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Abstract. We determine the Hausdorff limit-set of the Euclidean hypersurfaces with large λ_1 or small extrinsic radius. The result depends on the L^p norm of the curvature that is assumed to be bounded a priori, with a critical behaviour for p equal to the dimension minus 1.

1. Introduction

For any $A \subset \mathbb{R}^{n+1}$ and any $\varepsilon > 0$, we set A_ε the tubular neighbourhood of radius ε of A ($A_\varepsilon = \{x \in \mathbb{R}^{n+1} / \, d(A, x) \leqslant \varepsilon\}$). $d_H(A, B) = \inf\{\varepsilon > 0 \, A \subset B_\varepsilon$ and $B \subset A_\varepsilon\}$ is called the Hausdorff distance on closed subsets of $\mathbb{R}^{n+1}$. Let $(M_k^m)_{k\in\mathbb{N}}$ be a sequence of immersed submanifolds of dimension m in $\mathbb{R}^{n+1}$. We say that it converges weakly to a subset $Z \subset \mathbb{R}^{n+1}$ in Hausdorff topology if there exists a sequence of subsets $A_k \subset M_k$ such that $d_H(A_k, Z) \to 0$ and $\mathrm{Vol}\,(M_k \setminus A_k)/\mathrm{Vol}\, M_k \to 0$.

Of course, weak Hausdorff convergence does not imply Hausdorff convergence without supplementary assumption. Our first aim will be to determine which L^p-norm of the mean curvature has to be bounded so that weak convergence implies convergence. More precisely, we will study the limit-set for the Hausdorff distance of a weakly converging sequence of submanifolds with L^p norm of the mean curvature uniformly bounded and show that it depends essentially on the value of p. As an application, we derive some new results on the metric shape of Euclidean hypersurfaces with small extrinsic radius or large λ_1.

1.1. Weak Hausdorff convergence vs Hausdorff convergence.

In the paper, the L^p-norms are defined by $\|f\|_p^p = \frac{1}{v_M}\int_M |f|^p dv$. We denote by m_1 the 1-dimensional Hausdorff measure on $\mathbb{R}^{n+1}$. We denote by B the second fundamental form and $\mathrm{H} = \frac{1}{m}\mathrm{tr}\,\mathrm{B}$ the mean curvature of Euclidean m-submanifolds.

Our main result says that if $\mathrm{Vol}\, M_k \|\mathrm{H}\|_p^{m-1}$ remain bounded for some $p > m-1$ (resp. for $p = m-1$) then weak Hausdorff convergence implies Hausdorff convergence (resp. up to a set of bounded 1-dimensional Hausdorff measure).

Theorem 1.1. *Let $(M_k)_{k\in\mathbb{N}}$ be a sequence of immersed, compact submanifolds of dimension m which weakly converges to $Z \subset \mathbb{R}^{n+1}$.*

If there exist $p > m-1$ and $A > 0$ such that $\mathrm{Vol}\,(M_k)\|\mathrm{H}\|_p^{m-1} \leqslant A$ for any k, then $d_H(M_k, Z) \to 0$.

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There exists a constant $C(m)$ such that if $\mathrm{Vol}\,(M_k)\|\mathrm{H}\|_{m-1}^{m-1} = \int_{M_k} |\mathrm{H}|^{m-1} \leqslant A$ *for any k, then the limit-set of $(M_k)_{k\in\mathbb{N}}$ for the Hausdorff distance is not empty and any limit point is a closed, connected subset $Z \cup T \subset \mathbb{R}^{n+1}$ such that $m_1(T) \leqslant C(m)A$.*

Note that it derives from the proof that in the case $p = m-1$, we have $m_1(T) \leqslant C(m)\max_\varepsilon \liminf_k \int_{M_k\setminus Z_\varepsilon} |\mathrm{H}|^{m-1}$.

The previous result is rather optimal as shows the following result.

Theorem 1.2. *Let $M_1, M_2 \hookrightarrow \mathbb{R}^{n+1}$ be two immersed compact submanifolds with the same dimension m, $M_1\#M_2$ be their connected sum and T be any closed subset of $\mathbb{R}^{n+1}$ such that $M_1 \cup T$ is connected. Then there exists a sequence of immersions $i_k : M_1\#M_2 \hookrightarrow \mathbb{R}^{n+1}$ such that*

1) $i_k(M_1\#M_2)$ weakly converges to M_1 and converges to $M_1 \cup T$ in Hausdorff topology,

2) the curvatures of $i_k(M_1\#M_2)$ satisfy

$$\int_{i_k(M_1\#M_2)} |\mathrm{H}|^{m-1} \to \int_{M_1} |\mathrm{H}|^{m-1} + (\frac{m-1}{m})^{m-1}\mathrm{Vol}\,\mathbb{S}^{m-1} m_1(T),$$

$$\int_{i_k(M_1\#M_2)} |\mathrm{B}|^{m-1} \to \int_{M_1} |\mathrm{B}|^{m-1} + \mathrm{Vol}\,\mathbb{S}^{m-1} m_1(T),$$

$$\int_{i_k(M_1\#M_2)} |\mathrm{H}|^{\alpha} \to \int_{M_1} |\mathrm{H}|^{\alpha} \qquad \text{for any } \alpha \in [1, m-1),$$

$$\int_{i_k(M_1\#M_2)} |\mathrm{B}|^{\alpha} \to \int_{M_1} |\mathrm{B}|^{\alpha} \qquad \text{for any } \alpha \in [1, m-1),$$

3) $\lambda_p(i_k(M_1\#M_2)) \to \lambda_p(M_1)$ for any $p \in \mathbb{N}$,

4) $\mathrm{Vol}\,(i_k(M_1\#M_2)) \to \mathrm{Vol}\,M_1$*.*

Conditions 3) and 4) imposed to our sequence of immersions in Theorem 1.2 are designed on purpose for our study of almost extremal Euclidean hypersurfaces for the Reilly or Hasanis-Koutroufiotis Inequalities.

Theorem 1.1 proves that for $p > m-1$ the Hausdorff limit-set of a weakly convergent sequence is reduced to the weak limit. On the contrary, Theorem 1.2 shows that for $p < m-1$, the Hausdorff limit-point of a weakly convergent sequence can be any closed, connected Euclidean subset containing the weak-limit. For the critical exponent $p = m-1$, the Hausdorff limit-set can contain any $Z \cup T$ with $m_1(T) \leqslant C_2(m)A$ (by Theorem 1.2) and contains only $Z \cup T$ with $m_1(T) \leqslant C_1(m)A$ (by Theorem 1.1). Unfortunately, our constants $C_1(m)$ and $C_2(m)$ are different. We conjecture that is is only due to lake of optimality of the constant in Theorem 1.1.

Note that the two previous theorems can be easily extended to the case where $\mathbb{R}^{n+1}$ is replaced by any fixed Riemannian manifold (N, g).

1.2. **Application to hypersurfaces with large λ_1 or small Extrinsic radius.** Let $X\colon M^n \to \mathbb{R}^{n+1}$ be a closed, connected, immersed Euclidean hypersurface (with $n \geqslant 2$). We set v_M its volume and $\overline{X} := \frac{1}{v_M}\int_M X dv$ its center of mass.

The Hasanis-Koutroufiotis inequality is the following lower bound on the extrinsic radius r_M of M (i.e. the least radius of the Euclidean balls containing M)

$$r_M\|\mathrm{H}\|_2 \geqslant 1. \tag{1.1}$$

This inequality is optimal since we have equality for any Euclidean sphere. Moreover, if an immersed hypersurface M satisfies the equality case then M is the Euclidean sphere $S_M = \overline{X} + \frac{1}{\|\mathrm{H}\|_2}\mathbb{S}^n$ with center $\overline{X}$ and radius $\frac{1}{\|\mathrm{H}\|_2}$.

The Reilly inequality is the following upper bound on the first non zero eigenvalue λ_1^M of M

$$\lambda_1^M \leqslant n\|\mathrm{H}\|_2^2, \tag{1.2}$$

once again we have equality if and only if M is the sphere S_M.

Our aim is to study the metric shape of the Euclidean hypersurfaces with almost extremal extrinsic radius or λ_1.

1.2.1. *Almost extremal hypersurfaces weakly converge to S_M.* Our first result describes some volume and curvature concentration properties of almost extremal hypersurfaces that imply weak convergence to S_M. Note that in this result we do not assume any bound on the mean curvature.

We set $B_x(r)$ the closed ball with center x and radius r in $\mathbb{R}^{n+1}$ and A_η the annulus $\big\{X \in \mathbb{R}^{n+1} / \big|\|X - \bar{X}\| - \frac{1}{\|\mathrm{H}\|_2}\big| \leqslant \frac{\eta}{\|\mathrm{H}\|_2}\big\}$. Throughout the paper we shall adopt the notation that $\tau(\varepsilon|n, p, h, \cdots)$ is a positive function which depends on $n, p, h, \cdots$ and which converges to zero as $\varepsilon \to 0$. These functions τ will always be explicitly computable.

