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Facultat de Matemàtiques
i Informàtica

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Treball final de grau

Gauge theories and spontaneous symmetry breaking

Autor: Marta Alerany Solé

Director: Dr. Ricardo García López

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Abstract

In this work, we present a formulation of gauge theories from a mathematical perspective, through bundles and connections on them. We begin by providing the necessary background in differential geometry, covering topics such as Lie groups, Lie algebras, principal bundles and their connection $1-$forms. Then, we examine how classical Yang-Mills theories, a type of gauge theories, can be interpreted using the presented concepts. In particular, we focus on the explanation of the electroweak interaction as a Yang-Mills theory, following the Standard Model of particle physics. Finally, we study some of the problems of this formalism and the way they can be solved via the Higgs mechanism, based on the notion of spontaneous symmetry breaking.

Resum

En aquest treball es presenta una formulació de les teories de gauge des d'un punt de vista matemàtic, emprant fibrats i connexions sobre aquests. En primer lloc, es proporcionen els coneixements previs necessaris de geometria diferencial. Entre d'altres, es tracten temes com grups de Lie, àlgebres de Lie, fibrats principals i les seves corresponents connexions 1-formes. A continuació, s'analitza com les teories de Yang-Mills clàssiques, un tipus especial de teories de gauge, poden ser interpretades mitjançant els conceptes prèviament introduïts. En particular, es mostra que la interacció electrofeble pot ser explicada com una teoria de Yang-Mills, seguint l'esquema usat en el Model Estàndard de física de partícules. Finalment, s'estudien alguns dels problemes d'aquest formalisme i de quina manera es poden solucionar mitjançant el mecanisme de Higgs, basat en la idea de la ruptura espontània de la simetria.

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Contents

Introduction

The concept of gauge invariance appeared for the first time in 1929 in a paper published by the mathematician H. Weyl [Weyl29]. At the beginning of the 20$^{\text{th}}$ century, a few decades after J.C. Maxwell presented his acclaimed set of equations describing classical electromagnetism, many attempts to unveil the symmetries hidden in these equations were taking place.[1] At that time, Weyl was in pursuit of extending general relativity and unifying gravitation and electromagnetism within the same geometrical framework.

Even though he failed in his attempt, in the aforementioned paper he explicitly defined the notion of gauge transformation within quantum electrodynamics. Further, he showed how Maxwell's theory as a quantum mechanical theory is invariant under such a transformation. Weyl had discovered a new symmetry of electromagnetism, today known as gauge symmetry. This was the first time that the terms gauge transformation and gauge invariance appeared.

Apart from asserting the importance of gauge invariance as a symmetry principle from which electromagnetism can be derived, in his work Weyl formally presents virtually all the mathematical aspects that gave rise to non-abelian gauge theories. Gauge theories are a certain type of field theories. Classically, in these theories some physical quantities are represented by a function, called field, that has a value for each point in spacetime. In particular, gauge theories describe matter using fields. Many eminent theories in physics, apart from Maxwell's theory, are gauge theories.

However, it was not until the early 1950s when non-abelian gauge theories were formally introduced by C.N. Yang and R.L. Mills. At that time, experiments had yield the discovery of several new strange particles. As many others, Yang and Mills intended to develop a field theory which could describe weak and strong force interactions considering all the new experimental insights within the field of particle physics. In a renowned paper from 1954 [YM54], both physicists proposed an extended class of classical field theories satisfying a generalized type of gauge symmetry, inspired by electromagnetism.

In its beginnings, the physics community was reluctant to support Yang and Mills' non-abelian field theory, since it apparently predicted the existence of massless charged particles which seemed not to exist in nature. In the 1960s, this issue was solved, among others, by the physicist P. Higgs who presented the nowadays called Higgs mechanism which employed the recently introduced concept of spontaneous symmetry breaking. This allowed to develop a model to explain electroweak theory. Afterwards, a gauge theory to model strong interactions also emerged, which led to the Standard Model that we know nowadays.

[1] Given a physical system, we define a symmetry as a feature of the system which remains unchanged after applying some transformation.

Due to its influence, gauge theories which share essential elements with the theory developed by Yang and Mills are currently known as Yang-Mills theories. All known fundamental interactions can be described in terms of gauge theories and, excluding gravitation, all the significant theories of modern physics are quantized versions of the Yang-Mills theory. Actually, the currently prevalent theory for explaining how matter works, namely the Standard Model of particle physics, is a Yang-Mills theory.

Simultaneously but completely unrelated to these advancements, during the first half of the 20$^{\text{th}}$ century a mathematical theory called fiber-bundle theory was developed. This theory nurtures from different disciplines, such as differential geometry, topology or connection theory.

Both Yang-Mills theory and fiber-bundle theory were developed by physicists and mathematicians for entirely different reasons. Nonetheless, throughout the 1970s, links between the two points of view were revealed. It was discovered that the mathematics of gauge theories, both abelian and non-abelian, are exactly the same as those of bundle theory. This was shown by T.T. Wu and Yang (1975) [WY75] as well as by A. Trautman (1980) [Tra80] and, following their works, many others started to formulate electromagnetism and other gauge theories in terms of the mathematics of fiber bundles.

Therefore, fiber-bundle theory not only has yield new results in mathematics, but it has been key to gaining a better understanding of the structure of gauge theories. Even nowadays, when physicists are struggling to deal with the incompleteness of the Standard Model, modern theories still rely on the tremendous success of this mathematical formalism.

Structure of this work

In this work we aim to develop a classical Yang-Mills theory following the mathematical formulation of fiber bundles.

Before addressing fiber bundles, in the first chapter we introduce Lie groups, Lie algebras, and their corresponding representations, which will be necessary to develop the subsequent theory. In the second chapter, we examine bundle theory focusing our attention in principal fiber bundles and geometrical concepts with an analog in Yang-Mills theories such as connection 1-forms and curvatures. Besides, we define the so-called gauge transformations that will allow us to determine whether a theory is gauge invariant or not.

Finally, the last chapter is devoted to explaining how this mathematical theory is related with some of the fundamentals behind the Standard Model of particle physics. Firstly, we present a general Yang-Mills theory from a classical perspective and we briefly show how classical electromagnetism can be derived from it. Then, we proceed to develop the Higgs mechanism, used to solve the inconsistencies of the theory, and we demonstrate how it allows us to model electroweak interactions as stated in the Standard Model. In particular, we analyze how certain bosons and their interactions are characterized within this theory.

Chapter 1

Lie groups and Lie algebras

This chapter is aimed to introduce Lie groups and Lie algebras and how both concepts can be related. We explore some properties of these mathematical objects and we present several functions involving the two notions, such as representations and the exponential map. Finally, we explain in what manner Lie groups can act on certain differential structures.

1.1 Lie groups

Hereafter, during the whole work, we consider manifolds without boundary.

Definition 1.1.1. A **Lie group** is a differentiable manifold G which is at the same time a group so that the maps

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto g \cdot h \end{aligned} \qquad\qquad \begin{aligned} G &\to G \\ g &\mapsto g^{-1}, \end{aligned}$$

called **multiplication** and **inversion**, are smooth. Here, $G \times G$ has the canonical structure of a product manifold determined by the differential structure of G.

Remark 1.1.2. It can be shown that this definition is actually redundant. Indeed, if the multiplication is smooth, the inversion is automatically smooth (see [Bry18, Sect. 2]). Moreover, it is in fact not necessary to ask for smoothness, since it was proven that a topological manifold which is at the same time a group with continuous multiplication and inversion already has the structure of a Lie group. This is the solution of Hilbert's 5th problem.

Example 1.1.3. Let us now go through some examples that will be crucial in the following chapters:

1. $\mathrm{GL}(n; \mathbb{R}) = \{A \in \mathrm{Mat}(n \times n; \mathbb{R}) : \det(A) \neq 0\} \subset \mathrm{Mat}(n \times n; \mathbb{R}) = \mathbb{R}^{n^2}$ is an open subset, since it is the preimage of an open subset by the continuous function $\det\colon \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}$. Moreover, the multiplication map $(A, B) \mapsto A \cdot B$ and the inversion map $A \mapsto A^{-1}$ are smooth, as the matrix coefficients of $A \cdot B$ and

A^{-1} are polynomial and rational functions on the matrix coefficients of A and B. Hence, $\mathrm{GL}(n;\mathbb{R})$ is a Lie group.

2. The same way, we can see that $\mathrm{GL}(n;\mathbb{C}) \subset \mathrm{Mat}(n \times n;\mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$ is also a Lie group.

There are several ways to construct Lie groups based on other Lie groups. The following theorem provides an example of one of these procedures:

Theorem 1.1.4 (Cartan's theorem, Closed subgroup theorem). *Let G be a Lie group and let $H \subset G$ be a subgroup in the algebraic sense which is a closed subset. Then, H is a submanifold and an embedded Lie Group.*

Remark 1.1.5. We will not prove this theorem here, but there is a whole section in [Ham17, Sect. 1.8] devoted to it.

Example 1.1.6. The previous theorem can be used to find some important Lie groups out of the ones explained in Example 1.1.3:

1. $\mathrm{O}(n) := \{A \in \mathrm{GL}(n;\mathbb{R}) : A^t \cdot A = \mathbb{1}_n\}$, let us check that this is indeed a Lie group:

   i) $\mathbb{1}_n \in \mathrm{O}(n)$.
   ii) $\forall A, B \in \mathrm{O}(n),\ (AB)^t \cdot (AB) = B^t A^t \cdot AB = B^t \cdot B = \mathbb{1}_n \Rightarrow AB \in \mathrm{O}(n)$.
   iii) $\forall A \in \mathrm{O}(n),\ A^t = A^{-1} \Rightarrow (A^{-1})^t \cdot A^{-1} = A \cdot A^{-1} = \mathbb{1}_n \Rightarrow A^{-1} \in \mathrm{O}(n)$.

   Hence, $\mathrm{O}(n) \subset \mathrm{GL}(n;\mathbb{R})$ is a subgroup. In addition, $\mathrm{O}(n)$ is a closed subset, as it is the preimage of a closed subset by the continuous map $A \mapsto A^t \cdot A$. Thus, by Theorem 1.1.4, $\mathrm{O}(n)$ is a Lie group called the **orthogonal group**.

The same reasoning works for the following groups:

2. **Special linear group**: $\mathrm{SL}(n;\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) : \det(A) = 1\}$.
3. **Special orthogonal group**: $\mathrm{SO}(n) := \mathrm{O}(n) \cap \mathrm{SL}(n;\mathbb{R})$.
4. **Unitary group**: $\mathrm{U}(n) := \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) : A^\dagger \cdot A = \mathbb{1}_n\}$, where $A^\dagger = \bar{A}^t$.
5. **Special linear group**: $\mathrm{SL}(n;\mathbb{C}) := \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) : \det(A) = 1\}$.
6. **Special unitary group**: $\mathrm{SU}(n) := \mathrm{U}(n) \cap \mathrm{SL}(n;\mathbb{C})$.

Remark 1.1.7. Lie groups can also be constructed using products: let G and G' be Lie groups, then the product manifold $G \times G'$ with the direct product structure as a group is a Lie group.

Definition 1.1.8. Let G and G' be Lie groups. A **Lie group homomorphism** is a smooth group homomorphism $\phi : G \to G'$. The map ϕ is called a **Lie group isomorphism** if it is invertible and the inverse is again a Lie group homomorphism. In this case, G and G' are called **isomorphic**.

Remark 1.1.9. Given two Lie groups G and G', a continuous group homomorphism $\phi : G \to G'$ is automatically smooth, i.e. is a Lie group homomorphism (see [Ham17, Sect. 1.3]).

1.2 Lie algebras

Definition 1.2.1. A **Lie algebra** is a vector space V together with a map $[\cdot,\cdot] : V \times V \to V$ such that:

i) $[\cdot,\cdot]$ is bilinear.

ii) $[\cdot,\cdot]$ is antisymmetric: $[v,w] = -[w,v] \quad \forall v,w \in V$.

iii) $[\cdot,\cdot]$ satisfies the Jacobi identity:

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0 \quad \forall u,v,w \in V.$$

The map $[\cdot,\cdot]$ is called the **Lie bracket**.

Example 1.2.2. Let us check some important examples of Lie algebras:

1. Every vector space together with the trivial Lie bracket $[\cdot,\cdot] \equiv 0$ is a Lie algebra (called **abelian**).

2. The vector space $\mathrm{Mat}(n \times n;\mathbb{K})$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the bracket $[\cdot,\cdot]$ defined as the commutator of matrices $[A,B] := A \cdot B - B \cdot A$ for all $A,B \in \mathrm{Mat}(n \times n;\mathbb{K})$ is a real or complex Lie algebra, respectively.

3. Let M be a differentiable manifold and $\mathfrak{X}(M)$ the set of differentiable vector fields on M, which has the structure of a real vector space. Consider the bracket that assigns to $X,Y \in \mathfrak{X}(M)$ the differentiable vector field $[X,Y]$ defined by $[X,Y]f = X(Yf) - Y(Xf)$ where $f \in \mathcal{F}(M) = \{f : M \to \mathbb{R} : f \text{ is smooth}\}$.[1] The set $\mathfrak{X}(M)$ together with this bracket is a real Lie algebra.

Definition 1.2.3. Let $(V,[\cdot,\cdot])$ be a Lie algebra. A **Lie subalgebra** of V is a vector subspace $W \subset V$ together with the map $[\cdot,\cdot]_{|W\times W}$ such that W is closed under the bracket, that is, $[w,w'] \in W$ for all $w,w' \in W$.

Definition 1.2.4. Let V and V' be Lie algebras. A **Lie algebra homomorphism** is a linear map $\phi : V \to V'$ verifying $\phi([X,Y]) = [\phi(X),\phi(Y)]$ for all $X,Y \in V$. The map ϕ is called a **Lie algebra isomorphism** if it is bijective, and it is a **Lie algebra automorphism** if it is a Lie algebra isomorphism and $V' = V$.

Let us now construct a correlation between Lie groups and Lie algebras. First of all, we introduce some maps that will be useful throughout this process:

Definition 1.2.5. Let G be a Lie group. Fix $g \in G$ and consider the following maps:

i) **Left translation** by g: $L_g : G \to G$ such that $L_g(h) := g \cdot h$.

ii) **Right translation** by g: $R_g : G \to G$ such that $R_g(h) := h \cdot g$.

[1] Here, the vector field $[X,Y]$ is defined by its action as a derivation on $\mathcal{F}(M)$. Given $X \in \mathfrak{X}(M)$ and $f \in \mathcal{F}(M)$, then $X(f) : M \to \mathbb{R}$ is a differentiable map defined by $X(f)(p) = X_p(f)$ for all $p \in M$. Actually, every $X \in \mathfrak{X}(M)$ can be identified as a derivation on $\mathcal{F}(M)$ and vice versa (see [Cur09, Sect. 4.1]).

iii) **Conjugation** by g: $\alpha_g : G \to G$ such that $\alpha_g(h) := (L_g \circ R_{g^{-1}})(h) = g \cdot h \cdot g^{-1}$.

Notice that L_g and R_g are diffeomorphisms but not group homomorphisms, while α_g is a Lie group isomorphism.

The previous maps allow us to define a subset of $\mathfrak{X}(G)$ fulfilling some properties for which the following relations are needed:

Definition 1.2.6. Let M and N be smooth manifolds, $F : M \to N$ a differentiable map, and $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. Then, X and Y are F**-related**, denoted $X \simeq_F Y$, if $d_pF(X_p) = Y_{F(p)}$ for all $p \in M$, we also write $dF(X) = Y$.

Remark 1.2.7. If $F : M \to N$ is a diffeomorphism, then $dF(X)$ defines a vector field on N for all $X \in \mathfrak{X}(M)$ and it preserves the bracket, that is, $dF([X,Y]) = [dF(X), dF(Y)]$ for all $X, Y \in \mathfrak{X}(M)$ (see [Cur09, Sect. 4.3]).

We are interested in the case that F is a diffeomorphism and $N = M$ so that $dF(X)$ defines a differentiable vector field as $dF(X)(p) = d_{F^{-1}(p)}F(X_{F^{-1}(p)})$ for all $p \in M$, called the **pushforward** of X under F.

Definition 1.2.8. Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is **left-invariant** if $dL_g(X) = X$ for all $g \in G$, equivalently, $d_hL_g(X_h) = X_{gh}$ for all $g, h \in G$.

Remark 1.2.9. Given two left-invariant vector fields $X, Y \in \mathfrak{X}(G)$, since L_g is a diffeomorphism, we have that $dL_g([X,Y]) = [dL_g(X), dL_g(Y)] = [X,Y]$ i.e. $[X,Y]$ is left-invariant. Then, $\mathfrak{g} := \{X \in \mathfrak{X}(G) : X \text{ is left-invariant}\}$ is a Lie subalgebra of $\mathfrak{X}(G)$.

Definition 1.2.10. $\mathfrak{g}$ is called the **Lie algebra** of G.

There is a more intuitive way to understand the Lie algebra of a Lie group. A linear canonical isomorphism between $\mathfrak{g}$ and T_eG can be defined as the map that assigns $X \mapsto X_e$ where $e \in G$ is the neutral element of the group. To check that this is indeed an isomorphism consider the map

$$\begin{aligned} T_eG &\to \mathfrak{g} \\ X_0 &\mapsto X \end{aligned}$$

where $X_g := d_eL_g(X_0)$ for all $g \in G$. This defines a differentiable vector field for each $X_0 \in T_eG$. To see this, we need to compute the differential of the multiplication map (see [Ham17, Sect. 1.5.2]). Moreover, defined this way, X verifies $X_e = d_eL_e(X_0) = d_e\mathrm{id}(X_0) = \mathrm{id}(X_0) = X_0$ and it is left-invariant, since for all $g, h \in G$ we have

$$d_hL_g(X_h) = d_hL_g(d_eL_h(X_e)) = d_e(L_g \circ L_h)(X_e) = d_e(L_{gh})(X_e) = X_{gh}.$$

Therefore, this provides the inverse of the map $X \mapsto X_e$. This implies that a left-invariant vector field on a Lie group is completely determined by its value at one point. Furthermore, $\dim(\mathfrak{g})$ (as a real vector space) equals $\dim(G)$ (as a smooth manifold).

Example 1.2.11. Let us now compute the Lie algebras of some matrix Lie groups:

1. For $G = \mathrm{GL}(n;\mathbb{R})$, we get $\mathfrak{g} = T_{\mathbb{1}_n}\mathrm{GL}(n;\mathbb{R}) = \mathrm{Mat}(n \times n;\mathbb{R}) = \mathbb{R}^{n^2}$, and the same holds for $G = \mathrm{GL}(n;\mathbb{C})$.[2]

2. For $G = \mathrm{O}(n)$, we can express $\mathfrak{o}(n) := \mathfrak{g} = T_{\mathbb{1}_n}\mathrm{O}(n) = \{\dot{\gamma}(0) : \gamma : (-\epsilon,\epsilon) \to \mathrm{O}(n) \text{ smooth}, \gamma(0) = \mathbb{1}_n\}$.[3] From this we can compute a more intuitive expression for $\mathfrak{o}(n)$. First, let us see that $\mathfrak{o}(n) \subset \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) : A^t + A = 0\}$:

$$\forall \gamma(s) \in \mathrm{O}(n),\ \mathbb{1}_n = \gamma(s)^t \cdot \gamma(s) \Rightarrow$$
$$\Rightarrow 0 = \frac{d}{ds}\Big|_{s=0}(\gamma(s)^t \cdot \gamma(s)) = \dot{\gamma}(0)^t \cdot \gamma(0) + \gamma(0)^t \cdot \dot{\gamma}(0) = \dot{\gamma}(0)^t + \dot{\gamma}(0).$$

   Moreover,

$$\dim(\mathfrak{o}(n)) = \dim(\mathrm{O}(n)) = \frac{n(n-1)}{2} = \dim\{A \in \mathrm{Mat}(n \times n;\mathbb{R}) : A^t + A = 0\}.$$

   Hence, $\mathfrak{o}(n) = \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) : A^t + A = 0\}$.

The same reasoning works for the following groups:

3. For $G = \mathrm{SL}(n;\mathbb{R})$, we have $\mathfrak{sl}(n;\mathbb{R}) := \mathfrak{g} = \{A \in \mathrm{Mat}(n \times n;\mathbb{R}) : \mathrm{tr}(A) = 0\}$.

4. For $G = \mathrm{SO}(n)$, we get $\mathfrak{so}(n) := \mathfrak{g} = \mathfrak{o}(n) \cap \mathfrak{sl}(n;\mathbb{R}) = \mathfrak{o}(n)$, since $\mathfrak{o}(n) \subset \mathfrak{sl}(\mathrm{n};\mathbb{R})$.

5. For $G = \mathrm{U}(n)$, we have $\mathfrak{u}(n) := \mathfrak{g} = \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) : A^\dagger = -A\}$.

6. For $G = \mathrm{SL}(n;\mathbb{C})$, we have $\mathfrak{sl}(n;\mathbb{C}) := \mathfrak{g} = \{A \in \mathrm{Mat}(n \times n;\mathbb{C}) : \mathrm{tr}(A) = 0\}$.

7. For $G = \mathrm{SU}(n)$, we have $\mathfrak{su}(n) := \mathfrak{g} = \mathfrak{u}(n) \cap \mathfrak{sl}(n;\mathbb{C})$.

Remark 1.2.12. In all these examples of Lie algebras, the bracket corresponds to the matrix commutator defined in Example 1.2.2.

1.3 Representations

Definition 1.3.1. Let G be a Lie group. A **real** or **complex representation** of G is a Lie group homomorphism $\rho : G \to \mathrm{Aut}(V)$ where V is a finite dimensional $\mathbb{K}$-vector space with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, respectively.

Example 1.3.2. The following are some of the most basic examples of representations of Lie groups:

1. The **trivial representation**: $\rho(g) := \mathrm{id}_V\ \forall g \in G$.

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[2] Here, we use that $\mathrm{GL}(n;\mathbb{R})$ is an open subset of $\mathrm{Mat}(n \times n;\mathbb{R}) = \mathbb{R}^{n^2}$, such that the inclusion $\mathrm{GL}(n;\mathbb{R}) \hookrightarrow \mathbb{R}^{n^2}$ leads to an isomorphism between the tangent spaces, and $T_0\mathbb{R}^{n^2} = \mathbb{R}^{n^2}$.

