} \, dt
\end{aligned}$$

because the integrand is an even function. Substituting the variable

$$s = \frac{2r^2}{(1+r^2)^2}(1-\cos t),$$

we get

$$\begin{aligned} ds &= \frac{2r^2}{(1+r^2)^2}\sin t\, dt \\ &= \frac{2r^2}{(1+r^2)^2}\frac{1+r^2}{r}\sqrt{s}\sqrt{1-\frac{(1+r^2)^2}{4r^2}s}\, dt \\ &= \frac{2r}{1+r^2}\sqrt{s}\sqrt{1-\frac{(1+r^2)^2}{4r^2}s}\, dt \end{aligned}$$

Also $s(0) = 0$ and $s(\pi) = \dfrac{4r^2}{(1+r^2)^2}$, so we obtain the final formula for the variance

$$\begin{aligned} \mathrm{Var}[n_F(D_c(a,r))] &= \frac{2N^2}{\pi}\left(\frac{r}{1+r^2}\right)^4 \int_0^{\frac{4r^2}{(1+r^2)^2}} \frac{(1-s)^N}{1-(1-s)^N}\frac{\frac{(1+r^2)^2}{2r^2}s}{1-s}\frac{1+r^2}{2r}\frac{1}{\sqrt{s}}\times \\ &\quad\times\frac{1}{\sqrt{1-\frac{(1+r^2)^2}{4r^2}s}}\,ds \\ &= \frac{N^2}{2\pi}\frac{r}{1+r^2}\int_0^{\frac{4r^2}{(1+r^2)^2}}\frac{(1-s)^N}{1-(1-s)^N}\frac{\sqrt{s}}{1-s}\frac{1}{\sqrt{1-\frac{(1+r^2)^2}{4r^2}s}}\,ds. \end{aligned}$$

□

3.7 Pairing of zeros and critical points

In this section we state Boris Hanin's result on pairing of zeros and critical points for the GAF.

Let p_N be a GAF in $\mathbb{S}^2$ and let the set of critical points

$$\mathcal{C}(p_N) = \mathcal{Z}(p_N') = \{z \in \mathbb{C} : p_N'(z) = 0\}.$$

Theorem 3.7.1. *(Theorem 1 in [Han15b]) Let p_N be a GAF in $\mathbb{S}^2$, and suppose that it is conditioned to have $p_N(\xi) = 0$ for a fixed $\xi \in \mathbb{C}\setminus\{0\}$. Then, for all $\varepsilon \in (0, \frac{1}{2})$ the probability that there exists a unique critical point in the annulus*

$$A_{N,\varepsilon}(\xi) = \left\{ z \in \mathbb{C} : \frac{1}{N^{1+\varepsilon}} < |z-\xi| < \frac{1}{N^{1-\varepsilon}} \right\}$$

and no critical points closer to ξ is at least $1 - o\left(\dfrac{1}{N^{3/2+3\varepsilon}}\right)$.

More formally: let the events

$$\begin{aligned} E^1_{N,\varepsilon}(\xi) &= \{w : \#\mathcal{C}(p_N) \cap A_{N,\varepsilon}(\xi) = 1\}, \\ E^2_{N,\varepsilon}(\xi) &= \left\{w : \#\mathcal{C}(p_N) \cap D\left(\xi, \frac{1}{N^{1+\varepsilon}}\right) = 0\right\}. \end{aligned}$$

Then there exists $C > 0$ independent of N such that

$$P\left(E^1_{N,\varepsilon}(\xi) \cap E^2_{N,\varepsilon}(\xi)\right) \geq 1 - \frac{C}{N^{\frac{3}{2}+3\varepsilon}}.$$

The exponent $N^{-3/2}$ shows that the pairing probability is significantly higher than the previous model ($N^{-\delta}$, $\delta < 1$). This is in accordance with the fact that GAF is more rigid.

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