Theorem 1.3. *Any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1+\varepsilon$) satisfies*

$$\big\||\mathrm{H}| - \|\mathrm{H}\|_2\big\|_2 \leqslant 100\sqrt[8]{\varepsilon}\|\mathrm{H}\|_2, \tag{1.3}$$

$$\mathrm{Vol}\,(M \setminus A_{\sqrt[8]{\varepsilon}}) \leqslant 100\sqrt[8]{\varepsilon} v_M. \tag{1.4}$$

Moreover, for any $r > 0$ and any $x \in S_M = \overline{X} + \frac{1}{\|\mathrm{H}\|_2}\cdot\mathbb{S}^n$, we have

(1.5)
$$\Big|\frac{\mathrm{Vol}\,\big(B_x(\frac{r}{\|\mathrm{H}\|_2}) \cap M\big)}{v_M} - \frac{\mathrm{Vol}\,\big(B_x(\frac{r}{\|\mathrm{H}\|_2}) \cap S_M\big)}{\mathrm{Vol}\,S_M}\Big| \leqslant \tau(\varepsilon|n, r)\frac{\mathrm{Vol}\,\big(B_x(\frac{r}{\|\mathrm{H}\|_2}) \cap S_M\big)}{\mathrm{Vol}\,S_M}.$$

Note that (1.5) implies not only that M goes near any point of the sphere S_M, but also that the density of M near each point of S_M converge to $v_M/\mathrm{Vol}\,S_M$ at any scale. However, the convergence is not uniform with respect to the scales r. We infer that $A_{\tau(\varepsilon|n)} \cap M$ is Hausdorff close to S_M, which implies weak convergence to S_M of almost extremal hypersurfaces.

Corollary 1.4. *For any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1+\varepsilon$) there exists a subset $A \subset M$ such that* $\mathrm{Vol}\,(M \setminus A) \leqslant \tau(\varepsilon|n)v_M$ *and* $d_H(A, S_M) \leqslant \frac{\tau(\varepsilon|n)}{\|\mathrm{H}\|_2}$.

In the case where M is the boundary of a convex boby in $\mathbb{R}^{n+1}$ with $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ (or with $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1^M} \leqslant 1+\varepsilon$), the previous result implies easily that $d_H(M, S_M) \leqslant \frac{\tau(\varepsilon|n)}{\|H\|_2}$ and even $d_L(M, S_M) \leqslant \frac{\tau(\varepsilon|n)}{\|H\|_2}$.

1.3. **Hausdorff limit-set of almost extremal hypersurfaces.** Corollary 1.4 and Theorems 1.1 and 1.2 (applied to $M_1 = \mathbb{S}^n$ and M_2 any immersible hypersurface) allow a description of the limit-set of almost extremal hypersurfaces under a priori bounds on the mean curvature.

Theorem 1.5. *Let M be any hypersurface immersible in $\mathbb{R}^{n+1}$ and T be a closed subset of $\mathbb{R}^{n+1}$, such that $\mathbb{S}^n \cup T$ is connected (resp. and $T \cup \mathbb{S}^n \subset B_0(1)$). There exists a sequence of immersions $j_i : M \hookrightarrow \mathbb{R}^{n+1}$ of M which satisfies*
1) $\lambda_1^{j_i(M)} \to \lambda_1(\mathbb{S}^n)$ (resp. $r_{j_i(M)} \to 1$),
2) $\|\mathrm{B}_i - \mathrm{Id}\|_p \to 1$ for any $p \in [2, n-1)$,
3) $\mathrm{Vol}\, j_i(M) \to \mathrm{Vol}\, \mathbb{S}^n$,
4) $j_i(M)$ converges to $\mathbb{S}^n \cup T$ in pointed Hausdorff distance,
5) $\mathrm{Vol}\, \mathbb{S}^n \|\mathrm{H}_i\|_{n-1}^{n-1} \to C(n) m_1(T) + \mathrm{Vol}\, \mathbb{S}^n$.

This result shows that we can expect no control on the topology of almost extremal hypersurfaces nor on the metric shape (even on the diameter) of the part $M \setminus A$ of Corollary 1.4 if we do not assume a strong enough upper bound on the curvature.

On the other hand, Theorem 1.1 implies the following Hausdorff stability result.

Theorem 1.6. *For any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $v_M \|\mathrm{H}\|_{n-1}^n \leqslant A$ and $r_M \|\mathrm{H}\|_2 \leq 1 + \varepsilon$ (or with $v_M \|\mathrm{H}\|_{n-1}^n \leqslant A$ and $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1} \leqslant 1 + \varepsilon$) there exists a subset T of 1-dimensional Haussdorff measure less than $C(n) \int_M |\mathrm{H}|^{n-1} \leqslant C(n) A \|\mathrm{H}\|_2^{-1}$ such that $T \cup S_M$ is connected and $d_H(M, S_M \cup T) \leqslant \tau(\varepsilon | n, A) \|\mathrm{H}\|_2^{-1}$.*

More precisely, for any sequence $(M_k)_{k \in \mathbb{N}}$ of immersed hypersurfaces normalized by $\|\mathrm{H}_k\|_2 = 1$ and $\overline{X}_k = 0$, which satisfies $v_{M_k} \|\mathrm{H}_k\|_{n-1}^n \leqslant A$ and $r_{M_k} \to 1$ (or $v_{M_k} \|\mathrm{H}_k\|_{n-1}^n \leqslant A$ and $\frac{n}{\lambda_1(M_k)} \to 1$) there exists a closed subset $T \subset \mathbb{R}^{n+1}$ such that $m_1(T) \leqslant C(n) A$, $T \cup \mathbb{S}^n$ is connected and a subsequence $M_{k'}$ such that $d_H(M_{k'}, \mathbb{S}^n \cup T) \to 0$.

Here also the constant $C(n)$ of this theorem is not the same as in Theorem 1.5. So we do not have an exact computation of the Hausdorff limit set in the case $p = n - 1$ but we conjecture that it is just a mater of non optimality of the constant $C(m)$ in the bound on $m_1(T)$ in Theorem 1.1.

Finally, as a direct consequence of Theorem 1.1, we get the following result.

Theorem 1.7. *Let $2 \leqslant n - 1 < p \leqslant +\infty$. Any immersed hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ with $v_M \|\mathrm{H}\|_p^n \leqslant A$ and $r_M \|\mathrm{H}\|_2 \leq 1 + \varepsilon$ (or with $v_M \|\mathrm{H}\|_p^n \leqslant A$ and $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1} \leqslant 1 + \varepsilon$) satisfies $d_H(M, S_M) \leqslant \tau(\varepsilon | n, p, A) \|\mathrm{H}\|_2^{-1}$.*

Theorem 1.7 was already proved in the case $p = +\infty$ and under the stronger assumption $(1 + \varepsilon)\lambda_1 \geqslant n \|\mathrm{H}\|_4^2$ in [5], and in the case $p = +\infty$ and under the stronger assumption $r_M \|\mathrm{H}\|_4 \leq 1 + \varepsilon$ in [12]. It is also proved in an unpublished previous version of this paper [3] in the case $p > n$. In all these papers, the Hausdorff convergence is obtained by first proving that $\|X\|$ is almost constant in L^2 norm and then by applying a Moser iteration technique to infer that $\|X\|$ is almost constant is L^∞-norm. However, this scheme of proof cannot be applied to get the optimal condition $p > n - 1$ since $p = n$ is the critical exponent for the iteration. In place of a Moser iteration scheme, we adapt a technique introduced by P.Topping [14] to control the diameter of M by $\int_M |\mathrm{H}|^{n-1} \, dv$.

Note that by Theorem 1.6, in the case $v_M\|\mathrm{H}\|_p^n \leqslant A$ with $p > n-1$, almost extremal hypersurfaces for the Reilly inequality are almost extremal hypersurfaces for the Hasanis-Koutroufiotis inequality. Actually, in that case, an hypersurface is Hausdorff close to a sphere if and only if it is almost extremal for the Hasanis-Koutroufiotis inequality. In [2], we prove that an hypersurface Hausdorff close to a sphere or almost extremal for the Hasanis-Koutroufiotis inequality is not necessarily almost extremal for the Reilly inequality, even under the assumption $v_M\|\mathrm{H}\|_p^n \leqslant A$, for any $p < n$.

The structure of the paper is as follows: in Section 2, we recall some concentration properties for the volume and the mean curvature of almost extremal hypersurfaces (in particular Inequalities (1.4) and (1.3)) and some estimates on the restrictions to hypersurfaces of the homogeneous, harmonic polynomials of $\mathbb{R}^{n+1}$, proved in [2]. They are used in Section 3 to prove Inequality (1.5). Theorem 1.1 is proved in Section 4. We end the paper in section 5 by the proof of Theorem 1.2.

Throughout the paper we adopt the notation that $C(n, k, p, \cdots)$ is function greater than 1 which depends on p, q, n, $\cdots$. It eases the exposition to disregard the explicit nature of these functions. The convenience of this notation is that even though C might change from line to line in a calculation it still maintains these basic features.

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2. Some estimates on almost extremal hypersurfaces

We recall some estimate on almost extremal hypersurfaces proved in [2]. From now on, we assume, without loss of generality, that $\bar{X} = 0$. We say that M satisfies the pinching $(P_{p,\varepsilon})$ when $\|\mathrm{H}\|_p\|X - \overline{X}\|_2 \leqslant 1 + \varepsilon$. Let $X^T(x)$ denote the orthogonal projection of $X(x)$ on the tangent space T_xM.

Lemma 2.1 ([2])**.** *If $(P_{2,\varepsilon})$ holds, then we have $\|X^T\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$ and $\|X - \frac{\mathrm{H}}{\|\mathrm{H}\|_2^2}\nu\|_2 \leqslant \sqrt{3\varepsilon}\|X\|_2$.*

We set $A_\eta = B_0(\frac{1+\eta}{\|\mathrm{H}\|_2}) \setminus B_0(\frac{1-\eta}{\|\mathrm{H}\|_2})$.