[3] Here, we apply that for all $X_g \in T_gG$ there exists a smooth curve $\gamma : (-\epsilon,\epsilon) \to G$ with $\gamma(0) = g$ such that $X_g = \dot{\gamma}(0)$ (see [Cur09, Sect. 4.2]).

2. The **adjoint representation**: in this case V is chosen as the Lie algebra $\mathfrak{g}$ of the Lie group G, we write Ad: $G \to \mathrm{Aut}(\mathfrak{g})$. This representation is defined via the conjugation α_g considering its differential at the neutral element e:

$$\mathrm{Ad}_g := d_e\alpha_g : \mathfrak{g} \cong T_eG \to T_eG \cong \mathfrak{g}$$

where $\mathrm{Ad}_g = \mathrm{Ad}(g)$. It is easy to see that Ad is a Lie group homomorphism, since $\mathrm{Ad}_{g_1 \cdot g_2} = \mathrm{Ad}_{g_1} \circ \mathrm{Ad}_{g_2}$, $\mathrm{Ad}_e = \mathrm{id}_{\mathfrak{g}}$, $(\mathrm{Ad}_g)^{-1} = \mathrm{Ad}_{g^{-1}}$, and it is smooth (for a detailed proof see [Bär11, Sect. 1.3]).

Notice that if G is an abelian Lie group, then $\alpha_g = \mathrm{id}_G$ for all $g \in G$. Hence, $\mathrm{Ad}_g = d_e\alpha_g = \mathrm{id}_{\mathfrak{g}}$ for all $g \in G$.

It will be useful to compute the expression of Ad for the matrix groups of the Examples 1.1.1 and 1.1.4. Given $X \in \mathfrak{g}$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to G$ such that $\gamma(0) = e$ and $\dot{\gamma}(0) = X_e$, we have

$$\mathrm{Ad}_g(X) = d_e\alpha_g(X) = \frac{d}{ds}\Big|_{s=0} \alpha_g(\gamma(s)) = \frac{d}{ds}\Big|_{s=0} (g \cdot \gamma(s) \cdot g^{-1}) = g \cdot X \cdot g^{-1}.$$

Hence, for this groups Ad is matrix conjugation.

Let us now consider some methods that allow us to obtain new representations out of given ones, along with a mechanism to correlate different representations:

Definition 1.3.3. Let G be a Lie group, V_i finite dimensional vector spaces, and $\rho_i : G \to \mathrm{Aut}(V_i)$ representations of G for $i = 1, 2$. Then, we define:

i) The **direct sum representation** $\rho_1 \oplus \rho_2 : G \to \mathrm{Aut}(V_1 \oplus V_2)$ as

$$(\rho_1 \oplus \rho_2)(g)(v_1 \oplus v_2) := \rho_1(g)(v_1) \oplus \rho_2(g)(v_2).$$

ii) The **tensor product**[4] **representation** $\rho_1 \otimes \rho_2 : G \to \mathrm{Aut}(V_1 \otimes V_2)$ as

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) := \rho_1(g)(v_1) \otimes \rho_2(g)(v_2).$$

iii) The **antisymmetric tensor product**[5] (or **wedge product**) **representation** $\rho_1 \wedge \rho_2 : G \to \mathrm{Aut}(V_1 \wedge V_2)$ as

$$(\rho_1 \wedge \rho_2)(g)(v_1 \wedge v_2) := \rho_1(g)(v_1) \wedge \rho_2(g)(v_2).$$

Furthermore, ρ_1 and ρ_2 are called **equivalent** if there exists an isomorphism $T : V_1 \to V_2$ so that $T \circ \rho_1(g) \equiv \rho_2(g) \circ T$ for all $g \in G$.

[4] The tensor product is the unique vector space $V_1 \otimes V_2$ together with a bilinear map $b : V_1 \times V_2 \to V_1 \otimes V_2$ satisfying the universal property that any bilinear map $V_1 \times V_2 \to V$ factorizes through $V_1 \otimes V_2$ by a unique linear map $V_1 \otimes V_2 \to V$ (for more details see [Con21a]).

[5] The antisymmetric product $V_1 \wedge V_2$ is the equivalent of the tensor product for skew-symmetric bilinear maps $V_1 \times V_2 \to V$ (for more details see [Con21b]).

Let us now proceed likewise for Lie algebras:

Definition 1.3.4. Let $\mathfrak{g}$ be a Lie algebra. A **real** or **complex representation** of $\mathfrak{g}$ is a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathrm{End}(V)$ where V is a finite dimensional $\mathbb{K}$-vector space with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, respectively.

Example 1.3.5. As for Lie groups, we have:

1. The **trivial representation**: $\lambda(X) := 0 \ \forall X \in \mathfrak{g}$.

2. The **adjoint representation**: as for Lie groups, $V = \mathfrak{g}$ and we write $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. It is defined as $\mathrm{ad}(X)(Y) := [X, Y]$ for all $X, Y \in \mathfrak{g}$. This is indeed a Lie algebra homomorphism, since the bracket is bilinear and $\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$ for all $X, Y \in \mathfrak{g}$ (see [Bär11, Sect. 1.3]). Let us now compute this representation for some special unitary groups:

   Consider $\mathfrak{su}(2) = \{A \in \mathrm{Mat}(2 \times 2; \mathbb{C}) : A^\dagger = -A,\ \mathrm{tr}(A) = 0\} = \left\{ \begin{pmatrix} i\alpha & z \\ -\bar{z} & \bar{\alpha} \end{pmatrix} : \alpha \in \mathbb{R},\ z \in \mathbb{C} \right\}$, notice that $\{i\sigma_1, -i\sigma_2, i\sigma_3\}$ is a basis of $\mathfrak{su}(2)$, where σ_i are the Pauli matrices
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   Then, it suffices to find the expression of the image by ad of each element of the basis, i.e. we just need to compute $[u_i, u_j]$ for each $u_i, u_j \in \{i\sigma_1, -i\sigma_2, i\sigma_3\}$. We get
$$\mathrm{ad}(i\sigma_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}, \quad \mathrm{ad}(-i\sigma_2) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(i\sigma_3) = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
   Consider now
$$\mathfrak{su}(3) = \left\{ \begin{pmatrix} i\alpha & u & v \\ -\bar{u} & -i(\alpha + \beta) & w \\ -\bar{v} & -\bar{w} & i\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R},\ u, v, w \in \mathbb{C} \right\}.$$
   Notice that $\{u_j = -i\lambda_j/2\}_{j=1,\dots 8}$ is a basis of $\mathfrak{su}(3)$, where $\{\lambda_j\}_{j=1,\dots,8}$ are the Gell-Mann matrices (see [Ham17, Sect. 1.5.5]). Taking into account that $[\lambda_j, \lambda_k] = 2i \sum_{l=1}^{8} f^{jkl} \lambda_l$, we get
$$[u_j, u_k] = \frac{-i}{2} \cdot \frac{-i}{2} [\lambda_j, \lambda_k] = \frac{-1}{4} \cdot 2i \sum_{l=1}^{8} f^{jkl} \lambda_l = \frac{-i}{2} \sum_{l=1}^{8} f^{jkl} \cdot \frac{-2}{i} u_l = \sum_{l=1}^{8} f^{jkl} u_l.$$
   Hence, the image by ad of each element of the basis is an endomorphism of $\mathfrak{g}$ that can be expressed in a matrix form as
$$\left(\mathrm{ad}_{\frac{-i}{2}\lambda_j} \right)_{kl} = f_{jkl}$$
   where $f_{123} = 1$, $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$, $f_{458} = f_{678} = \sqrt{3}/2$, and $f_{jkl} = 0$ for any other combination of jkl.

Remark 1.3.6. Analogously as we did for Lie groups, equivalence between Lie algebra representations can be defined.

1.4 The exponential map

Theorem 1.4.1. *Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider a left-invariant vector field $X \in \mathfrak{g}$, an interval $0 \in I \subset \mathbb{R}$, and the maximal integral curve of X through the neutral element $e \in G$, denoted $\gamma_X : I \to G$, with $\gamma_X(0) = e$. Then, the following holds:*

i) *γ_X is defined on all of $\mathbb{R}$.*

ii) *$\gamma_X : \mathbb{R} \to G$ is a Lie group homomorphism.*

iii) *$\gamma_{sX}(t) = \gamma_X(st)$ $\forall s, t \in \mathbb{R}$.*

Proof. See [Ham17, Sect. 1.7.1]. □

Remark 1.4.2. It can also be shown that if a smooth curve $\gamma : \mathbb{R} \to G$ with $\gamma(0) = e$ is a group homomorphism, then γ is an integral curve to some $X \in \mathfrak{g}$ (see [Bär11, Sect. 1.4]).

Taking this into consideration, we can define the following map:

Definition 1.4.3. The map $\exp : \mathfrak{g} \to G$, $\exp(X) := \gamma_X(1)$ is called the **exponential map** of G.

Applying the properties of $\gamma_X(t)$ stated in Theorem 1.4.1, we find that $\gamma_X(s) = \gamma_X(s \cdot 1) = \gamma_{sX}(1) = \exp(sX)$ for all $s \in \mathbb{R}$. Hence, $\exp(tX)$ corresponds to the integral curve γ_X of X. Moreover, exp satisfies $\exp((s+t)X) = \exp(sX) \cdot \exp(tX)$, $\exp(0) = e$, and $\exp(-X) = (\exp(X))^{-1}$.

Remark 1.4.4. The exponential map for the matrix Lie groups coincides with the usual exponential map for matrices i.e. $\exp(A) = \sum_{k \geq 0} \frac{A^k}{k!}$.

The exponential map and its properties allow us to prove several results, these are two examples related with the previous sections (see [Hall15, Sect. 1.4] for proofs):

Proposition 1.4.5. *Let $\phi : G \to G'$ be a Lie group homomorphism. Then, $\phi_* := d_e\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism.*

Corollary 1.4.6. *Let $\rho : G \to \mathrm{Aut}(V)$ be a Lie group representation. Then, $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra representation. In particular, $\mathrm{Ad}_* = \mathrm{ad}$.*

1.5 Group actions

The concept of action of groups on sets can be considered in Lie groups:

Definition 1.5.1. Let G be a Lie group and M a smooth manifold. A **right action** of G on M is a smooth map $M \times G \to M$, $(x, g) \mapsto x \cdot g$, such that:

i) $x \cdot (g \cdot g') = (x \cdot g) \cdot g' \quad \forall x \in M, g, g' \in G$.

ii) $x \cdot e = x \quad \forall x \in M$.

Remark 1.5.2. Similarly, we can define a **left action** of G on M.

Example 1.5.3. Let us go through some basic examples of actions:

1. The **trivial action**: it is given by $x \cdot g := x$ for all $g \in G, x \in M$.
2. The action of G on V associated to a representation $\rho : G \to \mathrm{Aut}(V)$: it is given by $g \cdot v := \rho(g) \cdot v$ for all $g \in G, v \in V$. Observe that this is a left action.

Right (or left) actions can satisfy some useful properties:

Definition 1.5.4. A right action of G on M is called:

i) **Effective** if for all $g \in G$ we have: $(\forall x \in M,\ x \cdot g = x) \Rightarrow g = e$.

ii) **Free** if for all $g \in G$ we have: $(\exists x \in M:\ x \cdot g = x) \Rightarrow g = e$.

iii) **Transitive** if for all $x, x' \in M$ we have: $(\exists g \in G:\ x \cdot g = x')$.

Remark 1.5.5. Notice that if G acts from the right on M, for a fixed $g \in G$, the map $R_g : M \to M$, $x \mapsto x \cdot g$ is a diffeomorphism with inverse $(R_g)^{-1} = R_{g^{-1}}$. Moreover, since $R_g \circ R_{g'} = R_{g \cdot g'}$ for all $g, g' \in G$, this results in a group homomorphism $\phi : G \to \mathrm{Diff}(M)$ given by $\phi(g) = R_g$ for all $g \in G$.

Analogously, given an element $x \in M$, we can consider the map $L_x : G \to M$, $g \mapsto x \cdot g$. Note that its differential on the neutral element is a linear map $d_e L_x : \mathfrak{g} \cong T_e G \to T_x M$. This yields a relation between the elements of the Lie algebra $\mathfrak{g}$ and the vectors fields of M:

Definition 1.5.6. Let M be a smooth manifold, G a Lie group with a right action on M, and $X \in \mathfrak{g}$. The **fundamental vector field** associated with X is the vector field $\overline{X} \in \mathfrak{X}(M)$ defined by $\overline{X}_x := d_e L_x(X)$ for all $x \in M$.

Remark 1.5.7. It can be shown that the map that assigns to each $X \in \mathfrak{g}$ its fundamental vector field $\overline{X}$ is a Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{X}(M)$. Actually, this is the Lie algebra homomorphism $\phi_* := d_e\phi$ induced by the group homomorphism ϕ defined in Remark 1.5.5 (see [Ham17, Sect. 3.4]).

Chapter 2

Bundle theory

In this chapter we present fiber bundles, starting with a general notion and moving on to principal bundles and their associated vector bundles. Then, we focus on several geometrical objects related to these structures that will be key in the last chapter, such as connection 1-forms and their corresponding curvature 2-forms. We end by defining a special type of diffeomorphisms on principal bundles, namely gauge transformations.

2.1 Fiber Bundles

Definition 2.1.1. Let E and B be smooth manifolds and $\pi : E \to B$ a surjective differentiable map. Then:

i) The **fiber** of π over x, for $x \in B$, is the non-empty subset $E_x := \pi^{-1}(\{x\}) \subset E$, which we also denote as $\pi^{-1}(x)$. Given a subset $U \subset B$, we denote $E_U = \pi^{-1}(U) \subset E$. Notice that in this case $E_U = \bigcup_{x \in U} E_x$.

ii) A **global section** of π is a differentiable map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$, and a **local section** of π is a differentiable map $s : U \to E$ such that $\pi \circ s = \mathrm{id}_U$ where $U \subset B$ is an open subset.

Remark 2.1.2. Observe that a differentiable map $s : U \to E$ is a local section if and only if $s(x) \in E_x$ for all $x \in U$.

Definition 2.1.3. Let E, B and F be smooth manifolds and $\pi : E \to B$ a surjective differentiable map. A **fiber bundle** with **typical fiber** F is a triple (E, π, B) so that for all $x \in B$ there exists an open neighborhood $U \subset B$ with a diffeomorphism $\psi_U : \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \psi_U = \pi_{|\pi^{-1}(U)}$, where $\mathrm{pr}_1 : U \times F \to U$ is the usual projection.

We say that E is the **total space** of the fiber bundle, B is the **base**, and ψ_U is a **local trivialization** over U.

Remark 2.1.4. The fibers of a fiber bundle satisfy two remarkable properties that do not always hold for fibers of a general map:

1. All fibers E_x of a fiber bundle are diffeomorphic to the typical fiber F, as for all $x \in U \subset B$ the restriction of a local trivialization ψ_U induces a diffeomorphism $\psi_U|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times F \cong F$.

2. All fibers E_x of a fiber bundle are embedded submanifolds of E by the Regular Value Theorem, since there is a local trivialization ψ_U such that $x \in U \subset B$ which is a diffeomorphism and pr_1 is a submersion, so $\pi_{|E_U} = \mathrm{pr}_1 \circ \psi_U$ is a submersion.

Example 2.1.5. Let B and F be smooth manifolds. A straightforward example of fiber bundle is **the trivial bundle** given by the Cartesian product $(B \times F, \mathrm{pr}_1, B)$.

Remark 2.1.6. Note that for the trivial bundle $(B \times F, \mathrm{pr}_1, B)$ there exists a bijection between global sections $s : B \to B \times F$ and smooth maps $\phi : B \to F$.

In the following, we discuss how two fiber bundles are related when there exist maps between the manifolds that form them. First, we consider a map between their total spaces and then between their bases.

Definition 2.1.7. Let (E, π, B) and (E', π', B) be fiber bundles with typical fibers F and F', respectively. A **bundle map** (or **bundle morphism**) between them is a smooth map $\phi : E \to E'$ such that $\pi' \circ \phi = \pi$. A **bundle isomorphism** is a bundle map which is also a diffeomorphism, in this case the fiber bundles are called **isomorphic**.

Remark 2.1.8. Notice that a bundle map $\phi : E \to E'$ maps a point in the fiber E_x to a point in E'_x for all $x \in B$, since $\pi(E_x) = x = \pi'(\phi(E_x))$ implies $\phi(E_x) \subset (\pi')^{-1}(x) = E'_x$. Moreover, a bundle isomorphism yields a diffeomorphism between the fibers $E_x \cong E'_x$ for all $x \in B$.

This means that isomorphic bundles have diffeomorphic typical fibers, as $F \cong E_x \cong E'_x \cong F'$ for $x \in B$. The converse is not true in general.

Definition 2.1.9. A bundle is called **trivial** if it is isomorphic to the trivial bundle or, equivalently, if there exists a global trivialization $\psi_B : E \to B \times F$.

Remark 2.1.10. Suppose given a fiber bundle (E, π, B) with typical fiber F and an open subset $U \subset B$. Note that a local trivialization ψ_U is a bundle isomorphism between the restricted bundle $\pi_{|E_U} : E_U \to U$ and the trivial bundle $\mathrm{pr}_1 : U \times F \to U$. Thus, every fiber bundle is said to be locally trivial.

This implies that every fiber bundle has smooth local sections, as it is locally trivial and every trivial bundle has global sections.

Let us now consider a smooth map between differentiable manifolds $\lambda : B' \to B$, where B is the base of a fiber bundle (E, π, B) with typical fiber F. Then, we set:

i) $E' := \{(b', p) \in B' \times E : \lambda(b') = \pi(p)\}$.

ii) $\pi' := \mathrm{pr}_1|_{E'} : E' \to B'$.

It can be seen that this defines a fiber bundle (E', π', B') with typical fiber F. To find a local trivialization for $b' \in B'$, it suffices to consider an open neighborhood $U \subset B$ of $\lambda(b') \in U$ and a local trivialization ψ_U. Then, if we take $U' := \lambda^{-1}(U)$ as a neighborhood of $b' \in B'$, we get $(\pi')^{-1}(U') \cong U' \times F$ (for more details see [Bär11, Sect. 2.1]).

Definition 2.1.11. The fiber bundle $\lambda^*(E, \pi, B) := (E', \pi', B')$ is called the **pull-back** of (E, π, B) along λ.

Remark 2.1.12. It is easy to show that the projection $\mathrm{pr}_2 : E' \to E$ yields a diffeomorphism $E'_{b'} \cong E_{\lambda(b')}$ of fibers.

2.2 Vector bundles

Definition 2.2.1. A (**real** or **complex**) **vector bundle of rank n** is a fiber bundle (V, π, B) with typical fiber $\mathbb{K}^n$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, respectively, such that:

i) Each fiber V_x is a $\mathbb{K}$-vector space of dimension n.

ii) Each local trivialization ψ_U can be chosen so that $\psi_U|_{\pi^{-1}(x)} : V_x \to \{x\} \times \mathbb{K}^n \cong \mathbb{K}^n$ is a linear isomorphism.

Definition 2.2.2. Let (V, π, B) be a $\mathbb{K}$-vector bundle of rank n. A **vector subbundle** of rank k is a subset $W \subset V$ such that for every $x \in B$ there exists an open neighbourhood $U \subset B$ of x together with a local trivialization ψ_U of V satisfying $\psi_U(V_U \cap W) = U \times \mathbb{K}^k \subset U \times \mathbb{K}^n$.

Notice that (W, π_W, B) is a $\mathbb{K}$-vector bundle of rank k.

Remark 2.2.3. Observe that every vector bundle (V, π, B) has global sections. It suffices to consider the zero section $s(x) := 0_x \in V_x$ for all $x \in B$.

Furthermore, due to the vector space structure of each fiber, for vector bundles we can add any two sections and multiply them with a scalar and we will obtain another section. Hence, the set $\Gamma(V) := \{\text{global sections } s : B \to V\}$ is a $\mathbb{K}$-vector space.

Remark 2.2.4. New vector bundles can be obtained out of given ones using linear algebra constructions. For instance, let V and V' be vector bundles with base B over the same field $\mathbb{K}$. Then, we can canonically define vector bundles such as $V \oplus V'$, $V \otimes V'$, V^* or $\bigwedge^k V$, among others, all of them with base B. Their fibers are given by $(V \oplus V')_x = V_x \oplus V'_x$, and similarly in the other cases.

Example 2.2.5. Let B be a smooth manifold. Then, (TB, pr_1, B) where $TB = \bigcup_{x \in B}\{x\} \times T_xB$ is a vector bundle called the **tangent bundle** of B. It is easy to construct local trivializations for this fiber bundle using charts (U, ϕ_U) of B (see [Ham17, Sect. 4.5.1]).

For this vector bundle and linear algebraic constructions of it, we can identify the set of its sections with different objects from geometry. For example, $\Gamma(TB) = \mathfrak{X}(B)$ i.e. the sections of TB are the same as the vector fields on B (see [Bär11, Sect. 2.1]).

2.3 Principal bundles

Definition 2.3.1. A G**-principal bundle** is a fiber bundle (P, π, B) with typical fiber a Lie group G together with a right action of G on P such that:

i) The group action is free.

ii) The group action preserves the fibers of the bundle and is transitive on each of them.

iii) The local trivializations $\psi_U : \pi^{-1}(U) \to U \times G$ can be chosen such that

$$\psi_U(p \cdot g) = \psi_U(p) \cdot g$$

for all $p \in \pi^{-1}(U)$, $g \in G$, where the action of G on $(p, g) \in U \times G$ is defined by $(p, g) \cdot g' = (p, g \cdot g')$ for all $g' \in G$. In this case, ψ_U is called G**-equivariant**.

We say that G is the **structure group** of the principal bundle.

Remark 2.3.2. Note that *ii*) implies that for all $p \in P$ we have $p \cdot G = P_{\pi(p)}$. In addition, if for $p \in P$ we consider the map $L_p : G \to P_{\pi(p)}$ that sends $g \mapsto p \cdot g$, it is injective by *i*) and surjective by *ii*), hence it is a bijection. Moreover, it can be shown that L_p is actually a diffeomorphism (see [Bär11, Sect. 2.2]).

