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Houssem Haddar, Armin Lechleiter

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HOUSSEM HADDAR* AND ARMIN LECHLEITER†

Abstract. We consider scattering of time-harmonic electromagnetic waves from an unbounded penetrable dielectric layer mounted on a perfectly conducting infinite plate. This model describes for instance propagation of monochromatic light through dielectric photonic assemblies mounted on a metal plate. We give a variational formulation for the electromagnetic scattering problem in a suitable Sobolev space of functions defined in an unbounded domain containing the dielectric structure. Further, we derive a Rellich identity for a solution to the variational formulation. For simple material configurations and under suitable non-trapping and smoothness conditions, this integral identity allows to prove an a-priori estimate for such a solution. A-priori estimates for solutions to more complicated material configurations are then shown using a perturbation approach. While the estimates derived from the Rellich identity show that the electromagnetic rough surface scattering problem has at most one solution, a limiting absorption argument finally implies existence of a solution to the problem.

1. Introduction. Consider an antenna placed over an unbounded penetrable dielectric layer of finite height mounted on a planar metal substrate. Assuming that the antenna operates at fixed frequency, the electromagnetic field caused by the antenna solves a source problem for the time-harmonic Maxwell's equations in the half-space above the substrate. The problem to find this solution to Maxwell's equations when given the source and the dielectric is what we call the electromagnetic rough layer scattering problem. Figure 1.1 illustrates the setting of this problem.

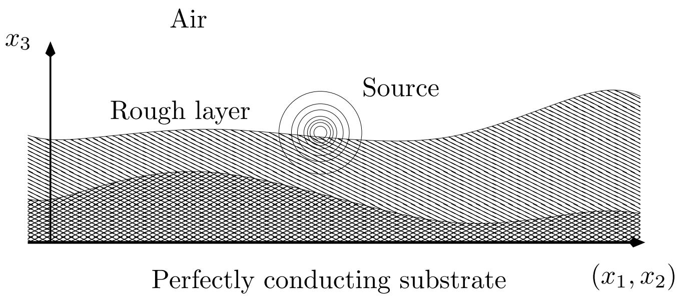

FIGURE 1.1. *Setting for the rough layer scattering problem. Points in $\mathbb{R}^3$ are denoted as $x = (x_1, x_2, x_3)^\top$. The penetrable layer has finite distance to the hyperplane $\{x_3 = 0\}$. We seek the wave field due to a time-harmonic source situated in a neighborhood of the layer.*

Due to the unboundedness of the domain, the rough layer scattering problem is quite involved from the point of view of mathematical analysis. This is the case at least if the dielectric material is a function of all three variables without decay or periodicity constraint, so that the problem to find the electromagnetic wave field cannot be reduced, e.g., to a bounded domain. Actually, already for the corresponding scalar wave problems the unbounded domain causes difficulties because Rellich's embedding lemma does not hold, Fredholm theory does not apply, and the arguments needed to establish existence and uniqueness of solution to such problems are in general far from trivial. Mathematical theory for scattering from rough unbounded structures was developed from the mid-nineties on by Chandler-Wilde and co-workers starting with theory

*INRIA Saclay–Ile-de-France / Ecole Polytechnique, CMAP, 91128 Palaiseau Cedex, France.
E-mail: haddar@cmap.polytechnique.fr
†INRIA Saclay–Ile-de-France / Ecole Polytechnique, CMAP, 91128 Palaiseau Cedex, France.
E-mail: alechle@cmap.polytechnique.fr

on integral equations for Dirichlet and impedance rough surface problems for the Helmholtz equation, see, e.g., [5,10,21]. Corresponding results for the same problem in three dimensions are very recent [6–8]. Scalar scattering problems involving penetrable media have been considered in [9, 11, 12, 15] both in two and three dimensions. However, rigorous mathematical solution theory for the full Maxwell's equations in unbounded penetrable media seems, to the best of our knowledge, not to exist in the literature.

Time-harmonic Maxwell's equations and electromagnetic wave propagation have been an important research area in the last years. For bounded scattering objects the mathematical theory for this system of partial differential equations is quite well developed, see, e.g., [17]. However, there are numerous problems that do not seem to be adequately modeled by problems posed in a bounded domain. Those include for instance electromagnetic wave propagation above unbounded rippled surfaces or interfaces, modeling for instance wave propagation related to ground penetrating radar. As a further example consider light propagation in a dielectric optical device. Such structures are often mounted on a substrate and thus their extension along the substrate is very large when compared to their thickness in the direction orthogonal to the substrate. Under periodicity assumptions on the structure wave scattering problems are usually reduced to a bounded domain and again a rather complete solution theory is available, see, e.g., [1, 14]. However, if the dielectric assembly lacks periodicity, for instance due to imperfections, doping, or combination of different sub-modules, then such a reduction does not work. Further, setting the scattering problem on a bounded domain seems inadequate since the dielectric structure has two different scales. Indeed, the thickness of such surface structures typically is of the order of the wave length whereas the transverse length of the structure is several orders of magnitudes larger. Thus, a natural way to model such wave propagation problems is to pose them on an unbounded domain of finite height containing the penetrable dielectric layer.

In this paper, we set up a variational formulation for the electromagnetic scattering problem in a suitably defined Sobolev space. This Hilbert space involves a singular spectral weight to be able to integrate the radiation condition into the variational formulation. Somewhat related spectral weights related to variational formulations for wave propagation problems have been introduced earlier in [4, 19]. For any solution of the variational formulation we derive an integral identity, a form of a Rellich identity, which gives an a-priori bound on the solution, at least if the dielectric material parameter satisfies certain non-trapping conditions. The appearance of non-trapping conditions at this point is not surprising, since they are well-known to play a crucial role in the solution theory of corresponding scalar problems, see [3, 11, 12, 15]. However, for the vectorial problem, the non-trapping assumptions are considerably more involved. Under suitable assumptions on the dielectric material, the Rellich identity yields uniqueness of solution to the variational problem.

Derivation of the Rellich identity is quite a technical matter, and we go through this procedure in detail. A special problem with definiteness of the variational formulation (to be explained below) forces us to prove the Rellich identity also for solutions to the variational problem in the case where the real wave number is replaced by an artificial complex wave number with small positive imaginary part corresponding to absorption. The result is that the a-priori estimate which follows from the Rellich identity is stable as the absorption parameter tends to zero. For the problem with absorption existence and uniqueness of solution are clear. Hence, by a limiting absorption process we obtain existence of solution to our variational problem. The main result on solvability of the rough layer scattering problem is formulated in Theorem 8.2. We should mention that the limiting absorption principle we require is not necessary to prove existence of solution to the corresponding scalar Helmholtz problem, see [15]. The reason is that the variational form of the scalar problem is an L^2-perturbation of a coercive form. Our variational form for the electromagnetic problem is not coercive modulo L^2-perturbations because the boundary terms incorporating the radiation condition have no definite real part. This is essentially a feature caused by the nature of the Maxwell's equations.

We have to mention a further restriction of our result. All right-hand sides appearing as source terms in our analysis have to be divergence free. The treatment of right hand sides that are not divergence free would require a Hodge decomposition (as it is done for a bounded domain in [17] for instance). However, due to the unbounded setting such a procedure would introduce even more technical difficulties that are not central in our present contribution and are postponed to a future work.

This paper is organized as follows. In Sections 2 we present the strong formulation of our scattering problem. In Section 3 we derive suitable variational formulations set in appropriate function spaces. Section 4 contains a few technical lemmas. The Rellich identity is presented in Section 5. Sections 6 and 7 derive a-priori estimates from the latter identity for two classes of dielectric material parameters. Finally, in Section 8 we prove existence and uniqueness of solution to our variational scattering problem.

Notation: Standard L^2 based Sobolev spaces defined in a domain Ω or on a surface Γ are denoted as $H^s(\Omega)$ or $H^s(\Gamma)$ for $s \in \mathbb{R}$ (see [16] for instance). The spaces $H(\operatorname{curl}, \Omega)$ and $H(\operatorname{div}, \Omega)$ contain those functions in $L^2(\Omega)$ whose curl or divergence belongs to $L^2(\Omega)$, respectively (see, e.g., [17]). For $s \in \mathbb{N}$, $W^{s,\infty}(\Omega)$ denotes the function space of s times weakly differentiable functions with essentially bounded derivatives up to order s (see [16] for instance).

2. Problem Setting. We consider the time harmonic linear Maxwell's equations at frequency $\omega > 0$ in $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3,\, x_3 > 0\}$ for a rough layered geometry which models a dielectric layer on top of some metallic plate. The electric permittivity $\varepsilon > 0$ and conductivity $\sigma \geq 0$ vary inside a layer of finite height $\Omega = \{0 < x_3 < h\}$ and they are constant above this layer. The magnetic permeability μ is assumed to be constant. The two boundaries of Ω are denoted as $\Gamma_0 = \{x_3 = 0\}$ and $\Gamma_h = \{x_3 = h\}$. Since we assume the dielectric layer to be mounted on a perfectly conducting plate, the electric field $\boldsymbol{E}$ satisfies perfectly conducting boundary conditions on Γ_0. Consider a function $\boldsymbol{g}_0$ such that the support of $\boldsymbol{g}_0$ is included in $\overline{\Omega}$. Maxwell's equations describing the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ due to the source $\boldsymbol{g}_0$ read

$$\operatorname{curl} \boldsymbol{E} - \mathrm{i}\omega\mu \boldsymbol{H} = 0, \quad \operatorname{curl} \boldsymbol{H} + \mathrm{i}\omega(\varepsilon + \mathrm{i}\sigma/\omega)\boldsymbol{E} = \boldsymbol{g}_0 \quad \text{in } \mathbb{R}^3_+, \quad \boldsymbol{E} \times \boldsymbol{e}_3 = 0 \quad \text{on } \Gamma_0. \tag{2.1}$$

Here, $\boldsymbol{e}_3 = (0, 0, 1)^\top$. The system of partial differential equations (2.1) has to be complemented by a radiation condition that we will later on set up using an adaption of the angular spectrum representation. We eliminate the magnetic field from (2.1) to obtain a second-order equation for the electric field,

$$\operatorname{curl}^2 \boldsymbol{E} - \omega^2\mu(\varepsilon + \mathrm{i}\sigma/\omega)\boldsymbol{E} = \mathrm{i}\omega\mu\boldsymbol{g}_0 \quad \text{in } \mathbb{R}^3_+, \quad \boldsymbol{E} \times \boldsymbol{e}_3 = 0 \quad \text{on } \Gamma_0. \tag{2.2}$$

The later variational analysis does by the way not work for the magnetic field, since we exploit that $\mathbf{E} \times \boldsymbol{e}_3$ vanishes on Γ_0.

Since we deal with a layered geometry, we assume that $\varepsilon = \varepsilon_+ > 0$ and $\sigma = 0$ in $\{x_3 > h - \eta\}$ for some constant $\varepsilon_+ > 0$, $0 < \eta \ll 1$. The relative material parameter is then defined by $\varepsilon_\mathrm{r} := (\varepsilon + \mathrm{i}\sigma/\omega)/\varepsilon_+$, $k^2 := \omega^2\varepsilon_+\mu$, and (2.2) becomes

$$\operatorname{curl}^2 \boldsymbol{E} - k^2\varepsilon_\mathrm{r}\boldsymbol{E} = \boldsymbol{g} \quad \text{in } \mathbb{R}^3_+, \quad \boldsymbol{E} \times \boldsymbol{e}_3 = 0 \quad \text{on } \Gamma_0, \tag{2.3}$$

where we have set $\boldsymbol{g} := \mathrm{i}\omega\mu\boldsymbol{g}_0$. We shall restrict ourselves to divergence free source terms,

$$\operatorname{div} \boldsymbol{g} = 0 \quad \text{in } \mathbb{R}^3_+.$$

The wave number k is physically real, but for mathematical reasons, more precisely for a limiting absorption argument, we also need to consider complex wave numbers $k \in \mathbb{C}$ such that

$$k^2 \in \{z \in \mathbb{C},\, \mathrm{Re}(z) > 0,\, \mathrm{Im}(z) \geq 0\}.$$

We need to complement Maxwell's equations (2.3) by outgoing conditions in the half space above Ω to have any chance to obtain a well-posed problem. To formulate these conditions we use the two-dimensional Fourier transform $\mathcal{F}$, defined for an integrable vector field $\boldsymbol{\phi}$,

$$\mathcal{F}(\boldsymbol{\phi})(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \boldsymbol{\phi}(x) \exp(-\mathrm{i}\,\tilde{x}\cdot\xi)\,\mathrm{d}x.$$

We recall that this integral transform can be extended to a unitary operator on $L^2(\mathbb{R}^2)^3$ or on $L^2(\mathbb{R}^2)$. Since the Fourier transform acts component-wise on $\boldsymbol{\phi}$ we do not distinguish between transforms of scalar and vector-valued functions.

Since the material parameter ε_r is constant in $\{x_3 > h\}$, and since the support of the source term $\boldsymbol{g}$ is included in Ω, the electric field $\boldsymbol{E}$ satisfies a vector Helmholtz equation $\Delta \boldsymbol{E} + k^2 \boldsymbol{E} = 0$ for $x_3 > h$. Writing $\tilde{x} = (x_1, x_2)^\top$ for $x = (x_1, x_2, x_3)^\top \in \mathbb{R}^3$, separation of variables shows that

$$\boldsymbol{E}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \exp\left(\mathrm{i}\left((x_3 - h)\sqrt{k^2 - |\xi|^2} + \tilde{x}\cdot\xi\right)\right) \mathcal{F}\left(\boldsymbol{E}|_{\Gamma_h}\right)(\xi)\,\mathrm{d}\xi, \quad x_3 > h, \tag{2.4}$$

is a representation of the upwards radiating field $\boldsymbol{E}$, where the square root in the latter expression is defined by a branch cut in the complex plane along the negative imaginary axis. In particular, for real k and $k^2 < |\xi|^2$, we have $(k^2 - |\xi|^2)^{1/2} = \mathrm{i}(|\xi|^2 - k^2)^{1/2}$.

The problem we aim to solve is hence the following: Find $\boldsymbol{E} : \mathbb{R}^3_+ \to \mathbb{C}$ that belongs to $H(\mathrm{curl}, \{0 < x_3 < H\})$ for all $H > 0$ and that satisfies $\boldsymbol{E} \times \boldsymbol{e}_3 = 0$ on Γ_0, $\mathrm{curl}\,\mathrm{curl}\,\boldsymbol{E} - k^2 \varepsilon_\mathrm{r} \boldsymbol{E} = \boldsymbol{g}$ in the sense of $L^2(\mathbb{R}^3_+)^3$, and the expansion (2.4).

3. Variational Formulation. Assuming that enough regularity holds, we formally multiply (2.3) by the complex conjugate of a test function $\boldsymbol{\psi}$. Partial integration leads to

$$\int_\Omega \left(\mathrm{curl}\,\boldsymbol{E}\cdot\mathrm{curl}\,\overline{\boldsymbol{\psi}} - k^2\varepsilon_\mathrm{r}\boldsymbol{E}\cdot\overline{\boldsymbol{\psi}}\right)\mathrm{d}x + \int_{\partial\Omega} (\nu \times \mathrm{curl}\,\boldsymbol{E})\cdot\overline{\boldsymbol{\psi}}\,\mathrm{d}s = \int_\Omega \boldsymbol{g}\cdot\overline{\boldsymbol{\psi}}\,\mathrm{d}x. \tag{3.1}$$

The task is now to replace the boundary term $\nu \times \mathrm{curl}\,\boldsymbol{E}$ by a suitable operator that acts on the tangential component of $\boldsymbol{E}|_{\Gamma_h}$, and that takes into account the upwards radiation condition. We shall construct for that purpose the so-called Calderón operator, the natural analogue of the exterior Dirichlet-to-Neumann operator known from the variational theory for the Helmholtz equation [17]. Formally taking the normal derivative of $\boldsymbol{E}$ from (2.4) on Γ_h, we obtain

$$\left.\frac{\partial \boldsymbol{E}}{\partial x_3}\right|_{\Gamma_h} = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2}\sqrt{k^2 - |\xi|^2}\exp(\mathrm{i}\tilde{x}\cdot\xi)\,\mathcal{F}\left(\boldsymbol{E}|_{\Gamma_h}\right)(\xi)d\xi, \tag{3.2}$$

a formula which defines a Dirichlet-to-Neumann operator T^+ on Γ_h,

$$(T^+_{k^2}\boldsymbol{\phi})(\tilde{x}) = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^2}\sqrt{k^2 - |\xi|^2}\exp(\mathrm{i}\tilde{x}\cdot\xi)\,\mathcal{F}\boldsymbol{\phi}(\xi)\,\mathrm{d}\xi. \tag{3.3}$$