Lemma 2.2 ([2])**.** *If $(P_{p,\varepsilon})$ (for $p > 2$), or $n\|\mathrm{H}\|_2^2/\lambda_1^M \leqslant 1+\varepsilon$, or $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ holds (with $\varepsilon \leqslant \frac{1}{100}$), then we have $\big\|\|X\| - \frac{1}{\|\mathrm{H}\|_2}\big\|_2 \leqslant \frac{C}{\|\mathrm{H}\|_2}\sqrt[8]{\varepsilon}$, $\||\mathrm{H}| - \|\mathrm{H}\|_2\|_2 \leq C\sqrt[8]{\varepsilon}\|\mathrm{H}\|_2$ and $\mathrm{Vol}\,(M \setminus A_{\sqrt[8]{\varepsilon}}) \leq C\sqrt[8]{\varepsilon}v_M$, where $C = 6 \times 2^{\frac{2p}{p-2}}$ in the case $(P_{p,\varepsilon})$ and $C = 100$ in the other cases.*

We set $\mathcal{H}^k(M)$ the set of functions $\{P_{|M}\}$, where P is any harmonic, homogeneous polynomials of degree k of $\mathbb{R}^{n+1}$. We also set ψ:$[0,\infty) \to [0,1]$ a smooth function, which is 0 outside $[\frac{(1-2\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}, \frac{(1+2\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}]$ and 1 on $[\frac{(1-\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}, \frac{(1+\sqrt[16]{\varepsilon})^2}{\|\mathrm{H}\|_2^2}]$, and φ the function on M defined by $\varphi(x) = \psi(|X_x|^2)$.

Lemma 2.3 ([2])**.** *For any hypersurface $M \hookrightarrow \mathbb{R}^{n+1}$ isometrically immersed with $r_M\|\mathrm{H}\|_2 \leqslant 1+\varepsilon$ (or $\frac{n\|\mathrm{H}\|_2^2}{\lambda_1} \leqslant 1+\varepsilon$) and for any $P \in \mathcal{H}^k(M)$, we have*

$$\big|\|\mathrm{H}\|_2^{2k}\|\varphi P\|_2^2 - \|P\|_{\mathbb{S}^n}^2\big| \leqslant C\,\sqrt[32]{\varepsilon}\,\|P\|_{\mathbb{S}^n}^2\,,$$

where $C = C(n,k)$.

If moreover $\varepsilon \leqslant \frac{1}{(2C)^{32}}$, then we have $\big\|\Delta(\varphi P) - \mu_k^{S_M}\varphi P\big\|_2 \leqslant C\,\sqrt[16]{\varepsilon}\mu_k^{S_M}\|\varphi P\|_2$.

3. Proof of Inequality 1.5

By a homogeneity, we can assume $\|\mathrm{H}\|_2 = 1$. Let $\theta \in (0,1)$, $x \in \mathbb{S}^n$ and set $V^n(s) = \mathrm{Vol}\,(B(x,s)\cap\mathbb{S}^n)$. Let $\beta(\theta,r) > 0$ small enough so that $(1+\theta/2)V^n\big((1+2\beta)r\big) \leqslant (1+\theta)V^n(r)$ and $(1-\theta/2)V^n\big((1-2\beta)r\big) \geqslant (1-\theta)V^n(r)$. Let $f_1 : \mathbb{S}^n \to [0,1]$ (resp. $f_2 : \mathbb{S}^n \to [0,1]$) be a smooth function such that $f_1 = 1$ on $B_x\big((1+\beta)r\big)\cap\mathbb{S}^n$ (resp. $f_2 = 1$ on $B_x\big((1-2\beta)r\big)\cap\mathbb{S}^n$) and $f_1 = 0$ outside $B_x\big((1+2\beta)r\big)\cap\mathbb{S}^n$ (resp. $f_2 = 0$ outside $B_x\big((1-\beta)r\big)\cap\mathbb{S}^n$). There exist an integer $N(\theta,r)$ and a family $(P_k^i)_{k\leqslant N}$ such that $P_k^i \in \mathcal{H}^k(\mathbb{R}^{n+1})$ and $A = \sup_{\mathbb{S}^n}\big|f_i - \sum_{k\leqslant N}P_k^i\big| \leqslant \|f_i\|_{\mathbb{S}^n}\theta/18$. We extend f_i to $\mathbb{R}^{n+1}\setminus\{0\}$ by $f_i(X) = f_i\big(\frac{X}{|X|}\big)$. Then we have

$$\Big|\|\varphi f_i\|_2^2 - \frac{1}{\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}|f_i|^2\Big| \leqslant I_1 + I_2 + I_3$$

where

$$I_1 := \Big|\frac{1}{v_M}\int_M\Big(|\varphi f_i|^2 - \varphi^2\big(\sum_{k\leqslant N}|X|^{-k}P_k^i\big)^2\Big)dv\Big|$$

$$I_2 := \Big|\frac{1}{v_M}\int_M\varphi^2\big(\sum_{k\leqslant N}|X|^{-k}P_k^i\big)^2dv - \sum_{k\leqslant N}\|P_k^i\|_{\mathbb{S}^n}^2\Big|$$

and

$$I_3 := \Big|\frac{1}{\mathrm{Vol}\,\mathbb{S}^n}\int_{\mathbb{S}^n}\Big(\big(\sum_{k\leqslant N}P_k^i\big)^2 - f_i^2\Big)\Big|.$$

On $\mathbb{S}^n$ we have $\big|f_i^2 - (\sum_{k\leqslant N}P_k^i)^2\big| \leqslant A\big(2\sup_{\mathbb{S}^n}|f_i| + A\big) \leqslant \|f_i\|_{\mathbb{S}^n}^2\theta/6$ and on M we have

$$\varphi^2\Big|f_i^2(X) - \big(\sum_{k\leqslant N}|X|^{-k}P_k^i(X)\big)^2\Big| \leqslant \Big|f_i^2\big(\frac{X}{|X|}\big) - \Big(\sum_{k\leqslant N}P_k^i\big(\frac{X}{|X|}\big)\Big)^2\Big| \leqslant \|f_i\|_{\mathbb{S}^n}^2\theta/6$$

.

Hence $I_1 + I_3 \leqslant \|f_i\|_{\mathbb{S}^n}^2 \theta/3$. Now

$$\begin{aligned}
I_2 \leqslant & \Big| \frac{1}{v_M} \int_M \varphi^2 \sum_{k \leqslant N} \frac{(P_k^i)^2}{|X|^{2k}} dv - \sum_{k\leqslant N} \|P_k^i\|_{\mathbb{S}^n}^2 \Big| + \frac{1}{v_M} \Big| \int_M \varphi^2 \sum_{1\leqslant k\neq k' \leqslant N} \frac{P_k^i P_{k'}^i}{|X|^{k+k'}} dv \Big| \\
\leqslant & \frac{1}{v_M} \int_M \varphi^2 \sum_{k\leqslant N} \Big| \frac{1}{|X|^{2k}} - \|\mathrm{H}\|_2^{2k} \Big| (P_k^i)^2 dv \\
& + \frac{1}{v_M} \int_M \sum_{1\leqslant k\neq k'\leqslant N} \varphi^2 \Big| \frac{1}{|X|^{k+k'}} - \|\mathrm{H}\|_2^{k+k'} \Big| |P_k^i P_{k'}^i| dv \\
& + \sum_{k\leqslant N} \big| \|\mathrm{H}\|_2^{2k} \|\varphi P_k^i\|_2^2 - \big\|P_k^i\big\|_{\mathbb{S}^n}^2 \big| + \sum_{1\leqslant k\neq k'\leqslant N} \frac{\|\mathrm{H}\|_2^{k+k'}}{v_M} \Big| \int_M \varphi^2 P_k^i P_{k'}^i dv \Big|
\end{aligned}$$

We have $\varphi^2 \big| \frac{1}{|X|^{k+k'}} - \|\mathrm{H}\|_2^{k+k'} \big| \leqslant \varphi^2 (k+k') 2^{k+k'+2} \sqrt[16]{\varepsilon} \|\mathrm{H}\|_2^{k+k'}$ by assumption on φ. From this and Lemma 2.3, we have

$$\begin{aligned}
I_2 \leqslant & N^2 4^{N+1} \sqrt[16]{\varepsilon} \sum_{k\leqslant N} \|\mathrm{H}\|_2^{2k} \|\varphi P_k^i\|_2^2 + \sqrt[32]{\varepsilon} \sum_{k\leqslant N} C(n,k) \big\|P_k^i\big\|_{\mathbb{S}^n}^2 \\
& + \sum_{1\leqslant k\neq k'\leqslant N} \frac{\|\mathrm{H}\|_2^{k+k'}}{v_M} \Big| \int_M \varphi^2 P_k^i P_{k'}^i dv \Big| \\
\leqslant & C(n,N) \sqrt[32]{\varepsilon} + \sum_{1\leqslant k\neq k'\leqslant N} \frac{\|\mathrm{H}\|_2^{k+k'}}{v_M} \Big| \int_M \varphi^2 P_k^i P_{k'}^i dv \Big|
\end{aligned}$$