Remark 2.3.3. Although each fiber of a G-principal bundle is diffeomorphic to G and a local trivialization yields a group structure on the fiber, this structure is not canonical, as it depends on the choice of a point on the fiber. We cannot canonically assign a group structure to the fibers.

Remark 2.3.4. Observe that given a vector space $\mathbb{K}^n$ that is also a Lie group, such as $\mathbb{R}^n$, the conditions for vector and principal bundles with typical fiber $\mathbb{K}^n$ look quite similar. However, the definitions are not equivalent. Firstly, we have seen that for vector bundles there always exists a global section, but we will later see that a principal bundle has global sections if and only if it is trivial. Moreover, for a $\mathbb{K}^n$-principal bundle (P, π, B) an action of $\mathbb{K}^n$ on the whole total space P transitive on the fibers is needed, which we cannot define in general for vector bundles.

Example 2.3.5. Let us now go through several examples of principal bundles involving some of the Lie groups studied in Chapter 1:

1. Suppose (V, π, B) is a $\mathbb{K}$-vector bundle of rank n. Since for any $b \in B$ the fiber V_b is an n-dimensional $\mathbb{K}$-vector space, we can consider the set $P_b := \{$(ordered) basis $(b_1, \dots, b_n)$ of $V_b\}$. An action of $\mathrm{GL}(n; \mathbb{K})$ on P_b can be defined as

$$(b_1, \dots, b_n) \cdot A = \left(\sum_{i=1}^{n} a_{i1} b_i, \dots, \sum_{i=1}^{n} a_{in} b_i \right) \in P_b$$

where $A = (a_{ij})_{i,j=1,\dots,n}$. It is easy to see that this action is free and transitive.

Set $P := \bigsqcup_{b \in B} P_b$ and $\pi' : P \to B$ defined such that $\pi'|_{P_b} \equiv b$. Observe that for any local trivialization ψ_U of the vector bundle (V, π, B), a local trivialization ψ'_U for (P, π', B) can be constructed as

$$\begin{aligned} (\pi')^{-1}(U) &\to U \times \mathrm{GL}(n; \mathbb{K}) \\ p = (p_1, \dots, p_n) &\mapsto (\pi'(p), (\psi_U(p_1) \dots \psi_U(p_n))). \end{aligned}$$

Since ψ_U is a linear isomorphism, we have $\psi'_U(p \cdot A) = \psi'_U(p) \cdot A$ for all $p \in P$ and $A \in \mathrm{GL}(n; \mathbb{K})$. Therefore, (P, π', B) is a $\mathrm{GL}(n; \mathbb{K})$-principal bundle.

2. The same way, considering vectors bundles with different structures we can construct principal bundles for several matrix Lie groups. For instance, an $\mathrm{O}(n)$- or $\mathrm{U}(n)$-principal bundle can be obtained starting with a $\mathbb{K}$-vector bundle of rank n with a Riemannian or Hermitian metric and taking its orthonormal bases (see [Bär11, Sect. 2.2]).

Definition 2.3.6. Let B be a smooth manifold and $V = TB$, so that (V, π, B) is the tangent bundle of B. The **frame bundle** of B is the $\mathrm{GL}(n; \mathbb{K})$-principal bundle (P, π', B) constructed as in the previous example.

Remark 2.3.7. Let (P, π, B) be a G-principal bundle and $\lambda : B' \to B$ a smooth map between differentiable manifolds. The pull-back $\lambda^*(P, \pi, B)$ together with the right action of G on $\lambda^* P$ defined by $(b', p) \cdot g = (b', p \cdot g)$ is a G-principal bundle too.

The same way as we did for the total space and the base of a fiber bundle in Section 2.1, let us now consider a map between Lie groups $\varphi : G \to H$, where G is the structure group of a principal bundle (P, π, B).

Suppose that φ is a Lie group homomorphism. Then, it can be shown that $P' := (P \times H)/G$ is a smooth manifold, where two elements $(p, h), (p', h') \in P \times H$ are related if and only if $(p', h') = (p \cdot g, \varphi(g^{-1}) \cdot h)$ for some $g \in G$. Furthermore, it is possible to define a right action of H on P' as $[p, h] \cdot h' = [p, h \cdot h']$ for all $[p, h] \in P'$, $h' \in H$. It is easy to see that this action is free using that the action of G on P is free.

Now, set $\pi' : P' \to B$ given by $\pi'([p, h]) = \pi(p)$. This map is well-defined as the image of a class does not depend on the choice of p, since the first component of two related elements satisfy $p' = p \cdot g$ for some $g \in G$ and G acts transitively along the fibers of π. Moreover, the action of H on P' is transitive over the fibers of π'.

Finally, given any local trivialization ψ_U of P such that $\psi_U(p) = (\pi(p), \alpha_U(p)) \in U \times G$, where $\alpha_U : \pi^{-1}(U) \to G$ is a smooth map, we can construct a local trivialization of P' as the following diffeomorphism:

$$\begin{aligned}\psi'_U : (\pi')^{-1}(U) &\to U \times H \\ [p, h] &\mapsto (\pi(p), \varphi(\alpha_U(p)) \cdot h).\end{aligned}$$

Note that ψ'_U is H-equivariant. Hence, (P', π', B) is an H-principal bundle (for more details of this construction see [Bär11, Sect. 2.2]).

Definition 2.3.8. (P', π', B) is called the **H-principal bundle associated to (P, π, B) with respect to φ.**

Remark 2.3.9. We also write P' as $P \times_\varphi H$.

Definition 2.3.10. An **associated vector bundle** of a G-principal bundle (P, π, B) is a vector bundle $(P \times_\rho V, \pi', B)$ where $\rho : G \to \mathrm{Aut}(V)$ is a representation of G and the action of G on $P \times V$ is given by $(p, v) \cdot g = (p \cdot g, \rho(g^{-1})(v))$.

Remark 2.3.11. For an associated vector bundle $(P \times_\rho V, \pi', B)$ we can build a local trivialization ψ'_U out of any local trivialization ψ_U of (P, π, B) proceeding analogously as in the construction of an associated principal bundle. If $\psi_U(p) = (\pi(p), \alpha_U(p))$ for all $p \in \pi^{-1}(U)$, we set ψ'_U as

$$\begin{aligned}(\pi')^{-1}(U) &\to U \times V\\ [p, v] &\mapsto (\pi(p), \rho(\alpha_U(p))(v)).\end{aligned}$$

Remark 2.3.12. Note that for every G-principal bundle (P, π, B) and every vector space V, the vector bundle associated to the trivial representation, $\rho(g) \equiv \mathrm{id}_V$ for all $g \in G$, is a trivial vector bundle.

Now, we explain how principal bundles can be described via local sections.

Theorem 2.3.13. *Let P and B be differentiable manifolds, $\pi : P \to B$ a smooth surjective map, and G a Lie group with a smooth right action on P. Then, (P, π, B) is a G-principal bundle if and only if the following holds:*

i) The fibers of π are invariant under the action of G on P and the induced action on them is free and transitive.

ii) There exists an open covering $\{U_i\}_{i\in I}$ of B with local sections $s_i : U_i \to P$ of π.

Proof. Here, we only show a sketch of the proof, for a detailed proof see [Ham17, Sect. 4.2].

First, suppose that (P, π, B) is a G-principal bundle. By definition, *i)* holds and for each $x \in B$ there exists an open neighborhood $U \subset B$ with a local trivialization ψ_U. For each ψ_U, a local section of π can be constructed as the following smooth map:

$$\begin{aligned}s_U : U &\to P\\ x &\mapsto \psi_U^{-1}(x, e)\end{aligned}$$

where $e \in G$ is the neutral element. Thus, (P, π, B) satisfies *ii)*.

Conversely, assume that (P, π, B) verifies *i)* and *ii)*. Then, given a local section $s_i : U_i \to P$, since the group action is free and transitive on the fibers, for each $p \in P_{U_i}$ there exists a unique $g(p) \in G$ such that $p = s_i(\pi(p)) \cdot g(p)$. Then, we set

$$\begin{aligned}\psi_i : \pi^{-1}(U_i) &\to U_i \times G\\ p &\mapsto (\pi(p), g(p)).\end{aligned}$$

It is easy to see that ψ_i is a G-equivariant local trivialization of (P, π, B). Hence, (P, π, B) is a G-principal bundle. □

Remark 2.3.14. We have proven above that for a principal bundle there is a bijection between local trivializations and local sections. In particular, a principal bundle has global sections if and only if it is trivial.

Let us now consider the case of associated vector bundles.

Proposition 2.3.15. *Let (P, π, B) be a G-principal bundle, (P', π', B) an associated vector bundle where $P' := P \times_\rho V$, and $s : U \to P$ a local section of π. Then, there exists a bijection between local sections $s' : U \to P'$ of π' and smooth maps $f : U \to V$ given by $s'(x) = [s(x), f(x)]$ for all $x \in U$.*

Proof. First, assume that $s' : U \to P'$ is a local section of π'. Then, if we take $s(x) \in P$ as the first component of $s'(x)$, there exists a unique $f(x) \in V$ such that $s'(x) = [s(x), f(x)]$. Indeed, suppose $[p, v_1] = [p, v_2] \in P'$, since the action of G is free, the only $g \in G$ that can relate (p, v_1) and (p, v_2) is the neutral element $g = e$. Then, $v_1 = \rho(e)(v_2) = \mathrm{id}_V(v_2) = v_2$. This means that, once we fix the first component of an element of P', the second component is uniquely determined.

It is easy to see that this yields a smooth map $f : U \to V$ using the local trivialization of the principal bundle $\psi_U : \pi^{-1}(U) \to U \times G$ defined in Theorem 2.3.13 and the one of the associated vector bundle $\psi'_U : (\pi')^{-1}(U) \to U \times V$ given in Remark 2.3.11 (see [Ham17, Sect. 4.7.1]).

Conversely, consider a smooth map $f : U \to V$. Then, the map $s' : U \to P'$ defined by $s'(x) = [s(x), f(x)]$ for all $x \in U$ is smooth . Moreover, it satisfies $\pi'(s'(x)) = \pi'([s(x), f(x)]) = \pi(s(x)) = x$ for all $x \in U$. Therefore, s' is a local section of π'. □

Finally, we introduce the concept of transition functions.

Suppose a G-principal bundle (P, π, B) and an open covering $\{U_i\}_{i \in I}$ of B such that P_{U_i} is trivial and there exist local sections $s_i : U_i \to P$ for all $i \in I$. Consider the intersection $U_{ij} := U_i \cap U_j$ for $i, j \in I$. Then, for all $x \in U_{ij}$ there exists a unique $g_{ij}(x) \in G$ such that $s_j(x) = s_i(x) \cdot g_{ij}(x)$. From this we obtain smooth maps $g_{ij} : U_{ij} \to G$ verifying the following conditions:

i) $g_{ii} = e$.

ii) $g_{ij} = g_{ji}^{-1}$.

iii) $g_{ij} g_{jk} g_{ki} = e$.

Definition 2.3.16. The maps $\{g_{ij}\}_{i,j \in I}$ are called **transition functions** of the principal bundle and we say that the conditions *i), ii)* and *iii)* are the **cocycle conditions**.

Let $\widetilde{s}_i : U_i \to P$ be a second set of local sections with corresponding transition functions $\widetilde{g}_{ij} : U_{ij} \to G$. Then, for all $x \in U_i$ there exists a unique $h_i(x) \in G$ such that $\widetilde{s}_i(x) = s_i(x) \cdot h_i(x)$. This yields smooth maps $h_i : U_i \to G$ which relate g_{ij} and $\widetilde{g}_{ij}$ satisfying the so-called **coboundary condition**: $g_{ij} = h_i \widetilde{g}_{ij} h_j^{-1}$.

Remark 2.3.17. It can be shown that given an open covering $\{U_i\}_{i \in I}$ of a smooth manifold B and smooth maps $g_{ij} : U_{ij} \to G$ verifying the cocycle conditions, we can construct a G-principal bundle with g_{ij} as transition functions (see [Nab11, Sect. 4.3]).

Moreover given two sets $\{g_{ij}\}_{i,j \in I}$ and $\{\widetilde{g}_{ij}\}_{i,j \in I}$ of transition functions and smooth maps $\{h_i\}_{i \in I}$ satisfying the coboundary conditions, the corresponding G-principal bundles P and $\widetilde{P}$ are isomorphic (see [Bär11, Sect. 2.2]).

2.4 Connections

For introducing the next concepts, some notions from differential geometry will be required.

Definition 2.4.1. Let M be a differentiable manifold. A **differential 1-form** on M is a smooth section of the vector bundle (T^*M, pr_1, M) where $T^*M = \bigcup_{x\in M}\{x\} \times T_x^*M$. This bundle is called the **cotangent bundle**.

We denote $\Omega^1(M) = \{\text{differential 1-forms on } M\} = \Gamma(T^*M)$, this is an $\mathcal{F}(M)$-module. Note that an element $\omega \in \Omega^1(M)$ is a smooth map $\omega : M \to T^*M$, $x \mapsto (x, \omega_x)$ such that $\omega_x \in T_x^*M$ for all $x \in M$.

Remark 2.4.2. Observe that, given a smooth map between differentiable manifolds $F : M \to N$, the dual of its differential on each point $d_x^*F : T^*_{F(x)}N \to T_x^*M$ results in a map between the 1-forms on each manifold. Indeed, if $\omega \in \Omega^1(N)$, we define $F^*\omega \in \Omega^1(M)$ as the 1-form that at each point is given by $F^*\omega(x) = d_x^*F(\omega_{F(x)}) \in T_x^*M$. Notice that, for $X \in T_xM$, we get $F^*\omega(x)(X) = \omega_{F(x)}(d_xF(X)) \in \mathbb{R}$.

Therefore, we have constructed a map $F^* : \Omega^1(N) \to \Omega^1(M)$, which is called **pull-back of 1-forms**.

Remark 2.4.3. It can be seen that there is a bijection between 1-forms $\Omega^1(M)$ and the set of $\mathcal{F}(M)$-linear maps between $\mathfrak{X}(M)$ and $\mathcal{F}(M)$, which we denote as $\mathrm{Hom}_{\mathcal{F}(M)}(\mathfrak{X}(M), \mathcal{F}(M))$ (see [Cur09, Sect. 5.3]).

Definition 2.4.4. Let M be a differentiable manifold and take the vector bundle $(\Lambda^k T^*M, \mathrm{pr}_1, M)$ for some $k \in \mathbb{N}$, where $\Lambda^k T^*M = \bigcup_{x\in M}\{x\} \times \Lambda^k T_x^*M$. A **differentiable k-form** on M is an element of $\Omega^k(M) := \Gamma(\Lambda^k T^*M)$. We have $\Omega^k(M) = \Lambda^k\Omega^1(M)$. Given $\omega \in \Omega^k(M)$, we get a map $\omega_x : \Lambda^k T_xM \to \mathbb{R}$ for all $x \in M$.

Moreover, for every finite-dimensional real vector space V, we can also consider the set of **differentiable k-forms with values in V** which corresponds to $\Omega^k(M, V) := \Gamma(\Lambda^k T^*M \otimes_{\mathbb{R}} V)$. Note that each $\omega \in \Omega^k(M, V)$ defines a map $\omega_x : \Lambda^k T_xM \to V$ for all $x \in M$.

Finally, given a vector bundle (V, π, B), the set of **differentiable k-forms with values in V** can be defined as $\Omega^k(B, V) := \Gamma(\Lambda^k T^*B \otimes V)$. Then, for $\omega \in \Omega^k(B, V)$, we have $\omega_x : \Lambda^k T_xB \to V_x$ for all $x \in B$.

Remark 2.4.5. As for $\Omega^1(M)$, there is a bijection between $\Omega^k(M)$ and the set of alternating[1] $\mathcal{F}(M)$-multilinear maps $\mathfrak{X}(M) \times \overset{k}{\ldots} \times \mathfrak{X}(M) \to \mathcal{F}(M)$, which can be written as $\mathrm{Hom}_{\mathcal{F}(M)}(\Lambda^k\mathfrak{X}(M), \mathcal{F}(M))$. The same way, $\Omega^k(M, V)$ is isomorphic to $\mathrm{Hom}_{\mathcal{F}(M)}(\Lambda^k\mathfrak{X}(M), \mathcal{F}(M, V))$, where $\mathcal{F}(M, V) = \{\text{smooth maps } M \to V\}$. Thus, in the following, we will indistinctly apply forms on points of the manifold or on its vector fields, as in some cases one of the two choices considerably reduces the difficulty of the calculations.

[1] A multilinear map is alternating if it vanishes whenever two of its arguments are equal.

2.4.1 Connection 1-forms

There exists a special type of 1-forms on principal bundles that presents several useful properties:

Definition 2.4.6. Let (P, π, B) be a G-principal bundle. A **connection** 1**-form** is a 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ that fulfills the following conditions:

i) $R_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \quad \forall g \in G$.

ii) $\omega_p(\overline{X}_p) = X$ for all $X \in \mathfrak{g}$, $p \in P$, where $\overline{X}_p = d_e L_p(X)$ is the fundamental vector field associated with X (as defined in Section 1.5).

We denote $\mathcal{C}(P)$ the set of connection 1-forms on P.

The following theorem states that connection 1-forms are preserved by pull-back:

Theorem 2.4.7. *Let (P, π, B) and (P', π', B') be G-principal bundles, $F : P \to P'$ a G-equivariant smooth map together with a smooth map $\overline{F} : B \to B'$ so that $\pi' \circ F = \overline{F} \circ \pi$, and $\omega \in \Omega^1(P', \mathfrak{g})$ a connection 1-form on P'. Then, $F^*\omega \in \Omega^1(P, \mathfrak{g})$ is a connection 1-form on P.*

Proof. We need to see that $F^*\omega$ verifies the two conditions of a connection 1-form. We denote R_g and L_p the actions on P, and R'_g and $L'_{F(p)}$ the actions on P'. Then, by the properties of ω, we get:

i) $R_g^*(F^*\omega_{F(p)g}) = F^*(R'^*_g \omega_{F(p)g}) = F^*(Ad_{g^{-1}} \circ \omega_{F(p)}) = Ad_{g^{-1}} \circ (F^*\omega_{F(p)}) \quad \forall g \in G$.

ii) $(F^*\omega_{F(p)})(\overline{X}_p) = \omega_{F(p)}(d_p F \circ d_e L_p(X)) = \omega_{F(p)}(d_e L'_{F(p)}(X)) = X \quad \forall X \in \mathfrak{g}$.

Here, we have applied that $F \circ R_g = R'_g \circ F$ and $F \circ L_p = L'_{F(p)}$ for all $g \in G$, $p \in P$, since F is G-equivariant. See [Son15, Sect. 10.2] for more details. □

Let us now introduce the notion of covariant derivative for vector bundles:

Definition 2.4.8. Let (V, π, B) be a vector bundle. A **connection** on V is a map

$$\begin{aligned} \nabla : \mathfrak{X}(B) \times \Gamma(V) &\to \Gamma(V) \\ (X, s) &\mapsto \nabla_X s \end{aligned}$$

such that ∇ is $\mathcal{F}(B)$-linear on X and $\nabla_X(f \cdot s) = X(f) \cdot s + f \cdot \nabla_X s$ for all $f \in \mathcal{F}(B)$, $s \in \Gamma(V)$.

For a given $X \in \mathfrak{X}(B)$, the map $\nabla_X : \Gamma(V) \to \Gamma(V)$, $s \mapsto \nabla_X s$ is called a **covariant derivative**.

Connection 1-forms allow us to define a covariant derivative on associated vector bundles. To do so, suppose that (P, π, B) is a G-principal bundle with ω a connection 1-form, $\rho : G \to \mathrm{Aut}(V)$ a representation of G, and (P', π', B) is the corresponding associated vector bundle with $P' := P \times_\rho V$. Given a section $s \in \Gamma(P')$ defined by $s(x) = [p(x), v(x)]$ for smooth maps $p : B \to P$ and $v : B \to V$, we set

$$\nabla_X^\omega [p(x), v(x)] := [p(x), d_x v(X) + d_e \rho(p^*\omega(X))(v(x))] \quad \forall X \in T_x B.$$

It is easy to check that ∇_X^ω is well-defined and a covariant derivative. Moreover, this yields a map $\nabla^\omega : \Gamma(P') \to \Omega^1(B, P')$ such that $\nabla^\omega s(x)(X) = \nabla_X^\omega s(x)$ for all $s \in \Gamma(P')$.

Remark 2.4.9. Notice that, even though a covariant derivative is defined for elements of $\mathfrak{X}(B)$, here an element of T_xB is taken instead. This is correct since for every $X \in T_xB$ there exists a vector field $\widetilde{X} \in \mathfrak{X}(B)$ such that $\widetilde{X}_x = X$, and the value of $\nabla_{\widetilde{X}}$ on a given point $x \in B$ only depends on $\widetilde{X}_x$.

Considering sections on a principal bundle, a local description of connection 1-forms can be constructed as follows.

Let (P, π, B) be a G-principal bundle and $\{U_i\}_{i\in I}$ an open covering of B with local sections $s_i : U_i \to P$ and transition functions $g_{ij} : U_{ij} \to G$ for all $i, j \in I$. Given a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, we define local connection forms $\omega_i := s_i^*\omega \in \Omega^1(U_i, \mathfrak{g})$. These 1-forms verify the following relation (see [Bär11, Sect. 2.3]):

$$\omega_{j_{|u}} = \mathrm{Ad}_{g_{ij}^{-1}(u)} \circ \omega_{i_{|u}} + d_u(g_{ij}^{-1}(u) \cdot g_{ij}) \quad \forall i, j \in I,\ u \in U_{ij}.$$

Remark 2.4.10. It can be seen that the connection 1-form ω is completely determined by the set of local connection forms $\{\omega_i\}_{i\in I}$. It is sufficient to take a partition of unity[2] $\{\varphi_i\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$, i.e. satisfying $\mathrm{supp}(\varphi_i) \subset U_i$ for all $i \in I$, and express $\omega = \sum_{i\in I} \varphi_i\omega_i$. Then, we can check that this combination defines an element of $\Omega^1(P, \mathfrak{g})$ using that there exists a bijection between connection 1-forms $\Omega^1(P_{U_i}, \mathfrak{g})$ and the local connections $\{\omega_i\}_{i\in I} \subset \Omega^1(U_i, \mathfrak{g})$ (see [Dup03, Chapt. 6])

Since a subordinate partition of unity always exists (see [Cur09, Sect. 1.3.2]), it can be seen that every principal bundle has a connection 1-form (see [Wen08, Sect. 3.4.2]).