It is obvious from Fourier expressions that $T^+_{k^2} : H^{1/2}(\Gamma_h) \to H^{-1/2}(\Gamma_h)$ is bounded and, by Parseval's identity,

$$-\mathrm{Re}\left\langle T^+_{k^2}\phi,\, \phi\right\rangle = \int_{\mathbb{R}^2}\mathrm{Im}\left(\sqrt{k^2 - |\xi|^2}\right)|\mathcal{F}\phi(\xi)|^2\;\mathrm{d}\xi \geq 0, \tag{3.4}$$

$$\mathrm{Im}\left\langle T^+_{k^2}\phi,\, \phi\right\rangle = \int_{\mathbb{R}^2}\mathrm{Re}\left(\sqrt{k^2 - |\xi|^2}\right)|\mathcal{F}\phi(\xi)|^2\;\mathrm{d}\xi \geq 0, \quad \phi \in H^{1/2}(\Gamma_h), \tag{3.5}$$

where $\langle\, ,\, \rangle$ denotes the $H^{-1/2}(\Gamma_h) - H^{1/2}(\Gamma_h)$ duality product that extends the $L^2(\Gamma_h)$ scalar product. These properties directly transfer to vector fields $\boldsymbol{\phi}$. For real k we have that

$$
\begin{aligned}
-\operatorname{Re}\left\langle T^+_{k^2}\phi,\, \phi\right\rangle &= \int_{|\xi|>k} \sqrt{|\xi|^2 - k^2}|\mathcal{F}\phi(\xi)|^2 \,\mathrm{d}\xi \geq 0,\\
\operatorname{Im}\left\langle T^+_{k^2}\phi,\, \phi\right\rangle &= \int_{|\xi|<k} \sqrt{k^2 - |\xi|^2}|\mathcal{F}\phi(\xi)|^2 \,\mathrm{d}\xi \geq 0, \quad \phi \in H^{1/2}(\Gamma_h).
\end{aligned}
$$

To avoid lengthy component-wise calculations we shall make use of surface differential operators. We have in mind that x_3 is the vertical coordinate and thus we denote the transverse part of a vector field $\boldsymbol{u} = (u_1, u_2, u_3)^\top$ by $\boldsymbol{u}_T = (u_1, u_2, 0)^\top$. For a scalar function v we set

$$
\nabla_T v := (\partial v/\partial x_1, \partial v/\partial x_2, 0)^\top \quad \text{and} \quad \vec{\operatorname{curl}}_T v := (\nabla_T v) \times \boldsymbol{e}_3 = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}, 0\right)^\top .
$$

For $\boldsymbol{u} = (u_1, u_2, 0)^\top$ we set $\operatorname{div}_T \boldsymbol{u} := \partial u_1/\partial x_1 + \partial u_2/\partial x_2$ and, again for a general vector field $\boldsymbol{u} = (u_1, u_2, u_3)^\top$, $\operatorname{curl}_T \boldsymbol{u} := \operatorname{div}_T(\boldsymbol{u} \times \boldsymbol{e}_3) = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$. Then there holds

$$
\operatorname{curl} \boldsymbol{u} = (\operatorname{curl}_T \boldsymbol{u}_T)\, \boldsymbol{e}_3 + \vec{\operatorname{curl}}_T u_3 - \frac{\partial}{\partial x_3}\boldsymbol{u} \times \boldsymbol{e}_3 \quad \text{for } \boldsymbol{u} : \mathbb{R}^3 \to \mathbb{R}^3. \tag{3.6}
$$

We remark that $\operatorname{div}_T \vec{\operatorname{curl}}_T = 0$ and $\operatorname{curl}_T \nabla_T = 0$. Moreover, $\vec{\operatorname{curl}}_T$ and curl_T are adjoint to each other in the sense that, formally, $\int_{\mathbb{R}^2} \operatorname{curl}_T \boldsymbol{u}\, v \,\mathrm{d}\tilde{x} = \int_{\mathbb{R}^2} \boldsymbol{u}\, \vec{\operatorname{curl}}_T v \,\mathrm{d}\tilde{x}$ for scalar function v and a vector field $\boldsymbol{u}$. From (3.6) it follows that

$$
\boldsymbol{e}_3 \times \operatorname{curl} \boldsymbol{E} = \nabla_T(\boldsymbol{E} \cdot \boldsymbol{e}_3) - \frac{\partial \boldsymbol{E}_T}{\partial x_3} \quad \text{on } \partial\Omega. \tag{3.7}
$$

Due to (3.3) we know that

$$
\frac{\partial \boldsymbol{E}_T}{\partial x_3} = T^+_{k^2}(\boldsymbol{E}_T) \quad \text{on } \Gamma_h. \tag{3.8}
$$

To express $\nabla_T(E_3)$ we use the divergence condition satisfied by $\boldsymbol{E}$ in a neighborhood of Γ_h,

$$
0 = \operatorname{div} \boldsymbol{E} = \operatorname{div}_T \boldsymbol{E}_T + \frac{\partial E_3}{\partial x_3} \quad \text{on } \Gamma_h. \tag{3.9}
$$

Taking the Fourier transform and using (3.2) shows that

$$
i\sqrt{k^2 - |\xi|^2}\mathcal{F}\left(E_3|_{\Gamma_h}\right) + i\xi \cdot \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right) = 0. \tag{3.10}
$$

By abuse of notation, we write $\xi \cdot \mathcal{F}(\boldsymbol{E}_T|_{\Gamma_h})$ instead of $\xi \cdot \mathcal{F}(\tilde{\boldsymbol{E}}|_{\Gamma_h})$ and ignore that strictly speaking $\mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)$ has three components, because the third one is anyway zero. Consequently,

$$
i\xi\mathcal{F}\left(E_3|_{\Gamma_h}\right) = -\frac{i\xi}{\sqrt{k^2 - |\xi|^2}}\left(\xi \cdot \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)\right).
$$

Let us then introduce the operator $N^+_{k^2}$ defined for a tangential vector field $\boldsymbol{u}$ by

$$
N^+_{k^2}(\boldsymbol{u})(\tilde{x}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{i\xi}{\sqrt{k^2 - |\xi|^2}}(\xi \cdot \mathcal{F}\boldsymbol{u}(\xi)) \exp\left(\mathrm{i}\tilde{x} \cdot \xi\right) \mathrm{d}\xi. \tag{3.11}
$$

Then one can formally write

$$\operatorname{curl} \boldsymbol{E} \times \boldsymbol{e}_3 = T^+_{k^2}(\boldsymbol{E}_T|_{\Gamma_h}) + N^+_{k^2}(\boldsymbol{E}_T|_{\Gamma_h}). \tag{3.12}$$

Let us introduce a Hilbert subspace of $TH^{1/2}(\Gamma_h)$ (the space of tangential vector fields in $H^{1/2}(\Gamma_h)$),

$$T\breve{H}^{1/2}(\Gamma_h) = \Big\{ \boldsymbol{u} \in TH^{1/2}(\Gamma_h), \text{ s.t. } (\xi \cdot \mathcal{F}\boldsymbol{u}(\xi))/|k^2 - |\xi|^2|^{1/4} \in L^2(\mathbb{R}^2) \\ \text{and } |k^2 - |\xi|^2|^{1/4} \mathcal{F}\boldsymbol{u}(\xi) \in L^2(\mathbb{R}^2)^2 \Big\}, \tag{3.13}$$

equipped with the norm

$$\|\boldsymbol{u}\|^2_{T\breve{H}^{1/2}(\Gamma_h)} = \int_{\mathbb{R}^2} \left(\frac{1}{|k^2 - |\xi|^2|^{1/2}} |\xi \cdot \mathcal{F}(\boldsymbol{u}_T|_{\Gamma_h})(\xi)|^2 + |k^2 - |\xi|^2|^{1/2} |\mathcal{F}(\boldsymbol{u}_T|_{\Gamma_h})(\xi)|^2 \right) \mathrm{d}\xi.$$

For real $k^2 > 0$ the weight in this norm has a singularity. As one can easily deduce from expression (3.11) and working in the Fourier domain, $N^+_{k^2} : T\breve{H}^{1/2}(\Gamma_h) \to T\breve{H}^{1/2}(\Gamma_h)^*$ is continuous and satisfies

$$\operatorname{Re} \langle N^+_{k^2} \boldsymbol{u}, \boldsymbol{u} \rangle = -\int_{\mathbb{R}^2} \operatorname{Im} \left(\frac{1}{\sqrt{k^2 - |\xi|^2}} \right) |\xi \cdot \mathcal{F}\boldsymbol{u}(\xi)|^2 \, \mathrm{d}\xi \geq 0,$$
$$\operatorname{Im} \langle N^+_{k^2} \boldsymbol{u}, \boldsymbol{u} \rangle = \int_{\mathbb{R}^2} \operatorname{Re} \left(\frac{1}{\sqrt{k^2 - |\xi|^2}} \right) |\xi \cdot \mathcal{F}\boldsymbol{u}(\xi)|^2 \, \mathrm{d}\xi \geq 0 \quad \text{for } \boldsymbol{u} \in T\breve{H}^{1/2}(\Gamma_h).$$

For real k, it holds that

$$\operatorname{Re} \langle N^+_{k^2} \boldsymbol{u}, \boldsymbol{u} \rangle = \int_{|\xi| > k} \frac{1}{\sqrt{|\xi|^2 - k^2}} |\xi \cdot \mathcal{F}\boldsymbol{u}(\xi)|^2 \, \mathrm{d}\xi \geq 0,$$
$$\operatorname{Im} \langle N^+_{k^2} \boldsymbol{u}, \boldsymbol{u} \rangle = \int_{|\xi| < k} \frac{1}{\sqrt{k^2 - |\xi|^2}} |\xi \cdot \mathcal{F}\boldsymbol{u}(\xi)|^2 \, \mathrm{d}\xi \geq 0 \quad \text{for } \boldsymbol{u} \in T\breve{H}^{1/2}(\Gamma_h).$$

The variational problem associated with (2.3) is derived after substituting (3.12) in the boundary term on Γ_h appearing in (3.1). Due to the continuity properties of $T^+_{k^2}$ and $N^+_{k^2}$ the problem is set in the Hilbert space

$$X_{k^2} := \{ \boldsymbol{u} \in L^2(\Omega)^3, \operatorname{curl} \boldsymbol{u} \in L^2(\Omega)^3, \operatorname{div} \varepsilon_{\mathrm{r}} \boldsymbol{u} = 0 \text{ in } \Omega, \\ \boldsymbol{u} \times \boldsymbol{e}_3 = 0 \text{ on } \Gamma_0 \text{ and } \boldsymbol{u}_T|_{\Gamma_h} \in T\breve{H}^{1/2}(\Gamma_h) \}, \tag{3.14}$$

equipped with the norm

$$\|\boldsymbol{u}\|^2_{X_{k^2}} = \|\boldsymbol{u}\|^2_{L^2(\Omega)^3} + \|\operatorname{curl} \boldsymbol{u}\|^2_{L^2(\Omega)^3} \\ + \int_{\mathbb{R}^2} \left(\frac{1}{|k^2 - |\xi|^2|^{1/2}} |\xi \cdot \mathcal{F}(\boldsymbol{u}_T|_{\Gamma_h})(\xi)|^2 + |k^2 - |\xi|^2|^{1/2} |\mathcal{F}(\boldsymbol{u}_T|_{\Gamma_h})(\xi)|^2 \right) \mathrm{d}\xi.$$

The variational problem then reads as follows: Given

$$\boldsymbol{g} \in L^2_0(\mathrm{div}0, \Omega)^3 := \left\{ \boldsymbol{f} \in L^2(\Omega)^3, \operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega, \ \boldsymbol{f} \cdot \boldsymbol{e}_3 = 0 \text{ on } \Gamma_h \right\},$$

find $\boldsymbol{E} \in X_{k^2}$ such that

$$\int_\Omega \left(\operatorname{curl} \boldsymbol{E} \cdot \operatorname{curl} \overline{\boldsymbol{\psi}} - k^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} \cdot \overline{\boldsymbol{\psi}}\right) \mathrm{d}x - \left\langle T_{k^2}^+(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T \right\rangle - \left\langle N_{k^2}^+(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T \right\rangle = \int_\Omega \boldsymbol{g} \cdot \overline{\boldsymbol{\psi}} \,\mathrm{d}x \quad (3.15)$$

for all $\boldsymbol{\psi} \in X_{k^2}$. In the last equation we abbreviated $T_{k^2}^+(\boldsymbol{E}_T)$ and $N_{k^2}^+(\boldsymbol{E}_T)$ instead of writing $T_{k^2}^+(\boldsymbol{E}_T|_{\Gamma_h})$ and $N_{k^2}^+(\boldsymbol{E}_T|_{\Gamma_h})$, respectively, and we will for notational simplicity continue to do so in the sequel. We will consider problem (3.15) for $\varepsilon_{\mathrm{r}} \in W^{1,\infty}(\Omega)$ such that $\varepsilon_{\mathrm{r}} = 1$ in a neighborhood of Γ_h. Even though we are interested in real wave numbers we consider $k^2 \in \{z \in \mathbb{C},\, \mathrm{Re}(z) > 0,\, \mathrm{Im}(z) \geq 0\}$ for mathematical reasons: the solution for real k^2 will be found by a limiting absorption argument.

After manipulating the variational form, we need to investigate the relation between the variational and the original strong formulation of the problem. For that purpose we shall need the following lemma, where we make use of the space

$$\tilde{X}_{k^2} := \{\boldsymbol{u} \in L^2(\Omega)^3,\, \operatorname{curl} \boldsymbol{u} \in L^2(\Omega)^3,\, \boldsymbol{u} \times \boldsymbol{e}_3 = 0 \text{ on } \Gamma_0 \text{ and } \boldsymbol{u}_T|_{\Gamma_h} \in T\breve{H}^{1/2}(\Gamma_h)\}.$$

LEMMA 3.1. *Assume that $\varepsilon_{\mathrm{r}} \in L^\infty(\Omega)$ has a positive real part bounded away from zero and that $k \in \{z \in \mathbb{C},\, \mathrm{Re}(z) > 0,\, \mathrm{Im}(z) \geq 0\}$. Then any field $\tilde{\boldsymbol{\psi}} \in \tilde{X}_{k^2}$ can be decomposed in the form $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi} + \nabla p$ with $\boldsymbol{\psi} \in X_{k^2}$ and $p \in H_0^1(\Omega)$.*

Proof. Let $p \in H_0^1(\Omega)$ be the unique solution to

$$\int_\Omega \varepsilon_{\mathrm{r}} \nabla p \cdot \nabla \varphi \,\mathrm{d}x = \int_\Omega \varepsilon_{\mathrm{r}} \tilde{\boldsymbol{\psi}} \cdot \nabla \varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in H_0^1(\Omega).$$

The function $\boldsymbol{\psi} = \tilde{\boldsymbol{\psi}} - \nabla p$ belongs to $L^2(\Omega)^3$ and satisfies $\operatorname{curl} \boldsymbol{\psi} = \operatorname{curl} \tilde{\boldsymbol{\psi}}$. On the other hand, by construction of p, $\operatorname{div}(\varepsilon_{\mathrm{r}} \boldsymbol{\psi}) = 0$ and $\boldsymbol{\psi}_T = \tilde{\boldsymbol{\psi}}_T$ on $\partial\Omega$. Consequently $\boldsymbol{\psi} \in X_{k^2}$ and the lemma is proven. ∎

LEMMA 3.2. *Assume that $\varepsilon_{\mathrm{r}} \in L^\infty(\Omega)$ has a positive real part bounded away from zero and that $k \in \{z \in \mathbb{C},\, \mathrm{Re}(z) > 0,\, \mathrm{Im}(z) \geq 0\}$. A solution $\boldsymbol{E} \in X_{k^2}$ to (3.15) is a distributional solution of*

$$\operatorname{curl} \operatorname{curl} \boldsymbol{E} - k^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} = \boldsymbol{g} \quad \textit{in } \Omega, \quad \textit{and} \quad \boldsymbol{E} \times \boldsymbol{e}_3 = 0 \ \textit{on } \Gamma_0. \qquad (3.16)$$

The partial differential equation holds in $L^2(\Omega)^3$ and

$$\operatorname{curl} \boldsymbol{E} \times \boldsymbol{e}_3|_{\Gamma_h} = T_{k^2}^+(\boldsymbol{E}_T|_{\Gamma_h}) + N_{k^2}^+(\boldsymbol{E}_T|_{\Gamma_h}) \quad \textit{holds in } H^{-1/2}(\Gamma_h)^3. \qquad (3.17)$$

Proof. Using Lemma 3.1 and the observation that equation (3.15) is satisfied for $\boldsymbol{\psi} = \nabla p$ with $p \in H_0^1(\Omega)$ one deduces that equation (3.15) is satisfies for all $\boldsymbol{\psi} \in \tilde{X}_{k^2}$, i.e. without the divergence constraint. Then using $\boldsymbol{\psi} \in C_0^\infty(\Omega)$ the solution $\boldsymbol{E}$ to the variational formulation (3.15) satisfies the differential equation in the distributional sense. Since the right-hand side and the zeroth order term in (3.16) belong to $L^2(\Omega)^3$, $\operatorname{curl} \operatorname{curl} \boldsymbol{E}$ is also in $L^2(\Omega)^3$ and hence the equation holds in $L^2(\Omega)^3$. To obtain the boundary equation in (3.17) one can take arbitrary $C_0^\infty(\mathbb{R}^3)$ functions ψ_1 and ψ_2 such $\psi_1 = \psi_2 = 0$ on Γ_0 and consider the test function (without the divergence free condition) $\boldsymbol{\psi}^\eta$ such that $\psi_3^\eta = 0$ in Ω and the two other components are constructed in the Fourier domain as follows

$$\mathcal{F}(\psi_i^\eta)(\xi, x_3) = \chi((|\xi| - k)/\eta)\, \mathcal{F}(\psi_i)(\xi, x_3), \qquad i = 1, 2,$$

where χ is a $C^\infty(\mathbb{R})$ function vanishing in a neighborhood of 0 and equals 1 outside a bounded neighborhood of 0. Application of the Stokes formula, together with the equation satisfied by $\boldsymbol{E}$ inside Ω, shows that

$$\langle \operatorname{curl} \boldsymbol{E} \times \boldsymbol{e}_3,\, \boldsymbol{\psi}_T^\eta \rangle - \langle T_{k^2}^+(\boldsymbol{E}_T) + N_{k^2}^+(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T^\eta \rangle = 0.$$

Then one concludes by letting $\eta \to 0$, since $\boldsymbol{\psi}_T^\eta|_{\Gamma_h} \to \boldsymbol{\psi}_T|_{\Gamma_h}$ in any Sobolev norm. □

The next lemma shows that a solution to the variational problem (3.15) is not only a weak solution of Maxwell's equations (2.3) but also satisfies the upwards radiation condition (2.4).