and, by Lemma 2.3, we have

$$\begin{aligned}
& \Big| \frac{\|\mathrm{H}\|_2^2 (\mu_k - \mu_{k'})}{v_M} \int_M \varphi^2 P_k^i P_{k'}^i dv \Big| \\
\leqslant & \int_M \frac{|\varphi P_k^i \big(\Delta(\varphi P_{k'}^i) - \|\mathrm{H}\|_2^2 \mu_{k'} \varphi P_{k'}^i\big)|}{v_M} dv + \int_M \frac{|\varphi P_{k'}^i \big(\Delta(\varphi P_k^i) - \|\mathrm{H}\|_2^2 \mu_k \varphi P_k^i\big)|}{v_M} dv \\
\leqslant & \|\varphi P_k^i\|_2 \big\| \Delta(\varphi P_{k'}^i) - \|\mathrm{H}\|_2^2 \mu_{k'} \varphi P_{k'}^i \big\|_2 + \|\varphi P_{k'}^i\|_2 \big\| \Delta(\varphi P_k^i) - \|\mathrm{H}\|_2^2 \mu_k \varphi P_k^i \big\|_2 \\
\leqslant & C(n,N) \sqrt[16]{\varepsilon} \|\mathrm{H}\|_2^2 \|\varphi P_{k'}^i\|_2 \|\varphi P_k^i\|_2
\end{aligned}$$

under the condition $\varepsilon \leqslant (\frac{1}{2C(n,N)})^{32}$. Since $\mu_k - \mu_{k'} \geqslant n$ when $k \neq k'$, we get

$$\sum_{1\leqslant k\neq k'\leqslant N} \Big| \frac{1}{v_M} \int_M \varphi^2 P_k^i P_{k'}^i dv \Big| \leqslant \sum_{1\leqslant k\neq k'\leqslant N} C(n,N) \sqrt[16]{\varepsilon} \|\varphi P_{k'}^i\|_2 \|\varphi P_k^i\|_2 \leqslant \frac{C(n,N) \sqrt[16]{\varepsilon}}{\|\mathrm{H}\|_2^{k+k'}}$$

hence $I_2 \leqslant C(n,N) \sqrt[32]{\varepsilon}$ and

$$\Big| \|\varphi f_i\|_2^2 - \frac{1}{\operatorname{Vol} \mathbb{S}^n} \int_{\mathbb{S}^n} f_i^2 \Big| \leqslant C(n,N) \sqrt[32]{\varepsilon} + \frac{\theta}{3} \|f_i\|_{\mathbb{S}^n}^2.$$

We infer that if $\sqrt[32]{\varepsilon} \leqslant \frac{V^n((1-2\beta)r)\theta}{6C(n,N) \operatorname{Vol} \mathbb{S}^n} \leqslant \frac{\|f_i\|_{\mathbb{S}^n}^2 \theta}{6C(n,N)}$, then we have

$$\Big| \|\varphi f_i\|_2^2 - \frac{1}{\operatorname{Vol} \mathbb{S}^n} \int_{\mathbb{S}^n} |f_i|^2 \Big| \leqslant \theta \|f_i\|_{\mathbb{S}^n}^2 / 2$$

Note that N depends on r and θ but not on x since $O(n+1)$ acts transitively on $\mathbb{S}^n$. By assumption on f_1 and f_2, we have

$$\begin{aligned}\frac{\mathrm{Vol}\,(B_x((1+\beta)r-\sqrt[16]{\varepsilon}))\cap M\cap A_{\sqrt[16]{\varepsilon}})}{v_M}&\leqslant\|\varphi f_1\|_2^2\leqslant(1+\frac{\theta}{2})\|f_1\|^2_{\mathbb{S}^n}\\&\leqslant(1+\frac{\theta}{2})\frac{V^n((1+2\beta)r)}{\mathrm{Vol}\,\mathbb{S}^n}\leqslant(1+\theta)\frac{V^n(r)}{\mathrm{Vol}\,\mathbb{S}^n}\\\frac{\mathrm{Vol}\,(B_x((1-\beta)r+2\sqrt[16]{\varepsilon})\cap M\cap A_{2\sqrt[16]{\varepsilon}})}{v_M}&\geqslant\|\varphi f_2\|_2^2\geqslant(1-\frac{\theta}{2})\|f_2\|^2_{\mathbb{S}^n}\\&\geqslant(1-\frac{\theta}{2})\frac{V^n((1-2\beta)r)}{\mathrm{Vol}\,\mathbb{S}^n}\geqslant(1-\theta)\frac{V^n(r)}{\mathrm{Vol}\,\mathbb{S}^n}\end{aligned}$$

In the second estimates, we can replace ε by $\varepsilon/2^{16}$ as soon as we assume that $\varepsilon\leqslant\big(\min(\frac{1}{4^{16}},\frac{1}{(2C(n,N))^{32}},(\beta r)^{16},(\frac{\|f_i\|^2_{\mathbb{S}^n}\theta}{6(C(n,N)})^{32}\big)=K(\theta,r,n)$. Then we have $(1-\beta)r+\sqrt[16]{\varepsilon}\leqslant r\leqslant(1+\beta)r-\sqrt[16]{\varepsilon}$ and get

$$\Big|\frac{\mathrm{Vol}\,(B_x(r)\cap M\cap A_{\sqrt[16]{\varepsilon}})}{v_M}-\frac{V^n(r)}{\mathrm{Vol}\,\mathbb{S}^n}\Big|\leqslant\theta\frac{V^n(r)}{\mathrm{Vol}\,\mathbb{S}^n}$$

Combined with Lemma 2.2, we get the result with $\tau(\varepsilon|r,n)=\min\{\theta/\,2^{16}\varepsilon\leqslant K(\theta,r,n)\}$.

4. Proof of Theorem 1.1

In this section, we extend the technique developed by P.Topping in [14] to get an upper bound of $\mathrm{Diam}(M)$ by $\int_M|\mathrm{H}|^{n-1}$.

4.1. **Decomposition lemma.** We begin by a general result on approximation of Euclidean submanifolds in Hausdorff distance by the union of a subset of large volume and a finite family of geodesic subtrees of total length bounded by $\int_M|\mathrm{H}|^{n-1}$.

Lemma 4.1. *Let M^m be an Euclidean compact submanifold of $\mathbb{R}^{n+1}$ and $A\subset M$ a closed subset. There exists a constant $C(m)$ and a finite family of geodesic trees $T_i\subset M$ such that $A\cap T_i\neq\emptyset$, $d_H\big(A\cup(\cup_iT_i),M\big)\leqslant C\big(\mathrm{Vol}\,(M\setminus A)\big)^{\frac{1}{m}}$ and $\sum_i m_1(T_i)\leqslant C^{m(m-1)}\int_{M\setminus A}|\mathrm{H}|^{m-1}$.*

Proof. In [14], using the Michael-Simon Sobolev inequality as a differential inequation on the volume of intrinsic spheres, P.Topping prove the following lemma (slightly modified for our purpose).

Lemma 4.2 ([14]). *Suppose that M^m is a submanifold smoothly immersed in $\mathbb{R}^{n+1}$, which is complete with respect to the induced metric. Then there exists a constant $\delta(m)>0$ such that for any $x\in M$ and $R>0$, at least one of the following is true:*

(i) $M(x,R):=\sup_{r\in(0,R]}\int_{B_x(r)}|\mathrm{H}|^{m-1}/r>\delta^{m-1}$;

(ii) $\kappa(x,R):=\inf_{r\in(0,R]}\frac{\mathrm{Vol}\,B_x(r)}{r^m}>\delta$.

Where $B_x(r)$ is the geodesic ball in M for the intrinsic distance.

In this section, d stands for the intrinsic distance on M.

If $d_H(A,M)\leqslant10(\frac{\mathrm{Vol}\,M\setminus A}{\delta(m)})^{\frac{1}{m}}$, then we set $T=\emptyset$.

Otherwise, there exists $x_0\in M$ such that $d(A,x_0)=d_H(A,M)\geqslant10(\frac{\mathrm{Vol}\,M\setminus A}{\delta(m)})^{\frac{1}{m}}$. Let $\gamma_0:[0,l_0]\to M\setminus A$ be a normal minimizing geodesic from x_0 to A. For any

$t\in I_0=[0,l_0-(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}]$, we have $B_{\gamma_0(t)}\big((\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}\big)\subset M\setminus A$ and by the previous lemma, there exists $r_{0,t}\leqslant(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}$ such that $r_{0,t}\leqslant\frac{1}{\delta^{m-1}}\int_{B_{\gamma_0(t)}(r_{0,t})}|\mathrm{H}|^{m-1}$. By compactness of $\gamma_0(I_0)$ and by Wiener's selection principle, there exists a finite family $(t_j)_{j\in J_0}$ of elements of I_0 such that the balls of the family $\mathcal{F}_0=\big(B_{\gamma_0(t_j)}(r_{0,t_j})\big)_{j\in J_0}$ are disjoint and $\gamma(I_0)\subset\cup_{j\in J_0}B_{\gamma_0(t_j)}(3r_{0,t_j})$. Hence we have

$$\frac{\delta^{m-1}(l_0-(\frac{\operatorname{Vol}M\backslash A}{\delta})^{\frac{1}{m}})}{6}\leqslant\delta^{m-1}\sum_{j\in J_0}r_{0,t_j}\leqslant\sum_{j\in J_0}\int_{B_{\gamma_0(t_j)}(r_{0,t_j})}|\mathrm{H}|^{m-1}$$

And by assumption on l_0, we get $10(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}\leqslant l_0\leqslant\frac{10}{\delta^{m-1}}\sum_{j\in J_0}\int_{B_{\gamma_0(t_j)}(r_{0,t_j})}|\mathrm{H}|^{m-1}$.