Now, we show that the set of connection 1-forms $\mathcal{C}(P)$ can also be seen as an affine space over the 1-forms of an associated vector bundle. To do so, let us state some properties about k-forms.

Definition 2.4.11. Let (P, π, B) be a G-principal bundle and $\rho : G \to \mathrm{Aut}(V)$ a representation of G. A differentiable k-form $\overline{\omega} \in \Omega^k(P, V)$ is called:

i) **Horizontal** if $\overline{\omega}_p(X_1, \ldots, X_k) = 0$ for all $p \in P$ whenever at least one $X_i \in T_pP$ satisfies $X_i \in T_pP_{\pi(p)}$, which means that X_i belongs to the tangent space of the fiber $P_{\pi(p)}$.[3]

ii) **Of type** ρ if $R_g^*\overline{\omega} = \rho(g^{-1}) \circ \overline{\omega}$ for all $g \in G$.

We denote the set of k-forms fulfilling both conditions by $\Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$.

Theorem 2.4.12. *There exists an isomorphism* $\Omega^k_{hor}(P, V)^{(G,\rho)} \cong \Omega^k(B, P')$, *where* $P' = P \times_\rho V$.

Proof. First of all, consider $\overline{\omega} \in \Omega^k_{\mathrm{hor}}(P, V)^{(G,\rho)}$. We define $\omega \in \Omega^k(B, P')$ as the family $\{\omega_x\}_{x\in B}$ given by

$$\omega_x(X_1, \ldots, X_k) := [p, \overline{\omega}_p(Y_1, \ldots, Y_k)] \quad \forall x \in B,\ X_i \in T_xB$$

[2] A partition of unity of B is a set of smooth maps $\{\varphi_i : B \to [0, 1]\}_{i\in I}$ such that the set $\{\mathrm{supp}(\varphi_i)\}_{i\in I}$, where $\mathrm{supp}(\varphi_i) = \mathrm{Cl}\{x \in B : \varphi_i(x) \neq 0\}$, is locally finite and $\sum_{i\in I} \varphi_i(x) = 1$ for all $x \in B$.

[3] Note that this tangent space $T_pP_{\pi(p)}$ is well-defined since the fibers are embedded submanifolds of P.

where $p \in P_x$ and $Y_i \in T_pP$ is such that $d_p\pi(Y_i) = X_i$ for all $i = 1, \dots, k$. We denote $\omega = \overline{\omega}_B$.

This ω is well-defined. Indeed, it does not depend on the choice of $p \in P_x$, as G is transitive on the fibers. Besides, it is independent of the vectors $Y_i \in T_pP$, which can be proven considering the fact that $d_p\pi(Y_i - Y_i') = 0$ for every $Y_i' \in T_pP$.

On the other hand, if we take $\omega \in \Omega^k(B, P')$, we define $\overline{\omega} \in \Omega^k_{\text{hor}}(P, V)^{(G,\rho)}$ as:

$$\overline{\omega}_p(Y_1, \dots, Y_k) := [p]^{-1}(\omega_{\pi(p)}(d_p\pi(Y_1), \dots, d_p\pi(Y_k))) \quad \forall p \in P,\ Y_i \in T_pP$$

where $[p] : V \to P_x'$ is the smooth map defined as $[p](v) = [p, v]$ for all $v \in V$, $p \in P_x$. See [Ham17, Sect. 5.13] for more details. □

From the defining properties of connection 1-forms, it follows that for all $\omega, \omega' \in \mathcal{C}(P)$ we have $\omega - \omega' \in \Omega^1_{\text{hor}}(P, \mathfrak{g})^{(G,\text{Ad})}$, where $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ is the adjoint representation. Moreover, all $\omega \in \Omega^1(P, \mathfrak{g})$ and $\overline{\omega} \in \Omega^1_{\text{hor}}(P, \mathfrak{g})^{(G,\text{Ad})}$ satisfy $\omega + \overline{\omega} \in \Omega^1(P, \mathfrak{g})$.

Hence, even though $\mathcal{C}(P)$ is not a vector space as $0 \notin \mathcal{C}(P)$, by Theorem 2.4.12, it is an affine space over the vector space of 1-forms $\Omega^1(B, P \times_{\text{Ad}} \mathfrak{g})$.

Remark 2.4.13. Observe that, given two connection 1-forms $\omega, \omega' \in \Omega^1(P, \mathfrak{g})$ and local sections $s_i : U_i \to P$, we can define the form $(\omega - \omega')_B \in \Omega^1(B, P \times_{\text{Ad}} \mathfrak{g})$ using the corresponding local connections $\omega_i = s_i^*\omega$ and $\omega_i' = s_i^*\omega'$ as

$$(\omega - \omega')_{B_{|x}}(X) = [s_i(x), (\omega_i - \omega_i')_{|x}(X)] \quad \forall x \in U_i,\ X \in T_xB.$$

2.4.2 Ehresmann connections

In the following, an alternative and equivalent description of connections on a principal bundle is presented. With this aim, it is necessary to introduce some new concepts related to fiber bundles.

Definition 2.4.14. Let M be a smooth manifold. A **distribution** on M is a vector subbundle of the tangent bundle TM.

Definition 2.4.15. Let (P, π, B) be a G-principal bundle, $x \in B$ a point, and $p \in P_x$ an element of the fiber. The **vertical tangent space** V_p of P in p is the tangent space of the fiber T_pP_x, which is a subspace of the tangent space T_pP.

Proposition 2.4.16. *Let (P, π, B) be a G-principal bundle. Given $p \in P$, its vertical tangent space satisfies the following properties:*

i) $V_p = \ker(d_p\pi)$.

ii) The following map is a vector space isomorphism:

$$\begin{aligned} d_eL_p : \mathfrak{g} &\to V_p \\ X &\mapsto \overline{X}_p. \end{aligned}$$

iii) The set $V = \bigcup_{p \in P} \{p\} \times V_p$ *is a distribution on* P *with rank* $k = \dim(G)$ *and a global trivialization with inverse given by*

$$\begin{aligned}\psi^{-1} : P \times \mathfrak{g} &\to V \\ (p, X) &\mapsto \overline{X}_p.\end{aligned}$$

*V is called the **vertical tangent bundle**.*

iv) V_p *is right-invariant:* $d_p R_g(V_p) = V_{pg}\ \forall g \in G$.

Proof. Let us go through each property:

i) For every $X \in V_p$ there exists a curve $\gamma : I \to P_{\pi(p)}$ such that $\gamma(0) = p$, $\dot{\gamma}(0) = X$. Then, $\pi(\gamma(t)) = \pi(p) = \text{ct}$ for all $t \in I$, which implies $d_p\pi(X) = d_p\pi(\dot{\gamma}(0)) = (\pi \dot{\circ} \gamma)(0) = 0$. Thus, $V_p \subset \ker(d_p\pi)$.

   Moreover, by the Regular Value Theorem, we have $\dim(\ker(d_p\pi)) = \dim P - \dim B = \dim P_{\pi(p)} = \dim V_p$. Therefore, $V_p = \ker(d_p\pi)$.

ii) First, observe that $d_e L_p(\mathfrak{g}) \subset \ker(d_p\pi) = V_p$. We can see this using the same argument than in condition *i*) taking a curve $\gamma : I \to G$ satisfying $\gamma(0) = e$, $\dot{\gamma}(0) = X$ so that $L_p(\gamma(t)) = p \cdot \gamma(t) \in P_{\pi(p)}$ for all $t \in I$, since G is transitive on the fibers. This way, we get $\pi(L_p(\gamma(t))) = \pi(p) = \text{ct}$ for all $t \in I$.

   Furthermore, since the action of G is free, $d_e L_p : \mathfrak{g} \to V_p$ is injective (see [Ham17, Sect. 3.4]). Finally, we have $\dim \mathfrak{g} = \dim G = \dim V_p$. Hence, the map described in property *ii*) is an isomorphism.

iii) Consider $p \in P$ and a local trivialization of the tangent bundle over $U \subset P$ such that $TP_U \cong U \times \mathbb{R}^{\dim(P)}$. By property *ii*), we have $V \cap TP_U = \bigcup_{p \in U} \{p\} \times V_p \cong \bigcup_{p \in U} \{p\} \times \mathfrak{g} \cong U \times \mathbb{R}^{\dim G}$. Hence, V is a vector subbundle of TP of rank $k = \dim G$. In addition, it follows from condition *ii*) that ψ^{-1}, as given in the proposition, yields a global trivialization of this vector subbundle.

iv) By property *ii*), it is sufficient to see that $d_p R_g(\overline{X}_p) = d_p R_g(d_e L_p(X)) = d_e(R_g \circ L_p)(X) = d_e L_{gp}(X) = \overline{X}_{pg}$ for all $X \in \mathfrak{g}$.

□

Remark 2.4.17. Notice that condition *ii*) implies that the vectors tangent to the fibers $P_{\pi(p)}$ of a G-principal bundle are of the form $\overline{X}_p$ for some $X \in \mathfrak{g}$.

Now, we consider complements of V_p in T_pP for $p \in P$. This notion yields new distributions for principal bundles that allow us to connect the space tangent to the total space T_pP with the space tangent to the base of the bundle $T_{\pi(p)}B$.

Definition 2.4.18. Let (P, π, B) be a G-principal bundle and a point $p \in P$. A **horizontal tangent space** H_p in p is a subspace of T_pP complementary to the vertical tangent space in p, so that $T_pP = V_p \oplus H_p$.

Proposition 2.4.19. *Let (P, π, B) be a G-principal bundle and H_p a horizontal tangent space in $p \in P$. Then, the map $d_p\pi : H_p \to T_{\pi(p)}B$ is an isomorphism.*

Proof. It suffices to observe that the map is injective, as $H_p \cap \ker(d_p\pi) = H_p \cap V_p = \{0\}$, and that $\dim H_p = \dim T_pP - \dim T_pP_{\pi(p)} = \dim P - \dim P_{\pi(p)} = \dim B$. □

Definition 2.4.20. An **Ehresmann connection** H on P is a distribution on P consisting of horizontal tangent spaces that is right-invariant i.e. such that the horizontal spaces verify $d_pR_g(H_p) = H_{pg}$ for all $p \in P$, $g \in G$.

H is also called the **horizontal tangent bundle** given by the connection.

Remark 2.4.21. As they are right-invariant, the horizontal spaces H_p that form an Ehresmann connection H can be determined along a fiber $P_{\pi(p)}$ just by fixing an H_{p_0} for some $p_0 \in P_{\pi(p)}$, since G is transitive on the fibers.

Remark 2.4.22. Notice that horizontal tangent spaces are not uniquely determined. Thus, Ehresmann connections are not unique either. However, it can be proven that every principal bundle has at least one Ehresmann connection (see [Ham17, Sect. 5.1.2]).

As mentioned before, connection 1-forms are completely equivalent to Ehresmann connections on a principal bundle:

Theorem 2.4.23. *Let (P, π, B) be a G-principal bundle. Then, there exists a bijection between connections 1-forms and Ehresmann connections on P:*

i) Given a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, the elements $H_p = \ker(\omega_p)$ for $p \in P$ define an Ehresmann connection H on P.

ii) Given an Ehresmann connection H on P, the map $\omega_p(\overline{X}_p + Y_p) = X$ for $p \in P$, $X \in \mathfrak{g}$, and $Y_p \in H_p$, yields a connection 1-form on P.

Proof. See [Ham17, Sect. 5.2.1]. □

Example 2.4.24. Let us go through some trivial examples of connections:

1. Consider the trivial G-principal bundle $(P \times G, \mathrm{pr}_1, P)$. In this case we can write $V_{(p,g)} = T_{(p,g)}(\{p\} \times G) \cong T_gG$ for all $(p, g) \in P \times G$. Then, the collection of horizontal spaces $H_{(p,g)} = T_{(p,g)}(P \times \{g\}) \cong T_pP$ is an Ehresmann connection called the **canonical flat connection**.

2. Suppose a G-principal bundle (P, π, B) where $G = \{e\}$ is the trivial Lie group. This means $\mathfrak{g} = \{0\}$ and $B = P$. Then, $V_p = \{0\}$ and, as a consequence, $H_p = T_pP$ for all $p \in P$. Hence, there exists a unique connection 1-form for this bundle, since there is a unique map $\omega_p : T_pP \to \{0\}$ for each $p \in P$.

3. Consider a G-principal bundle (P, π, B) with a trivial base space $B = \{x\}$, which implies $P = G$. Thus, we have $V_p = \ker(d_p\pi) = T_pG$, since $T_xB = \{0\}$ for all $x \in B$, and $H_p = \{0\}$ for all $p \in P$. In this case, the map $\omega_g = d_gL_{g^{-1}}$ determines a connection 1-form of the bundle. Actually, it can be seen that this is the only possible connection 1-form of this bundle (see [Son15, Sect. 10.3]).

2.5 Curvature

In the following, we see how to construct a curvature 2-form on a principal bundle out of a connection 1-form. To do so, some new concepts regarding vector fields and differential forms are necessary:

Definition 2.5.1. Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ a vector field on M. The **Lie derivative** L_X is the map $L_X : \mathcal{F}(M) \to \mathcal{F}(M)$ given by

$$(L_X f)(p) = (d_p f)(X_p) \quad \forall f \in \mathcal{F}(M), p \in M.$$

Remark 2.5.2. It can be shown that this map is a derivation (see [Ham17, Sect. A.1.10]).

The Lie derivative can also be defined over a 1-form, as follows:

Definition 2.5.3. Let $\omega \in \Omega^1(M)$ be a 1-form and $X \in \mathfrak{X}(M)$ a differential vector field. Then, $L_X\omega$ is the unique 1-form that satisfies $(L_X\omega)(Y) = X(\omega(Y)) - \omega([X,Y])$ for every smooth vector field $Y \in \mathfrak{X}(M)$. This is a differentiable 1-form (see [Cur09, Sect. 5.3] for more properties).

Let us now consider a map that connects forms of different orders and state some of its properties (see [Ham17, Sect. A.2.7] for proofs):

Theorem 2.5.4. *Let M be a smooth manifold. For every $k \geq 0$ there exists a unique map $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ that fulfills the following conditions:*

i) d_k *is* $\mathbb{R}$*-linear.*

ii) $\forall f \in \mathcal{F}(M), X \in \mathfrak{X}(M)$: $d_0 f(X) = L_X f$.

iii) $d_k^2 = d_k \circ d_k \equiv 0$.

iv) $\forall \omega \in \Omega^k(M), \eta \in \Omega^l(M)$: $d_{k+l}(\omega \wedge \eta) = d_k\omega \wedge \eta + (-1)^k \omega \wedge d_l\eta$.

*If k is clear from the context, we write d instead of d_k. This map is called the **differential** or **exterior derivative**.*

Proposition 2.5.5. *Let M be a smooth manifold. Then:*

1. *Given a 1-form $\omega \in \Omega^1(M)$, the differential satisfies $d\omega(X,Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X,Y])$ for all $X, Y \in \mathfrak{X}(M)$.*

2. *Given a 2-form $\widetilde{\omega} \in \Omega^2(M)$, the differential satisfies $d\widetilde{\omega}(X,Y,Z) = L_X(\widetilde{\omega}(Y,Z)) + L_Y(\widetilde{\omega}(Z,X)) + L_Z(\widetilde{\omega}(X,Y)) - \widetilde{\omega}([X,Y],Z) - \widetilde{\omega}([Y,Z],X) - \widetilde{\omega}([Z,X],Y)$ for all $X, Y, Z \in \mathfrak{X}(M)$.*

Proposition 2.5.6. *Let $f : N \to M$ be a smooth function between differentiable manifolds and $\omega \in \Omega^k(M)$ a k-form. Then, $f^*(d\omega) = d(f^*\omega)$.*

Remark 2.5.7. There exist analogous theorem and propositions for the k-forms $\Omega^k(M, V)$ for any vector space V.

As mentioned before, the differential and the Lie derivative allow us to define a 2-form related with connection 1-forms. Suppose that (P, π, B) is a G-principal bundle, $\omega \in \Omega^1(P, \mathfrak{g})$ a connection 1-form, H the associated horizontal tangent bundle such that $T_pP = T_pP_{\pi(p)} \oplus H_p$ for all $p \in P$, and $\mathrm{pr}_H : T_pP \to H_p$ the horizontal projection. Then:

Definition 2.5.8. The **curvature 2-form** of ω is the 2-form $\Omega \in \Omega^2(P, \mathfrak{g})$ defined as $\Omega_p(X, Y) := d\omega_p(\mathrm{pr}_H(X), \mathrm{pr}_H(Y))$ for all $X, Y \in T_pP$, $p \in P$.

This curvature Ω can also be written as Ω^ω to emphasize the dependence on the connection 1-form ω.

Now, we provide an alternative expression for the curvature 2-form. With this aim, given two forms $\eta, \varphi \in \Omega^1(P, \mathfrak{g})$, we consider the differential form $[\eta, \varphi] \in \Omega^2(P, \mathfrak{g})$ given by $[\eta_p, \varphi_p](X, Y) := [\eta_p(X), \varphi_p(Y)] - [\eta_p(Y), \varphi_p(X)]$ for all $X, Y \in T_pP$, $p \in P$.[4] Notice that $[\omega_p, \omega_p](X, Y) = 2[\omega_p(X), \omega_p(Y)]$ for all $\omega \in \Omega^1(P, \mathfrak{g})$.

Lemma 2.5.9. *Let (P, π, B) be a G-principal bundle with ω a connection 1-form, $\overline{X}$ a fundamental vector field for $X \in \mathfrak{g}$, and Y a horizontal vector field. Then, $[\overline{X}, Y]$ is horizontal.*

Proof. See [Bär11, Sect. 2.4]. □

Proposition 2.5.10 (Structure equation). *Let (P, π, B) be a G-principal bundle with connection 1-form ω. Then, the curvature 2-form of ω satisfies*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

Proof. Recall that given $p \in P$ every element of T_pP can be uniquely written as $\overline{X}_p + Y_p$ where $X \in \mathfrak{g}$ and $Y_p \in H_p$. Then, since Ω is linear, it suffices to check the equality for vector fields that are either fundamental vector fields or elements of horizontal tangent spaces. We distinguish the following cases:

i) First, assume that X and Y are fundamental vector fields, so $X = \overline{X'}$ and $Y = \overline{Y'}$ where $X', Y' \in \mathfrak{g}$. Then, $\mathrm{pr}_H(X) = \mathrm{pr}_H(Y) = 0$, which implies $\Omega(X, Y) = 0$. Moreover, using part 1 of Proposition 2.5.5, we have

$$\begin{aligned} d\omega(X, Y) &= L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X, Y]) \\ &= L_X(Y') - L_Y(X') - \omega([\overline{X'}, \overline{Y'}]) \\ &= -[X', Y'] = -[\omega(X), \omega(Y)] = -\frac{1}{2}[\omega, \omega](X, Y) \;\Rightarrow \end{aligned}$$

$$\Rightarrow\; (d\omega + \frac{1}{2}[\omega, \omega])(X, Y) = -\frac{1}{2}[\omega, \omega](X, Y) + \frac{1}{2}[\omega, \omega](X, Y) = 0,$$

where in the third equality we use that X' and Y' are constant maps from P to $\mathfrak{g}$ and that the map that assigns $X \mapsto \overline{X}$, where $X \in \mathfrak{g}$, is a Lie algebra homomorphism, i.e. $[\overline{X}, \overline{Y}] = \overline{[X, Y]}$ for all $X, Y \in \mathfrak{g}$.

[4] The form $[\eta, \varphi] \in \Omega^{k+l}(P, \mathfrak{g})$ can be defined in general for any two arbitrary forms $\eta \in \Omega^k(P, \mathfrak{g})$ and $\varphi \in \Omega^l(P, \mathfrak{g})$ (see [Ham17, Sect. 5.5.2]).

ii) Suppose now that both X and Y are horizontal. Then, $\mathrm{pr}_H(X) = X$ and $\mathrm{pr}_H(Y) = Y$, which implies $\Omega(X,Y) = d\omega(X,Y)$. On the other hand, $[\omega,\omega](X,Y) = 2[\omega(X), \omega(Y)] = 0$. Hence, $(d\omega + \frac{1}{2}[\omega,\omega])(X,Y) = d\omega(X,Y)$.

iii) Finally, if $X = \overline{X}'$ is a fundamental vector for $X' \in \mathfrak{g}$ and Y is horizontal, we have that $\Omega(X,Y) = d\omega(0,Y) = 0$. Besides, applying part 1 of Proposition 2.5.5 again and Lemma 2.5.9, we get $d\omega[X,Y] = -\omega([X,Y]) = 0$ and $[\omega,\omega](X,Y) = 2[\omega(X), 0] = 0$.

□

Analogously as for connection 1-forms, the curvature 2-form satisfies the following equality:

Proposition 2.5.11. *If Ω is a curvature 2-form, then $R_g^*\Omega = \mathrm{Ad}_{g^{-1}} \circ \Omega$ for all $g \in G$.*

Proof. To check this formula, we use condition *ii*) of Definition 2.4.6 of a connection 1-form and Proposition 2.5.6. It is also necessary to take into account that $d_pR_g(H_p) = H_{pg}$ and $d_pR_g(V_p) = V_{pg}$, which means $\mathrm{pr}_H \circ d_pR_g = d_pR_g \circ \mathrm{pr}_H$ for all $g \in G, p \in P$, since d_pR_g is linear.

See [Bär11, Sect. 2.4] for a detailed proof. □

Remark 2.5.12. Observe that the definition of curvature 2-form together with Proposition 2.5.11 imply that every $\Omega \in \Omega^2(P,\mathfrak{g})$ is an element of $\Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{(G,\mathrm{Ad})}$. Then, according to Theorem 2.4.12, there exists a corresponding 2-form $\Omega_B \in \Omega^2(B, P \times_{\mathrm{Ad}} \mathfrak{g})$.