COROLLARY 3.3. *Assume that* $\varepsilon_{\rm r} \in L^\infty(\Omega)$ *has a positive real part bounded away from zero, that* $k^2 \in \{z \in \mathbb{C},\, \mathrm{Re}(z) > 0,\, \mathrm{Im}(z) \geq 0\}$*, and let* $\boldsymbol{E} \in X_{k^2}$ *be a solution to* (3.15)*. Then* $\boldsymbol{E}$ *extended to* $\{x \in \mathbb{R}^3,\, x_3 > h\}$ *by*

$$\begin{cases} \boldsymbol{E}_T^+(x) = \displaystyle\int_{\mathbb{R}^2} e^{\mathrm{i}\left((x_3-h)\sqrt{k^2-|\xi|^2}+\tilde{x}\cdot\xi\right)} \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)(\xi)\,\mathrm{d}\xi \\ E_3^+(x) = \displaystyle\int_{\mathbb{R}^2} e^{\mathrm{i}\left((x_3-h)\sqrt{k^2-|\xi|^2}+\tilde{x}\cdot\xi\right)} \left(\xi \cdot \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)(\xi)\right)/\sqrt{k^2-|\xi|^2}\,\mathrm{d}\xi \end{cases} \tag{3.18}$$

is in $X_{k^2} \cap H^1(\{h-\eta < x_3 < H\})^3$ *for all* $H > h$ *and satisfies*

$$\operatorname{curl}\operatorname{curl} \boldsymbol{E} - k^2 \varepsilon_{\rm r} \boldsymbol{E} = \boldsymbol{g} \quad \text{in } \mathbb{R}^3_+, \quad \text{and} \quad \boldsymbol{E} \times \boldsymbol{e}_3 = 0 \text{ on } \Gamma_0. \tag{3.19}$$

Proof. Let us first check that $\boldsymbol{E}^+$ is well defined and belongs to $H^1(\{h < x_3 < H\})^3$ for all $H > h$. Since $\boldsymbol{E}_T \in TH^{1/2}(\Gamma_h)$, using [8, Lemma 2.2] one obtains that $\boldsymbol{E}_T^+ \in H^1(\{h < x_3 < H\})^2$ for all $H > h$. Let χ_M be the characteristic function of a set M and let us set

$$f_1(\xi) := \left(\xi \cdot \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)(\xi)\right)/\sqrt{k^2-|\xi|^2}\chi_{|\xi|\leq 2k},$$
$$f_2(\xi) := \left(\xi \cdot \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)(\xi)\right)/\sqrt{k^2-|\xi|^2}\chi_{|\xi|\geq 2k},$$

so that

$$E_3^+(x) = \int_{\mathbb{R}^2} \exp\left(\mathrm{i}\left((x_3-h)\sqrt{k^2-|\xi|^2}+\tilde{x}\cdot\xi\right)\right)(f_1(\xi)+f_2(\xi))\,\mathrm{d}\xi =: E_3^{+,1}(x) + E_3^{+,2}(x).$$

One can easily check, using the Cauchy-Schwartz inequality, that $|\xi|^m f_1(\xi)$ is in $L^1(\mathbb{R}^2)$ for any integer m and therefore $E_3^{+,1} \in C^\infty(\{x_3 \geq h\})$. On the other hand $(1+|\xi|^{1/2})f_2(\xi) \in L^2(\mathbb{R}^2)$ and therefore $E_3^{+,1} \in H^1(\{h < x_3 < H\})$, $H > h$. We then conclude that $E_3^+ \in H^1(\{h < x_3 < H\})$. By taking the Fourier transform with respect to $\tilde{x}$ one checks that

$$\begin{cases} \operatorname{curl}\operatorname{curl} \boldsymbol{E}^+ - k^2 \boldsymbol{E}^+ = 0 \quad \text{in } \{x_3 > h\} \\ \boldsymbol{E}^+|_{\Gamma_h} \times \boldsymbol{e}_3 = \boldsymbol{E}|_{\Gamma_h} \times \boldsymbol{e}_3. \end{cases} \tag{3.20}$$

Using (3.17) one also checks (in the Fourier domain) that

$$\operatorname{curl} \boldsymbol{E}^+\big|_{\Gamma_h} \times \boldsymbol{e}_3 = \operatorname{curl} \boldsymbol{E}|_{\Gamma_h} \times \boldsymbol{e}_3. \tag{3.21}$$

It is then the a classical exercise (using the Stokes formula) to verify that (3.16), (3.20) and (3.21) combined with the local $H(\mathrm{curl})$ regularity imply that (3.19) is verified. In turn, (3.19) implies that $\operatorname{div} E = 0$ in $\{x_3 > h - \eta\}$ for $H > h$ and $\boldsymbol{E} \in H^1(\{h-\eta < x_3 < H\})^3$. □

Remark 3.4. *The corollary's statement that the extension by* (3.18) *belongs to* $H^1(\{h-\eta < x_3 < H\})$ *for* $H > h$ *implies that* $\left(\xi \cdot \mathcal{F}\left(\boldsymbol{E}_T|_{\Gamma_h}\right)(\xi)\right)/\sqrt{k^2-|\xi|^2}$ *is the inverse Fourier transform of* E_3*, due to the second equation in* (3.18)*. This shows that a solution* $\boldsymbol{E}$ *to* (3.15) *indeed satisfies the angular spectrum representation* (2.4).

In particular, if the variational problem (3.15) *has a unique solution, then this solution is independent of the truncation parameter* h *(chosen large enough for not touching the rough layer).*

The central technique to prove solvability of the variational problem (3.15) is a suitable Rellich identity for a solution of Maxwell's equations. This tool uses integrations by parts requiring additional regularity.

Lemma 3.5. *Assume that* $\varepsilon_{\mathrm{r}} \in W^{1,\infty}(\Omega)$ *has a positive real part bounded away from zero and let* $\boldsymbol{E} \in X_{k^2}$ *be a solution to* (3.15)*. Then* $\boldsymbol{E} \in H^1(\Omega)^3 \cap H^2(\{h-\eta/2 < x_3 < h\})^3$*. If* $\varepsilon_{\mathrm{r}} \in W^{2,\infty}(\Omega)$*, then* $\boldsymbol{E} \in H^2(\Omega)^3$*.*

Proof. Assume that $\varepsilon_{\mathrm{r}} \in W^{1,\infty}(\Omega)$. We shall first prove that $\boldsymbol{E} \in H^1(\Omega)^3$. Let $\boldsymbol{j} \in \mathbb{Z}^2$ and set $\Omega^1_{\boldsymbol{j}} := \{x \in \Omega, |x_1 - j_1| \le 1, |x_2 - j_2| \le 1\}$ and $\Omega^2_{\boldsymbol{j}} := \{x \in \Omega, |x_1 - j_1| \le 2, |x_2 - j_2| \le 2\}$. Let χ be a $C^\infty(\overline{\Omega})$ function such that $\chi = 1$ in $\Omega^1_{\boldsymbol{0}}$ and $\chi = 0$ outside $\Omega^2_{\boldsymbol{0}}$. Considering the function $\boldsymbol{E}_{\boldsymbol{j}} = \chi(\tilde{x} - \boldsymbol{j})\boldsymbol{E}$ and the fact that $\operatorname{div} \varepsilon_{\mathrm{r}} \boldsymbol{E} = 0$ in Ω and that the tangential components $\boldsymbol{E}_{\boldsymbol{j}T}$ of $\boldsymbol{E}_{\boldsymbol{j}}$ belong to $H^{1/2}(\partial\Omega^2_{\boldsymbol{j}})$ we get from standard results on $H(\mathrm{curl})$ spaces that $\boldsymbol{E}_{\boldsymbol{j}} \in H^1(\Omega^2_{\boldsymbol{j}})$ and that there exists a constant C such that

$$\|\boldsymbol{E}_{\boldsymbol{j}}\|^2_{H^1(\Omega^2_{\boldsymbol{j}})} \le C(\|\boldsymbol{E}_{\boldsymbol{j}}\|^2_{L^2(\Omega^2_{\boldsymbol{j}})} + \|\operatorname{curl} \boldsymbol{E}_{\boldsymbol{j}}\|^2_{L^2(\Omega^2_{\boldsymbol{j}})} + \|\boldsymbol{E}_{\boldsymbol{j}T}\|^2_{H^{1/2}(\partial\Omega^2_{\boldsymbol{j}})}).$$

The constant C depends only on χ and $\|\varepsilon_{\mathrm{r}}\|_{W^{1,\infty}}$ and is independent from $\boldsymbol{j}$ due to the invariance by translation of the norms. Setting $\Gamma^{\boldsymbol{j}}_h := \Gamma_h \cap \partial\Omega^2_{\boldsymbol{j}}$ we have that $\boldsymbol{E}_{\boldsymbol{j}T} = 0$ on $\partial\Omega^2_{\boldsymbol{j}} \setminus \Gamma^{\boldsymbol{j}}_h$. Using the definition of $\boldsymbol{E}_{\boldsymbol{j}}$ as well as the differentiation rule $\operatorname{curl} \chi \boldsymbol{u} = \nabla\chi \times \boldsymbol{u} + \chi \operatorname{curl} \boldsymbol{u}$ we deduce that

$$\|\boldsymbol{E}\|^2_{H^1(\Omega^1_{\boldsymbol{j}})} \le \tilde{C}(\|\boldsymbol{E}\|^2_{L^2(\Omega^2_{\boldsymbol{j}})} + \|\operatorname{curl} \boldsymbol{E}\|^2_{L^2(\Omega^2_{\boldsymbol{j}})} + \|\boldsymbol{E}_T\|^2_{H^{1/2}(\Gamma^{\boldsymbol{j}}_h)}).$$

where the again constant $\tilde{C}$ is independent from $\boldsymbol{j}$. Summing over $\boldsymbol{j} \in \mathbb{Z}^2$ shows that

$$\|\boldsymbol{E}\|^2_{H^1(\Omega)} \le 4\tilde{C}(\|\boldsymbol{E}\|^2_{L^2(\Omega)} + \|\operatorname{curl} \boldsymbol{E}\|^2_{L^2(\Omega)} + \|\boldsymbol{E}_T\|^2_{H^{1/2}(\Gamma_h)}).$$

We now prove that $\boldsymbol{E} \in H^2(\{h-\eta/2 < x_3 < h\})$. Set $U_1 = \{h - \eta/2 < x_3 < h\}$. and $U_2 = \{h - \eta < x_3 < h + \eta/2\}$. We extend $\boldsymbol{E} \in H^1(\Omega)^3$ to U_2 as in Corollary 3.3 and we denote the extension still by $\boldsymbol{E}$. We first remark that according to that Lemma the extension is continuous across Γ_h. We therefore deduce that $\boldsymbol{E}$ can also be represented in $\{x_3 \ge h\}$ by (2.4). From the first part one has that $\boldsymbol{E}|_{\Gamma_h} \in H^{1/2}(\Gamma_h)^3$. Consequently, using [8, Equation (2.19)] implies that $\boldsymbol{E} \in H^1(U_2)^3$. Define further

$$U^{\boldsymbol{j}}_1 = \{x \in U_1, |x_1 - j_1| < 1, |x_2 - j_2| < 1\}, \quad \text{and} \quad U^{\boldsymbol{j}}_2 = \{x \in U_2, |x_1 - j_1| < 2, |x_2 - j_2| < 2\}, \quad \boldsymbol{j} \in \mathbb{Z}^2.$$

Since $\boldsymbol{E} \in H^1(U_2)$ solves an inhomogeneous vector Helmholtz equation $\Delta\boldsymbol{E} + k_+^2\boldsymbol{E} = \boldsymbol{g}$ with constant coefficients, elliptic regularity results [16] yield

$$\|\boldsymbol{E}\|^2_{H^2(U^{\boldsymbol{j}}_1)} \le C\left(\|\boldsymbol{E}\|^2_{H^1(U^{\boldsymbol{j}}_2)} + \|\boldsymbol{g}\|^2_{L^2(U^{\boldsymbol{j}}_2)}\right), \quad \boldsymbol{j} \in \mathbb{Z}, \tag{3.22}$$

with a constant C independent of $\boldsymbol{j}$ since the geometry of $U^{\boldsymbol{j}}_{1,2}$ is independent of $\boldsymbol{j}$. Summation over $\boldsymbol{j} \in \mathbb{Z}^2$ shows that $\|\boldsymbol{E}\|_{H^2(U_1)}$ is bounded.

Assume now that $\varepsilon_{\rm r} \in W^{2,\infty}(\Omega)$. We proceed as above with $U_1 = \{\eta/2 < x_3 < h - \eta/2\}$ and $U_2 = \Omega$. Elliptic regularity implies that there exists a constant C that only depends on h, η and $\|\varepsilon_{\rm r}\|_{W^{2,\infty}}$ such that (3.22) still holds. We then conclude, using part (a), that $\|\boldsymbol{E}\|_{H^2(U_1)}$ is finite. Combining with part (a) we obtain that $\boldsymbol{E} \in H^2(\Omega)^3$. ▯

4. Technical Lemmas. We first prove some auxiliary lemmas that will be useful in establishing integral identities for Maxwell's equations. The next lemma recalls integration by parts formulas in the unbounded domain Ω.

LEMMA 4.1.

(a) For $\boldsymbol{u}, \boldsymbol{v} \in H(\operatorname{curl}, \Omega)$ *there holds* $\int_\Omega \operatorname{curl} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} \, \mathrm{d}x + \int_{\partial\Omega} (\nu \times \boldsymbol{u})\, \boldsymbol{v}_T \, \mathrm{d}s$.

(b) For $\phi, \psi \in H^1(\Omega)$ *there holds* $\int_\Omega \phi \, \partial\psi/\partial x_j \, \mathrm{d}x = -\int_\Omega \partial\phi/\partial x_j \psi \, \mathrm{d}x + \int_{\partial\Omega} \nu_j \phi \, \psi \, \mathrm{d}s$, $j = 1, 2, 3$.

(c) For $\phi \in W^{1,\infty}(\Omega)$ *and* $\psi \in H^1(\Omega)$ *there holds* $\int_\Omega \phi \, \partial|\psi|^2/\partial x_3 \, \mathrm{d}x = -\int_\Omega \partial\phi/\partial x_3 |\psi|^2 \, \mathrm{d}x + \int_{\partial\Omega} \nu_3 \phi \, |\psi|^2 \, \mathrm{d}s$.

(d) For $\phi, \psi \in H^{1/2}(\Gamma_h)$ *there holds* $\langle \phi, \, \partial\psi/\partial x_j \rangle = -\langle \partial\phi/\partial x_j, \, \psi \rangle$, $j = 1, 2$.