If $d_H\big(A\cup\gamma_0([0,l_0]),M\big)\leqslant10(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}$, we set $T=\gamma_0([0,l_0])$. Otherwise, we set x_1 a point of $M\backslash A$ at maximal distance l_1 from $A\cup\gamma_0([0,l_0])$ and γ_1 the corresponding minimal geodesic. We set $I_1=[2(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}},l_1-2(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}]$. Once again, by the Wiener Lemma applied to $\gamma_1(I_1)$ we get a family of disjoint balls $\mathcal{F}_1=\big(B_{\gamma_1(t_j)}(r_{1,t_j})\big)_{j\in J_1}$ such that

$$\frac{\delta^{m-1}(l_1-4(\frac{\operatorname{Vol}M\backslash A}{\delta})^{\frac{1}{\delta}})}{6}\leqslant\delta^{m-1}\sum_{j\in J_1}r_{1,t_j}\leqslant\sum_{j\in J_1}\int_{B_{\gamma_1(t_j)}(r_{1,t_j})}|\mathrm{H}|^{m-1}$$

which gives $10(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}\leqslant l_1\leqslant\frac{10}{\delta^{m-1}}\sum_{j\in J_1}\int_{B_{\gamma_1(t_j)}(r_{1,t_j})}|\mathrm{H}|^{m-1}$. Note also that the balls of the family $\mathcal{F}_1\cup\mathcal{F}_2$ are disjoint.

If $d_H\big(A\cup\gamma_0([0,l_0])\cup\gamma_1([0,l_1]),M\big)\leqslant10(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}$, we set $T=\gamma_0([0,l_0])\cup\gamma_1([0,l_1])$. Note that T is a geodesic tree (if $\gamma_1(l_1)\in\gamma_0([0,l_1])$) or the disjoint union of 2 geodesic trees.

If $d_H\big(A\cup\gamma_0([0,l_0])\cup\gamma_1([0,l_1]),M\big)\geqslant10(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}$, then by iteration of what was made for x_1, γ_1 and $\mathcal{F}_1$, we construct a family $(x_j)_j$ of points, a family $(\gamma_j)_j$ of geodesics and a family $(\mathcal{F}_j)_j$ of sets of disjoint balls. Since the $(x_j)_j$ are $10(\frac{\operatorname{Vol}M\backslash A}{\delta(m)})^{\frac{1}{m}}$-separated in M and since M is compact, the families are finite and only a finite step of iterations can be made. The set $T=\cup_j\gamma_j([0,l_j])$ is the disjoint union of a finite set of finite geodesic trees and we have

$$l(T)\leqslant\frac{10}{\delta^{m-1}}\sum_j\sum_{k\in J_j}\int_{B_{\gamma_j(t_j)}(r_{j,t_k})}|\mathrm{H}|^{m-1}\leqslant\frac{10}{\delta^{m-1}}\int_{M\backslash A}|\mathrm{H}|^{m-1}. \tag{4.1}$$

Moreover, the connected parts of T are geodesic trees whose number is bounded above by $\frac{\delta^{\frac{m+1-m^2}{m}}}{(\operatorname{Vol}M\backslash A)^{\frac{1}{m}}}\int_{M\backslash A}|\mathrm{H}|^{m-1}$. □

4.2. **Proof of Theorem 1.1.** We begin the proof by the case where $\int_{M_k}|\mathrm{H}|^{m-1}\leqslant A$. By Topping's upper bound on the diameter [14] and Blaschke selection theorem, the sequence M_k converges, in Hausdorff topology, to a closed, connected limit set M_∞, which contains Z.

It just remain to prove that $m_1(M_\infty\setminus Z)\leqslant C(m)A$. Let $\ell\in\mathbb{N}^*$ fixed. We set $Z_r=\{x\in\mathbb{R}^{n+1}/\,d(x,Z)\leqslant r\}$. By the Michael-Simon Sobolev inequality applied to a constant function, we have $(\operatorname{Vol}M_k)^{\frac{1}{m}}\leqslant C(m)\int_{M_k}|\mathrm{H}|^{m-1}$. By weak convergence of $(M_k)_k$ to Z, we have $\lim_k\operatorname{Vol}(M_k\setminus Z_{1/2\ell})/\operatorname{Vol}(M_k)=0$ and by Lemma 4.1, there exists a finite union of geodesic trees T_k^ℓ such that $\lim_k d_H\big((M_k\cap Z_{1/3\ell})\cup T_k^\ell, M_\infty\big)=0$ and $l(T_k^\ell)\leqslant C(m)\int_{M_k\setminus Z_{1/3\ell}}|\mathrm{H}|^{m-1}$ for any k. Moreover, by construction of the part T in the proof of Lemma 4.1, each connected part of T_k^ℓ is a geodesic tree intersecting $Z_{1/3\ell}$, and by Inequality (4.1), the number of such component leaving $Z_{2/3\ell}$ is bounded above by $3\ell C(m)\int_{M_k\setminus Z_{1/3\ell}}|\mathrm{H}|^{m-1}$. We can assume that this number is constant up to a subsequence. Their union forms a sequence of compact sets $(\tilde{T}_k^\ell)$ which, up to a subsequence, converges to a set Y that contains $M_\infty\setminus Z_{1/\ell}$. By lower semi-continuity of the m_1-measure for sequence of trees (see Theorem 3.18 in [7]), we get that $m_1(M_\infty\setminus Z_{1/\ell})\leqslant\liminf_k l(\tilde{T}_k^\ell)\leqslant C(m)\liminf_k\int_{M_k\setminus Z_{1/3\ell}}|\mathrm{H}|^{m-1}$. Since $M_\infty\setminus Z=\cup_{\ell\in\mathbb{N}^*}M_\infty\setminus Z_{1/\ell}$, we get the result. So $M_\infty=Z\cup T$ with T a 1-dimensional subset of $\mathbb{R}^{n+1}$ of measure less than $C(m)\liminf_k\int_{M_k\setminus Z}|\mathrm{H}|^{m-1}\leqslant C(m)A$.

In the case $\int_{M_k}|\mathrm{H}|^p\leqslant A$ with $p>m-1$, we have

$$\int_{M_k\setminus Z_{1/3\ell}}|\mathrm{H}|^{m-1}\leqslant\Big(\frac{\operatorname{Vol}M_k\setminus Z_{1/3\ell}}{\operatorname{Vol}M_k}\Big)^{\frac{p-m+1}{p}}\operatorname{Vol}M_k\|\mathrm{H}\|_p^{m-1}$$

So the weak convergence to Z implies that $m_1(M_\infty\setminus Z_{1/3\ell})=0$ for any ℓ. Since $M_\infty\setminus Z_{1/3\ell}\neq\emptyset$ implies $m_1(M_\infty\setminus Z)\geqslant 1/3\ell$ by what precedes, we get $M_\infty\subset Z$, hence $M_\infty=Z$.

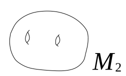
For the proof of Theorems 1.6 and 1.7, we can assume that $\overline{X}(M_k)=0$ and $\|\mathrm{H}\|_2=1\leqslant\|\mathrm{H}\|_p$ by scaling. Hence we have $v_{M_k}\|\mathrm{H}\|_p^{n-1}\leqslant v_{M_k}\|\mathrm{H}\|_p^n\leqslant A$ and $S_{M_k}=\mathbb{S}^n$ for any k. Inequality (1.4) and Lemma 4.1 give the Theorems.

5. Proof of Theorem 1.2

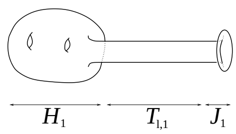
We first prove a weak version of Theorem 1.2, where the set T is a segment $[x_0,x_0+l\nu]$ with $x_0\in M_1$ and ν a normal vector to M_1 at x_0.

Adding of a segment and estimates on the curvature

We take off a small ball of M_2 and instead we glue smoothly a curved cylinder along one of its boundary and which is isometric to the product $[0,1]\times\frac{1}{10}\mathbb{S}^{m-1}$ at the neighbourhood of its other boundary component.



We note H_1 the resulting submanifold and $H_\varepsilon=\varepsilon H_1$. Let $c:[0,l]\to\mathbb{R}^+$ be a $\mathcal{C}^1$ positive function, constant equal to $\frac{1}{10}$ at the neighbourhoods of 0 and l, $T_{c,\varepsilon}$ be a cylinder of revolution isometric to $\{(t,u)\in[0,l]\times\mathbb{R}^m/|u|=\varepsilon c(t)\}$ and J_1 be a cylinder of revolution isometric to $[0,1/4]\times\frac{1}{10}\mathbb{S}^{m-1}$ at the neighbourhood of one of its boundary component and isometric to the flat annulus $B_0(\frac{3}{10})\setminus B_0(\frac{2}{10})\subset\mathbb{R}^m$) at the neighbourhood of its other boundary component. Note that in this paper we will only use the case $c\equiv\frac{1}{10}$ but the general case will be used in [2]. We also set $J_\varepsilon=\varepsilon J_1$ and $N_{c,\varepsilon}$ the submanifold obtained by gluing H_ε, $T_{c,\varepsilon}$ and J_ε.



Since the second fundamental form of $T_{c,\varepsilon}$ is given by $|B|^2 = \frac{(\varepsilon c'')^2}{(1+(\varepsilon c')^2)^3} + \frac{m-1}{\varepsilon^2 c^2 (1+(\varepsilon c')^2)}$, we have $\int_{N_{c,\varepsilon}} |\mathrm{B}|^\alpha dv = a(H_1, J_1)\varepsilon^{m-\alpha} + \mathrm{Vol}\,\mathbb{S}^{m-1}\varepsilon^{m-1-\alpha}(m-1)^{\frac{\alpha}{2}} \int_0^l c^{m-1-\alpha} + O_{c,\alpha}(\varepsilon^{m+1-\alpha})$, where $a(H_1, J_1)$ is a constant that depends only on H_1 and J_1 (not on c, l and ε).