Proposition 2.5.13 (Bianchi identity). *Let (P,π,B) be a G-principal bundle with connection 1-form ω and denote Ω its curvature 2-form. Then, $d\Omega$ vanishes on $H \times H \times H$.*

Proof. Using the structure equation and condition (iii) of Theorem 2.5.4, we get $d\Omega = dd\omega + \frac{1}{2}d[\omega,\omega] = \frac{1}{2}d[\omega,\omega]$. Thus, it suffices to prove that $d[\omega,\omega]$ vanishes on $H \times H \times H$.

We denote $\widetilde{\omega} := [\omega,\omega] \in \Omega^2(P,\mathfrak{g})$. Then, applying part 2 of Proposition 2.5.5 and using that $\widetilde{\omega}(X,Y) = 0$ if X or Y are horizontal, we get $d\widetilde{\omega}(Y_1,Y_2,Y_3) = 0$ for horizontal vectors Y_1, Y_2, Y_3. □

Remark 2.5.14. Notice that if G is abelian, then $\mathfrak{g}$ is abelian too, that is, $[\cdot,\cdot] \equiv 0$, so we get $\Omega = d\omega$. In this case, $d\Omega \equiv 0$.

Similarly as we did in Section 2.4 for connection 1-forms, local curvature 2-forms can be defined on the base of a G-principal bundle (P,π,B).

Consider an open covering $\{U_i\}_{i\in I}$ of B such that there exist local sections $s_i : U_i \to P$ and transition functions $g_{ij} : U_{ij} \to G$ for all $i,j \in I$. Let ω be a connection 1-form of the bundle and $\{\omega_i\}_{i\in I}$ its local 1-forms given by $\omega_i = s_i^*\omega$, we set $\Omega_i := s_i^*\Omega \in \Omega^2(U_i,\mathfrak{g})$. It is easy to see that for these local curvatures an equation analogous to the structure equation holds:

$$\Omega_i = d\omega_i + \frac{1}{2}[\omega_i,\omega_i].$$

Proceeding the same way as for connection 1-forms, the following relation between different local curvature 2-forms can be proven:

$$\Omega_j = \mathrm{Ad}_{g_{ij}^{-1}(u)} \circ \Omega_i \quad \forall u \in U_{ij},\ i, j \in I.$$

Remark 2.5.15. If G is abelian, the previous relation reads $\Omega_i = \Omega_j$ for all $i, j \in I$, which implies that Ω_i is independent of the choice of a local section. Therefore, each Ω_i is a restriction of a globally defined 2-form $\widetilde{\Omega} \in \Omega^2(B, \mathfrak{g})$ given by $\widetilde{\Omega}_{|U_i} = \Omega_i$ for all $i \in I$.

Remark 2.5.16. Notice that, as explained in Remark 2.4.13 for connection 1-forms, we can write $\Omega_B \in \Omega^2(B, P \times_{\mathrm{Ad}} \mathfrak{g})$ in terms of the local 2-forms Ω_i on B as

$$\Omega_{B|_x}(X, Y) = [s_i(x), \Omega_{i|_x}(X, Y)] \quad \forall x \in U_i,\ X, Y \in T_x B$$

where $s_i : U_i \to P$ is a local section.

2.6 Gauge transformations

Definition 2.6.1. Let (P, π, B) be a G-principal bundle. An **automorphism** of P is a diffeomorphism $f : P \to P$ that is G-equivariant, i.e.

$$f(p \cdot g) = f(p) \cdot g \quad \forall g \in G, p \in P.$$

We denote $\mathrm{Aut}(P) := \{\text{automorphisms of } P\}$ and call this set the **automorphism group** of P.

Remark 2.6.2. Note that $\mathrm{Aut}(P) \subset \mathrm{Diff}(P)$ is a subgroup. Moreover, as a group it acts from the right on the group of connection 1-forms $\mathcal{C}(P)$ with a right action given by $f^*\omega$ for all $\omega \in \mathcal{C}(P)$, $f \in \mathrm{Aut}(P)$. Indeed, this defines a right action since $(f \circ g)^* = g^* \circ f^*$ for all $f, g \in \mathrm{Aut}(P)$. Besides, $f^*\omega$ is a connection 1-form for all $\omega \in \mathcal{C}(P)$, this is a particular case of Theorem 2.4.7 for $P' = P$, $F = f$ and $\overline{F} = \bar{f}$, which will be defined in the following remark.

Remark 2.6.3. For each $f \in \mathrm{Aut}(P)$, we now construct a map $\bar{f}$ on the base of the bundle. In order to define it, observe that we have $f(P_{\pi(p)}) = P_{\pi(f(p))}$ for all $p \in P$, since G acts transitively on the fibers. This implies that there exists a unique smooth map $\bar{f} : B \to B$ satisfying $\bar{f} \circ \pi = \pi \circ f$, which can be defined as $\bar{f}(x) = \pi(f(P_x))$ for all $x \in B$.

Furthermore, the map $\mathrm{Aut}(P) \to \mathrm{Diff}(B)$ that assigns $f \mapsto \bar{f}$ is a group homomorphism, since $(\overline{f' \circ f})(x) = \pi(f'(f(P_x))) = \pi(f'(P_{\bar{f}(x)})) = \bar{f}'(\bar{f}(x))$ for all $x \in B$.

Definition 2.6.4. A **gauge transformation** of P is an automorphism $f \in \mathrm{Aut}(P)$ so that $\bar{f} = \mathrm{id}_B$. We denote $\mathcal{G}(P) := \{\text{gauge transformations on } P\}$, which is called the **gauge group** of P.

A **local gauge transformation** is a gauge transformation of the G-principal bundle (P_U, π_{P_U}, U) for some open subset $U \subset B$.

Remark 2.6.5. To show that $\mathcal{G}(P)$ is indeed a group, it suffices to observe that $\mathcal{G}(P) = \ker(f \mapsto \bar{f}) \subset \mathrm{Aut}(P)$. Moreover, as a subgroup of $\mathrm{Aut}(P)$, there exists a right action of $\mathcal{G}(P)$ on $\mathcal{C}(P)$.

For gauge transformations, it is also possible to define an action on an associated vector bundle as follows:

Theorem 2.6.6. *Let (P', π', B) be an associated vector bundle where $P' := P \times_\rho V$ for some representation $\rho : G \to \mathrm{Aut}(V)$. Then, the gauge group of P acts from the left on P' through the following morphism:*

$$\begin{aligned} \mathcal{G}(P) \times P' &\to P' \\ (f, [p, v]) &\mapsto f \cdot [p, v] = [f(p), v]. \end{aligned}$$

Proof. In order to check that the map is well-defined, assume that $[p, v] = [p', v']$. Then, by definition, $[p', v'] = [p \cdot g, \rho(g)^{-1}(v)]$ for some $g \in G$. Thus, we have $[f(p'), v'] = [f(p \cdot g), \rho(g)^{-1}(v)] = [f(p) \cdot g, \rho(g)^{-1}(v)] = [f(p), v]$, which implies that the action is well-defined. □

Note that gauge transformations preserve fibers, as $\bar{f}(x) = \pi(f(P_x)) = x$ implies $f(P_x) = P_x$ for all $x \in B$. Therefore, for each $p \in P$ there exists a unique $g(p) \in G$ such that $f(p) = p \cdot g(p)$. This yields a smooth function $g : P \to G$.

Conversely, if G is abelian, every smooth map $g : P \to G$ defines a gauge transformation of the form $f(p) := p \cdot g(p)$ for all $p \in P$.

Definition 2.6.7. A **physical gauge transformation** is a smooth map $\tau : B \to G$. A **local physical gauge transformation** is a smooth map $\tau : U \to G$ over an open subset $U \subset B$.

Remark 2.6.8. Given a map $g : P \to G$ associated to $f \in \mathcal{G}(P)$ such that $f(p) = p \cdot g(p)$ for all $p \in P$, the map $\tau_f := g \circ s$ is a physical gauge transformation for any section $s : B \to P$.

Analogously, we can assign a local physical gauge transformation $\tau_f : U \to G$ to a local gauge transformation $f \in \mathcal{G}(P_U)$ using a local section $s : U \to P$. Moreover, it can be seen that if we fix a local section $s : U \to P$, this determines a bijection between $f \in \mathcal{G}(P_U)$ and local physical gauge transformations $\tau : U \to G$ (see [Ham17, Sect. 5.3.2]).

The action of gauge transformations on an associated vector bundle (P', π', B), where $P' = P \times_\rho V$, can be described in terms of these physical gauge transformations. The following theorem is an example:

Theorem 2.6.9. *Let $\Phi : U \to P'$ be a local section of P' given by $\Phi(x) = [s(x), \phi(x)]$ where $s : U \to P$ is a local section of P and $\phi : U \to V$ is a smooth map. If we take a local gauge transformation $f \in \mathcal{G}(P_U)$ with associated physical gauge transformation $\tau_f : U \to G$, we have*

$$(f \cdot \Phi)(x) = [s(x), \rho(\tau_f(x))(\phi(x))] \quad \forall x \in U.$$

Proof. Using the map $g : P_U \to G$ associated to f, Theorem 2.6.6, and the definition of τ_f, it is easy to check the equality: $(f \cdot \Phi)(x) = f \cdot [s(x), \phi(x)] = [f(s(x)), \phi(x)] = [s(x) \cdot g(s(x)), \phi(x)] = [s(x) \cdot \tau_f(x), \phi(x)] = [s(x), \rho(\tau_f(x))(\phi(x))]$. □

Chapter 3

Applications to Physics

In the following we apply the mathematical concepts presented in the previous chapters to understand how the Standard Model of particle physics explains one of the fundamental forces of nature, that is, the electroweak interaction. Our aim is to develop the classical gauge theory presented by Yang and Mills and describe how its inconsistencies with the experimental observations are solved through the so-called Higgs mechanism.

We begin by defining some operators regarding k-forms that will be necessary in the subsequent sections.

3.1 Scalar products on forms

Hereafter in this section, we always assume (M, g) is an oriented pseudo-Riemannian manifold, meaning that the metric tensor is everywhere non-degenerate, symmetric, and smooth, but not necessarily positive definite. Besides, we assume that (M, g) is n-dimensional and g has signature (s, t).[1]

Moreover, throughout this section and the following ones, we employ Einstein summation convention.

Given $\omega \in \Omega^k(M)$, we can express this k-form in terms of smooth functions considering a smooth atlas $\{(U_i, \phi_i)\}_{i\in I}$ of M. For some U_i with local coordinates $\{x^1, \dots, x^n\}$, we can write locally

$$\omega = \sum_{1\leq i_1<\cdots<i_k\leq n} \omega_{i_1\dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where $\omega_{i_1\dots i_k} \in \mathcal{F}(M)$. Note that $\omega_{i_1\dots i_k} = \omega(\partial_{i_1}, \dots, \partial_{i_k})$ with $\partial_i = \partial/\partial x^i$.

Definition 3.1.1. The **scalar product of k-forms** is defined as the map:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \Omega^k(M) \times \Omega^k(M) &\to \mathcal{F}(M) \\ (\omega, \eta) &\mapsto \frac{1}{k!} \omega_{i_1\dots i_k} \eta^{i_1\dots i_k} \end{aligned}$$

Remark 3.1.2. This scalar product is well-defined independently of the choice of local chart and, given two charts, it coincides on the intersection.

[1] The same applies for (B, g) when considering a vector bundle (V, π, B).

Definition 3.1.3. Let $\{e_1, \dots, e_n\}$ be a positively oriented orthonormal basis of T_xM for some $x \in M$. The **canonical volume form on** M is the element $\mathrm{dvol}_g \in \Lambda^n T_x^*M$ determined by $\mathrm{dvol}_g(e_1, \dots, e_n) = +1$.

Remark 3.1.4. This form dvol_g does not depend on the choice of orthonormal basis $\{e_1, \dots, e_n\}$ up to sign, and the sign is determined by the orientation. Moreover, it can actually be defined on any n-dimensional real vector space (see [Bär11, Sect. 3.1]).

Lemma 3.1.5. *Let (U, ϕ) be an oriented chart on M with local coordinates $\{x^1, \dots, x^n\}$. Then, $\mathrm{dvol}_g = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$, where $|g| = |\det(g_{ij})|$ being $\det(g_{ij}) = \det(g(\partial_i, \partial_j))$ the determinant of the metric tensor.*

Proof. See [Ham17, Sect. 7.2.1]. □

The volume form allows us to define the following operator on k-forms:

Definition 3.1.6. The **Hodge star operator** is the linear map $* : \Omega^k(M) \to \Omega^{n-k}(M)$ characterized by $\langle \omega, \eta \rangle \mathrm{dvol}_g = \omega \wedge *\eta$ for all $\omega, \eta \in \Omega^k(M)$.

Proposition 3.1.7. *The Hodge star operator is given by*

$$*(e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*) = g^{i_1 i_1} \cdot \cdots \cdot g^{i_k i_k} \cdot \epsilon_{i_1 \dots i_n} \cdot e_{i_{k+1}}^* \wedge \cdots \wedge e_{i_n}^*,$$

where $\{i_1, \dots, i_k\}$ and $\{i_{k+1}, \dots, i_n\}$ are complementary sets, $g^{ii} = g(e_i, e_i)$, $\{e_1^*, \dots, e_n^*\}$ is the basis of T_x^*M dual to $\{e_1, \dots, e_n\}$, and ϵ is totally antisymmetric with $\epsilon_{1\dots n} = 1$.

Proof. See [Bär11, Sect. 3.1]. □

Remark 3.1.8. Given a metric and a choice of an orientation, it can be seen that $*$ is uniquely defined. For more properties of this operator see [Ham17, Sect. 7.2.1].

Definition 3.1.9. The **codifferential** is defined as the map $d^* : \Omega^{k+1}(M) \to \Omega^k(M)$ given by $d^* = (-1)^{t+nk+1} * d*$, where d is the differential defined in Theorem 2.5.4.

In what follows, we introduce the notion of integration of forms, beginning on $\mathbb{R}^k$ and generalizing to smooth manifolds.

Definition 3.1.10. Let $U \subset \mathbb{R}^k$ be an open subset with standard coordinates $\{x^1, \dots, x^k\}$ and $\omega = f(x)dx^1 \wedge \cdots \wedge dx^k \in \Omega^k(U)$ a differentiable k-form for some smooth function $f : U \to \mathbb{R}$. The **integral of** ω **over a subset** $A \subset U$ is the Riemann integral of $f(x)$ (if it exists):

$$\int_A \omega := \int_A f(x)dx^1 \dots dx^k.$$

In order to give a generalization for the integral of a k-form ω over a manifold M, we need several conditions:

i) The manifold M has to be oriented.

ii) If the dimension of M is n, only n-forms can be integrated.

iii) The n-form ω must belong to $\Omega_c^n(M) := \{\omega \in \Omega^n(M) : \text{supp}(\omega) \text{ is compact}\}$.[2]

Definition 3.1.11. Let M be a n-dimensional oriented manifold, $\{(U_i, \phi_i)\}_{i\in I}$ an oriented atlas, and $\omega \in \Omega_c^n(U_i)$ a n-form with compact support for some $i \in I$. Note that $(\phi_i^{-1})^*\omega \in \Omega_c^n(\phi(U_i))$, since $\phi_i : U_i \to \phi_i(U_i)$ is a diffeomorphism. The **integral of ω over U_i** is given by:

$$\int_{U_i} \omega := \int_{\phi_i(U_i)} (\phi_i^{-1})^*\omega.$$

Remark 3.1.12. The definition of this integral over U_i is independent of the choice of coordinates on U_i and it is linear on $\omega \in \Omega_c^n(U_i)$ (see [Tu11, Sect. 23.4]).

These two properties allow us to define the integral of a form $\omega \in \Omega_c^n(M)$ over the whole manifold. To do so, we consider a partition of unity $\{\varphi_i\}_{i\in I}$ subordinate to the open cover $\{U_i\}_{i\in I}$, that is, $\text{supp}(\varphi_i) \subset U_i$ for all $i \in I$. As mentioned in Section 2.4.1, a partition of unity verifying this condition always exists. Then, as the partition of unity has locally finite supports, we can express ω as a finite sum $\omega = \sum_{i\in I} \varphi_i\omega$. Moreover, it can be shown that the support of $\varphi_i\omega$ is compact for all $i \in I$, thus the integrals $\int_{U_i} \varphi_i\omega$ are well-defined. This yields the following definition:

Definition 3.1.13. Let $\omega \in \Omega_c^n(M)$ be a smooth n-form with compact support. The **integral of ω over M** is defined as:

$$\int_M \omega := \sum_{i\in I} \int_{U_i} \varphi_i\omega.$$

Remark 3.1.14. The integral $\int_M \omega$ is independent of the choice of oriented atlas and partition of unity. Besides, we have that $\int_{-M} \omega = -\int_M \omega$, where $-M$ corresponds to the same manifold with opposite orientation (see [Tu11, Sect. 23.4]).

With the presented concept of integration of forms, it is possible to define a new type of scalar product for forms with compact support.

Definition 3.1.15. The **L^2-scalar product of k-forms** is the map:

$$\begin{aligned} \langle\cdot,\cdot\rangle_{L^2} : \Omega_c^k(M) \times \Omega_c^k(M) &\to \mathbb{R} \\ (\omega, \eta) &\mapsto \int_M \langle\omega, \eta\rangle \text{dvol}_g. \end{aligned}$$

The scalar products presented above, as well as the codifferential, can be extended to k-forms with values in vector bundles.

Let (V, π, B) be a vector bundle of rank r where V carries a Riemannian metric $\langle\cdot,\cdot\rangle$. This induces a map $\langle\cdot,\cdot\rangle_V : V \times_B V \to B \times \mathbb{R}$ that assigns $(v_1, v_2) \mapsto (\pi(v_1), \langle v_1, v_2\rangle)$, where $V \times_B V = \{(v_1, v_2) \in V \times V : \pi(v_1) = \pi(v_2)\}$. We call $\langle\cdot,\cdot\rangle_V$ a **bundle metric**.

Now, consider an open cover $\{U_i\}_{i\in I}$ of B. For each $i \in I$, we can take locally defined smooth maps $v_j : U_i \to V$ for $j = 1, \dots, r$, such that for all $x \in U_i$ the set $\{v_1(x), \dots, v_r(x)\}$ is a basis of V_x. Then, any k-form $\omega \in \Omega^k(B, V)$ can be written locally as $\omega = \sum_{j=1}^r \omega_j \otimes v_j$, with $\omega_j \in \Omega^k(U_i)$ for all $j = 1, \dots, r$.

[2] The support of an n-form $\omega \in \Omega^n(M)$ is defined as $\text{supp}(\omega) = \text{Cl}\{x \in M : \omega_x \not\equiv 0\}$ (see [Tu11, Sect. 18]).

Definition 3.1.16. The **scalar product of k-forms with values in** V is the map:

$$\begin{aligned}\langle\cdot,\cdot\rangle_V : \Omega^k(B,V)\times\Omega^k(B,V) &\to \mathcal{F}(B)\\ (\omega,\eta) &\mapsto \sum_{i,j=1}^{r}\langle\omega_i,\eta_j\rangle\langle v_i,v_j\rangle_V.\end{aligned}$$

Definition 3.1.17. The **Hodge star operator on k-forms with values in** V is given by the map $* : \Omega^k(B,V)\to\Omega^{n-k}(B,V)$ defined by $*\omega = \sum_{j=1}^{r}(*\omega_j)\otimes v_j$ for all $\omega\in\Omega^k(B,V)$.

Definition 3.1.18. Let ∇ be a covariant derivative on the vector bundle V. The **covariant differential** or **exterior covariant derivative** is the following map:

$$\begin{aligned}d_\nabla : \Omega^k(B,V) &\to \Omega^{k+1}(B,V)\\ \omega &\mapsto \sum_{j=1}^{r}[d\omega_j\otimes v_j + (-1)^k\omega_j\wedge\nabla v_j].\end{aligned}$$

Given an associated vector bundle $(P\times_\rho V,\pi',B)$ and a connection 1-form $\omega\in\mathcal{C}(P)$, if we take its associated covariant derivative ∇^ω, we denote $d_\omega := d_{\nabla^\omega}$.

Remark 3.1.19. The scalar product of k-forms $\langle\cdot,\cdot\rangle_V$ and the covariant differential d_∇ are independent of the choice of the local maps $\{v_j\}_{j=1,\dots,r}$ (see [Ham17, Sect. 5.12]).

Definition 3.1.20. The **covariant codifferential** is the map $d^*_\nabla : \Omega^{k+1}(B,V)\to\Omega^k(B,V)$ defined by $d^*_\nabla = (-1)^{t+nk+1} * d_\nabla *$. If $\nabla = \nabla^\omega$, we write $d^*_\omega := d^*_{\nabla^\omega}$.

Definition 3.1.21. The **L^2-scalar product of k-forms with values in** V is the map:

$$\begin{aligned}\langle\cdot,\cdot\rangle_{V,L^2} : \Omega^k_c(B,V)\times\Omega^k_c(B,V) &\to \mathbb{R}\\ (\omega,\eta) &\mapsto \int_B\langle\omega,\eta\rangle_V \mathrm{dvol}_g.\end{aligned}$$

Theorem 3.1.22. *The equality $\langle d_\nabla\omega,\eta\rangle_{V,L^2} = \langle\omega,d^*_\nabla\eta\rangle_{V,L^2}$ holds for all $\omega\in\Omega^k_c(B,V)$, $\eta\in\Omega^{k+1}_c(B,V)$.*

Proof. See [Ham17, Sect. 7.2.2]. □

3.2 Yang-Mills theory

From a physical standpoint, we postulate that a physical system is determined by a function called Lagrangian and that the evolution of the system is modelled by the equations that minimize its corresponding action.

The aim of this section is to introduce the Yang-Mills Lagrangian and the associated Yang-Mills equation that can be derived from it.

Let (P,π,B) be a G-principal bundle with (B,g) an n-dimensional oriented pseudo-Riemannian manifold and G an r-dimensional compact Lie group. Consider a connection 1-form $\omega\in\mathcal{C}(P)$ and its curvature 2-form $\Omega^\omega\in\Omega^2(P,\mathfrak{g})$.