Proof. (a) We introduce a smooth cut-off function χ_A in the $\tilde{x}$ variables such that $\chi(\tilde{x}) = 1$ for $|\tilde{x}| < A$, $\chi(\tilde{x}) = 0$ for $|\tilde{x}| > A + 1$ and $0 \le \tilde{x} \le 1$. By choosing for instance a radial function $\chi_A(x) = \chi_A(|\tilde{x}|)$, one observes that we can furthermore impose that the maximum norm of the gradient of χ_A is uniformly bounded in $A > 0$. The integration by parts formula in Lipschitz domains [17, Theorem 3.31] implies

$$\int_\Omega \chi_A \operatorname{curl} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \boldsymbol{u} \cdot \operatorname{curl} \chi_A \boldsymbol{v} \, \mathrm{d}x + \int_{\partial\Omega} \chi_A (\nu \times \boldsymbol{u}) \cdot \boldsymbol{v}_T \, \mathrm{d}s$$
$$= \int_\Omega \chi_A \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} \, \mathrm{d}x + \int_\Omega \boldsymbol{u} \cdot (\nabla \chi_A \times \boldsymbol{v}) \, \mathrm{d}x + \int_{\partial\Omega} \chi_A (\nu \times \boldsymbol{u}) \cdot \boldsymbol{v}_T \, \mathrm{d}s.$$

By Lebesgue's dominated convergence theorem, $\int_\Omega \chi_A \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} \, \mathrm{d}x \to \int_\Omega \boldsymbol{u} \cdot \operatorname{curl} \boldsymbol{v} \, \mathrm{d}x$ as $A \to \infty$ and $\int_{\partial\Omega} \chi_A (\nu \times \boldsymbol{u}) \cdot \boldsymbol{v}_T \, \mathrm{d}s \to \int_{\partial\Omega} (\nu \times \boldsymbol{u}) \cdot \boldsymbol{v}_T \, \mathrm{d}s$ as $A \to \infty$ by continuity of the linear form $H(\operatorname{curl}, \Omega) \ni \psi \mapsto \int_{\partial\Omega} (\nu \times \boldsymbol{u}) \cdot \psi_T \, \mathrm{d}s$. Moreover,

$$\left| \int_\Omega \boldsymbol{u} \cdot (\nabla \chi_A \times \boldsymbol{v}) \, \mathrm{d}x \right| \le 2 \|\nabla \chi_A\|_{L^\infty(\Omega)} \int_{A < |x| < A+1} |\boldsymbol{u}| \, |\boldsymbol{v}| \, \mathrm{d}x \to 0 \quad (A \to \infty)$$

since $\boldsymbol{u}, \boldsymbol{v} \in L^2(\Omega)^3$. Parts (b) and (c) follow in a similar way and (d) can, for instance, be shown using Plancherel's theorem. ▯

LEMMA 4.2. *Assume that* $\varepsilon_{\rm r} \in W^{1,\infty}(\Omega)$ *has a positive real part bounded away from zero. Then the following identity holds for all* $\boldsymbol{E} \in H^2(\Omega)^3$*:*

$$2 \operatorname{Re} \int_\Omega \left(\boldsymbol{e}_3 \times \frac{\partial \boldsymbol{E}}{\partial x_3} \right) \cdot \operatorname{curl} \overline{\boldsymbol{E}} \, \mathrm{d}x = 2 \int_\Omega \left| \frac{\partial \boldsymbol{E}}{\partial x_3} \right|^2 \mathrm{d}x + 2 \operatorname{Re} \int_\Omega \frac{\nabla_T \varepsilon_{\rm r}}{\varepsilon_{\rm r}} \cdot \boldsymbol{E}_T \frac{\partial \overline{E}_3}{\partial x_3} \, \mathrm{d}x$$
$$+ \int_\Omega \frac{\partial \log(|\varepsilon_{\rm r}|)}{\partial x_3} \frac{\partial |E_3|^2}{\partial x_3} \, \mathrm{d}x - 2 \int_\Omega \operatorname{Im} \left(\frac{\partial \log(\varepsilon_{\rm r})}{\partial x_3} \right) \operatorname{Im} \left(\frac{\partial \overline{E}_3}{\partial x_3} E_3 \right) \mathrm{d}x - 2 \operatorname{Re} \int_\Omega \frac{\operatorname{div}(\varepsilon_{\rm r} \boldsymbol{E})}{\varepsilon_{\rm r}} \frac{\partial \overline{E}_3}{\partial x_3} \, \mathrm{d}x$$
$$- 2 \operatorname{Re} \int_{\Gamma_h} \left(\frac{\partial E_3}{\partial x_3} - \operatorname{div} \boldsymbol{E} \right) \overline{E_3} \, \mathrm{d}s - 2 \operatorname{Re} \int_{\Gamma_0} \operatorname{div}_T E_T \overline{E_3} \, \mathrm{d}s. \quad (4.1)$$

Proof. Using (3.6) one has

$$2 \operatorname{Re} \int_\Omega \left(\boldsymbol{e}_3 \times \frac{\partial \boldsymbol{E}}{\partial x_3} \right) \cdot \operatorname{curl} \overline{\boldsymbol{E}} \, \mathrm{d}x = 2 \int_\Omega \left| \frac{\partial \boldsymbol{E}_T}{\partial x_3} \right|^2 \mathrm{d}x - 2 \operatorname{Re} \int_\Omega \frac{\partial \boldsymbol{E}_T}{\partial x_3} \cdot \nabla_T \overline{E_3} \, \mathrm{d}x.$$

We observe that

$$
\begin{aligned}
-2\,\mathrm{Re}\int_\Omega \frac{\partial \boldsymbol{E}_T}{\partial x_3}\cdot \nabla_T \overline{E_3}\ \mathrm{d}x &= 2\,\mathrm{Re}\int_\Omega \left(\frac{\partial}{\partial x_3}\mathrm{div}_T \boldsymbol{E}_T\right)\overline{E_3}\ \mathrm{d}x \\
&= -2\,\mathrm{Re}\int_\Omega \mathrm{div}_T \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\ \mathrm{d}x + 2\,\mathrm{Re}\int_{\Gamma_h}\mathrm{div}_T \boldsymbol{E}_T \overline{E_3}\,\mathrm{d}s - 2\,\mathrm{Re}\int_{\Gamma_0}\mathrm{div}_T \boldsymbol{E}_T \overline{E_3}\,\mathrm{d}s.
\end{aligned}
$$

Using the identity

$$
\mathrm{div}_T \boldsymbol{E}_T = -\partial E_3/\partial x_3 - \nabla(\varepsilon_\mathrm{r})/\varepsilon_\mathrm{r}\cdot \boldsymbol{E} + \frac{\mathrm{div}(\varepsilon_\mathrm{r}\boldsymbol{E})}{\varepsilon_\mathrm{r}}
$$

in combination with the fact that ε_r is constant in a neighborhood of Γ_h, we find that

$$
\int_{\Gamma_h}\mathrm{div}_T \boldsymbol{E}_T \overline{E_3}\,\mathrm{d}s = -\int_{\Gamma_h}\left(\frac{\partial E_3}{\partial x_3} - \mathrm{div}\,\boldsymbol{E}\right)\overline{E_3}\ \mathrm{d}s.
$$

We also get

$$
-\ 2\,\mathrm{Re}\int_\Omega \mathrm{div}_T \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\ \mathrm{d}x \ = \ 2\,\mathrm{Re}\int_\Omega \left(\left|\frac{\partial E_3}{\partial x_3}\right|^2 + \frac{\nabla\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x - \frac{\mathrm{div}(\varepsilon_\mathrm{r}\boldsymbol{E})}{\varepsilon_\mathrm{r}}\,\frac{\partial \overline{E_3}}{\partial x_3}\right)\mathrm{d}x.
$$

Furthermore,

$$
\mathrm{Re}\left(\frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}E_3\,\frac{\partial \overline{E_3}}{\partial x_3}\right) = \frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\,\mathrm{Re}\left(\frac{\partial \overline{E_3}}{\partial x_3}E_3\right) - \mathrm{Im}\left(\frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\,\mathrm{Im}\left(\frac{\partial \overline{E_3}}{\partial x_3}E_3\right)
$$

and we conclude the proof by noting that

$$
\begin{aligned}
&2\,\mathrm{Re}\int_\Omega \frac{\nabla\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x = 2\,\mathrm{Re}\int_\Omega \frac{\nabla_T\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega \frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}E_3\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x = \\
&2\,\mathrm{Re}\int_\Omega \frac{\nabla_T\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x + \int_\Omega \frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\,\frac{\partial |E_3|^2}{\partial x_3}\,\mathrm{d}x - 2\int_\Omega \mathrm{Im}\left(\frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\,\mathrm{Im}\left(\frac{\partial \overline{E_3}}{\partial x_3}E_3\right)\,\mathrm{d}x.
\end{aligned}
$$

□

LEMMA 4.3. *Assume that* $\boldsymbol{E}\in H^2(\Omega)^3$. *Then*

$$
\begin{aligned}
2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot \mathrm{curl}\ \mathrm{curl}\,\overline{\boldsymbol{E}}\,\mathrm{d}x &= -\int_\Omega |\mathrm{curl}\,\boldsymbol{E}|^2\ \mathrm{d}x + h\int_{\Gamma_h}|\mathrm{curl}\,\boldsymbol{E}|^2\ \mathrm{d}s \\
&+ 2\,\mathrm{Re}\int_\Omega\left(\boldsymbol{e}_3\times\frac{\partial \boldsymbol{E}}{\partial x_3}\right)\cdot \mathrm{curl}\,\overline{\boldsymbol{E}}\,\mathrm{d}x + 2h\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial \boldsymbol{E}_T}{\partial x_3}\cdot(\boldsymbol{e}_3\times \mathrm{curl}\,\overline{\boldsymbol{E}})\,\mathrm{d}s. \qquad (4.2)
\end{aligned}
$$

Proof. Denote by ν the outward unit normal to Ω. Two integrations by parts imply that

$$
\begin{aligned}
&2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot \operatorname{curl}\operatorname{curl}\overline{\boldsymbol{E}}\,\mathrm{d}x\\
&\quad=2\,\mathrm{Re}\int_\Omega \operatorname{curl}\left(x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\right)\cdot\operatorname{curl}\overline{\boldsymbol{E}}\,\mathrm{d}x+2\,\mathrm{Re}\int_{\partial\Omega}x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot(\nu\times\operatorname{curl}\overline{\boldsymbol{E}})\,\mathrm{d}s\\
&\quad=\int_\Omega x_3\frac{\partial}{\partial x_3}\left|\operatorname{curl}\boldsymbol{E}\right|^2\,\mathrm{d}x+2\,\mathrm{Re}\int_\Omega\left(\boldsymbol{e}_3\times\frac{\partial \boldsymbol{E}}{\partial x_3}\right)\cdot\operatorname{curl}\overline{\boldsymbol{E}}\,\mathrm{d}x\\
&\qquad\qquad+2\,\mathrm{Re}\int_{\partial\Omega}x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot(\nu\times\operatorname{curl}\overline{\boldsymbol{E}})\,\mathrm{d}s\\
&\quad=-\int_\Omega\left|\operatorname{curl}\boldsymbol{E}\right|^2\,\mathrm{d}x+\int_{\partial\Omega}x_3\left|\operatorname{curl}\boldsymbol{E}\right|^2(\nu\cdot\boldsymbol{e}_3)\,\mathrm{d}s\\
&\qquad+2\,\mathrm{Re}\int_\Omega\left(\boldsymbol{e}_3\times\frac{\partial \boldsymbol{E}}{\partial x_3}\right)\cdot\operatorname{curl}\overline{\boldsymbol{E}}\,\mathrm{d}x+2\,\mathrm{Re}\int_{\partial\Omega}x_3\frac{\partial \boldsymbol{E}_T}{\partial x_3}\cdot(\nu\times\operatorname{curl}\overline{\boldsymbol{E}})\,\mathrm{d}s.
\end{aligned}
$$

One then concludes by noticing that $x_3=0$ on Γ_0 and $x_3=h$ and $\nu=\boldsymbol{e}_3$ on Γ_h. ◻

LEMMA 4.4. *Assume that $\varepsilon_{\mathrm{r}}\in W^{1,\infty}(\Omega)$ has a positive real part bounded away from zero. Then any solution $\boldsymbol{E}\in X_{k^2}$ to* (3.15) *satisfies the identity* (4.2) *and, moreover,*

$$
\begin{aligned}
2\,\mathrm{Re}\int_\Omega\left(\boldsymbol{e}_3\times\frac{\partial \boldsymbol{E}}{\partial x_3}\right)\cdot\operatorname{curl}\overline{\boldsymbol{E}}\,\mathrm{d}x&=2\int_\Omega\left|\frac{\partial \boldsymbol{E}}{\partial x_3}\right|^2\mathrm{d}x+2\,\mathrm{Re}\int_\Omega\frac{\nabla_T\varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}}\cdot\boldsymbol{E}_T\,\frac{\partial\overline{E_3}}{\partial x_3}\,\mathrm{d}x \qquad (4.3)\\
+\int_\Omega\frac{\partial\log(|\varepsilon_{\mathrm{r}}|)}{\partial x_3}\,\frac{\partial|E_3|^2}{\partial x_3}\,\mathrm{d}x&-2\int_\Omega\mathrm{Im}\left(\frac{\partial\log(\varepsilon_{\mathrm{r}})}{\partial x_3}\right)\mathrm{Im}\left(\frac{\partial\overline{E_3}}{\partial x_3}E_3\right)\mathrm{d}x-2\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial E_3}{\partial x_3}\,\overline{E_3}\,\mathrm{d}s.
\end{aligned}
$$

Proof. Let $\theta\in C^\infty(\mathbb{R}^3)$ be a smooth and non-negative function with support in the unit ball and $\int_{\mathbb{R}^3}\theta\,\mathrm{d}x=1$. For $\delta>0$ and $x\in\mathbb{R}^3$ let $\theta^\delta(x):=\delta^{-3}\theta(x/\delta)$. Let $\boldsymbol{E}\in X_{k^2}$ be a solution to (3.15) and note that Lemma 3.5 states that $\boldsymbol{E}\in H^1(\Omega)^3\cap H^2(\{h-\eta/2<x_3<h\})^3$. We extend $\boldsymbol{E}$ to a function defined in all of $\mathbb{R}^3$ so that the extension belongs to $H^1(\{x_3<h\})^3\cap H^2(\{x_3>h-\eta/2\})^3$ (see, e.g., [20] for a suitable extension operator for the half-space). By abuse of notation, the extended function is still denoted by $\boldsymbol{E}$. The convolution $\boldsymbol{E}^\delta:=\theta^\delta*\boldsymbol{E}$ belongs to $H^2(\Omega)$ and therefore satisfies identity (4.2). Moreover, $\boldsymbol{E}^\delta\to\boldsymbol{E}$ in $H^1(\Omega)^3\cap H^2(\{h-\eta/2<x_3<h\})^3$ and $\operatorname{curl}\operatorname{curl}\boldsymbol{E}^\delta\to\operatorname{curl}\operatorname{curl}\boldsymbol{E}$ in $L^2(\Omega)^3$ due to Lemma 3.2. Convergence in $H^2(\{h-\eta/2<x_3<h\})^3$ in particular implies that $\operatorname{curl}\boldsymbol{E}^\delta\to\operatorname{curl}\boldsymbol{E}$ in $L^2(\Gamma_h)^3$. Consequently, taking the limit as $\delta\to0$ implies that $\boldsymbol{E}$ also satisfies (4.2).

The smoothed function $\boldsymbol{E}^\delta$ also satisfies the identity (4.1) and again we consider the limit of this identity as $\delta\to0$. Since $\varepsilon_{\mathrm{r}}\in W^{1,\infty}(\Omega)$ it holds that $\operatorname{div}(\varepsilon_{\mathrm{r}}\boldsymbol{E}^\delta)\to\operatorname{div}(\varepsilon_{\mathrm{r}}\boldsymbol{E})=0$ in $L^2(\Omega)$. Moreover, $\operatorname{div}_T\boldsymbol{E}_T^\delta\to\operatorname{div}_T\boldsymbol{E}_T=0$ in $H^{-1/2}(\Gamma_0)$ holds due to the convergence in $H^1(\Omega)^3$ of $\boldsymbol{E}^\delta$ to $\boldsymbol{E}$ and since $\boldsymbol{E}_T=0$ on Γ_0. Convergence of $\boldsymbol{E}^\delta$ to $\boldsymbol{E}$ in $H^2(\{h-\eta/2<x_3<h\})^3$ implies that $\partial\boldsymbol{E}^\delta/\partial x_3\to\partial\boldsymbol{E}/\partial x_3$ in $H^{1/2}(\Gamma_h)$. Taking the limit as $\delta\to0$ yields (4.3). ◻

5. Inequalities Resulting from Rellich Identities. Various Rellich identities for the Helmholtz equation [3, 8, 15] motivate to multiply the Maxwell's equations (2.3) by $x_3\,\partial\boldsymbol{E}/\partial x_3$ and to integrate by parts to obtain an integral identity linking $|\partial\boldsymbol{E}/\partial x_3|^2$ with the right-hand side of the variational formulation. For a limiting absorption argument we need to establish this identity for solutions to a scattering problem with (artificial) complex wave number $k_\alpha\in\{z\in\mathbb{C},\,\mathrm{Re}(z)>0,\,\mathrm{Im}(z)\geq0\}$. For simplicity, we set

$$k_\alpha^2:=k^2+\mathrm{i}\alpha\quad\text{for }\alpha\in[0,1].$$

It is crucial that the estimates resulting from the Rellich identity are uniform in $\alpha \in [0,1]$. Note that Proposition 5.1 below only contains the estimate resulting from the Rellich identity. The identity itself appears in equation (5.7) in the proposition's proof.

We consider $\boldsymbol{E} \in X_{k_\alpha^2}$, a solution to

$$\int_\Omega \left(\operatorname{curl} \boldsymbol{E} \cdot \operatorname{curl} \overline{\boldsymbol{\psi}} - k_\alpha^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} \cdot \overline{\boldsymbol{\psi}}\right) \mathrm{d}x - \left\langle T^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T \right\rangle - \left\langle N^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T \right\rangle = \int_\Omega \boldsymbol{g} \cdot \overline{\boldsymbol{\psi}} \,\mathrm{d}x \quad (5.1)$$

for all $\boldsymbol{\psi} \in X_{k_\alpha^2}$.