We set M_1^ε the submanifold of $\mathbb{R}^{n+1}$ obtained by flattening M_1 at the neighbourhood of a point $x_0 \in M_1$ and taking out a ball centred at x_0 and of radius $\frac{3\varepsilon}{10}$: M_1 is locally equal to $\{x_0 + w + f(w),\ w \in B_0(\varepsilon_0) \subset T_{x_0}M_1\}$ where $f : B_0(\varepsilon_0) \subset T_{x_0}M_1 \to N_{x_0}M_1$ is a smooth function and $N_{x_0}M_1$ is the normal bundle M_1 at x_0. Let $\varphi : \mathbb{R}_+ \to [0,1]$ be a smooth function such that $\varphi = 0$ on $[0, \frac{\varepsilon_0}{3}]$ and $\varphi = 1$ on $[\frac{2\varepsilon_0}{3}, +\infty)$. We set M_1^ε the submanifold obtained by replacing the subset $\{x_0 + w + f(w),\ w \in B_0(\varepsilon_0) \subset T_{x_0}M_1\}$ by $\{x_0 + w + f_\varepsilon(w),\ w \in B_0(\varepsilon_0) \setminus B_0(3\varepsilon/10) \subset T_{x_0}M_1\}$, with $f_\varepsilon(w) = f\big(\varphi(\frac{\varepsilon_0\|w\|}{\varepsilon})w\big)$ for any $\varepsilon \leqslant 3\varepsilon_0/2$. Note that M_1^ε is a smooth deformation of M_1 in a neighbourhood of x_0 and its boundary has a neighbourhood isometric to a flat annulus $B_0(\varepsilon/3) \setminus B_0(3\varepsilon/10)$ in $\mathbb{R}^m$. Note that for ϵ small enough, $M_1^\varepsilon \setminus \{x \in M_1^\varepsilon / d(x, \partial M_1^\varepsilon) \leqslant 8\varepsilon\}$ is a subset of M_1. This fact will be used below. As a graph of a function, the curvatures of M_1^ε at the neighbourhood of x_0 are given by the formulae

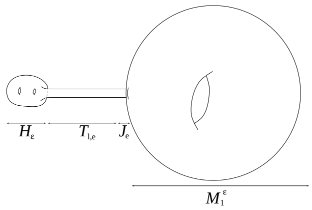
$$|\mathrm{B}_\varepsilon|^2 = \sum_{i,j,k,l=1}^{m} \sum_{p,q=m+1}^{n+1} Ddf_p(e_i, e_k) Ddf_q(e_j, e_l) H^{i,j} H^{k,l} G^{p,q}$$

$$\mathrm{H}_\varepsilon = \frac{1}{m} \sum_{k,l=m+1}^{n+1} \sum_{i,j=1}^{m} Ddf_k(e_i, e_j) H^{i,j} G^{k,l} (\nabla f_l - e_l)$$

where $(e_1, \cdots, e_m)$ is an ONB of $T_{x_0}M_1$, $(e_{m+1}, \cdots, e_{n+1})$ an ONB of $N_{x_0}M_1$, $f_\varepsilon(w) = \sum_{i=m+1}^{n+1} f_i(w) e_i$, $G_{kl} = \delta_{kl} + \langle \nabla f_k, \nabla f_l \rangle$ and $H_{kl} = \delta_{kl} + \langle df_\varepsilon(e_k), df_\varepsilon(e_l) \rangle$. Now f_ε converges in $\mathcal{C}^\infty$ norm to f on any compact subset of $B_0(\varepsilon_0) \setminus \{0\}$, while $|df_\varepsilon|$ and $|Ddf_\varepsilon|$ remain uniformly bounded on $B_0(\varepsilon_0)$ when ε tends to 0. By the Lebesgue convergence theorem, we have

$$\int_{M_1^\varepsilon} |\mathrm{H}_\varepsilon|^\alpha dv \to \int_{M_1} |\mathrm{H}|^\alpha dv \qquad \int_{M_1^\varepsilon} |\mathrm{B}_\varepsilon|^\alpha dv \to \int_{M_1} |\mathrm{B}|^\alpha dv$$

We set M_ε the m-submanifold of $\mathbb{R}^{n+1}$ obtained by gluing M_1^ε and $N_{c,\varepsilon}$ along their boundaries in a fixed direction $\nu \in N_{x_0}M_1$. Note that M_ε is a smooth immersion of $M_1 \# M_2$.



By the computations above, we get the announced limits 1), 2) and 4) for the sequence $i_k(M_1 \# M_2) = M_{\frac{1}{k}}$ as k tends to ∞.

Computation of the spectrum

We will adapt the method developed by C.Anné in [4]. We set $(\lambda_k)_{k\in\mathbb{N}}$ the spectrum with multiplicities obtained by union the spectrum of M_1 and of the spectrum $Sp(P_c)$ of the operator $P(f) = -f'' - (m-1)\frac{c'}{c} f'$ on $[0, l]$ with Dirichlet condition at 0 and Neumann condition at l. We denote by $(\mu_k)_{k\in\mathbb{N}}$ the eigenvalues of M_1 counted with multiplicities and by $(P_k)_{k\in\mathbb{N}}$ a L^2-ONB of eigenfunctions of M_1. We set $(\nu_k,h_k)_{k\in\mathbb{N}}$ and $(\lambda_k^\varepsilon,f_k^\varepsilon)_{k\in\mathbb{N}}$ the corresponding data on $([0,l],c^{n-1}(t)\,dt)$ and M_ε. We set $\tilde h_k^\varepsilon$ the function on M_ε obtained by considering h_k as a function on the cylinder $T_{c,\varepsilon}$, extending it continuously by 0 on J_ε and M_1^ε, and by $h_k(l)$ on H_ε. We also set $\tilde P_k^\varepsilon$ the function on M_ε which is equal to $\psi_\varepsilon\big(d(\partial M_1^\varepsilon,\cdot)\big)P_k$ on M_1^ε (with $\psi_\varepsilon(t)=0$ when $t\leqslant 8\varepsilon$, $\psi_\varepsilon(t)=\frac{\ln t-\ln(8\varepsilon)}{-\ln(8\sqrt{\varepsilon})}$ when $t\in[8\varepsilon,\sqrt{\varepsilon}]$ and $\psi_\varepsilon(t)=1$ otherwise) and is extended by 0 outside M_1^ε. Using the family $(\tilde h_k^\varepsilon,\tilde P_k^\varepsilon)$ as test functions, the min-max principle easily gives us

$$\lambda_k^\varepsilon\leqslant\lambda_k\big(1+\tau(\varepsilon|k,n,c,M_1)\big)\tag{5.1}$$

For any $k\in\mathbb{N}$, we set $\alpha_k=\liminf_{\varepsilon\to0}\lambda_k^\varepsilon$, $\varphi_{k,\varepsilon}^{(1)}(x)=\varepsilon^{\frac{m}{2}}(f_k^\varepsilon)_{|H_\varepsilon\cup J_\varepsilon}(\varepsilon x)$, seen as a function on $H_1\cup J_1$, $\varphi_{k,\varepsilon}^{(2)}(t,x)=\varepsilon^{\frac{m-1}{2}}(f_k^\varepsilon)_{|T_{c,\varepsilon}}(t,\varepsilon c(t)x)$ seen as a function on $[0,l]\times\mathbb{S}^{m-1}$ and $\varphi_{k,\varepsilon}^{(3)}$ the function on M_1 equal to f_k^ε on $\{x\in M_1^\varepsilon/\,d(x,\partial M_1^\varepsilon)\geqslant8\varepsilon\}$ and extended harmonically to M_1.

Easy computations give us

$$\int_{H_1\cup J_1}|\varphi_{k,\varepsilon}^{(1)}|^2=\int_{H_\varepsilon\cup J_\varepsilon}|f_k^\varepsilon|^2,\qquad\int_{H_1\cup J_1}|d\varphi_{k,\varepsilon}^{(1)}|^2=\varepsilon^2\int_{H_\varepsilon\cup J_\varepsilon}|df_k^\varepsilon|^2\tag{5.2}$$

$$\int_{T_{c,\varepsilon}}|f_k^\varepsilon|^2=\int_0^l\Big(\int_{\mathbb{S}^{m-1}}|\varphi_{k,\varepsilon}^{(2)}(t,u)|^2du\Big)\sqrt{1+\varepsilon^2(c'(t))^2}\,c^{m-1}(t)dt,\tag{5.3}$$

$$\int_{T_{c,\varepsilon}}|df_k^\varepsilon|^2=\int_0^l\Big[\frac{c^{m-1}}{\sqrt{1+\varepsilon^2(c')^2}}\int_{\mathbb{S}^{m-1}}\Big|\frac{\partial\varphi_{k,\varepsilon}^{(2)}}{\partial t}\Big|^2+\frac{\sqrt{1+\varepsilon^2(c')^2}\,c^{m-1}}{\varepsilon^2c^2}\int_{\mathbb{S}^{m-1}}|d_{\mathbb{S}^{m-1}}\varphi_{k,\varepsilon}^{(2)}|^2\Big].\tag{5.4}$$

The argument of C. Anne in [4] (or of Rauch and Taylor in [10]) can be adapted to get that there exists a constant $C(M_1)$ such that $\|\varphi_{k,\varepsilon}^{(3)}\|_{H^1(M_1)}\leqslant C\|f_k^\varepsilon\|_{H^1(M_\varepsilon)}$. Since we have $\|f_k^\varepsilon\|_{H^1(M_\varepsilon)}=1+\lambda_k^\varepsilon$, (5.1) gives us $\|\varphi_{k,\varepsilon}^{(3)}\|_{H^1(M_1)}\leqslant C(k,M_1,l)$ for $\varepsilon\leqslant\varepsilon_0(k,M_1,l)$. We infer that for any $k\in\mathbb{N}$ there is a subsequence $\varphi_{k,\varepsilon_i}^{(3)}$ which weakly converges to $\tilde f_k^{(3)}\in H^1(M_1)$ and strongly in $L^2(M_1)$ and such that $\lim_i\lambda_k^{\varepsilon_i}=\alpha_k$. By definitions of M_1^ε and $\varphi_{k,\varepsilon}^{(3)}$, and since $\mathcal{C}_0^\infty(M_1\setminus\{x_0\})$ is dense in $\mathcal{C}^\infty(M_1)$, it is easy to see that $\tilde f_k^{(3)}$ is a distributional (hence a strong) solution to $\Delta\tilde f_k^{(3)}=\alpha_k\tilde f_k^{(3)}$ on M_1 (see [13], p.206). In particular, either $\tilde f_k^{(3)}$ is 0 or α_k is an eigenvalue of M_1.