Since G is compact, there exists an Ad-invariant positive definite scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on its Lie algebra $\mathfrak{g}$ (see [Ham17, Sect. 2.2]). This scalar product yields a bundle metric $\langle\cdot,\cdot\rangle_{\mathrm{Ad}(P)}$ on the associated vector bundle $(\mathrm{Ad}(P),\pi',B)$, where $\mathrm{Ad}(P) := P\times_{\mathrm{Ad}}\mathfrak{g}$, satisfying $\langle[p_1,g_1],[p_2,g_2]\rangle_{\mathrm{Ad}(P)} = \langle g_1,g_2\rangle_{\mathfrak{g}}$ for all $[p_1,g_1],[p_2,g_2]\in\mathrm{Ad}(P)$ such that $\pi'(p_1)=\pi'(p_2)$. We also take an orthonormal basis $\{T_1,\dots,T_r\}$ of $\mathfrak{g}$.

Remark 3.2.1. With the scalar product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$, we can define a new scalar product on local 2-forms $\Omega^2(U,\mathfrak{g})$ for any local chart (U,ϕ) of B. To do so, if suffices to write the elements $\Theta\in\Omega^2(U,\mathfrak{g})$ as $\Theta=\sum_{a=1}^r\Theta_a\otimes T_a$ where $\Theta_a\in\Omega^2(U)$ for all $a\in\{1,\dots,r\}$. Then, in local coordinates $\{x^1,\dots,x^n\}$, we set $\Theta_{ij}=\Theta(\partial_i,\partial_j)=\Theta^a(\partial_i,\partial_j)\cdot T_a=\Theta^a_{ij}\cdot T_a$ for $i,j\in\{1,\dots,n\}$. With this, we define

$$\begin{aligned}\langle\cdot,\cdot\rangle_{\mathfrak{g}}:\Omega^2(U,\mathfrak{g})\times\Omega^2(U,\mathfrak{g})&\to\mathcal{F}(U)\\ (\Theta,\Xi)&\mapsto 1/2\cdot\Theta^a_{ij}\Xi^{ij}_a.\end{aligned}$$

In this framework, we can define the following Lagrangian using that, as pointed up in Remark 2.5.12, the curvature $\Omega^\omega\in\Omega^2_{\mathrm{hor}}(P,\mathfrak{g})^{(G,\mathrm{Ad})}$ has an associated form $\Omega^\omega_B\in\Omega^2(B,\mathrm{Ad}(P))$ defined as in Theorem 2.4.12.

Definition 3.2.2. The **Yang-Mills Lagrangian** is defined as the following map:

$$\begin{aligned}\mathcal{L}_{YM}:\mathcal{C}(P)&\to\mathcal{F}(B)\\ \omega&\mapsto-\frac{1}{2}\langle\Omega^\omega_B,\Omega^\omega_B\rangle_{\mathrm{Ad}(P)}.\end{aligned}$$

Theorem 3.2.3. *The Yang-Mills Lagrangian is gauge invariant, meaning that $\mathcal{L}_{YM}(f^*\omega)=\mathcal{L}_{YM}(\omega)$ for all $f\in\mathcal{G}(P)$, $\omega\in\mathcal{C}(P)$.*

Proof. Observe that, by Proposition 2.5.6 and the structure equation, we have $\Omega^{f^*\omega}=d(f^*\omega)+1/2[f^*w,f^*w]=f^*(d\omega)+1/2f^*([\omega,\omega])=f^*\Omega^\omega$. Besides, we can write $\Omega^{f^*\omega}_B=[s_i,\Omega^{f^*\omega}_i]$ as shown in Remark 2.5.16 for some local section $s_i:U_i\to P$ and $\Omega^{f^*\omega}_i\in\Omega^2(U_i,\mathfrak{g})$. Then, for $x\in U_i$ and $X,Y\in T_xB$, we have:

$$\begin{aligned}\Omega^{f^*\omega}_B(X,Y)|_x&=[s_i(x),\Omega^{f^*\omega}_i|_x(X,Y)]\\&=[s_i(x),s_i^*\Omega^{f^*\omega}|_x(X,Y)]\\&=[s_i(x),s_i^*(f^*\Omega^\omega)|_x(X,Y)]\\&=[s_i(x),(f\circ s_i)^*\Omega^\omega|_x(X,Y)]\\&=[(s_i')(x),\mathrm{Ad}_{(g^{-1}\circ s_i')(x)}(s_i')^*\Omega^\omega|_x(X,Y)]\end{aligned}$$

where $s_i'=f\circ s_i$ is a local section and $g:P\to G$ is the smooth map satisfying $f(p)=p\cdot g(p)$ for all $p\in P$. Note that in the last equality we have used that $s_i(x)=f^{-1}(s_i'(x))=s_i'(x)\cdot g^{-1}(s_i'(x))$ for all $x\in U_i$.

Now, using the obtained equality and the fact that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is Ad-invariant, we get $\langle\Omega^{f^*\omega}_B,\Omega^{f^*\omega}_B\rangle_{\mathrm{Ad}(P)}=\langle\mathrm{Ad}_{(g^{-1}\circ s_i')(x)}(s_i')^*\Omega^\omega,\mathrm{Ad}_{(g^{-1}\circ s_i')(x)}(s_i')^*\Omega^\omega\rangle_{\mathfrak{g}}=\langle(s_i')^*\Omega^\omega,(s_i')^*\Omega^\omega\rangle_{\mathfrak{g}}=\langle\Omega^\omega_B,\Omega^\omega_B\rangle_{\mathrm{Ad}(P)}$ by definition of $\langle\cdot,\cdot\rangle_{\mathrm{Ad}(P)}$.

Therefore, the equality $\mathcal{L}_{YM}(f^*\omega)=\mathcal{L}_{YM}(\omega)$ is fulfilled by all $\omega\in\mathcal{C}(P)$. See [Bär11, Sect. 3.3] for more details. □

Remark 3.2.4. A theory of this type such that the Lagrangian is locally gauge invariant is called a **gauge theory**, where *locally* means that a possibly different transformation is applied at each point of the manifold i.e. $f \in \mathcal{G}(P)$ may not induce the same transformation on all fibers.

Taking local sections of the principal bundle P, a local formula for the Lagrangian $\mathcal{L}_{YM}$ can be written as follows:

Let $s : U \to P$ be a local section. As mentioned in Remark 2.5.16, we can write locally $\Omega_B^\omega = [s, s^*\Omega^\omega]$. Thus, we have $\langle \Omega_B^\omega, \Omega_B^\omega \rangle_{\mathrm{Ad}(P)} = \langle [s, s^*\Omega^\omega], [s, s^*\Omega^\omega] \rangle_{\mathrm{Ad}(P)} = \langle s^*\Omega^\omega, s^*\Omega^\omega \rangle_{\mathfrak{g}}$ by definition of $\langle \cdot, \cdot \rangle_{\mathrm{Ad}(P)}$.

Using the scalar product on $\Omega^2(U, \mathfrak{g})$ defined in Remark 3.2.1, for $s^*\Omega^\omega \in \Omega^2(U, \mathfrak{g})$, we write $(s^*\Omega^\omega)_{ij} = (s^*\Omega^\omega)_{ij}^a \cdot T_a$ where $a \in \{1, \ldots, r\}$ and $i, j \in \{1, \ldots, n\}$. Then, locally:

$$\mathcal{L}_{YM}(\omega) = -\frac{1}{2} \langle s^*\Omega^\omega, s^*\Omega^\omega \rangle_{\mathfrak{g}} = -\frac{1}{4} \cdot (s^*\Omega^\omega)_{ij}^a (s^*\Omega^\omega)_a^{ij}.$$

Remark 3.2.5. It is also possible to relate the components $(s^*\Omega^\omega)_{ij}$ with the local connections $\omega_s = s^*\omega$ and rewrite the Lagrangian $\mathcal{L}_{YM}(\omega)$ in terms of these elements ω_s (see [Ham17, Sect. 7.3.1]).

In the following, we assume that B is also closed, i.e. compact and without boundary, so that the integral of n-forms is well-defined.

Definition 3.2.6. The **Yang-Mills action** is defined as the map $\mathcal{S}_{YM} : \mathcal{C}(P) \to \mathbb{R}$ given by

$$\mathcal{S}_{YM}(\omega) = -\frac{1}{2} \langle \Omega_B^\omega, \Omega_B^\omega \rangle_{\mathrm{Ad}(P), L^2} = -\frac{1}{2} \int_B \langle \Omega_B^\omega, \Omega_B^\omega \rangle_{\mathrm{Ad}(P)} \mathrm{dvol}_g = \int_B \mathcal{L}_{YM}(\omega) \mathrm{dvol}_g.$$

Definition 3.2.7. A **critical point** of the Yang-Mills action is a connection 1-form $\omega \in \mathcal{C}(P)$ that satisfies

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{S}_{YM}(\omega + t\eta) = 0 \quad \forall \eta \in \Omega^1_{\mathrm{hor}}(P, \mathfrak{g})^{(G, \mathrm{Ad})} \cong \Omega^1(B, \mathrm{Ad}(P)).$$

Theorem 3.2.8. *A connection 1-form $\omega \in \mathcal{C}(P)$ is a critical point of the Yang-Mills action if and only if its curvature 2-form verifies the following equality:*

$$d_\omega^* \Omega_B^\omega = 0,$$

*called the **Yang-Mills equation**.*

In order to prove this theorem, we need the following proposition:

Proposition 3.2.9. *Let (P, π, B) be a G-principal bundle, $\rho : G \to \mathrm{Aut}(V)$ a representation of G, $\omega \in \mathcal{C}(P)$ a connection 1-form, and $\eta \in \Omega^k_{hor}(P, V)^{(G, \rho)}$ a horizontal form of type ρ. Then, the following equation holds:*

$$d_\omega \eta = d\eta + \rho_*(\omega) \wedge \eta, \tag{3.1}$$

where $(\rho_*(\omega) \wedge \eta)_p(X_0, \ldots, X_k) := \sum_{i=0}^k (-1)^i \rho_*(\omega_p(X_i))(\eta_p(X_0, \ldots, \hat{X_i}, \ldots, X_k))$ for all $p \in P$ and $X_0, \ldots, X_k \in T_pP$.

Proof. The equality is shown distinguishing different cases depending on whether the vector fields X_i are vertical or horizontal (see [RS17, Sect. 1.4]). $\square$

Remark 3.2.10. In our case, Equation 3.1 simplifies as $d_\omega \eta = d\eta + [\omega, \eta]$. To check this, notice that if ρ is the adjoint representation, applying the definition of $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, we obtain:

$$\begin{aligned}
(\mathrm{Ad}_*(\omega) \wedge \eta)_p(X_0, X_1) &= (\mathrm{ad}(\omega) \wedge \eta)_p(X_0, X_1) \\
&= \mathrm{ad}(\omega_p(X_0))(\eta_p(X_1)) - \mathrm{ad}(\omega_p(X_1))(\eta_p(X_0)) \\
&= [\omega_p(X_0), \eta_p(X_1)] - [\omega_p(X_1), \eta_p(X_0)] \\
&= [\omega, \eta]_p(X_0, X_1)
\end{aligned}$$

for all $p \in P$ and $X_0, X_1 \in T_pP$. Hence, $\mathrm{Ad}_*(\omega) \wedge \eta = [\omega, \eta]$.

Proof (of Theorem 3.2.8). We need to compute

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{S}_{YM}(\omega + t\eta) = \frac{d}{dt}\bigg|_{t=0} \Big(-\frac{1}{2}\langle \Omega_B^{\omega+t\eta}, \Omega_B^{\omega+t\eta} \rangle_{\mathrm{Ad}(P),L^2} \Big).$$

With this aim, we first find an expression for the 2-form $\Omega_B^{\omega+t\eta}$. By the structure equation, we have:

$$\Omega^{\omega+t\eta} = d(\omega + t\eta) + \frac{1}{2}[\omega + t\eta, \omega + t\eta] = \Omega^\omega + t(d\eta + [\omega, \eta]) + \frac{1}{2}t^2[\eta, \eta].$$

Then, using Equation 3.1, written as in Remark 3.2.10, the 2-form on B can be expressed as $\Omega_B^{\omega+t\eta} = \Omega_B^\omega + t(d_\omega \eta_B) + 1/2 \cdot t^2[\eta_B, \eta_B]$. Hence, the derivative reads

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \Big(\langle \Omega_B^{\omega+t\eta}, \Omega_B^{\omega+t\eta} \rangle_{\mathrm{Ad}(P),L^2} \Big) &= \frac{d}{dt}\bigg|_{t=0} \Big(\langle \Omega_B^\omega, \Omega_B^\omega \rangle_{\mathrm{Ad}(P),L^2} \\
&\qquad + 2t \langle d_\omega \eta_B, \Omega_B^\omega \rangle_{\mathrm{Ad}(P),L^2} + \mathcal{O}(t^2) \Big) \\
&= 2\langle d_\omega \eta_B, \Omega_B^\omega \rangle_{\mathrm{Ad}(P),L^2} \\
&= 2\langle \eta_B, d_\omega^* \Omega_B^\omega \rangle_{\mathrm{Ad}(P),L^2},
\end{aligned}$$

where in the last equality we applied Theorem 3.1.22.

Therefore, for $\omega \in \mathcal{C}(P)$ to be a critical point of $\mathcal{S}_{YM}$, the 2-form Ω_B^ω has to satisfy $\langle \eta_B, d_\omega^* \Omega_B^\omega \rangle_{\mathrm{Ad}(P),L^2} = 0$ for all $\eta_B \in \Omega^1(B, \mathrm{Ad}(P))$. Since the scalar product is non-degenerate, this happens if and only if $d_\omega^* \Omega_B^\omega = 0$. $\square$

Remark 3.2.11. Notice that the Yang-Mills equation depends on the chosen metric g on B through the Hodge star operator. Thus, even if the equation is fulfilled for one metric, it does not necessarily hold for a different metric.

Definition 3.2.12. A **Yang-Mills connection** is a connection 1-form $\omega \in \mathcal{C}(P)$ that is a critical point of the Yang-Mills action.

Remark 3.2.13. Since the Yang-Mills equation does not depend on the choice of local sections $s : U \to P$, the action of the gauge group $\mathcal{G}(P)$ preserves the Yang-Mills connections. This means that if $\omega \in \mathcal{C}(P)$ is critical, then $f^*\omega$ is also critical for all $f \in \mathcal{G}(P)$, since we can consider $s^*(f^*\omega)$ as $(s')^*\omega$ for $s' = f \circ s$.

3.2.1 Electromagnetism

Let (P, π, B) be a $U(1)$-principal bundle, (B, g) a 4-dimensional oriented pseudo-Riemannian manifold, $\omega \in \mathcal{C}(P)$ a connection 1-form, and $\Omega^\omega \in \Omega^2(P, \mathfrak{u}(1))$ its curvature 2-form.

It is easy to see that $U(1) = \mathbb{S}^1$ and $\mathfrak{u}(1) = i\mathbb{R}$. Since $U(1)$ is abelian, we have that $d\Omega^\omega = 0$ and the curvature defines a global 2-form $\widetilde{\Omega}^\omega \in \Omega^2(B, \mathfrak{u}(1)) = \Omega^2(B, i\mathbb{R})$ as in Remark 2.5.15. Moreover, it can be shown that in this case the Yang-Mills equation reduces to $d(*\widetilde{\Omega}^\omega) = 0$.

In the following, we derive Maxwell's equations considering a 2-form defined in terms of the electromagnetic fields and we see how these relate to the Yang-Mills equation in this scenario.

For the whole section, we write the equations in Gaussian units, with c denoting the speed of light.

Let us now take the manifold $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ with the Minkowski metric[3]. We consider an open set $U \subset \mathbb{R}^3$ and choose local coordinates $\{ct, x, y, z\}$ on $\mathbb{R} \times U$. With respect to these coordinates, we define the following 2-form $\widetilde{\Omega} \in \Omega^2(\mathbb{R} \times U)$:

$$\widetilde{\Omega} := (E_x dx + E_y dy + E_z dz) \wedge cdt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

where $\vec{E} := (E_x, E_y, E_z) : \mathbb{R} \times U \to \mathbb{R}^3$ and $\vec{B} := (B_x, B_y, B_z) : \mathbb{R} \times U \to \mathbb{R}^3$ are time dependent vector fields corresponding to the electric and magnetic fields of classical electromagnetism, respectively.

Then, applying the definition of the differential, we get:

$$\begin{aligned} d\widetilde{\Omega} =& \Big(-\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x} + \frac{1}{c}\frac{\partial B_z}{\partial t} \Big) \cdot cdt \wedge dx \wedge dy + \Big(-\frac{\partial E_y}{\partial z} + \frac{\partial E_z}{\partial y} + \frac{1}{c}\frac{\partial B_x}{\partial t} \Big) \cdot cdt \wedge dy \wedge dz \\ &+ \Big(-\frac{\partial E_z}{\partial x} + \frac{\partial E_x}{\partial z} + \frac{1}{c}\frac{\partial B_y}{\partial t} \Big) \cdot cdt \wedge dz \wedge dx + \Big(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \Big) \cdot dx \wedge dy \wedge dz. \end{aligned}$$

Therefore, the identity $d\widetilde{\Omega} = 0$ is equivalent to the following two equations:

$$\begin{cases} \operatorname{div}(\vec{B}) = 0 \\ \frac{1}{c} \cdot \frac{\partial \vec{B}}{\partial t} + \operatorname{rot}(\vec{E}) = 0, \end{cases}$$

which correspond to Gauss' and Faraday's laws, respectively, i.e. two of the classical Maxwell's equations.

Remark 3.2.14. Observe that the definition of the vector fields $\vec{E}$ and $\vec{B}$ depends on the choice of local coordinates.

The chosen metric on $\mathbb{R}^4$ determines a Hodge star operator $*$. In the local coordinates $\{ct, x, y, z\}$ we have, by Proposition 3.1.7:

$$\begin{aligned} *(cdt \wedge dx) &= dy \wedge dz, \quad & *(dx \wedge dy) &= -cdt \wedge dz \\ *(cdt \wedge dy) &= dz \wedge dx, \quad & *(dz \wedge dx) &= -cdt \wedge dy \\ *(cdt \wedge dz) &= dx \wedge dy, \quad & *(dy \wedge dz) &= -cdt \wedge dx. \end{aligned}$$

[3]The Minkowski metric is a tensor g defined by a diagonal matrix with diagonal $(-1, +1, +1, +1)$.

Thus, using that $*$ is linear, we obtain (see [Bär11, Sect. 3.2] for more details):

$$\begin{aligned} d(*\widetilde{\Omega}) =& (-\mathrm{div}\vec{E})\cdot dx\wedge dy\wedge dz + \left(\mathrm{rot}\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)_x \cdot cdt\wedge dy\wedge dz \\ &+ \left(\mathrm{rot}\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)_y \cdot cdt\wedge dz\wedge dx + \left(\mathrm{rot}\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right)_z \cdot cdt\wedge dx\wedge dy. \end{aligned}$$

We now introduce a 3-form $J_\rho \in \Omega^3(\mathbb{R}\times U)$ defined as

$$J_\rho := \rho\cdot dx\wedge dy\wedge dz - J_x\cdot dt\wedge dy\wedge dz - J_y\cdot dt\wedge dz\wedge dx - J_z\cdot dt\wedge dx\wedge dy,$$

where $\rho : \mathbb{R}\times U \to \mathbb{R}$ and $\vec{J} := (J_x, J_y, J_z) : \mathbb{R}\times U \to \mathbb{R}^3$ correspond in classical electrodynamics to the electric charge density and the electric current density, respectively.

Then, it is straightforward that the equality $d(*\widetilde{\Omega}) + 4\pi J_\rho = 0$ is equivalent to the remaining two Maxwell's equations:

$$\begin{cases} \mathrm{div}(\vec{E}) = 4\pi\rho \\ \mathrm{rot}(\vec{B}) - \frac{1}{c}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J}, \end{cases}$$

corresponding to Coulomb's and Ampère's laws, respectively.

Hence, Maxwell's equations can be rewritten as follows:

$$\begin{cases} d\widetilde{\Omega} = 0 \\ d(*\widetilde{\Omega}) + 4\pi J_\rho = 0. \end{cases}$$

Now, if we choose a contractible set $U \subset \mathbb{R}^3$ and the initial $U(1)$-principal bundle (P, π, B) as a trivial bundle with $B = \mathbb{R}\times U$ and $P = B\times U(1)$, we can see the defined 2-form $\widetilde{\Omega}$ as an element of $\Omega^2(B)$.

Since it is defined on an abelian set, the 2-form verifies $d\widetilde{\Omega} = 0$. This, together with the fact the U is contractible implies, by the Poincaré lemma, that there exists $\widetilde{\omega} \in \Omega^1(B, i\mathbb{R}) = \Omega^1(B, \mathfrak{u}(1))$ satisfying $d\widetilde{\omega} = i\widetilde{\Omega}$ (see [RS13, Sect. 4.3]).

Remark 3.2.15. This 1-form $\widetilde{\omega}$ plays the role of the four-potential of classical electromagnetism, which is defined as a vector $A = (\phi, -\vec{A})$ where ϕ is an electric scalar potential and $\vec{A}$ is a magnetic vector potential satisfying $\vec{E} = -\vec{\nabla}\phi - \frac{\partial\vec{A}}{c\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$. Note that under a transformation $A_i \mapsto A_i + \partial_i\Lambda$ for $i \in \{ct, x, y, z\}$ where Λ is a scalar function, the electromagnetic fields remain unchanged. The same way, the 1-form $\widetilde{\omega}$ is not uniquely determined.

Moreover, the 2-form $\widetilde{\Omega}$ can be seen as the Faraday tensor F which describes the electromagnetic field in spacetime (see [Nie07]).

It can be shown that given a trivial bundle $(B\times G, \mathrm{pr}_1, B)$ for some Lie group G, there exists a bijection between the connection 1-forms $\mathcal{C}(B\times G)$ and the 1-forms $\Omega^1(B, \mathfrak{g})$ (see [Dup03, Chapt. 6]). In our case, this means that there exists a connection 1-form $\omega \in \mathcal{C}(P) = \mathcal{C}(B\times U(1))$ equivalent to $\widetilde{\omega} \in \Omega^1(B, \mathfrak{u}(1))$ such that $\widetilde{\Omega}^\omega = i\widetilde{\Omega} \in \Omega^2(B, \mathfrak{u}(1))$.