Proposition 5.1. *Assume that $\varepsilon_{\mathrm{r}} \in W^{1,\infty}(\Omega)$ has a positive real part bounded away from zero and a non negative imaginary part and that $\boldsymbol{g} \in L^2_0(\operatorname{div}0, \Omega)^3$. Let $k > 0$ and $\alpha \geq 0$, and set $k_\alpha^2 = k^2 + \mathrm{i}\alpha$. Assume that $\boldsymbol{E} \in X_{k_\alpha^2}$ is a solution to* (5.1)*. Then it holds*

$$\begin{aligned}
&\int_\Omega \left(2\left|\frac{\partial \boldsymbol{E}}{\partial x_3}\right|^2 + k^2 x_3 \frac{\partial \operatorname{Re}(\varepsilon_{\mathrm{r}})}{\partial x_3} |\boldsymbol{E}|^2 + \frac{\partial \log(|\varepsilon_{\mathrm{r}}|)}{\partial x_3} \frac{\partial |E_3|^2}{\partial x_3} - 2 \operatorname{Im}\left(\frac{\partial \log(\varepsilon_{\mathrm{r}})}{\partial x_3}\right) \operatorname{Im}\left(\frac{\partial \overline{E_3}}{\partial x_3} E_3\right)\right) \mathrm{d}x \\
&\qquad + 2 \operatorname{Re} \int_\Omega \frac{\nabla_T \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \cdot \boldsymbol{E}_T \frac{\partial \overline{E}_3}{\partial x_3} \,\mathrm{d}x + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi \\
&\leq \left(2h \left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|_{L^2(\Omega)^3} + (1 + 2\sqrt{2}kh) \|\boldsymbol{E}\|_{L^2(\Omega)^3} \right) \|\tilde{\boldsymbol{g}}\|_{L^2(\Omega)^3} + \sqrt{2\alpha} h \|\boldsymbol{E}\|_{L^2(\Omega)^3} \|\boldsymbol{g}\|_{L^2(\Omega)^3} \quad (5.2)
\end{aligned}$$

where we have set $\tilde{\boldsymbol{g}} := \boldsymbol{g} + \mathrm{i}\alpha\varepsilon_{\mathrm{r}}\boldsymbol{E} + k_\alpha^2 \operatorname{Im}(\varepsilon_{\mathrm{r}})\boldsymbol{E}$. Moreover,

$$\int_\Omega k^2 \operatorname{Im}(\varepsilon_{\mathrm{r}}) |\boldsymbol{E}|^2 \,\mathrm{d}x \leq \|\boldsymbol{g}\|_{L^2(\Omega)^3} \|\boldsymbol{E}\|_{L^2(\Omega)^3}. \quad (5.3)$$

Proof. From the variational formulation (5.1) we infer that

$$\int_\Omega \left(\operatorname{curl} \boldsymbol{E} \cdot \operatorname{curl} \overline{\boldsymbol{\psi}} - k^2 \operatorname{Re}(\varepsilon_{\mathrm{r}}) \boldsymbol{E} \cdot \overline{\boldsymbol{\psi}}\right) \mathrm{d}x - \left\langle T^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T \right\rangle - \left\langle N^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{\psi}_T \right\rangle = \int_\Omega \tilde{\boldsymbol{g}} \cdot \overline{\boldsymbol{\psi}} \,\mathrm{d}x \quad (5.4)$$

for all $\boldsymbol{\psi} \in X_{k_\alpha^2}$. Since $\boldsymbol{E} \in X_{k_\alpha^2}$ solves (5.1) we can apply Lemma 3.5 to obtain that $\boldsymbol{E} \in H^1(\Omega)^3 \cap H^2(\{h - \eta < x_3 < h\})$. Moreover, Lemmas 4.3 and 4.4 state that

$$\begin{aligned}
2 \operatorname{Re} \int_\Omega x_3 \frac{\partial \boldsymbol{E}}{\partial x_3} \cdot \operatorname{curl} \operatorname{curl} \overline{\boldsymbol{E}} \,\mathrm{d}x &= -\int_\Omega |\operatorname{curl} \boldsymbol{E}|^2 \,\mathrm{d}x + h \int_{\Gamma_h} |\operatorname{curl} \boldsymbol{E}|^2 \,\mathrm{d}s \\
&\quad + 2 \operatorname{Re} \int_\Omega \left(\boldsymbol{e}_3 \times \frac{\partial \boldsymbol{E}}{\partial x_3}\right) \cdot \operatorname{curl} \overline{\boldsymbol{E}} \,\mathrm{d}x + 2h \operatorname{Re} \int_{\Gamma_h} \frac{\partial \boldsymbol{E}_T}{\partial x_3} \cdot (\boldsymbol{e}_3 \times \operatorname{curl} \overline{\boldsymbol{E}}) \,\mathrm{d}s.
\end{aligned}$$

In Lemma 4.4 we have already treated the term $2 \operatorname{Re} \int_\Omega (\boldsymbol{e}_3 \times \partial \boldsymbol{E}/\partial x_3) \cdot \operatorname{curl} \overline{\boldsymbol{E}} \,\mathrm{d}x$, see (4.3). Due to identity (3.6), it holds that $\boldsymbol{e}_3 \times \operatorname{curl} \boldsymbol{E} = \nabla_T E_3 - \partial \boldsymbol{E}_T/\partial x_3$ and therefore

$$2h \operatorname{Re} \int_{\Gamma_h} \frac{\partial \boldsymbol{E}_T}{\partial x_3} \cdot (\boldsymbol{e}_3 \times \operatorname{curl} \overline{\boldsymbol{E}}) \,\mathrm{d}s = -2h \int_{\Gamma_h} |\partial \boldsymbol{E}_T/\partial x_3|^2 \,\mathrm{d}s + 2h \operatorname{Re} \int_{\Gamma_h} \frac{\partial \boldsymbol{E}_T}{\partial x_3} \cdot \nabla_T \overline{E}_3 \,\mathrm{d}s.$$

We then obtain

$$2\,\mathrm{Re}\int_\Omega x_3 \frac{\partial \boldsymbol{E}}{\partial x_3}\cdot \mathrm{curl}\,\mathrm{curl}\,\overline{\boldsymbol{E}}\,\mathrm{d}x = -\int_\Omega \left|\mathrm{curl}\,\boldsymbol{E}\right|^2\,\mathrm{d}x + h\int_{\Gamma_h}\left|\mathrm{curl}\,\boldsymbol{E}\right|^2\,\mathrm{d}s + 2\int_\Omega \left|\frac{\partial \boldsymbol{E}}{\partial x_3}\right|^2\,\mathrm{d}x$$
$$+ 2\,\mathrm{Re}\int_\Omega \frac{\nabla_T\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x + \int_\Omega \left(\frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\,\frac{\partial |E_3|^2}{\partial x_3} - 2\,\mathrm{Im}\left(\frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\,\mathrm{Im}\left(\frac{\partial \overline{E_3}}{\partial x_3}E_3\right)\right)\,\mathrm{d}x$$
$$- 2\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial E_3}{\partial x_3}\,\overline{E_3}\,\mathrm{d}s - 2h\int_{\Gamma_h}|\partial \boldsymbol{E}_T/\partial x_3|^2\,\mathrm{d}s + 2h\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial \boldsymbol{E}_T}{\partial x_3}\cdot\nabla_T\overline{E}_3\,\mathrm{d}s.$$

Now we exploit that $\boldsymbol{E}$ satisfies Maxwell's equations (2.3) in the L^2 sense,

$$\begin{aligned}2\,\mathrm{Re}\int_\Omega x_3 \frac{\partial \boldsymbol{E}}{\partial x_3}\cdot \mathrm{curl}\,\mathrm{curl}\,\overline{\boldsymbol{E}}\,\mathrm{d}x &= 2k^2\,\mathrm{Re}\int_\Omega x_3\,\mathrm{Re}(\varepsilon_\mathrm{r})\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\boldsymbol{E}}\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x\\
&= k^2\int_\Omega x_3\,\mathrm{Re}(\varepsilon_\mathrm{r})\frac{\partial}{\partial x_3}\left|\boldsymbol{E}\right|^2\,\mathrm{d}x + 2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x\\
&= -k^2\int_\Omega \frac{\partial(x_3\,\mathrm{Re}(\varepsilon_\mathrm{r}))}{\partial x_3}\left|\boldsymbol{E}\right|^2\,\mathrm{d}x + k^2 h\int_{\Gamma_h}|\boldsymbol{E}|^2\,\mathrm{d}s + 2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x.\end{aligned}$$

Together, the last two equations imply

$$-\int_\Omega\left(\left|\mathrm{curl}\,\boldsymbol{E}\right|^2 - k^2\frac{\partial(x_3\,\mathrm{Re}(\varepsilon_\mathrm{r}))}{\partial x_3}\left|\boldsymbol{E}\right|^2\right)\,\mathrm{d}x + 2\int_\Omega\left|\frac{\partial \boldsymbol{E}}{\partial x_3}\right|^2\,\mathrm{d}x$$
$$+ 2\,\mathrm{Re}\int_\Omega \frac{\nabla_T\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x + \int_\Omega \left(\frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\,\frac{\partial |E_3|^2}{\partial x_3}\,\mathrm{d}x - 2\,\mathrm{Im}\left(\frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\,\mathrm{Im}\left(\frac{\partial \overline{E_3}}{\partial x_3}E_3\right)\right)\,\mathrm{d}x$$
$$- 2\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial E_3}{\partial x_3}\,\overline{E_3}\,\mathrm{d}s - 2h\int_{\Gamma_h}|\partial \boldsymbol{E}_T/\partial x_3|^2\,\mathrm{d}s + 2h\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial \boldsymbol{E}_T}{\partial x_3}\cdot\nabla_T\overline{E}_3\,\mathrm{d}s$$
$$+ h\int_{\Gamma_h}\left(\left|\mathrm{curl}\,\boldsymbol{E}\right|^2 - k^2|\boldsymbol{E}|^2\right)\,\mathrm{d}s = 2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x. \quad (5.5)$$

Taking the real part of the variational formulation (5.4) with $\boldsymbol{\psi} = \boldsymbol{E}$ yields

$$\int_\Omega\left(\left|\mathrm{curl}\,\boldsymbol{E}\right|^2 - k^2\,\mathrm{Re}(\varepsilon_\mathrm{r})\left|\boldsymbol{E}\right|^2\right)\,\mathrm{d}x - \mathrm{Re}\left\langle T^+_{k_\alpha^2}(\boldsymbol{E}_T),\,\boldsymbol{E}_T\right\rangle$$
$$- \mathrm{Re}\left\langle N^+_{k_\alpha^2}(\boldsymbol{E}_T),\,\boldsymbol{E}_T\right\rangle = \mathrm{Re}\int_\Omega \tilde{\boldsymbol{g}}\cdot\overline{\boldsymbol{E}}\,\mathrm{d}x. \quad (5.6)$$

By adding (5.5) and (5.6) and ordering volumetric and boundary terms, we obtain

$$\int_\Omega\left(2\left|\frac{\partial \boldsymbol{E}}{\partial x_3}\right|^2 + k^2 x_3\frac{\partial\,\mathrm{Re}(\varepsilon_\mathrm{r})}{\partial x_3}\left|\boldsymbol{E}\right|^2 + \frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\,\frac{\partial |E_3|^2}{\partial x_3}\,\mathrm{d}x\right)$$
$$+ 2\,\mathrm{Re}\int_\Omega \frac{\nabla_T\varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}}\cdot \boldsymbol{E}_T\,\frac{\partial \overline{E_3}}{\partial x_3}\,\mathrm{d}x - \int_\Omega 2\,\mathrm{Im}\left(\frac{\partial \log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\,\mathrm{Im}\left(\frac{\partial \overline{E_3}}{\partial x_3}E_3\right)\,\mathrm{d}x$$
$$+ h\int_{\Gamma_h}\left(\left|\mathrm{curl}\,\boldsymbol{E}\right|^2 - k^2|\boldsymbol{E}|^2 - 2|\partial\boldsymbol{E}_T/\partial x_3|^2\right)\,\mathrm{d}s + 2h\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial \boldsymbol{E}_T}{\partial x_3}\cdot\nabla_T\overline{E}_3\,\mathrm{d}s$$
$$- 2\,\mathrm{Re}\int_{\Gamma_h}\frac{\partial E_3}{\partial x_3}\overline{E_3}\,\mathrm{d}s - \mathrm{Re}\left\langle T^+_{k_\alpha^2}(\boldsymbol{E}_T),\,\boldsymbol{E}_T\right\rangle - \mathrm{Re}\left\langle N^+_{k_\alpha^2}(\boldsymbol{E}_T),\,\boldsymbol{E}_T\right\rangle$$
$$= 2\,\mathrm{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x + \mathrm{Re}\int_\Omega \tilde{\boldsymbol{g}}\cdot\overline{\boldsymbol{E}}\,\mathrm{d}x. \quad (5.7)$$

In the sequel of this proof we abbreviate by $A(\boldsymbol{E})$ all the volumetric terms appearing on the left-hand side of the last equation. From the decomposition (3.6) of the curl operator we see that

$$
\begin{aligned}
|\operatorname{curl} \boldsymbol{E}|^2 &= |\operatorname{curl}_T \boldsymbol{E}_T|^2 + \left|\vec{\operatorname{curl}}_T E_3\right|^2 + \left|\frac{\partial \boldsymbol{E}_T}{\partial x_3}\right|^2 - 2\operatorname{Re}\left(\vec{\operatorname{curl}}_T E_3 \cdot \frac{\partial(\overline{\boldsymbol{E}} \times \boldsymbol{e}_3)}{\partial x_3}\right) \\
&= |\operatorname{curl}_T \boldsymbol{E}_T|^2 + \left|\vec{\operatorname{curl}}_T E_3\right|^2 + \left|\frac{\partial \boldsymbol{E}_T}{\partial x_3}\right|^2 - 2\operatorname{Re}\left(\nabla_T E_3 \cdot \frac{\partial \overline{\boldsymbol{E}}_T}{\partial x_3}\right).
\end{aligned}
$$

We use this relation to substitute the term $|\operatorname{curl} \boldsymbol{E}|^2$ in the boundary integral over Γ_h in (5.7),

$$
\begin{aligned}
A(\boldsymbol{E}) - h\int_{\Gamma_h} &\left(\left|\frac{\partial \boldsymbol{E}_T}{\partial x_3}\right|^2 + k^2|\boldsymbol{E}|^2 - |\operatorname{curl}_T \boldsymbol{E}_T|^2 - \left|\vec{\operatorname{curl}}_T E_3\right|^2\right) \mathrm{d}s \\
&- \operatorname{Re}\left\langle T^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle - \operatorname{Re}\left\langle N^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle - 2\operatorname{Re}\int_{\Gamma_h} \frac{\partial E_3}{\partial x_3}\overline{E_3}\,\mathrm{d}s \\
&= 2\operatorname{Re}\int_\Omega x_3 \frac{\partial \boldsymbol{E}}{\partial x_3} \cdot \overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x + \operatorname{Re}\int_\Omega \tilde{\boldsymbol{g}} \cdot \overline{\boldsymbol{E}}\,\mathrm{d}x. \qquad (5.8)
\end{aligned}
$$

We consider now the boundary terms appearing in the second line by computing their sum in the Fourier domain. Plancherel's theorem implies

$$
\begin{aligned}
\int_{\Gamma_h} &\left(k^2|\boldsymbol{E}|^2 + \left|\frac{\partial \boldsymbol{E}_T}{\partial x_3}\right|^2 - |\operatorname{curl}_T \boldsymbol{E}_T|^2 - \left|\vec{\operatorname{curl}}_T E_3\right|^2\right) \mathrm{d}s \\
&= \int_{\mathbb{R}^2} \left[k^2|\hat{\boldsymbol{E}}|^2 + \left|\sqrt{k_\alpha^2 - |\xi|^2}\,\hat{\boldsymbol{E}}_T\right|^2 - |\xi_1 \hat{E}_2 - \xi_2 \hat{E}_1|^2 - \xi_1^2|\hat{E}_3|^2 - \xi_2^2|\hat{E}_3|^2\right] \mathrm{d}\xi \\
&= \int_{\mathbb{R}^2} \left[\left(k^2 - |\xi|^2 + |k_\alpha^2 - |\xi|^2|\right)|\hat{\boldsymbol{E}}_T|^2 + (k^2 - |\xi|^2)|\hat{E}_3|^2 + \left|\xi_1 \hat{E}_1 + \xi_2 \hat{E}_2\right|^2\right] \mathrm{d}\xi \\
&= \int_{\mathbb{R}^2} \left[\left(k^2 - |\xi|^2 + |k^2 - |\xi|^2| + \zeta_\alpha(\xi)\right)|\hat{\boldsymbol{E}}_T|^2 + (k^2 - |\xi|^2)|\hat{E}_3|^2 + \left|\xi_1 \hat{E}_1 + \xi_2 \hat{E}_2\right|^2\right] \mathrm{d}\xi,
\end{aligned}
$$

where $\zeta_\alpha(\xi) := \alpha^2/(|k_\alpha^2 - |\xi|^2| + |k^2 - |\xi|^2|)$, $\xi \in \mathbb{R}^2$. The identity

$$
\int_{\mathbb{R}^2} \left|\xi_1 \hat{E}_1 + \xi_2 \hat{E}_2\right|^2 \mathrm{d}\xi = \|\operatorname{div}_T \boldsymbol{E}_T\|^2_{L^2(\Gamma_h)} = \|\partial E_3/\partial x_3\|^2_{L^2(\Gamma_h)} = \int_{\mathbb{R}^2} |k_\alpha^2 - |\xi|^2||\hat{E}_3|^2\,\mathrm{d}\xi
$$

and the radiation condition yield

$$
\begin{aligned}
\int_{\Gamma_h} &\left(k^2|\boldsymbol{E}|^2 + \left|\frac{\partial \boldsymbol{E}_T}{\partial x_3}\right|^2 - |\operatorname{curl}_T \boldsymbol{E}_T|^2 - \left|\vec{\operatorname{curl}}_T E_3\right|^2\right) \mathrm{d}s \\
&\le \int_{\mathbb{R}^2} \zeta_\alpha(\xi)|\boldsymbol{E}|^2\,\mathrm{d}\xi + 2\int_{|\xi|<k} (k^2 - |\xi|^2)|\hat{\boldsymbol{E}}_T|^2\,\mathrm{d}\xi + 2\int_{|\xi|<k} (k^2 - |\xi|^2)|\hat{E}_3|^2\,\mathrm{d}\xi \\
&\le \int_{\mathbb{R}^2} \zeta_\alpha(\xi)|\boldsymbol{E}|^2\,\mathrm{d}\xi + 2k\int_{|\xi|<k} \sqrt{k^2 - |\xi|^2}\,|\hat{\boldsymbol{E}}|^2\,\mathrm{d}\xi.
\end{aligned}
$$