By the same compactness argument, there exists a subsequence $\varphi_{k,\varepsilon_i}^{(1)}$ which weakly converges to $\tilde f_k^{(1)}$ in $H^1(H_1\cup J_1)$ and strongly in $L^2(H_1\cup J_1)$. By Equalities (5.2), we get that $\|d\tilde f_k^{(1)}\|_{L^2(H_1)}=0$ and so $\tilde f_k^{(1)}$ is constant on H_1 and on J_1 and $\varphi_{k,\varepsilon_i}^{(1)}$ strongly converges to $\tilde f_k^{(1)}$ in $H^1(H_1\cup J_1)$. Let $\eta:[0,10]\to[0,1]$ be a smooth function such that $\eta(x)=1$ for any $x\leqslant1/2$, $\eta(x)=0$ for any $x\geqslant1$ and $|\eta'|\leqslant4$. We set s_ε the distance function to $\partial S_\varepsilon=\{0\}\times\frac{\varepsilon}{10}\mathbb{S}^{m-1}$ in $S_\varepsilon=M_1^\varepsilon\cup J_\varepsilon$ and θ_ε the volume density of S_ε in normal coordinate to ∂S_ε. We set L the distance between the two boundary components of J_1. By construction of S_ε, we have $\frac{3}{10}\geqslant\theta_\varepsilon(s_\varepsilon,u)=\theta_1(s_\varepsilon/\varepsilon)\geqslant1$ for any

$s_\varepsilon \in [0, L\varepsilon]$ and any u normal to ∂S_ε, and $c(M_1)(\frac{s_\varepsilon}{\varepsilon})^{m-1} \geqslant \theta_\varepsilon(s_\varepsilon, u) \geqslant \frac{1}{c(M_1)}(\frac{s_\varepsilon}{\varepsilon})^{m-1}$ for $s_\varepsilon \in [\varepsilon L, 8\varepsilon]$. Hence, if we denote by $S_{\partial S_\varepsilon}(r)$ the set of points in S_ε at distance r from ∂S_ε, we get for any $r \leqslant 8 + L$ that

$$\int_{S_{\partial S_\varepsilon}(\varepsilon r)} (f_k^\varepsilon)^2 = \int_{\frac{\varepsilon}{10}\mathbb{S}^{m-1}} \Big(\int_{\varepsilon r}^1 \frac{\partial}{\partial s_\varepsilon}[\eta(\cdot) f_k^\varepsilon(\cdot, u)] ds_\varepsilon\Big)^2 \theta_\varepsilon(r\varepsilon, u) du$$

$$= \frac{\varepsilon^{m-1}}{10^{m-1}} \int_{\mathbb{S}^{m-1}} \Big(\int_{\varepsilon r}^1 \frac{\partial}{\partial s_\varepsilon}[\eta(\cdot) f_k^\varepsilon(\cdot, \frac{\varepsilon}{10}u)] ds_\varepsilon\Big)^2 \theta_\varepsilon(r\varepsilon, \frac{\varepsilon}{10}u) du$$

$$\leqslant \frac{c(M_1)\varepsilon^{m-1}}{10^{m-1}} \int_{\mathbb{S}^{m-1}} \Big(\int_0^1 \big(\frac{\partial}{\partial s_\varepsilon}[\eta(\cdot) f_k^\varepsilon(\cdot, \frac{\varepsilon}{10}u)]\big)^2 \theta_\varepsilon(s_\varepsilon, \frac{\varepsilon}{10}u) ds_\varepsilon\Big)\Big(\int_0^1 \frac{1}{\theta_\varepsilon(s_\varepsilon, \frac{\varepsilon}{10}u)} ds_\varepsilon\Big) du$$

$$\int_{S_{\partial S_\varepsilon}(\varepsilon r)} (f_k^\varepsilon)^2 \leqslant c(M_1) \|f_k^\varepsilon\|^2_{H^1(S_\varepsilon)} \varepsilon |\ln \varepsilon| \tag{5.5}$$

which gives us $\varepsilon_i \int_{\partial S_{\varepsilon_i}} (f_k^{\varepsilon_i})^2 = \int_{\partial S_1} (\varphi_{k,\varepsilon_i}^{(1)})^2 \to \int_{\partial S_1} (\tilde{f}_k^{(1)})^2 = 0$ (by the trace inequality and the compactness of the trace operator) and so $\tilde{f}_k^{(1)}$ is null on J_1.

By (5.4), and since c is positive and $\mathcal{C}^1$ on $[0, l]$, there exists a subsequence $\varphi_{k,\varepsilon_i}^{(2)}$ which converges weakly to $\tilde{f}_k^{(2)}$ in $H^1([0, l] \times \mathbb{S}^{m-1})$ and strongly in $L^2([0, l] \times \mathbb{S}^{m-1})$. By the trace inequality applied on $[0, l] \times \mathbb{S}^{m-1}$, we also have that $\|\varphi_{k,\varepsilon_i}^{(2)}\|_{L^2(\{l\} \times \mathbb{S}^{m-1})}$ is bounded. Now, since

$$10^{1-m}\varepsilon_i \int_{\{l\} \times \mathbb{S}^{m-1}} |\varphi_{k,\varepsilon_i}^{(2)}|^2 = \varepsilon_i \int_{\{l\} \times \frac{\varepsilon_i}{10}\mathbb{S}^{m-1}} |f_k^{\varepsilon_i}|^2 = \varepsilon_i \int_{\partial H_{\varepsilon_i}} |f_k^{\varepsilon_i}|^2 = \int_{\partial H_1} |\varphi_{k,\varepsilon_i}^{(1)}|^2$$

we get that $\tilde{f}_k^{(1)} = 0$ on H_1.

We set $h_i(t) = \int_{\mathbb{S}^{m-1}} \varphi_{k,\varepsilon_i}^{(2)}(t, x) dx$ and $h(t) = \int_{\mathbb{S}^{m-1}} \tilde{f}_k^{(2)}(t, x) dx$, we have $h, h_i \in H^1([0, l])$ (with $h_i'(t) = \int_{\mathbb{S}^{m-1}} \frac{\partial \varphi_{k,\varepsilon_i}^{(2)}}{\partial t}(t, x) dx$), $h_i \to h$ strongly in $L^2([0, l])$ and weakly in $H^1([0, l])$. For any $\psi \in \mathcal{C}^\infty([0, l])$ with $\psi(0) = 0$ and $\psi'(l) = 0$, seen as a function on $T_{c,\varepsilon}$ and extended by 0 to S_ε and by $\psi(l)$ to H_ε, we have

$$\int_0^l h'(\psi c^{m-1})' \, dt - (m-1)\int_0^l h' \frac{c'}{c} \psi c^{m-1} \, dt = \int_0^l h' \psi' c^{m-1} \, dt$$

$$= \lim_i \int_0^l h_i'(t) \psi'(t) \frac{c^{m-1}}{\sqrt{1 + \varepsilon_i^2 (c')^2}} \, dt = \lim_i \int_{M_{\varepsilon_i}} \varepsilon_i^{\frac{1-m}{2}} \langle df_k^{\varepsilon_i}, d\psi \rangle = \lim_i \int_{M_{\varepsilon_i}} \varepsilon_i^{\frac{1-m}{2}} \lambda_k^{\varepsilon_i} f_k^{\varepsilon_i} \psi$$

$$= \alpha_k \lim_i \Big(\int_{[0,l] \times \mathbb{S}^{m-1}} \varphi_{k,\varepsilon_i}^{(2)} \psi c^{m-1} \sqrt{1 + \varepsilon_i^2 (c')^2} + \psi(l) \varepsilon_i^{\frac{1-m}{2}} \int_{H_{\varepsilon_i}} f_k^{\varepsilon_i}\Big)$$

$$= \alpha_k \int_0^l h \psi c^{m-1} \, dt$$

where we have used that $\varepsilon_i^{\frac{1-m}{2}} |\int_{H_{\varepsilon_i}} f_k^{\varepsilon_i}| \leqslant \sqrt{\varepsilon_i} \sqrt{\mathrm{Vol}\,(H_1) \int_{H_{\varepsilon_i}} (f_k^{\varepsilon_i})^2}$. Since c is positive, we get that h is a weak solution to $y'' + (m-1)\frac{c'}{c} y' + \alpha_k y = 0$ on $[0, l]$ and that $h'(l) = 0$. Since we have $10^{m-1} \int_{\partial S_{\varepsilon_i}} (f_k^{\varepsilon_i})^2 = \int_{\{0\} \times \mathbb{S}^{m-1}} (\varphi_{k,\varepsilon_i}^{(2)})^2 \to \int_{\{0\} \times \mathbb{S}^{m-1}} (\tilde{f}_k^{(2)})^2$ (by compactness of the trace operator) and $\int_{\partial S_{\varepsilon_i}} (f_k^{\varepsilon_i})^2 \to 0$ by (5.5), we get $|h(0)|^2 \leqslant$