Therefore, $\widetilde{\Omega}$ corresponds to the curvature on B related to a connection 1-form ω on the $U(1)$-principal bundle P. Moreover, if we consider a source-free electromagnetic field such that $\rho = 0$ and $\vec{J} = 0$, Maxwell's equations read:

$$\begin{cases} d\widetilde{\Omega} = 0 \\ d(*\widetilde{\Omega}) = 0, \end{cases}$$

where the second equation is equal to the Yang-Mills equation for this principal bundle.

Remark 3.2.16. With this formalism, it is possible to recover other formulas from classical electromagnetism in terms of the 2-form $\widetilde{\Omega}$, such as the continuity equation or the Lorentz force law (see [Bär11, Sect. 3.2]).

3.2.2 Massive gauge bosons

We have seen that the Yang-Mills theory successfully describes classical electromagnetism. However, if we try to apply this theory to model other fundamental forces, as it is done in the Standard Model, we face a problem when trying to characterize certain types of particles.

When describing the elementary particles using gauge theories, it is postulated that bosons are associated to connection 1-forms on a principal bundle (also called **gauge fields**), except for Higgs bosons.[4] In addition, physically, it can be shown that massive gauge bosons with mass m appear in the corresponding Lagrangian as terms of the form:

$$\frac{1}{2}m^2(s^*\omega)^a_{ij}(s^*\omega)^{ij}_a$$

where, as for the curvature 2-form, we set locally $(s^*\omega)(\partial_i, \partial_j) = (s^*\omega)_{ij} = (s^*\omega)^a_{ij} \cdot T_a$ with $T_1, \dots, T_r$ a basis of the Lie algebra $\mathfrak{g}$.

It is easy to check, with the local expression $\mathcal{L}_{YM}(\omega) = -\frac{1}{4} \cdot (s^*\Omega^\omega)^a_{ij}(s^*\Omega^\omega)^{ij}_a$, that the Yang-Mills Lagrangian does not present any terms of this form. Furthermore, there is no easy way to add an equivalent term while maintaining the gauge invariance (see [Ham17, Sect. 7.3.3]). Hence, just by means of the Yang-Mills Lagrangian, it is not possible to describe the massive bosons found experimentally, namely $W^\pm$ and Z.

In the sections that follow, we explain how this issue is addressed within the Standard Model in order to describe the electroweak interaction as a gauge theory.

3.3 The Higgs Lagrangian

All the assumptions made throughout this section hold for the following sections.

Let (P, π, B) be a G-principal bundle where (B, g) is an m-dimensional oriented pseudo-Riemannian manifold and G is an r-dimensional compact Lie group. Also, let

[4] Bosons associated to connection 1-forms, also called gauge bosons, are a type of elementary particles which mediate the fundamental interactions of nature. On the other hand, Higgs bosons are responsible for the non-zero mass of some gauge bosons, as we will later see.

(P', π', B) be an associated complex vector bundle where $P' := P \times_\rho W$ for a complex representation $\rho : G \to \mathrm{Aut}(W)$.

Consider a G-invariant Hermitian[5] scalar product $\langle \cdot, \cdot \rangle_W$ on W with an action of G given by the representation ρ, we write $g \cdot w := \rho(g)(w)$ for all $g \in G$, $w \in W$. This yields a bundle metric $\langle \cdot, \cdot \rangle_{P'}$ on P'. Furthermore, we choose a basis of W so we can assume that $W = \mathbb{C}^n$ with the standard Hermitian product defined by $\langle w, w' \rangle_W = w^\dagger \cdot w'$ for all $w, w' \in W$, where $w^\dagger = \overline{w}^t$.

Remark 3.3.1. Observe that, with the considered scalar product, the representation ρ is unitary. Indeed, if we take $\rho(g) = A \in \mathrm{GL}(W)$ for some $g \in G$, we have $w^\dagger \cdot w' = \langle w, w' \rangle_W = \langle \rho(g)(w), \rho(g)(w') \rangle_W = \langle Aw, Aw' \rangle_W = w^\dagger A^\dagger \cdot Aw'$ for all $w, w' \in W$ since the product is G-invariant. Therefore, $\rho(g)^\dagger \cdot \rho(g) = A^\dagger \cdot A = \mathbb{1}_n$ for all $g \in G$, which means $\rho(G) \subset \mathrm{U}(W)$.

Definition 3.3.2. A **potential** is a smooth map $V : \mathbb{R} \to \mathbb{R}$.

Given a potential V, we have an action over the sections of the associated vector bundle P' defined as $V(\Phi) = V(\langle \Phi, \Phi \rangle_{P'})$ for all $\Phi \in \Gamma(P')$. We assume V to be gauge invariant, that is, $V(f\Phi) = V(\Phi)$ for all $f \in \mathcal{G}(P)$, $\Phi \in \Gamma(P')$.

From now on, we use the following notation: the **Higgs vector space** is the complex vector space W, the **Higgs bundle** is the associated vector bundle P', the **Higgs field** is a section $\Phi \in \Gamma(P')$, and the **Higgs potential** is the potential V.

Remark 3.3.3. From a physics perspective, the sections $\Phi \in \Gamma(P')$ are interpreted as matter fields, which describe certain types of particles. Moreover, this fields can interact (couple) with a gauge field $\omega \in \mathcal{G}(P)$, which represents bosons.

Definition 3.3.4. The **Higgs Lagrangian** is given by the following map:[6]

$$\begin{aligned} \mathcal{L}_H : \Gamma(P') \times \mathcal{C}(P) &\to \mathcal{F}(B) \\ (\Phi, \omega) &\mapsto \langle d_\omega \Phi, d_\omega \Phi \rangle_{P'} - V(\Phi). \end{aligned}$$

Theorem 3.3.5. *The Higgs Lagrangian is gauge invariant, meaning that* $\mathcal{L}_H(f^{-1}\Phi, f^*\omega) = \mathcal{L}_H(\Phi, \omega)$ *for all* $f \in \mathcal{G}(P)$, $\omega \in \mathcal{C}(P)$, *and* $\Phi \in \Gamma(P')$.

Proof. Since the potential $V(\Phi)$ is chosen gauge invariant, we just need to check the term $\langle d_\omega \Phi, d_\omega \Phi \rangle_{P'}$.

First, we have to prove the equality $d_{f^*\omega}(f^{-1}\Phi) = f^{-1} d_\omega \Phi$. To do so, we express $\Phi = [s, \phi]$ where $s : B \to P$ is a smooth section of P and $\phi : B \to W$ is a smooth map. Observe that over $\Gamma(P') = \Omega^0(B, P')$ we have $d_\omega|_{\Gamma(P')} = \nabla^\omega$ (see [Ham17, Sect. 5.12]). Then, we can write:

$$d_{f^*\omega}(f^{-1}\Phi) = \nabla^{f^*\omega}([f^{-1}s, \phi]) = \nabla^{f^*\omega}[s, \tau_f^{-1} \cdot \phi] = [s, \nabla^{f^*\omega} \tau_f^{-1} \cdot \phi]$$

[5] An Hermitian scalar product on a complex vector space W is a map $\langle \cdot, \cdot \rangle : W \times W \to \mathbb{C}$ that is linear in the first argument, antilinear in the second argument, conjugate symmetric, and positive definite.

[6] Note that $\Gamma(P') = \Omega^0(B, P')$, so we can apply the exterior covariant derivative d_ω defined as in Section 3.1 to sections of the associated vector bundle.

where $\tau_f : B \to G$ is the physical gauge transformation associated with $f \in \mathcal{G}(P)$.

Now, using the definition of covariant derivative on an associated vector bundle given in Section 2.4.1, we get:[7]

$$\begin{aligned}
\nabla^{f^*\omega}(\tau_f^{-1} \cdot \phi) &= d(\tau_f^{-1} \cdot \phi) + s^*(f^*\omega)(\tau_f^{-1} \cdot \phi) \\
&= \tau_f^{-1} \cdot d\phi + d\tau_f^{-1} \cdot \phi + ((f \circ s)^*\omega)(\tau_f^{-1} \cdot \phi) \\
&= \tau_f^{-1} \cdot d\phi + d\tau_f^{-1} \cdot \phi + ((s \cdot \tau_f)^*\omega)(\tau_f^{-1} \cdot \phi) \\
&= \tau_f^{-1}\Big[d + \tau_f \cdot d\tau_f^{-1} + \tau_f \cdot ((s \cdot \tau_f)^*\omega) \cdot \tau_f^{-1}\Big](\phi) \\
&= \tau_f^{-1}(d + s^*\omega)(\phi) \\
&= \tau_f^{-1}\nabla^\omega\phi,
\end{aligned}$$

where we use that connection 1-forms satisfy $s^*\omega = \mathrm{Ad}_{\bar{g}} \circ((s \cdot \bar{g})^*\omega) + \bar{g} \cdot d\bar{g}^{-1}$ for all $\bar{g} : B \to G$. Since $[s, \tau_f^{-1}\nabla^\omega\phi] = f^{-1}[s, \nabla^\omega\phi] = f^{-1}\nabla^\omega([s, \phi]) = f^{-1}d_\omega\Phi$, this proves the equality $d_{f^*\omega}(f^{-1}\Phi) = f^{-1}d_\omega\Phi$.

Finally, as the scalar product $\langle\cdot,\cdot\rangle_W$ is G-invariant, we have:

$$\langle d_{f^*\omega}(f^{-1}\Phi), d_{f^*\omega}(f^{-1}\Phi)\rangle_{P'} = \langle f^{-1}d_\omega\Phi, f^{-1}d_\omega\Phi\rangle_{P'} = \langle d_\omega\Phi, d_\omega\Phi\rangle_{P'}$$

for all $f \in \mathcal{G}(P)$, $\omega \in \mathcal{C}(P)$, and $\Phi \in \Gamma(P')$. Thus, $\mathcal{L}_H$ is gauge invariant. □

Definition 3.3.6. The **Yang-Mills-Higgs Lagrangian** is the combined Lagrangian that describes the dynamics of a Higgs field Φ with the Higgs potential V coupled to a gauge field ω. It can be expressed as

$$\mathcal{L}_{YMH}(\Phi, \omega) = \mathcal{L}_H(\Phi, \omega) + \mathcal{L}_{YM}(\omega) = \langle d_\omega\Phi, d_\omega\Phi\rangle_{P'} - V(\Phi) - \frac{1}{2}\langle\Omega_B^\omega, \Omega_B^\omega\rangle_{\mathrm{Ad}(P)}.$$

Remark 3.3.7. Note that a theory governed by this Lagrangian is a gauge theory.

From now on, we consider the Yang-Mills-Higgs Lagrangian to describe the electroweak interactions as it is done in the Standard Model.

3.4 Spontaneous symmetry breaking

For the following definitions, we keep the same assumptions as in the previous section.

Definition 3.4.1. A **vacuum configuration** or **vacuum** for the Yang-Mills-Higgs Lagrangian is a pair $(\Phi_0, \omega_0) \in \Gamma(P') \times \mathcal{C}(P)$ verifying the following conditions:

i) The connection 1-form ω_0 is **flat**: $\Omega^{\omega_0} \equiv 0$.

ii) The Higgs field Φ_0 is **covariantly constant**: $d_{\omega_0}\Phi_0 = \nabla^{\omega_0}\Phi_0 \equiv 0$.

iii) The element $\Phi_0(x)$ is a minimum of the potential V for all $x \in B$.

[7] For simplicity, we denote the action of $\mathfrak{g}$ on W as $X \cdot w := \rho_*(X)(w)$ for $X \in \mathfrak{g}$, $w \in W$.

Definition 3.4.2. A **vacuum vector** is an element $w_0 \in W$ which is a minimum of the map $V : W \to \mathbb{R}$, $w \mapsto V(\langle w, w \rangle_W)$. The **space of vacua** for V is the set of vacuum vectors in the Higgs vector space W.

Proposition 3.4.3. *Let (P, π, B) be a trivial principal bundle such that B is connected and simply connected. Then, given a vacuum configuration (Φ_0, ω_0), there exists a global section $s_0 : B \to P$ satisfying*

i) $s_0^* \omega_0 \equiv 0$,

ii) $\Phi_0 = [s_0, w_0]$ *where $w_0 \in W$ is a constant vacuum vector.*

The section s_0 is called the ***vacuum gauge****. Conversely, for an arbitrary fixed global section $s_0 \in \Gamma(P)$, each vacuum vector $w_0 \in W$ yields a unique vacuum configuration (Φ_0, ω_0) verifying conditions (i) and (ii).*

Proof. See [Ham17, Sect. 8.1.2]. □

Since our aim is to develop the theory within the Standard Model, in the following we can assume that B is connected and simply connected. In this case, it can be seen that if there exists a flat connection 1-form $\omega \in \Omega(P, \mathfrak{g})$, the principal bundle has to be trivial (see [Ham17, Sect. 5.15]). Hence, we also assume that (P, π, B) is a trivial G-principal bundle.

Remark 3.4.4. Even though P is trivial, there is no preferred trivialization. In order to fix a trivialization, we have to choose a global section $s \in \Gamma(P)$ that determines an isomorphism $B \times G \cong P$ given by $(x, g) \mapsto s(x) \cdot g$.

We also suppose that there exists a vacuum vector $w_0 \in W$ and we fix it together with a global vacuum gauge $s_0 \in \Gamma(P)$. Besides, we take (Φ_0, ω_0) as their associated vacuum configuration.

Definition 3.4.5. The **unbroken subgroup** of the vacuum configuration is the isotropy group of the vacuum vector: $H = G_{w_0} = \{g \in G : \rho(g)(w_0) = w_0\} \subset G$.

A gauge theory is **spontaneously broken** if the unbroken subgroup H is a proper subgroup of G, that is, $H \subsetneq G$.

Remark 3.4.6. The isotropy group H is a Lie subgroup of G (see [Ham17, Sect. 3.2]). Its explicit embedding in G depends on the choice of the vacuum vector w_0.

In the Standard Model we assume that a **spontaneous symmetry breaking** process occurred because the potential V does not have a minimum at $w = 0$ but at $w_0 \neq 0$ instead, so that $H = G_{w_0} \subsetneq G_0 = G$. We will see that this spontaneous process is the reason why some elementary particles have non-zero mass.

Definition 3.4.7. The **Higgs condensate** is the nowhere vanishing field $\Phi_0 = [s_0, w_0]$ for the vacuum vector $w_0 \neq 0$.

Remark 3.4.8. Observe that in a spontaneously broken gauge theory the Lagrangian and, as a consequence, the laws of physics, are still invariant under all physical gauge transformations $\tau : B \to G$. However, in this case, the Higgs condensate Φ_0 is only invariant under transformations $\tau : B \to H$ with values in the smaller group $H \subsetneq G$.

In general, a spontaneous symmetry breaking implies that a problem presents a symmetry which is not held by its solutions.

Let $\Phi = [s_0, \phi]$ be a Higgs field defined by the previously fixed vacuum gauge $s_0 \in \Gamma(P)$ and a smooth map $\phi : B \to W$. In the following, we derive an approximation of the Higgs potential V as a Taylor series around the vacuum vector w_0. To do so, we consider a shifted Higgs field $\Delta\phi = \phi - w_0$, which takes small values.

Let $\mathcal{O}_{w_0}$ be the orbit of G through the vacuum vector w_0, that is, $\mathcal{O}_{w_0} = G \cdot w_0 = \{g \cdot w_0 : g \in G\}$. Since G is compact, we have $\mathcal{O}_{w_0} \cong G/H$ (see [Ham17, Sect. 3.8.3]). We denote $d = \dim(\mathcal{O}_{w_0}) = \dim(G) - \dim(H)$.

Remark 3.4.9. Note that, if G acts transitively on the space of vacua, the space of vacua is equal to the orbit $\mathcal{O}_{w_0}$.

We can rewrite the Higgs vector space W as $W = T_{w_0}W = T_{w_0}\mathcal{O}_{w_0} \oplus (T_{w_0}\mathcal{O}_{w_0})^\perp$, where the orthogonality is with respect to the associated scalar product $\langle\langle \cdot, \cdot \rangle\rangle_W := \mathrm{Re}(\langle \cdot, \cdot \rangle_W)$. With this splitting, the following holds:

Proposition 3.4.10. *There exists a real orthonormal basis $u_1, \dots, u_d, v_1, \dots, v_{2n-d}$ of $T_{w_0}W$ consisting of eigenvectors of the Hessian $\mathrm{Hess}(V)_{w_0}$ such that:*

i) *$u_1, \dots, u_d$ is a basis of $T_{w_0}\mathcal{O}_{w_0}$ with common eigenvalue $\lambda = 0$.*

ii) *$v_1, \dots, v_{2n-d}$ is a basis of $(T_{w_0}\mathcal{O}_{w_0})^\perp$ with non-negative eigenvalues, since w_0 is a local minimum, which we denote as $\lambda_{v_j} = 2m_{v_j}^2$ where $m_{v_j} \geq 0$.*

Proof. We can see the Hessian of V at a certain point $w \in W$ as the linear map $\mathrm{Hess}(V)_w : T_wW \to T_wW$ given in standard coordinates $\{x^1, \dots, x^{2n}\}$ on $W = T_wW$ by the symmetric matrix of second partial derivatives $(\partial_i\partial_j V(w))_{i,j}$. This map satisfies $\langle\langle \mathrm{Hess}(V)_w(X), Y \rangle\rangle_W = \langle\langle X, \mathrm{Hess}(V)_w(Y) \rangle\rangle_W$ for all $X, Y \in T_wW$.

Since V is G-invariant, all the elements of the orbit $\mathcal{O}_{w_0} = G \cdot w_0$ are minima of V. Thus, the Hessian verifies $\mathrm{Hess}(V)_{w_0}(X) = 0$ for all $X \in T_{w_0}\mathcal{O}_{w_0}$. This implies that the Hessian $\mathrm{Hess}(V)_{w_0}$ preserves the orthogonal splitting $T_{w_0}W = T_{w_0}\mathcal{O}_{w_0} \oplus (T_{w_0}\mathcal{O}_{w_0})^\perp$. It suffices to observe that:

$$\langle\langle \mathrm{Hess}(V)_{w_0}(X), Y \rangle\rangle_W = \langle\langle X, \mathrm{Hess}(V)_{w_0}(Y) \rangle\rangle_W = \langle\langle X, 0 \rangle\rangle_W = 0$$

for all $X \in (T_{w_0}\mathcal{O}_{w_0})^\perp$, $Y \in T_{w_0}\mathcal{O}_{w_0}$. Hence, $\mathrm{Hess}(V)_{w_0}(T_{w_0}\mathcal{O}_{w_0})^\perp \subset (T_{w_0}\mathcal{O}_{w_0})^\perp$.

Therefore, there exists a diagonalization of $\mathrm{Hess}(V)_{w_0}$ adapted to the splitting of $W = T_{w_0}W$ as stated in the proposition. See [Ham17, Sect. 8.1.3] for more details. $\square$

Using the previous splitting of $T_{w_0}W$, we can express the map $\Delta\phi : B \to W \cong T_{w_0}W$ as

$$\Delta\phi = \frac{1}{\sqrt{2}} \sum_{i=1}^{d} a_i u_i + \frac{1}{\sqrt{2}} \sum_{j=1}^{2n-d} b_j v_j$$

where $a_i, b_j : B \to \mathbb{R}$ are real scalar fields for $i = 1, \dots, d$, $j = 1, \dots, 2n-d$.

Definition 3.4.11. The maps $\{a_i\}_{i=1,\dots,d}$ are called the **Nambu-Goldstone bosons** and the maps $\{b_j\}_{j=1,\dots,2n-d}$ are the **Higgs bosons**.

Now, we can write the Higgs potential as a Taylor series around w_0 using these scalar maps:

Theorem 3.4.12. *Up to second order in $\Delta\phi$, we have:*

$$V(\phi) \approx V(w_0) + \frac{1}{2} \sum_{j=1}^{2n-d} m_{v_j}^2 b_j^2.$$

Proof. By definition of the Taylor series up to second order, we get:

$$\begin{aligned} V(\phi) = V(w_0 + \Delta\phi) &\approx V(w_0) + \langle\langle \Delta\phi, \operatorname{grad}(V)(w_0) \rangle\rangle_W + \frac{1}{2} \langle\langle \Delta\phi, \operatorname{Hess}(V)_{w_0} \Delta\phi \rangle\rangle_W \\ &\approx V(w_0) + \langle\langle \Delta\phi, 0 \rangle\rangle_W + \frac{1}{2} \langle\langle \Delta\phi, \frac{1}{\sqrt{2}} \sum_{j=1}^{2n-d} b_j \cdot 2m_{v_j}^2 \cdot v_j \rangle\rangle_W \\ &\approx V(w_0) + \frac{1}{2} \sum_{j=1}^{2n-d} m_{v_j}^2 b_j^2. \end{aligned}$$

□

Remark 3.4.13. From a physical standpoint, according to Theorem 3.4.12, we associate a zero mass to the Nambu-Goldstone bosons a_i and a non-negative mass $m_{v_j} \geq 0$ to the Higgs bosons b_j.

Example 3.4.14. Let us now apply this approach to explain the electroweak interactions as in the Standard Model, considering the Yang-Mills-Higgs Lagrangian. To do so, we take the base manifold (B, g) as a 4-dimensional flat spacetime with the Minkowski metric and $W = \mathbb{C}^2$. We also set $G = \mathrm{SU}(2) \times \mathrm{U}(1)$. It is easy to check that $\mathrm{SU}(2) \cong \mathbb{S}^3$ and $\mathrm{U}(1) \cong \mathbb{S}^1$. Besides, we consider the following unitary representation where $n \in \mathbb{N} \setminus \{0\}$:

$$\begin{aligned} \rho : (\mathrm{SU}(2) \times \mathrm{U}(1)) \times W &\to W \\ ((A, e^{i\alpha}), (w_1, w_2)) &\mapsto A \cdot \begin{pmatrix} e^{in\alpha} & 0 \\ 0 & e^{in\alpha} \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \end{aligned}$$

In this case, due to several restrictions including that it has to be G-invariant, the Higgs potential has to be of the form[8]

$$V(w) = -\mu \cdot ||w||^2 + \lambda \cdot ||w||^4$$

[8]The physical restrictions imposed on the potential are beyond the scope of this work. For the specific conditions that V has to fulfill, see [Ham17, Sect. 8.1.2].

for all $w \in W$ where $\mu, \lambda > 0$ are constant parameters, the exact values of which are determined experimentally. It is easy to see that this potential V reaches a minimum value at vectors $w \neq 0$, which means that the theory is spontaneously broken. In particular, the vacuum vectors are the elements $w_0 \in W$ such that $||w_0|| = \sqrt{\frac{\mu}{2\lambda}}$. Thus, the space of vacua is a 3-sphere centered at the origin with radius $||w_0||$.