Further, Corollary 3.3 implies that $\partial E_3/\partial x_3|_{\Gamma_h} = T^+_{k_\alpha^2}(E_3)$ and that

$$
\left\langle N^+_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle = \mathrm{i}\int_{\mathbb{R}^2} \frac{|\xi \cdot \hat{\boldsymbol{E}}_T|^2}{\sqrt{k_\alpha^2 - |\xi|^2}}\,\mathrm{d}\xi = \mathrm{i}\int_{\mathbb{R}^2} \sqrt{k_\alpha^2 - |\xi|^2}\,|\hat{E}_3|^2\,\mathrm{d}\xi.
$$

Hence,

$$
\begin{aligned}
&-\operatorname{Re}\left\langle T^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle - \operatorname{Re}\left\langle N^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle - 2\operatorname{Re}\int_{\Gamma_h} \frac{\partial E_3}{\partial x_3}\overline{E_3}\,\mathrm{d}s \\
&= \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\hat{\boldsymbol{E}}_T|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^2} \operatorname{Im}\left(\overline{\sqrt{k_\alpha^2-|\xi|^2}}\right)|\hat{E}_3|^2\,\mathrm{d}\xi + 2\int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\hat{\boldsymbol{E}}_3|^2\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\hat{\boldsymbol{E}}|^2\,\mathrm{d}\xi \geq 0.
\end{aligned}
$$

Returning to (5.8), we find

$$
\begin{aligned}
A(\boldsymbol{E}) - 2kh\int_{|\xi|<k}\sqrt{k^2-|\xi|^2}\,|\hat{\boldsymbol{E}}|^2\,\mathrm{d}\xi - h\int_{\mathbb{R}^2}\zeta_\alpha(\xi)|\boldsymbol{E}|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^2}\operatorname{Im}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\hat{\boldsymbol{E}}|^2\,\mathrm{d}\xi \\
\leq 2\operatorname{Re}\int_\Omega x_3\frac{\partial \boldsymbol{E}}{\partial x_3}\cdot\overline{\tilde{\boldsymbol{g}}}\,\mathrm{d}x + \operatorname{Re}\int_\Omega \tilde{\boldsymbol{g}}\cdot\overline{\boldsymbol{E}}\,\mathrm{d}x. \quad (5.9)
\end{aligned}
$$

The two negative terms from the last expression can be estimated a-priori by taking the imaginary part of the variational formulation (3.15) with $\boldsymbol{\psi} = \boldsymbol{E}$, yielding

$$
\int_\Omega(\alpha\operatorname{Re}(\varepsilon_\mathrm{r}) + k^2\operatorname{Im}(\varepsilon_\mathrm{r}))|\boldsymbol{E}|^2\,\mathrm{d}x + \operatorname{Im}\left\langle T^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle + \operatorname{Im}\left\langle N^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle = -\operatorname{Im}\int_\Omega \boldsymbol{g}\cdot\overline{\boldsymbol{E}}\,\mathrm{d}x. \quad (5.10)
$$

All quantities on the left of (5.10) are non-negative. We recall that

$$
\begin{aligned}
\operatorname{Im}\left\langle T^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \overline{\boldsymbol{E}_T}\right\rangle = \int_{\mathbb{R}^2}\operatorname{Re}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\hat{\boldsymbol{E}_T}|^2\,\mathrm{d}\xi \quad \text{and} \\
\operatorname{Im}\left\langle N^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \overline{\boldsymbol{E}_T}\right\rangle = \int_{\mathbb{R}^2}\operatorname{Re}(\sqrt{k_\alpha^2-|\xi|^2})|\hat{E}_3|^2\,\mathrm{d}\xi. \quad (5.11)
\end{aligned}
$$

From Lemma 5.2 below it follows that $\zeta_\alpha(\xi) \leq \sqrt{2\alpha}\operatorname{Re}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)$. Therefore, using (5.10) and (5.11),

$$
\begin{aligned}
\int_{\mathbb{R}^2}\zeta_\alpha(\xi)|\boldsymbol{E}|^2\,\mathrm{d}\xi &\leq \int_{\mathbb{R}^2}\sqrt{2\alpha}\operatorname{Re}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\boldsymbol{E}|^2\,\mathrm{d}\xi \\
&= \sqrt{2\alpha}\left(\operatorname{Im}\left\langle T^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle + \operatorname{Im}\left\langle N^{+}_{k_\alpha^2}(\boldsymbol{E}_T),\, \boldsymbol{E}_T\right\rangle\right) \leq \sqrt{2\alpha}\|\boldsymbol{g}\|_{L^2(\Omega)^3}\|\boldsymbol{E}\|_{L^2(\Omega)^3}. \quad (5.12)
\end{aligned}
$$

Inserting the last two estimates into (5.9), we obtain (5.2). Estimate (5.3) is a direct consequence of (5.10). □

LEMMA 5.2. *For $k > 0$, $\alpha \geq 0$ and $\xi \in \mathbb{R}^2$,*

$$
\zeta_\alpha(\xi) = \frac{\alpha^2}{|k_\alpha^2-|\xi|^2| + |k^2-|\xi|^2|} \leq \sqrt{2\alpha}\operatorname{Re}\left(\sqrt{k_\alpha^2-|\xi|^2}\right).
$$

Proof. We recall that $k_\alpha^2 = k^2 + \mathrm{i}\alpha$ and therefore $\left|k_\alpha^2-|\xi|^2\right|^2 = |k^2-|\xi|^2|^2+\alpha^2$. For $|\xi|^2 > k^2$ it holds

$$
\begin{aligned}
\sqrt{2}\operatorname{Re}\sqrt{k_\alpha^2-|\xi|^2} &= \sqrt{|k_\alpha^2-|\xi|^2| + k^2-|\xi|^2} \\
&= \sqrt{|k_\alpha^2-|\xi|^2| - |k^2-|\xi|^2|} = \frac{\alpha}{(|k_\alpha^2-|\xi|^2| + |k^2-|\xi|^2|)^{1/2}}.
\end{aligned}
$$

Hence,

$$\sqrt{2\alpha}\,\mathrm{Re}\,\sqrt{k_\alpha^2-|\xi|^2}=\alpha\,\underbrace{\frac{\sqrt{\alpha}}{(|k_\alpha^2-|\xi|^2|+|k^2-|\xi|^2|)^{1/2}}}_{\le 1}\ge\frac{\alpha^2}{|k_\alpha^2-|\xi|^2|+|k^2-|\xi|^2|},\quad |\xi|^2>k^2.$$

Let now $|\xi|^2\le k^2$. Then $\sqrt{2\alpha}\,\mathrm{Re}\,\sqrt{k_\alpha^2-|\xi|^2}\ge\sqrt{2\alpha}\sqrt{\alpha/2}\ge\alpha^2/(\left|k_\alpha^2-|\xi|^2\right|+\left|k^2-|\xi|^2\right|)$. □

6. A-priori Bounds for 1D Structures. The simplest situation to prove an a-priori estimate for solutions to the electromagnetic rough layer scattering problem is when ε_r only depends on x_3. One might argue whether such a material parameter still corresponds to a *rough* layer. Nevertheless, in this section we investigate this simplified setting, because we rely on the corresponding a-priori estimates when considering more general situations in the next section. The following lemma is a direct consequence of Proposition 5.1.

LEMMA 6.1. *Let ε_r, $\boldsymbol{g}$ and k_α^2 be as in Proposition 5.1 and further assume that ε_r does only depend on x_3. Then any variational solution $\boldsymbol{E}\in X_{k_\alpha^2}$ of* (5.1) *satisfies the a-priori estimate*

$$\int_\Omega\left(2\left|\frac{\partial\boldsymbol{E}}{\partial x_3}\right|^2+k^2x_3\frac{\partial\,\mathrm{Re}(\varepsilon_\mathrm{r})}{\partial x_3}|\boldsymbol{E}|^2+\frac{\partial\log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\frac{\partial|E_3|^2}{\partial x_3}\right)\mathrm{d}x+\int_{\mathbb{R}^2}\mathrm{Im}\left(\sqrt{k_\alpha^2-|\xi|^2}\right)|\hat{\boldsymbol{E}}|^2\,\mathrm{d}\xi$$
$$\le\left(2(h+1)\left\|\frac{\partial\boldsymbol{E}}{\partial x_3}\right\|_{L^2(\Omega)^3}+(1+2\sqrt{2}hk+\sqrt{2\alpha}h)\|\boldsymbol{E}\|_{L^2(\Omega)^3}\right)\Bigg[\|\boldsymbol{g}\|_{L^2(\Omega)^3}+\alpha\|\varepsilon_\mathrm{r}\|_{L^\infty}\|\boldsymbol{E}\|_{L^2(\Omega)^3}$$
$$+\left(\frac{(k^2+\alpha)^2}{k^2}\|\,\mathrm{Im}(\varepsilon_\mathrm{r})\|_{L^\infty}\|\boldsymbol{g}\|_{L^2(\Omega)^3}\|\boldsymbol{E}\|_{L^2(\Omega)^3}\right)^{1/2}+\left\|\mathrm{Im}\left(\frac{\partial\log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\boldsymbol{E}\right\|_{L^2(\Omega)^3}\Bigg].\quad(6.1)$$

REMARK 6.2. *Let δ be a positive constant such that $0<\delta<h/2$. We denote by a "tubular domain of thickness δ" of Ω any open domain $D_\delta:=\{x\in\Omega,\ r(\tilde{x})-\delta/2<x_3<r(\tilde{x})+\delta/2\}$ where $r:\mathbb{R}^2\to\mathbb{R}$ is a piecewise Lipschitz continuous function that satisfies $\delta<r(\tilde{x})<h-\delta$.*

The following Poincaré-like result is well-known (see [15, Lemma 4.3] and [8, Lemma 3.4]).

LEMMA 6.3. *Let δ be a positive constant and let D_δ be a tubular domain of thickness δ of Ω. Then*

$$\delta\|u\|^2_{L^2(\Omega)}\le 4h\|u\|^2_{L^2(D_\delta)}+8h^3\left\|\frac{\partial u}{\partial x_3}\right\|^2_{L^2(\Omega)}\qquad\textit{for all } u\in H^1(\Omega).$$

LEMMA 6.4. *For $u\in\{v\in H^1(\Omega),\ v|_{\Gamma_0}=0\}$ there holds $\|u\|_{L^2(\Omega)}\le h/\sqrt{2}\,\|\partial u/\partial x_3\|_{L^2(\Omega)}$.*

Our assumptions on ε_r are as follows:

$$\begin{cases}(a)\quad \varepsilon_\mathrm{r}\in W^{1,\infty}(\Omega)\text{ only depends on }x_3.\\ (b)\quad \mathrm{Re}(\varepsilon_\mathrm{r})\text{ is positive and bounded away from zero,}\\ \qquad\partial\,\mathrm{Re}(\varepsilon_\mathrm{r})/\partial x_3\ge 0,\ \text{and }\ \mathrm{Im}(\varepsilon_\mathrm{r})\ge 0\text{ in }\Omega.\\ (c)\quad \text{There exists a tubular domain } D_\delta\subset\Omega\text{ of thickness }\delta>0\\ \qquad\text{and constants }\gamma>1\text{ and }c>0\text{ with}\\ \qquad\qquad k^2x_3\dfrac{\partial\,\mathrm{Re}(\varepsilon_\mathrm{r})}{\partial x_3}-\gamma^2\left(\dfrac{\partial\log(|\varepsilon_\mathrm{r}|)}{\partial x_3}\right)^2\ge\begin{cases}0&\text{in }\Omega,\\ c>0&\text{in }D_\delta.\end{cases}\\ (d)\quad \text{There exist constants }\beta\ge 0\text{ and }\theta\ge 1/2\text{ such that}\\ \qquad\qquad\left|\mathrm{Im}\left(\dfrac{\partial\log(\varepsilon_\mathrm{r})}{\partial x_3}\right)\right|\le\beta\,\mathrm{Im}(\varepsilon_\mathrm{r})^\theta\quad\text{in }\Omega.\end{cases}\qquad(6.2)$$

Two remarks are in order.

REMARK 6.5. *Assumption (d) in* (6.2) *is not too restrictive compared to the conditions (a)-(c), since it can be written as*

$$\frac{1}{|\varepsilon_{\rm r}|^2}\left|\operatorname{Im}(\varepsilon_{\rm r})\frac{\partial \operatorname{Re}(\varepsilon_{\rm r})}{\partial x_3} - \operatorname{Re}(\varepsilon_{\rm r})\frac{\partial \operatorname{Im}(\varepsilon_{\rm r})}{\partial x_3}\right| \le \beta \operatorname{Im}(\varepsilon_{\rm r})^\theta$$

and therefore roughly means that the imaginary part of $\varepsilon_{\rm r}$ should not vary more than exponentially. This condition is for instance verified if $\operatorname{Im}(\varepsilon_{\rm r}) = 0$.

REMARK 6.6. *To show that there are material parameters that satisfy* (6.2) *we construct a real-valued example. For real-valued $\varepsilon_{\rm r} = \varepsilon_{\rm r}(x_3)$, we reformulate* (6.2)*(c) as*

$$k^2 x_3\, \varepsilon_{\rm r}^2 \ge \gamma^2 \varepsilon_{\rm r}' \text{ in } \Omega \quad \text{and} \quad (k^2 x_3 - c)\varepsilon_{\rm r}^2 \ge \gamma^2 \varepsilon_{\rm r}' \text{ in } D_\delta. \tag{6.3}$$

We construct in the following a piecewise linear function that satisfies these conditions. For parameters $0 < h_1 < h_2 < h$ and $0 < \varepsilon_- < 1$, we define $\varepsilon_{\rm r}(x_3) = \varepsilon_-$ in $(0, h_1)$, $\varepsilon_{\rm r}(x_3) = \varepsilon_- + (x_3 - h_1)(1-\varepsilon_-)/(h_2 - h_1)$ in (h_1, h_2), and $\varepsilon_{\rm r}(x_3) = 1$ in (h_2, h). Then $\varepsilon_{\rm r}$ is an increasing function in (h_1, h_2), it possesses a bounded weak derivative, and it satisfies (6.3) *for $D_\delta = \{h_1 < x_3 < h_2\}$ and $\delta = 1/2$ and if and only if*

$$k^2 h_1\, \varepsilon_-^2 \ge \gamma^2 \frac{1-\varepsilon_-}{h_2 - h_1} \quad \text{and} \quad (k^2 h_1 - c)\varepsilon_-^2 \ge \gamma^2 \frac{1-\varepsilon_-}{h_2 - h_1}.$$

The latter conditions hold, e.g., for $\varepsilon_- = 0.5$, $h_1 = 0.25$, $h_2 = 0.75$, $k = 6$, $\gamma = 3/\sqrt{8}$, and $c = k^2/8$. Condition (6.3) *can be interpreted as a bound on the growth of $\varepsilon_{\rm r}$. Indeed, dividing the left inequality of* (6.3) *by $\varepsilon_{\rm r}^2$ and integrating between* 0 *and x_3 we obtain the necessary condition*

$$\varepsilon_{\rm r}(0)\left[1 + \frac{k^2}{2\gamma^2} x_3^2 \varepsilon_{\rm r}(x_3)\right] \ge \varepsilon_{\rm r}(x_3) \ge \varepsilon_{\rm r}(0), \quad x_3 \in (0, h).$$

Next we use the assumptions (6.2) on the material parameter $\varepsilon_{\rm r}$ to prove a-priori bounds on a solution to (5.1).