$\operatorname{Vol}\mathbb{S}^{m-1}\int_{\{0\}\times\mathbb{S}^{m-1}}(\tilde f_k^{(2)})^2=0$, and so $h(0)=0$. Since $d_{\mathbb{S}^{m-1}}\varphi^{(2)}_{k,\varepsilon_i}$ converges weakly to $d_{\mathbb{S}^{m-1}}\tilde f_k^{(2)}$ in $L^2([0,l]\times\mathbb{S}^{m-1})$, Inequality (5.4) gives $\|d_{\mathbb{S}^{m-1}}\tilde f_k^{(2)}\|_{L^2([0,l]\times\mathbb{S}^{m-1})}=0$, i.e. $\tilde f_k^{(2)}$ is constant on almost every sphere $\{t\}\times\mathbb{S}^{m-1}$ of $[0,l]\times\mathbb{S}^{m-1}$. We infer that $\tilde f_k^{(2)}$ is equal to $\frac{1}{\operatorname{Vol}\mathbb{S}^{m-1}}h$ seen as a function on $[0,l]\times\mathbb{S}^{m-1}$ and so, either $\tilde f_k^{(2)}=0$ or α_k is an eigenvalue of P_c for the Dirichlet condition at 0 and the Neumann condition at l.

To conclude, we have

$$\begin{aligned}
&\int_{M_1}\tilde f_k^{(3)}\tilde f_l^{(3)}+\int_{[0,l]\times\mathbb{S}^{m-1}}\tilde f_k^{(2)}\tilde f_l^{(2)}c^{m-1}\\
&=\lim_i\int_{M_1}\varphi^{(3)}_{k,\varepsilon_i}\varphi^{(3)}_{l,\varepsilon_i}+\int_{J_1\cup H_1}\varphi^{(1)}_{k,\varepsilon_i}\varphi^{(1)}_{l,\varepsilon_i}+\int_{[0,l]\times\mathbb{S}^{m-1}}\varphi^{(2)}_{k,\varepsilon_i}\varphi^{(2)}_{l,\varepsilon_i}c^{m-1}\sqrt{1+\varepsilon_i^2(c')^2}\\
&=\lim_i\int_{M_{\varepsilon_i}}f_k^{\varepsilon_i}f_l^{\varepsilon_i}-\lim_i\int_{M_1^{\varepsilon_i}\cap B(\partial M_1^{\varepsilon_i},8\varepsilon_i)}f_k^{\varepsilon_i}f_l^{\varepsilon_i}+\lim_i\int_{M_1\setminus\left(M_1^{\varepsilon_i}\setminus B(\partial M_1^{\varepsilon_i},8\varepsilon_i)\right)}\varphi^{(3)}_{k,\varepsilon_i}\varphi^{(3)}_{l,\varepsilon_i}\\
&=\delta_{kl},
\end{aligned}$$

where, in the last equality, we have used that $\varphi^{(3)}_{k,\varepsilon_i}$ and $\varphi^{(3)}_{l,\varepsilon_i}$ converge strongly to $\tilde f_k^{(3)}$ and $\tilde f_l^{(3)}$ in $L^2(M_1)$, that $\operatorname{Vol}\left(M_1\setminus(M_1^{\varepsilon_i}\setminus B(\partial M_1^{\varepsilon_i},8\varepsilon_i)\right)$ tends to 0 with ε_i, and the inequality

$$\int_{M_1^{\varepsilon_i}\cap B(\partial M_1^{\varepsilon_i},8\varepsilon_i)}(f_k^{\varepsilon_i})^2\leqslant c(M_1)\|f_k^{\varepsilon_i}\|_{H^1(M_{\varepsilon_i})}\varepsilon_i^2|\ln\varepsilon_i|$$

which is obtained by integration of Inequality (5.5) with respect on $r\in[L,L+8]$. Note that we need the inclusion Hence, by the min-max principle, we have $\alpha_k\geqslant\lambda_k$ for any $k\in\mathbb{N}$. We conclude that $\lim_{\varepsilon\to0}\lambda_k(M_\varepsilon)=\lambda_k$ for any $k\in\mathbb{N}$. Note that in the case $c\equiv\frac{1}{10}$, the spectrum of P_c with Dirichlet condition at 0 and Neumann condition at l is $\{\frac{\pi^2}{l^2}(k+\frac12)^2,\,k\in\mathbb{N}\}$ with all the multiplicities equal to 1.

End of the proof of Theorem 1.2

In the sequence of immersions constructed above we have all the properties announced for $T=[x_0,x_0+l\nu]$, except the point 3) since all the eigenvalues of $[0,l]$ appear in the spectrum of the limit. To get the result for $T=[x_0,x_0+l\nu]$, we fix $k\in\mathbb{N}$ and l_k small enough such that $\lambda_1([0,l_k])>2k$ and with $l/l_k\in\mathbb{N}$. We then consider an immersion of $N_1=M_1\#\mathbb{S}^m$ such that $d_H(M_1\cup[x_0,x_0+l_k\nu],N_1)\leqslant2^{-\frac{l}{l_k}}$, $|\lambda_p(N_1)-\lambda_p(M_1)|\leqslant2^{-\frac{l}{l_k}}$ for any p such that $\lambda_p(M_1)\leqslant k$ (it is possible for this choice of l_k according to the weak version of the theorem proved above) and the same for the point 4) and 2) of the theorem (equality up to an error bounded by $2^{-\frac{l}{l_k}}$). We now iterate the procedure to get a sequence of $\frac{l}{l_k}$ immersions $N_2=N_1\#\mathbb{S}^m,\cdots,\ N_{\frac{l}{l_k}-1}=N_{\frac{l}{l_k}-2}\#\mathbb{S}^m$, $N_{\frac{l}{l_k}}=N_{\frac{l}{l_k}-1}\#M_2$

such that

$$\begin{aligned}
&d_H(N_i, M_1 \cup [x_0, x_0 + il_k\nu]) \leqslant i2^{-\frac{l}{l_k}}, \quad |\mathrm{Vol}\, N_{i+1} \setminus N_i| \leqslant 2^{-\frac{l}{l_k}} \mathrm{Vol}\, M_1,\\
&\int_{N_{i+1}\setminus N_i} |\mathrm{B}|^{(m-1)\frac{k-1}{k}} \leqslant 2^{-\frac{l}{l_k}} \int_{M_1} |\mathrm{H}|^{(m-1)\frac{k-1}{k}},\\
&|\int_{N_{i+1}\setminus N_i} |\mathrm{B}|^{m-1} - \mathrm{Vol}\,\mathbb{S}^{m-1} l_k| \leqslant 2^{-\frac{l}{l_k}} \int_{M_1} |\mathrm{B}|^{m-1},\\
&|\int_{N_{i+1}\setminus N_i} |\mathrm{H}|^{m-1} - \mathrm{Vol}\,\mathbb{S}^{m-1} (\frac{m-1}{m})^{m-1} l_k| \leqslant 2^{-\frac{l}{l_k}} \int_{M_1} |\mathrm{H}|^{m-1},\\
&|\lambda_p(N_i) - \lambda_p(N_{i+1})| \leqslant 2^{-\frac{l}{l_k}} \text{ for any } i \leqslant \frac{l}{l_k} - 1 \text{ and any } p \leqslant N.
\end{aligned}$$

The sequence $i_k(M_1 \# M_2) = N_{\frac{l}{l_k}}$ satisfies Theorem 1.2 for $T = [x_0 + l\nu]$.

In the procedure to get the theorem for $T = [x_0, x_0 + l\nu]$ from its weak version, we add at each step the new small cylinder T_{ε, l_N} along the same axis $x_0 + \mathbb{R}_+\nu$. This can be easily generalized to get the lemma for $T = \cup_i T_i$ any finite union of finite trees, each intersecting M_1, and such that $\sum_i m_1(T_i) \leqslant l$. Finally, if T is a closed subset such that $m_1(T) \leqslant C(m)\int |\mathrm{H}|^{m-1}$ and $M_1 \cup T$ is connected, then each connected component of T intersects M_1. Arguing as in the proof of Theorem 1.1, we get that T has only a finite number of connected component leaving $(M_1)_{\frac{1}{k}}$. Since any closed, connected $F_i \subset \mathbb{R}^{n+1}$ with $m_1(F_i)$ finite can be approximated in Hausdorff distance by a sequence of finite trees $T_{i,k}$ (such that $m_1(T_{i,k}) \to_k m_1(F_i)$ see [7]), by the same kind of diagonal procedure, Theorem 1.1 is obtained for any T with finite $m_1(T)$. In the case $m_1(T) = \infty$ then the L^{m-1} control of the curvature in condition 2) are automatically satisfied and the other conditions are fulfilled as above by approximating $M_1 \cup T$ by a finite number of finite trees. Finally, if $M_1 \cup T$ is not bounded, then we replace it by $\big((M_1 \cup T) \cap B_0(k)\big) \cup k\mathbb{S}^n$, which is closed, connected and bounded. We then get the result in its whole generality by a diagonal procedure.

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