For all the vacuum vectors, the unbroken subgroup is $H \cong \mathrm{U}(1)$. In order to define an embedding of this subgroup in G, we need to choose a specific vacuum vector. For convenience, we fix

$$w_0 = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{2\lambda}} \end{pmatrix} \in \mathbb{C}^2, \quad \text{then} \quad H = \left\{ \left(\begin{pmatrix} e^{i\delta/2} & 0 \\ 0 & e^{-i\delta/2} \end{pmatrix}, e^{i\delta/2n} \right) : \delta \in \mathbb{R} \right\} \subset G.$$

As we will later see, this choice allows us to find a so-called unitary gauge for this theory.

Remark 3.4.15. Notice that all the elements in the space of vacua are equivalent via the action of G, even though they are invariant only under the action of H.

Let us now consider how many bosons are obtained within this framework. Since $\dim(G) = 4$ and $\dim(H) = 1$, we get $d = 4 - 1 = 3$. Besides, $2n = \dim(W) = 4$. Hence, with the chosen manifolds and Higgs potential, there appear $d = 3$ Nambu-Goldstone bosons $\{a_1, a_2, a_3\}$ and $2n - d = 1$ Higgs boson $\{b\}$.

Moreover, computing the Hessian for the chosen vacuum vector $\mathrm{Hess}(V)_{w_0}$ in standard coordinates for $\mathbb{C}^2$, we get a diagonal matrix with null eigenvalues for the Nambu-Goldstone bosons, as expected, and eigenvalue $2m_v^2 = 4\mu$ for the Higgs boson (see [Ham17, Sect. 8.1.4]). This implies that the Higgs boson b has as associated mass $m_v = \sqrt{2\mu}$.

Remark 3.4.16. With the presented approach, we have been able to obtain massive bosons, contrarily to what we found for the Yang-Mills Lagrangian. However, these results are not yet consistent with the experimental observations for the electroweak interaction. Experimentally, three massive bosons are found, apart from the Higgs boson, and there is no evidence of the existence of non-massive Nambu-Goldstone bosons.

In the following section, we see how this problem is addressed within the Standard Model exploiting the gauge symmetry of the theory.

3.5 Unitary gauge

Let $f \in \mathcal{G}(P)$ be a gauge transformation. With respect to the fixed vacuum gauge $s_0 \in \Gamma(P)$, there exists an associated physical gauge transformation $\tau_f : B \to G$. Recall that in Section 2.6 we defined an action of the maps τ_f over the Higgs fields $\Phi = [s_0, \phi] \in \Gamma(P')$ given by $(\tau_f \cdot \Phi)(x) = [s_0(x), \tau_f(x) \cdot \phi(x)]$ for all $x \in B$, where we set $\tau_f(x) \cdot \phi(x) := \rho(\tau_f(x))(\phi(x))$.

Since the theory that we are considering is gauge invariant, meaning that gauge transformations map solutions to solutions, we may be able to find different expressions for the Higgs field Φ with different scalar fields $\{a_i\}_{i=1,\dots,d}$, $\{b_j\}_{j=1,\dots,2n-d}$. Our aim is to check if the inconsistencies found with respect to the experimental results are a consequence of our choice of gauge. To do so, we use another gauge, the so-called unitary gauge.

Definition 3.5.1. Let $\Phi \in \Gamma(P')$ be a Higgs field and $w_0 \in W$ a vacuum vector. A **unitary gauge** with respect to w_0 is a physical gauge transformation $\tau : B \to G$ such that all the Nambu-Goldstone bosons of the transformed field $\tau \cdot \Phi$ with respect to w_0 vanish identically, that is, $a_i \equiv 0$ for all $i = 1, \cdots, d$. In this case, the Higgs field $\tau \cdot \Phi$ is said to be **in unitary gauge** with respect to w_0.

The existence of unitary gauge is non-trivial in general (see [Ham20, Sect. 6]). Here, we only provide an equivalent condition for the case that we are considering.

Proposition 3.5.2. *Let $\Phi = [s_0, \phi] \in \Gamma(P')$ be a Higgs field and $w_0 \in W$ a vacuum vector. Assume that the action of G is defined by a unitary representation. Then, a gauge transformation $\tau : B \to G$ is a unitary gauge for Φ with respect to w_0 if and only if $\langle\langle \tau(x) \cdot \phi(x), X \cdot w_0 \rangle\rangle_W = 0$ for all $x \in B$, $X \in \mathfrak{g}$ or, equivalently, the image of $\tau \cdot \phi$ is orthogonal to the tangent space of the orbit $\mathscr{O}_{w_0}$, which is $T_{w_0}\mathscr{O}_{w_0} = \{X \cdot w_0 : X \in \mathfrak{g}\}$.*

Proof. First, observe that the action of $\mathfrak{g}$ on W is defined by skew-Hermitian matrices, since the action of G on W is unitary and the Lie algebra of $\mathrm{U}(n)$ is $\mathfrak{u}(n) = \{A \in \mathrm{Mat}(n \times n; \mathbb{C}) : A^\dagger = -A\}$. Then, $\langle X \cdot w, w' \rangle_W = -\langle w, X \cdot w' \rangle_W$ for all $w, w' \in W$, $X \in \mathfrak{g}$.

Moreover, as $\langle\langle \cdot, \cdot \rangle\rangle_W = \mathrm{Re}(\langle \cdot, \cdot \rangle_W)$ is symmetric, we get $\langle\langle w_0, X \cdot w_0 \rangle\rangle_W = -\langle\langle X \cdot w_0, w_0 \rangle\rangle_W = -\langle\langle w_0, X \cdot w_0 \rangle\rangle_W$, which implies $\langle\langle w_0, X \cdot w_0 \rangle\rangle_W = 0$ for all $X \in \mathfrak{g}$.

Then, if we write $\phi = w_0 + \Delta\phi$ around w_0 and express $\Delta\phi$ in terms of the basis $u_1, \dots, u_d, v_1, \dots, v_{2n-d}$ of $T_{w_0}W$, we find:

$$\begin{aligned}
\langle\langle \phi(x), X \cdot w_0 \rangle\rangle_W &= \langle\langle w_0, X \cdot w_0 \rangle\rangle_W + \langle\langle \Delta\phi(x), X \cdot w_0 \rangle\rangle_W \\
&= 0 + \langle\langle \frac{1}{\sqrt{2}} \sum_{i=1}^{d} a_i(x) u_i + \frac{1}{\sqrt{2}} \sum_{j=1}^{2n-d} b_j(x) v_j, X \cdot w_0 \rangle\rangle_W \\
&= \frac{1}{\sqrt{2}} \sum_{i=1}^{d} \langle\langle a_i(x) u_i, X \cdot w_0 \rangle\rangle_W
\end{aligned}$$

for all $x \in B$, $X \in \mathfrak{g}$. In the last equality we use that the elements $b_j(x)v_j$ are orthogonal to $T_{w_0}\mathscr{O}_{w_0}$ for all $j = 1, \dots, 2n - d$.

Therefore, given a certain physical gauge transformation τ, the equality $\langle\langle \tau(x) \cdot \phi(x), X \cdot w_0 \rangle\rangle_W = 0$ holds for all $x \in B$, $X \in \mathfrak{g}$ if and only if $a_i \equiv 0$ for all $i = 1, \dots, d$, since $a_i(x)u_i \in T_{w_0}\mathscr{O}_{w_0}$. By the definition of unitary gauge, this proves the proposition. □

Example 3.5.3. In the case of the electroweak theory, for the chosen vacuum vector $w_0 = \begin{pmatrix} 0 \\ \sqrt{\frac{\mu}{2\lambda}} \end{pmatrix}$, we get $(T_{w_0}\mathscr{O}_{w_0})^\perp = < \begin{pmatrix} 0 \\ 1 \end{pmatrix} >$. Thus, in order to be orthogonal to $T_{w_0}\mathscr{O}_{w_0}$, a unitary gauge τ must satisfy $(\tau \cdot \phi)(x) = \binom{0}{\psi(x)}$ for some smooth function $\psi : B \to \mathbb{R}$. It can be shown that such a gauge transformation exists (see [Ham17, Sect. 8.1.5]).

Therefore, by means of a unitary gauge, we are able to find a Lagrangian which does not predict the existence of Nambu-Goldstone bosons. Hence, this solves one of the discrepancies between the theory and the experimental results.

3.6 The Higgs mechanism

Throughout this section we explore how, by writing the Yang-Mills-Higgs Lagrangian in terms of a Higgs field in unitary gauge, we can predict the existence of the massive bosons found experimentally without adding any extra ones.

In the same framework than in the previous sections, let $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be an Ad-invariant positive definite scalar product on the Lie algebra $\mathfrak{g}$, which exists because G is compact, as mentioned in Section 3.2.

Definition 3.6.1. The **mass form** is the positive semi-definite bilinear symmetric form given by:

$$\begin{aligned} m : \mathfrak{g} \times \mathfrak{g} &\to \mathbb{R} \\ (X, Y) &\mapsto \langle\langle X \cdot w_0, Y \cdot w_0 \rangle\rangle_W. \end{aligned}$$

Consider the Lie algebra of the unbroken subgroup $H = G_{w_0}$, which we denote $\mathfrak{h} \subset \mathfrak{g}$, and its $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$-orthogonal complement $\mathfrak{h}^\perp \subset \mathfrak{g}$, which verifies $\dim(\mathfrak{h}^\perp) = \dim(\mathfrak{g}) - \dim(\mathfrak{h}) = \dim(G) - \dim(H) = d$.

Proposition 3.6.2. *There exists a $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$-orthonormal basis $\alpha_1, \dots, \alpha_r$ of $\mathfrak{g}$ on which the matrix of the mass form is diagonal with $m(\alpha_a, \alpha_a) = \frac{1}{2} M_a^2$, and satisfying that:*

*i) $\alpha_1, \dots, \alpha_d$ is a basis of $\mathfrak{h}^\perp$ with $M_a > 0$ (these α_a are called **broken generators**).*

*ii) $\alpha_{d+1} \dots, \alpha_r$ is a basis of $\mathfrak{h}$ with $M_a = 0$ (these α_a are called **unbroken generators**).*

*The elements M_a are called the **masses of the gauge bosons**.*

Proof. It suffices to observe that $X \cdot w_0 = 0$ for all $X \in \mathfrak{h}$, since the action of H on w_0 is the identity. Hence, we have that $m|_{\mathfrak{h} \times \mathfrak{g}} = m|_{\mathfrak{g} \times \mathfrak{h}} \equiv 0$. Moreover, it can be seen that $Y \cdot w_0 \neq 0$ for all $Y \in \mathfrak{h}^\perp \setminus \{0\}$ (see [Ham17, Sect. 3.2]). Thus, the restriction $m|_{\mathfrak{h}^\perp \times \mathfrak{h}^\perp}$ is positive definite, by definition of the scalar product $\langle\langle \cdot, \cdot \rangle\rangle_W$. $\square$

Now, we aim to write the Yang-Mills-Higgs Lagrangian in terms of these elements. In particular, we are interested in the expression of the Lagrangian when we have a unitary gauge.

First of all, we need to introduce some notation in order to express the Lagrangian in local coordinates. To do so, we consider a local section $s : U \to B$ for some open subset $U \subset B$ with local coordinates $\{x^1, \dots, x^m\}$ and we denote $\partial_\mu = \partial/\partial x^\mu$. Then:

i) Given a connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, we write $\omega_\mu := (s^*\omega)(\partial_\mu)$. These elements $\omega_\mu : U \to \mathfrak{g}$ are also called **gauge fields**.

   Moreover, with the basis $\alpha_1, \dots, \alpha_r$ of $\mathfrak{g}$ of Proposition 3.6.2, we can decompose $\omega_\mu = \sum_{a=1}^r \omega_\mu^a \cdot \alpha_a$. The elements $\omega_\mu^a : U \to \mathbb{R}$ are called **broken** and **unbroken gauge bosons**, for $a \in \{1, \dots, d\}$ and $a \in \{d+1, \dots, r\}$, respectively.

ii) Let $\Phi = [s, \phi]$ be a local section of the associated vector bundle P'. With the local elements ω_μ, we can express the local covariant derivative given by ω as $\nabla_\mu^\omega \Phi = [s, \nabla_\mu^\omega \phi] = [s, (\partial_\mu + \omega_\mu)\phi]$, by definition of the covariant derivative on associated vector bundles.

iii) Given the curvature 2-form $\Omega^\omega \in \Omega^2(P, \mathfrak{g})$ of ω, we set $\Omega_{\mu\nu} := (s^*\Omega^\omega)(\partial_\mu, \partial_\nu)$. By the structure equation, it can be shown that locally we have $\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu]$.

   Besides, using the previous basis of $\mathfrak{g}$, we get $\Omega_{\mu\nu} = \sum_{a=1}^r \Omega_{\mu\nu}^a \cdot \alpha_a$.

With these definitions, we can prove the following theorem:

Theorem 3.6.3. *Let $\Phi = [s_0, \phi]$ be a Higgs field in unitary gauge with respect to w_0 after spontaneous symmetry breaking. Then, up to terms of second order in $\Delta\phi$ and ω_μ, we can express the Lagrangian $\mathcal{L}_{YMH}$ as:*

$$\begin{aligned}
\mathcal{L}_{YMH}(\Phi, \omega) \approx\ & \frac{1}{2}\sum_{j=1}^{2n-d} (\partial^\mu b_j)(\partial_\mu b_j) - \frac{1}{2}\sum_{j=1}^{2n-d} m_{v_j}^2 b_j^2 \\
& + \frac{1}{2}\sum_{a=1}^{d} M_a^2 \omega_a^\mu \omega_\mu^a - \frac{1}{4}\sum_{a=1}^{d} (\partial^\mu\omega_a^\nu - \partial^\nu\omega_a^\mu)(\partial_\mu\omega_\nu^a - \partial_\nu\omega_\mu^a) \\
& - \frac{1}{4}\sum_{a=d+1}^{r} (\partial^\mu\omega_a^\nu - \partial^\nu\omega_a^\mu)(\partial_\mu\omega_\nu^a - \partial_\nu\omega_\mu^a).
\end{aligned}$$

Proof. To begin with, by definition of $\mathcal{L}_{YMH}$ and of the aforementioned local elements we can write:

$$\begin{aligned}
\mathcal{L}_{YMH}(\Phi, \omega) &= \langle d_\omega\Phi, d_\omega\Phi\rangle_{P'} - V(\Phi) - \frac{1}{2}\langle \Omega_B^\omega, \Omega_B^\omega\rangle_{\mathrm{Ad}(P)} \\
&= \langle \nabla^{\omega\mu}\phi, \nabla_\mu^\omega\phi\rangle_W - V(\phi) - \frac{1}{2}\langle \Omega^{\mu\nu}, \Omega_{\mu\nu}\rangle_{\mathfrak{g}} \\
&= (\nabla^{\omega\mu}\phi)^\dagger \cdot (\nabla_\mu^\omega\phi) - V(\phi) - \frac{1}{4}\Omega_a^{\mu\nu}\Omega_{\mu\nu}^a.
\end{aligned}$$

Now, we consider the shifted Higgs field with $\phi = w_0 + \Delta\phi$ around the vacuum vector $w_0 \in W$. Then, using the definition $\nabla_\mu^\omega = \partial_\mu + \omega_\mu$, up to second order in $\Delta\phi$ and ω_μ we get:

$$\begin{aligned}
\mathcal{L}_{YMH}(\Phi, \omega) \approx\ & (\partial^\mu\Delta\phi)^\dagger(\partial_\mu\Delta\phi) + 2 \cdot \mathrm{Re}[(\partial^\mu\Delta\phi)^\dagger(\omega_\mu \cdot w_0)] + (\omega^\mu \cdot w_0)^\dagger(\omega_\mu \cdot w_0) \\
& - V(\phi) - \frac{1}{4}(\partial^\mu\omega_a^\nu - \partial^\nu\omega_a^\mu)(\partial_\mu\omega_\nu^a - \partial_\nu\omega_\mu^a).
\end{aligned}$$

Note that we have not yet used a unitary gauge. If we assume that Φ is in unitary gauge with respect to w_0 so that the Nambu-Goldstone bosons are identically 0, the following is satisfied:

i) We can write $\Delta\phi = \frac{1}{\sqrt{2}}\sum_{j=1}^{2n-d} b_j v_j$ where b_j are the Higgs bosons. Hence, $(\partial^\mu \Delta\phi)^\dagger(\partial_\mu \Delta\phi) = \frac{1}{2}\sum_{j=1}^{2n-d}(\partial^\mu b_j)(\partial_\mu b_j)$.

ii) By definition of the basis $v_1, \ldots, v_{2n-d}$, we have $\Delta\phi(x) \in (T_{w_0}\mathcal{O}_{w_0})^\perp$ for all $x \in B$. Thus, $\mathrm{Re}[(\partial^\mu \Delta\phi)^\dagger(\omega_\mu \cdot w_0)] \equiv 0$, since $(\omega_\mu \cdot w_0)(x) \in \mathfrak{g} \cdot w_0 = T_{w_0}\mathcal{O}_{w_0}$.

Moreover, by definition of the mass form, we can express:

$$(\omega^\mu \cdot w_0)^\dagger(\omega_\mu \cdot w_0) = m(\omega^\mu, \omega_\mu) = m\left(\sum_{a=1}^{r} \omega_a^\mu \alpha^a, \sum_{b=1}^{r} \omega_\mu^b \alpha_b\right) = \frac{1}{2}\sum_{a=1}^{d} M_a^2 \omega_a^\mu \omega_\mu^a.$$

Applying the previous equalities together with the expression of the Higgs potential given in Theorem 3.4.12, we obtain the desired Lagrangian.

See [Ham17, Sect. 8.2] for a more detailed proof. □

From the presented expression of the Lagrangian $\mathcal{L}_{YMH}$, we observe the following:

i) The d Nambu-Goldstone bosons a_i have disappeared, while there are still $2n - d$ Higgs bosons b_j with associated mass m_{v_j}.

ii) There appear d massive broken gauge bosons $\omega_\mu^1, \ldots, \omega_\mu^d$ of mass M_a and $r - d$ massless unbroken gauge bosons $\omega_\mu^{d+1}, \ldots, \omega_\mu^r$.

Remark 3.6.4. From a physics perspective, the term $\langle d_\omega\Phi, d_\omega\Phi\rangle_{P'}$ describes an interaction (or coupling) between the Higgs field Φ and the gauge bosons w_μ^a. This is interpreted as an interaction between the particles of the matter field mediated by gauge bosons. Furthermore, the appearance of massive gauge bosons is a consequence of this coupling and the choice of a non-zero value of the vacuum vector w_0, obtained after spontaneous symmetry breaking.

In addition, the terms of order higher than two, which we have not considered in here, are also interpreted as a direct interaction between Higgs bosons and gauge bosons, also self-interactions.

Definition 3.6.5. The presented method of creating masses for gauge bosons without changing the invariance under gauge transformations is called the **Brout-Englert-Higgs mechanism**.

Example 3.6.6. Let us now consider again the case of the electroweak interactions. Recall that with our choice of principal bundle and Lie group we had $2n = \dim(W) = 4$, $r = \dim(G) = 4$, and $d = \dim(G) - \dim(H) = 3$. Then, the theory predicts $d = 3$ massive gauge bosons (which correspond to W^+, W^-, and Z), $r - d = 1$ massless gauge boson (corresponding to the photon), and $2n - d = 1$ Higgs boson (see [Ham20, Sect. 8] for a detailed derivation of the bosons). Thus, the theory is finally consistent with the experimental results.

Summary and conclusions

Throughout the last century, gauge theories turned out to be the cornerstone of the development of modern physical theories that aim to encompass all the known interactions of the physical universe, even though there is still a long way to go to achieve this goal. In particular, the Standard Model has repeatedly proven its undeniable success with the extreme accuracy shown in many of its experimental predictions.

In this work, we tried to offer a glimpse of some of the intricacies of this theory. Step by step, we provided a brief but thorough mathematical overview of bundle theory. The constructions presented constitute the fundamental objects of interest in gauge theories. Then, beginning with the exposition of a general, classical Yang-Mills theory, we saw how Maxwell's equations of classical electromagnetism can be obtained within this formalism. After stating some of its shortcomings, we examined the methods that have been developed to explain electroweak interactions as a Yang-Mills theory, in a way compatible with real-world observations. With this aim, we introduced the Higgs mechanism and analyzed its usage of spontaneous symmetry breaking and unitary gauges.

However, in order to present an exhaustive explanation of these interactions, as done in the Standard Model, there is still a long list of topics that need to be covered. Starting with the particles complementary to bosons, namely fermions, which require concepts such as the Dirac Lagrangian or Yukawa couplings to be added to the theory (see [Ham20]). Moreover, only a quantized version of these theories is applicable to the real world. But the process of quantization involves techniques, as the process of renormalization, which are far beyond the scope of this work.

In addition, the Standard Model itself remains an incomplete theory, mainly because it has not still integrated gravity with the other fundamental interactions and there are still some inconsistencies with the experimental results, such as the mass of certain particles (an overview of these subjects can be found in [Ham17, Chapt. 9]).

On the other hand, from a purely mathematical point of view, there are also some interesting questions that still need to be addressed. For instance, the existence of unitary gauges in more general theories.

Even though I would have very much liked to continue exploring these topics and they can serve as ideas for subsequent work, the subjects that I have delved into appeared to be equally interesting and indispensable.

To conclude, this work has given me an insight into one fraction of the profound connection between mathematics and physics, and has allowed me to approach the world of particle physics from a totally unknown perspective, which has captivated me. I can just hope that I have been able to depict in a comprehensible manner what represents the germ of an extremely complex formalism.

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