LEMMA 6.7. *Let $\boldsymbol{g} \in L_0^2(\operatorname{div}0, \Omega)^3$ and $k > 0$. Assume that $\varepsilon_{\rm r}$ satisfies* (6.2) *and set $k_\alpha^2 = k^2 + {\rm i}\alpha$ for $\alpha \in [0,1]$. Then any variational solution $\boldsymbol{E} \in X_{k_\alpha^2}$ of* (5.1) *satisfies the a-priori estimate*

$$\left(\left\|\frac{\partial \boldsymbol{E}}{\partial x_3}\right\|_{L^2(\Omega)^3}^2 + \|\boldsymbol{E}\|_{L^2(\Omega)^3}^2 + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,{\rm d}\xi\right)^{1/2} \le C\, L_\alpha(\boldsymbol{g}, \boldsymbol{E}), \tag{6.4}$$

with $C = (3 + \sqrt{2}h + 2h + 2\sqrt{2}hk)[1 + (\gamma(4hc + h^2\delta c)/(\gamma^2 - 1) + 8h^3)/(\delta c)]$ and where

$$L_\alpha(\boldsymbol{g}, \boldsymbol{E}) = \|\boldsymbol{g}\|_{L^2(\Omega)^3} + \alpha\|\varepsilon_{\rm r}\|_{L^\infty}\|\boldsymbol{E}\|_{L^2(\Omega)^3} + \left(\frac{2(k^2+1)^2\|\operatorname{Im}(\varepsilon_{\rm r})\|_{L^\infty} + 2\beta^2\|\operatorname{Im}\varepsilon_{\rm r}\|^{2\theta-1}}{k^2}\|\boldsymbol{g}\|_{L^2(\Omega)^3}\|\boldsymbol{E}\|_{L^2(\Omega)^3}\right)^{1/2}. \tag{6.5}$$

Proof. From Assumption (6.2)(d) and (5.3) we infer that

$$\left\|\operatorname{Im}\left(\frac{\partial \log(\varepsilon_{\rm r})}{\partial x_3}\right)\boldsymbol{E}\right\|_{L^2(\Omega)^3} \le \frac{\beta\|\operatorname{Im}\varepsilon_{\rm r}\|_{L^\infty}^{\theta-1/2}}{k}\sqrt{\|\boldsymbol{g}\|_{L^2(\Omega)^3}\|\boldsymbol{E}\|_{L^2(\Omega)^3}}.$$

Therefore (6.1) implies

$$\int_\Omega \left(\left| \frac{\partial \boldsymbol{E}}{\partial x_3} \right|^2 + k^2 x_3 \frac{\partial \operatorname{Re}(\varepsilon_\mathrm{r})}{\partial x_3} |\boldsymbol{E}|^2 + \frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3} \frac{\partial |E_3|^2}{\partial x_3} \right) \mathrm{d}x + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi$$
$$\le \left(2(h+1) \left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|_{L^2(\Omega)^3} + (1 + 2\sqrt{2}hk + \sqrt{2\alpha}h) \|\boldsymbol{E}\|_{L^2(\Omega)^3} \right) L_\alpha(\boldsymbol{g}, \boldsymbol{E}). \quad (6.6)$$

The term $\partial |E_3|^2/\partial x_3$ can be rewritten as $2\operatorname{Re}(E_3 \partial \overline{E_3}/\partial x_3)$. Hence, setting $\tilde{\gamma} = 1 - \gamma^{-2} > 0$ and complementing the square, the first integral of (6.6) equals

$$\int_\Omega \left(2 \left| \frac{\partial \boldsymbol{E}_T}{\partial x_3} \right|^2 + k^2 x_3 \frac{\partial \operatorname{Re}(\varepsilon_\mathrm{r})}{\partial x_3} |\boldsymbol{E}_T|^2 + \tilde{\gamma} \left| \frac{\partial E_3}{\partial x_3} \right|^2 + \left| \gamma^{-1} \frac{\partial E_3}{\partial x_3} + \gamma \frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3} E_3 \right|^2 \right.$$
$$\left. + \left(k^2 x_3 \frac{\partial \operatorname{Re}(\varepsilon_\mathrm{r})}{\partial x_3} - \gamma^2 \left(\frac{\partial \log(|\varepsilon_\mathrm{r}|)}{\partial x_3} \right)^2 \right) |\boldsymbol{E}_3|^2 \right) \mathrm{d}x$$

Now we exploit assumption (6.2)(c) to estimate the latter term from below, and thereby we obtain from (6.6) that

$$\tilde{\gamma} \left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|^2_{L^2(\Omega)^3} + c\|E_3\|^2_{L^2(D_\delta)} + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi$$
$$\le (3 + \sqrt{2}h + 2h + 2\sqrt{2}hk) \left(\left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|_{L^2(\Omega)^3} + \|\boldsymbol{E}\|_{L^2(\Omega)^3} \right) L_\alpha(\boldsymbol{g}, \boldsymbol{E}). \quad (6.7)$$

From Lemma 6.3 we see that

$$\|E_3\|^2_{L^2(\Omega)} \le \frac{4hc + 8h^3\tilde{\gamma}}{\delta c \tilde{\gamma}} \left(\tilde{\gamma} \left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|^2_{L^2(\Omega)^3} + c\|E_3\|^2_{L^2(D_\delta)} \right)$$

and Lemma 6.4 states that $\|\boldsymbol{E}_T\|^2_{L^2(\Omega)} \le h^2/2 \|\partial \boldsymbol{E}/\partial x_3\|^2_{L^2(\Omega)^3}$, since $\boldsymbol{E} \in X_{k_\alpha^2}$. Combining the last two inequalities for $\boldsymbol{E}_T$ and E_3 shows that

$$\|\boldsymbol{E}\|^2_{L^2(\Omega)} \le \frac{4hc + 8h^3\tilde{\gamma} + h^2\delta c}{\delta c \tilde{\gamma}} \left(\tilde{\gamma} \left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|^2_{L^2(\Omega)^3} + c\|E_3\|^2_{L^2(D_\delta)} \right).$$

Using again (6.7) we obtain

$$\|\boldsymbol{E}\|^2_{L^2(\Omega)} \le \frac{4hc + 8h^3\tilde{\gamma} + h^2\delta c}{\delta c \tilde{\gamma}} (3 + \sqrt{2}h + 2h + 2\sqrt{2}hk) \left(\left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|_{L^2(\Omega)^3} + \|\boldsymbol{E}\|_{L^2(\Omega)^3} \right) L_\alpha(\boldsymbol{g}, \boldsymbol{E})$$

and, additionally, (6.6) trivially implies that $\|\partial \boldsymbol{E}/\partial x_3\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^2} \operatorname{Im}(\sqrt{k_\alpha^2 - |\xi|^2}) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi$ is bounded by the left hand side of (6.6). Adding the last two inequalities finally yields (6.4). □

An immediate consequence of the last lemma is the following uniqueness result.

COROLLARY 6.8. *Let* $\boldsymbol{g} \in L^2_0(\operatorname{div} 0, \Omega)^3$, $k > 0$, *and assume that* ε_r *satisfies* (6.2). *Then problem* (3.15) *(or, equivalently, problem* (5.1) *with real wave number, that is, for* $\alpha = 0$*) possesses at most one solution* $\boldsymbol{E} \in X_{k^2}$.

A further consequence of Lemma 6.7 is an a-priori estimate in $H(\mathrm{curl}, \Omega)$.

LEMMA 6.9. *Let* $\boldsymbol{g} \in L_0^2(\mathrm{div}0, \Omega)^3$ *and* $k > 0$. *Assume that* ε_r *satisfies* (6.2) *and set* $k_\alpha^2 = k^2 + \mathrm{i}\alpha$ *for* $\alpha \in [0, 1]$. *Then any variational solution* $\boldsymbol{E} \in X_{k_\alpha^2}$ *of* (5.1) *satisfies the a-priori estimate*

$$\left(\|\boldsymbol{E}\|_{H(\mathrm{curl},\Omega)}^2 + \int_{\mathbb{R}^2} \mathrm{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi \right)^{1/2} \\ \leq \sqrt{((k^2+1)\|\varepsilon_\mathrm{r}\|_{L^\infty} + 1)C^2 + C}\; L_\alpha(\boldsymbol{g}, \boldsymbol{E}), \quad (6.8)$$

where C *is the constant from Lemma 6.7 and* $L_\alpha(\boldsymbol{g}, \boldsymbol{E})$ *is given by* (6.5).

Proof. From the real part of the variational formulation (5.1) with $\boldsymbol{\psi} = \boldsymbol{E}$ we infer that

$$\int_\Omega |\mathrm{curl}\,\boldsymbol{E}|^2 - \mathrm{Re}\left\langle T_{k_\alpha^2}^+(\boldsymbol{E}_T),\, \boldsymbol{E}_T \right\rangle \\ \leq \left\langle N_{k_\alpha^2}^+(\boldsymbol{E}_T),\, \boldsymbol{E}_T \right\rangle + |k_\alpha^2| \|\varepsilon_\mathrm{r}\|_{L^\infty} \|\boldsymbol{E}\|_{L^2(\Omega)^3}^2 + \|\boldsymbol{g}\|_{L^2(\Omega)^3} \|\boldsymbol{E}\|_{L^2(\Omega)^3}.$$

Since $\boldsymbol{E}$ solves (5.1) and exploiting Corollary 3.3 we obtain that $\mathrm{div}_T\, \boldsymbol{E}_T = -\partial E_3 / \partial x_3$. Thus, the radiation condition implies

$$\mathrm{Re}\left\langle N_{k_\alpha^2}^+(\boldsymbol{E}_T),\, \boldsymbol{E}_T \right\rangle = -\int_{\mathbb{R}^2} \mathrm{Im}\left(\frac{|\xi \cdot \hat{\boldsymbol{E}}_T|^2}{\sqrt{k_\alpha^2 - |\xi|^2}} \right) \mathrm{d}\xi = \int_{\mathbb{R}^2} \mathrm{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{E}_3|^2 \,\mathrm{d}\xi.$$

Since $\mathrm{Re}\left\langle T_{k_\alpha^2}^+(\boldsymbol{E}_T),\, \boldsymbol{E}_T \right\rangle \leq 0$, using (6.4), we arrive at

$$\|\mathrm{curl}\,\boldsymbol{E}\|_{L^2(\Omega)^3}^2 + \|\boldsymbol{E}\|_{L^2(\Omega)^3}^2 + \int_{\mathbb{R}^2} \mathrm{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi \\ \leq ((k^2+1)\|\varepsilon_\mathrm{r}\|_{L^\infty} + 1)\|\boldsymbol{E}\|_{L^2(\Omega)^3}^2 + \|\boldsymbol{g}\|_{L^2(\Omega)^3} \|\boldsymbol{E}\|_{L^2(\Omega)^3} + 2\int_{\mathbb{R}^2} \mathrm{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi \\ \leq (((k^2+1)\|\varepsilon_\mathrm{r}\|_{L^\infty} + 1)C^2 + C) L_\alpha(\boldsymbol{g}, \boldsymbol{E})^2$$

where C is the constant from Lemma 6.7. ◻

7. A-Priori Bounds for Rough Structures. In this section we formulate a-priori estimates for solutions to the scattering problem (3.15) for dielectrics ε_r that are allowed to depend on all three variables x_1, x_2, and x_3. Using a perturbation approach we bound the dependence of ε_r on the transverse variables $\tilde{x}$ to preserve the a-priori estimate gained from the Rellich identity. To this end, we shall impose that the term

$$\left| \int_\Omega \frac{\nabla_T \varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}} \cdot \boldsymbol{E}_T \frac{\partial \overline{E}_3}{\partial x_3} \,\mathrm{d}x \right| \leq \left\| \frac{\nabla_T \varepsilon_\mathrm{r}}{\varepsilon_\mathrm{r}} \right\|_{L^\infty(\Omega)^2} \|\boldsymbol{E}_T\|_{L^2(\Omega)^2} \left\| \frac{\partial E_3}{\partial x_3} \right\|_{L^2(\Omega)} \quad (7.1)$$

is small. Here, the L^∞ vector norm is defined as the square root of the sum of all squares of the maximum norm of the components.

Analogously to Assumption (6.2) we suppose that ε_{r} satisfies the following assumptions:

$$\begin{cases} (a) & \varepsilon_{\mathrm{r}} \in W^{1,\infty}(\Omega),\ \operatorname{Re}(\varepsilon_{\mathrm{r}}) \text{ is positive and bounded away from zero,} \\ & \dfrac{\partial \operatorname{Re}(\varepsilon_{\mathrm{r}})}{\partial x_3} \geq 0 \text{ and } \operatorname{Im}(\varepsilon_{\mathrm{r}}) \geq 0 \text{ in } \Omega. \\ (b) & \text{There exists a tubular domain } D_\delta \subset \Omega \text{ of thickness } \delta > 0 \\ & \text{and constants } \gamma > 1 \text{ and } c > 0 \text{ with} \\ & k^2 x_3 \dfrac{\partial \operatorname{Re}(\varepsilon_{\mathrm{r}})}{\partial x_3} - \gamma^2 \left(\dfrac{\partial \log(|\varepsilon_{\mathrm{r}}|)}{\partial x_3} \right)^2 \geq \begin{cases} 0 & \text{in } \Omega, \\ c > 0 & \text{in } D_\delta. \end{cases} \\ (c) & \text{There exist constants } \beta \geq 0 \text{ and } \theta \geq 1/2 \text{ such that} \\ & \left| \operatorname{Im}\left(\dfrac{\partial \log(\varepsilon_{\mathrm{r}})}{\partial x_3} \right) \right| \leq \beta \operatorname{Im}(\varepsilon_{\mathrm{r}})^\theta \quad \text{in } \Omega. \\ (d) & \|\nabla_T \varepsilon_{\mathrm{r}} / \varepsilon_{\mathrm{r}}\|_{L^\infty(\Omega)^2} \leq \sqrt{2}/h. \end{cases} \tag{7.2}$$

REMARK 7.1. *It is possible to construct examples of contrasts that satisfy these requirements. For instance, one can take the piecewise linear profile constructed in Remark 6.6 for parameters* $0 < h_1 < h_2 < h$ *and* $0 < \varepsilon_- < 1$, *and add a sufficiently small function that vanishes for* $x_3 \notin (h_1, h_2)$: $\varepsilon_{\mathrm{r}}(x) = \varepsilon_-$ *for* $x_3 \in (0, h_1)$, $\varepsilon_{\mathrm{r}}(x) = 1$ *for* $x_3 \in (0, h_1)$, *and*

$$\varepsilon_{\mathrm{r}}(x) = \varepsilon_- + (x_3 - h_1)\frac{1 - \varepsilon_-}{h_2 - h_1} + \delta f_1(x_1, x_2) f_2(x_3) \qquad \text{for } x_3 \in (h_1, h_2),$$

where $\delta > 0$ *is a (small) parameter,* $f_1 : \mathbb{R}^2 \to \mathbb{R}$ *possesses bounded weak partial derivatives, and* $f_2 : (0, h) \to \mathbb{R}$ *possesses one bounded weak derivative and vanishes in* $(0, h_1) \cup (h_2, h_3)$. *For* $\delta > 0$ *small enough,* ε_{r} *is positive and increasing in* (h_1, h_2) *with respect to* x_3 *and* $\|\nabla_T \varepsilon_{\mathrm{r}} / \varepsilon_{\mathrm{r}}\|_{L^\infty(\Omega)^2} \leq \sqrt{2}/h$. *Condition (b) from* (7.2) *is also satisfied if* $\delta > 0$ *is chosen small enough.*

LEMMA 7.2. *Let* $\boldsymbol{g} \in L^2_0(\operatorname{div} 0, \Omega)^3$ *and* $k > 0$. *Assume that* ε_{r} *satisfies* (7.2) *and set* $k_\alpha^2 = k^2 + \mathrm{i}\alpha$ *for* $\alpha \in [0, 1]$. *Then any variational solution* $\boldsymbol{E} \in X_{k_\alpha^2}$ *of* (5.1) *satisfies the a-priori estimate*

$$\left(\left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|^2_{L^2(\Omega)^3} + \|\boldsymbol{E}\|^2_{L^2(\Omega)^3} + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2} \right) |\hat{\boldsymbol{E}}|^2 \,\mathrm{d}\xi \right)^{1/2} \leq C\, L_\alpha(\boldsymbol{g}, \boldsymbol{E}), \tag{7.3}$$

where $L_\alpha(\boldsymbol{g}, \boldsymbol{E})$ *is given by* (6.5) *and* C *is the constant from Lemma 6.7.*

Proof. By complementing the square, we note that Lemma 6.4 implies that

$$\begin{aligned} &\left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|^2_{L^2(\Omega)^3} - 2 \left\| \frac{\nabla_T \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \right\|_{L^\infty(\Omega)^2} \|\boldsymbol{E}_T\|_{L^2(\Omega)^2} \left\| \frac{\partial E_3}{\partial x_3} \right\|_{L^2(\Omega)} \\ &\geq \left\| \frac{\partial E_3}{\partial x_3} \right\|^2_{L^2(\Omega)^3} + \frac{2}{h^2} \|\boldsymbol{E}_T\|^2_{L^2(\Omega)^2} - 2 \left\| \frac{\nabla_T \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \right\|_{L^\infty(\Omega)^2} \|\boldsymbol{E}_T\|_{L^2(\Omega)^2} \left\| \frac{\partial E_3}{\partial x_3} \right\|_{L^2(\Omega)} \\ &= \left(\left\| \frac{\partial E_3}{\partial x_3} \right\|_{L^2(\Omega)^3} - \left\| \frac{\nabla_T \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \right\|_{L^\infty(\Omega)^2} \|\boldsymbol{E}_T\|_{L^2(\Omega)^2} \right)^2 + \left(\frac{2}{h^2} - \left\| \frac{\nabla_T \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \right\|^2_{L^\infty(\Omega)^2} \right) \|\boldsymbol{E}_T\|^2_{L^2(\Omega)^2}. \end{aligned}$$

If $\|\nabla_T \varepsilon_{\mathrm{r}} / \varepsilon_{\mathrm{r}}\|_{L^\infty(\Omega)^2} \leq \sqrt{2}/h$, then

$$\left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|^2_{L^2(\Omega)^3} \geq 2 \left\| \frac{\nabla_T \varepsilon_{\mathrm{r}}}{\varepsilon_{\mathrm{r}}} \right\|_{L^\infty(\Omega)^2} \|\boldsymbol{E}_T\|_{L^2(\Omega)^2} \left\| \frac{\partial E_3}{\partial x_3} \right\|_{L^2(\Omega)}. \tag{7.4}$$

In consequence, inequality (5.2) implies that

$$\int_\Omega \left(\left| \frac{\partial \boldsymbol{E}}{\partial x_3} \right|^2 + k^2 x_3 \frac{\partial \operatorname{Re}(\varepsilon_{\mathrm{r}})}{\partial x_3} |\boldsymbol{E}|^2 + \frac{\partial \log(|\varepsilon_{\mathrm{r}}|)}{\partial x_3} \frac{\partial |E_3|^2}{\partial x_3} \right) \mathrm{d}x + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \, \mathrm{d}\xi$$

$$\leq \left(2(h+1) \left\| \frac{\partial \boldsymbol{E}}{\partial x_3} \right\|_{L^2(\Omega)^3} + (1 + 2hk + \sqrt{2\alpha}) \|\boldsymbol{E}\|_{L^2(\Omega)^3} \right) L_\alpha(\boldsymbol{g}, \boldsymbol{E}) \quad (7.5)$$

where $L_\alpha(\boldsymbol{g}, \boldsymbol{E})$ is given by (6.5). The last estimate corresponds to (6.6), and the rest of the proof follows as in proof of Lemma (6.7). ∎

As in the last section, we deduce a uniqueness result and an a-priori estimate in $H(\mathrm{curl}, \Omega)$.

COROLLARY 7.3. *Let $\boldsymbol{g} \in L^2_0(\mathrm{div}0, \Omega)^3$, $k > 0$, and assume that ε_{r} satisfies* (7.2). *Then problem* (5.1) *(or, equivalently, problem* (3.15) *with real wave number, that is, for $\alpha = 0$) possesses at most one solution $\boldsymbol{E} \in X_{k^2}$.*

LEMMA 7.4. *Let $\boldsymbol{g} \in L^2_0(\mathrm{div}0, \Omega)^3$ and $k > 0$. Assume that ε_{r} satisfies* (7.2) *and set $k_\alpha^2 = k^2 + \mathrm{i}\alpha$ for $\alpha \in [0, 1]$. Then any variational solution $\boldsymbol{E} \in X_{k_\alpha^2}$ of* (5.1) *satisfies the a-priori estimate*

$$\left(\|\boldsymbol{E}\|^2_{H(\mathrm{curl},\Omega)} + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}|^2 \, \mathrm{d}\xi \right)^{1/2}$$

$$\leq \sqrt{((k^2+1)\|\varepsilon_{\mathrm{r}}\|_{L^\infty} + 1)C^2 + C} \; L_\alpha(\boldsymbol{g}, \boldsymbol{E}), \quad (7.6)$$

where C is the constant from Lemma 6.7 and $L_\alpha(\boldsymbol{g}, \boldsymbol{E})$ is given by (6.5).

The proof is analogous to the proof of Lemma 6.9, it suffices to replace estimate (6.4) by (7.3).

8. Solvability of the Variational Formulation in $\boldsymbol{X_{k^2}}$. In this section we show solvability of the variational problem (3.15) for real wave number $k > 0$ and right-hand sides in $L^2_0(\mathrm{div}0, \Omega)^3$ by combining the a-priori estimate from Sections 6 and 7 with a limiting absorption approach. This approach consists in considering the scattering problem first for complex wave number, say, $k_\alpha^2 = k^2 + \mathrm{i}\alpha$. The corresponding variational problem in the entire upper half-space is easily seen to be coercive. As $\alpha \to 0$ we exploit the a-priori estimates from the last two sections to conclude that the solutions $\boldsymbol{E}_\alpha \in X_{k_\alpha^2}$ remain bounded in $H(\mathrm{curl}, \Omega)$, thus, we can extract a weakly convergent subsequence and a weak limit $\boldsymbol{E} \in H(\mathrm{curl}, \Omega)$. This limit satisfies the differential equation $\mathrm{curl}^2 \boldsymbol{E} - k^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} = \boldsymbol{g}$. Finally, we show that $\boldsymbol{E}$ belongs to X_{k^2} and solves the variational formulation (3.15).

LEMMA 8.1. *Assume that $\varepsilon_{\mathrm{r}} \in W^{1,\infty}(\Omega)$ satisfies* (6.2) *or* (7.2). *Let $k > 0$, $\alpha \in (0, 1]$ and set $k_\alpha^2 = k^2 + \mathrm{i}\alpha$. Then there is a unique solution $\boldsymbol{E}_\alpha \in X_{k_\alpha^2}$ to the variational problem* (3.15) *with complex wave number k_α^2 for any right-hand side $\boldsymbol{g} \in L^2_0(\mathrm{div}0, \Omega)^3$. This solution belongs to $H^1(\Omega)^3$ and it satisfies*

$$\|\boldsymbol{E}_\alpha\|^2_{H(\mathrm{curl},\Omega)} + \int_{\mathbb{R}^2} \operatorname{Im}(\sqrt{k_\alpha^2 - |\xi|^2}) |\boldsymbol{E}_\alpha|^2 \, \mathrm{d}\xi \leq C \|\boldsymbol{g}\|^2_{L^2(\Omega)^3}$$

with a constant C independent of $\alpha \in (0, \alpha^*]$ for some $\alpha^* > 0$. For any sequence $\alpha_n \to 0$ as $n \to \infty$ there exists a subsequence, also denoted by α_n, such that $\boldsymbol{E}_{\alpha_n}$ converges weakly in $H(\mathrm{curl}, \Omega)$ and $\boldsymbol{E}_{\alpha_n}|_{\Gamma_h}$ converges weakly in $H^{1/2}(\Gamma_h)^3$. The limit element $\boldsymbol{E} \in H(\mathrm{curl}, \Omega)$ belongs to X_{k^2} and is the unique solution of the variational problem (3.15) *with wave number $k > 0$.*

Proof. Problem (3.15) with complex wave number k_α^2 is equivalent to the following problem in the upper half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3, \, x_3 > 0\}$: Find $\boldsymbol{E} \in H_0(\mathrm{curl}, \mathbb{R}^3_+) = \{\boldsymbol{u} \in H_0(\mathrm{curl}, \mathbb{R}^3_+), \, \boldsymbol{u} \times$

$e_3 = 0$ on $\Gamma_0\}$ such that

$$\int_{\mathbb{R}^3_+} \left(\operatorname{curl} \boldsymbol{E} \cdot \operatorname{curl} \overline{\boldsymbol{\psi}} - k_\alpha^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} \cdot \boldsymbol{\psi}\right) \, \mathrm{d}x = \int_{\mathbb{R}^3_+} \boldsymbol{g} \cdot \boldsymbol{\psi} \, \mathrm{d}x \tag{8.1}$$

for all $\boldsymbol{\psi} \in H_0(\operatorname{curl}, \mathbb{R}^3_+)$. Indeed, the restriction of a solution of (8.1) to Ω solves (3.15) and the extension of a solution of (3.15) by means of (3.18) solves (8.1). Note that the complex wave number k_α^2 implies that the extension by means of (3.18) is exponentially decaying in the x_3 direction. Problem (8.1) is coercive, thus, there exists a unique solution $\boldsymbol{E}_\alpha \in H(\operatorname{curl}, \mathbb{R}^3_+)$. By abuse of notation, we also denote the restriction of this solution to Ω by $\boldsymbol{E}_\alpha$ and note that the differential equation $\operatorname{curl}^2 \boldsymbol{E}_\alpha - k_\alpha^2 \varepsilon_{\mathrm{r}} \boldsymbol{E}_\alpha = \boldsymbol{g}$, together with $\boldsymbol{g} \in L_0^2(\operatorname{div}0, \Omega)^3$ yields that $\boldsymbol{E}_\alpha \in X_{k_\alpha^2}$. Lemma 3.5 yields that $\boldsymbol{E}_\alpha \in H^1(\Omega)^3$. The Rellich identity from Proposition 5.1 is applicable and yields, by means of Lemma 6.9 or Lemma 7.4, the existence of three constants C_1, C_2 and C_3, each independent of α, such that

$$\|\boldsymbol{E}_\alpha\|^2_{H(\operatorname{curl},\Omega)} + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}_\alpha|^2 \, \mathrm{d}\xi \\ \leq C_1 \|\boldsymbol{g}\|^2_{L^2(\Omega)^3} + C_2 \|\boldsymbol{g}\|_{L^2(\Omega)^3} \|\boldsymbol{E}_\alpha\|_{L^2(\Omega)^3} + C_3 \alpha \|\boldsymbol{E}_\alpha\|^2_{L^2(\Omega)}.$$

Thus, for $C_3 \alpha < 1/2$ one deduces the existence of a constant C independent of α such that

$$\|\boldsymbol{E}_\alpha\|_{H(\operatorname{curl},\Omega)} + \int_{\mathbb{R}^2} \operatorname{Im}\left(\sqrt{k_\alpha^2 - |\xi|^2}\right) |\hat{\boldsymbol{E}}_\alpha|^2 \, \mathrm{d}\xi \leq C \|\boldsymbol{g}\|_{L^2(\Omega)^3}.$$

In particular, the norms $\|\boldsymbol{E}_\alpha\|_{H(\operatorname{curl},\Omega)}$ are uniformly bounded with respect to α and each subsequence $\boldsymbol{E}_{\alpha_n}$, $\alpha_n \to 0$ as $n \to \infty$, contains a weakly convergent subsequence. Let $\boldsymbol{E}$ denote the weak limit of such a subsequence. Since $\operatorname{div}(\varepsilon_{\mathrm{r}} \boldsymbol{E}_\alpha) = 0$ in Ω we also have $\operatorname{div}(\varepsilon_{\mathrm{r}} \boldsymbol{E}) = 0$ in Ω. On the other hand,

$$\begin{aligned} \|(\boldsymbol{E}_{\alpha_n})_T\|^2_{H^{1/2}(\Gamma_h)^2} &= \int_{\mathbb{R}^2} (1 + |\xi|^2)^{1/2} |(\hat{\boldsymbol{E}}_{\alpha_n})_T|^2 \, \mathrm{d}\xi \\ &\leq \frac{\max(1, 1/k^2)}{\sqrt{2}} \int_{||\xi| - k| > 2k} \operatorname{Im}(\sqrt{k_\alpha^2 - |\xi|^2}) |(\hat{\boldsymbol{E}}_{\alpha_n})_T|^2 \, \mathrm{d}\xi \\ &\qquad + (1 + 9k^2)^{-1} \int_{||\xi| - k| < 2k} (1 + |\xi|^2)^{-1/2} |(\hat{\boldsymbol{E}}_{\alpha_n})_T|^2 \, \mathrm{d}\xi. \end{aligned}$$

This estimate combined with the trace theorem for the tangential component from $H(\operatorname{curl}, \Omega)$ into $H^{-1/2}(\Gamma_h)^2$ implies that $(\boldsymbol{E}_{\alpha_n})_T$ is uniformly bounded in $H^{1/2}(\Gamma_h)^2$. Thus, eventually extracting a further subsequence, $(\boldsymbol{E}_{\alpha_n})_T$ converges weakly in $H^{1/2}(\Gamma_h)^2$ to $\boldsymbol{E}_T$. Further, the trace theorem for the normal component from $H(\operatorname{div}, \Omega)$ into $H^{-1/2}(\Gamma_h)$ yields in the same way that, eventually extracting a further subsequence, also $(E_{\alpha_n})_3$ converges weakly in $H^{1/2}(\Gamma_h)$.

From the partial differential equation satisfied by $\boldsymbol{E}_\alpha$ we infer that $\operatorname{curl}^2 \boldsymbol{E} - k^2 \varepsilon_{\mathrm{r}} \boldsymbol{E} = \boldsymbol{g}$ in Ω. The same trace argument shows that Γ_0, $\boldsymbol{E} \times \boldsymbol{e}_3 = 0$ on Γ_0. However, we still need to check whether $\boldsymbol{E}$ satisfies the radiation condition (2.4), or, in other words, whether $\boldsymbol{E} \in X_{k^2}$ and whether $\operatorname{curl} \boldsymbol{E} \times \boldsymbol{e}_3|_{\Gamma_h} = T^+_{k^2}(\boldsymbol{E}_T|_{\Gamma_h}) + N^+_{k^2}(\boldsymbol{E}_T|_{\Gamma_h})$.

We also know that the third component $(E_\alpha)_3$ converges weakly in $H^{1/2}(\Gamma_h)$, that is, by means of (3.18), the inverse Fourier transform of

$$\frac{\xi \cdot \mathcal{F}((\boldsymbol{E}_\alpha)_T)}{\sqrt{k_\alpha^2 - |\xi|^2}} = -\mathcal{F}((E_\alpha)_3) \tag{8.2}$$

converges weakly in $H^{1/2}(\Gamma_h)$ to $-E_3$. Let $\chi \in C^\infty(\mathbb{R})$ be a smooth cut-off function that equals one in a neighborhood of k and vanishes outside a bounded neighborhood of k. On the one hand, the inverse Fourier transform of $(1-\chi(|\xi|))(\xi\cdot\mathcal{F}((\boldsymbol{E}_\alpha)_T))/\sqrt{k_\alpha^2-|\xi|^2}$ converges weakly to the inverse Fourier transform of $(1-\chi(|\xi|))(\xi\cdot\mathcal{F}(\boldsymbol{E}_T))/\sqrt{k^2-|\xi|^2}$ in $H^{1/2}(\mathbb{R}^2)$. On the other hand, using Lebesgue's dominated convergence theorem one can check that $1/\sqrt{k_\alpha^2-|\xi|^2} = 1/\sqrt{k^2+\mathrm{i}\alpha-|\xi|^2}$ converges in $L^1(\mathbb{R}^2)$ to $1/\sqrt{k^2-|\xi|^2}$. Hence, $\chi(|\xi|)(\xi\cdot\mathcal{F}((\boldsymbol{E}_\alpha)_T))/\sqrt{k_\alpha^2-|\xi|^2}$ converges in the distributional sense to $\chi(|\xi|)(\xi\cdot\mathcal{F}(\boldsymbol{E}_T))/\sqrt{k^2-|\xi|^2}$. Combined with the previous statement we obtain that the inverse Fourier transform of $(\xi\cdot\mathcal{F}((\boldsymbol{E}_\alpha)_T))/\sqrt{k_\alpha^2-|\xi|^2}$ converges in the distributional sense to the inverse Fourier transform of $(\xi\cdot\mathcal{F}(\boldsymbol{E}_T))/\sqrt{k^2-|\xi|^2}$. By uniqueness of the limit, this convergence holds also weakly in $H^{1/2}(\mathbb{R}^2)$. We conclude that $\boldsymbol{E}$ belongs to X_{k^2}.

Concerning the boundary condition satisfied by $\boldsymbol{E}$, we note that

$$\operatorname{curl}\boldsymbol{E}_\alpha\times\boldsymbol{e}_3|_{\Gamma_h} = T^+_{k_\alpha^2}((\boldsymbol{E}_\alpha)_T|_{\Gamma_h}) + N^+_{k_\alpha^2}((\boldsymbol{E}_\alpha)_T|_{\Gamma_h}) \tag{8.3}$$

holds in $H^{-1/2}(\Gamma_h)^3$. Since $\operatorname{curl}^2\boldsymbol{E}_\alpha$ is bounded in $L^2(\Omega)^3$ we can assume that up to extracting a subsequence $\operatorname{curl}\boldsymbol{E}_\alpha\times\boldsymbol{e}_3$ converges weakly to $\operatorname{curl}\boldsymbol{E}\times\boldsymbol{e}_3$ in $H^{-1/2}(\Gamma_h)^3$. Using weak convergence of the trace $(\boldsymbol{E}_\alpha)_T$ in $H^{1/2}(\Gamma_h)^3$ and the previous results one can check by working in the Fourier domain that the right-hand side in (8.3) converges in $H^{-1/2}(\Gamma_h)^3$ to $T^+_{k^2}(\boldsymbol{E}_T|_{\Gamma_h}) + N^+_{k^2}(\boldsymbol{E}_T|_{\Gamma_h})$.
□

A simple corollary of the last convergence result and the a-priori estimate (7.6) is the following existence and uniqueness result for the variational problem (3.15).

THEOREM 8.2. *Assume that* ε_r *satisfies* (6.2) *or* (7.2) *and let* $k>0$. *Then there is a unique solution* $\boldsymbol{E}\in X_{k^2}$ *to the variational problem* (3.15) *for any right-hand side* $\boldsymbol{g}\in L^2_0(\mathrm{div}0,\Omega)^3$. *This solution satisfies the a-priori estimate given in* (6.8) *or* (7.6) *for* $\alpha=0$, *depending on which of the assumptions* (6.2) *or* (7.2) *is supposed.*

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