A satellite imaging chain based on the Compressed Sensing technique

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A SATELLITE IMAGING CHAIN BASED ON THE COMPRESSED SENSING TECHNIQUE

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1. INTRODUCTION

Satellite imaging has been the focus of intense work in remote sensing for the last years. The ability of satellite optical systems to produce high-resolution images has indeed been of a great interest in applications such as change detection or image classification. It has however outcomed to be quite challenging for the design of satellite acquisition chains. Due to the increase of the imaging resolution and the conservation of the swath, images acquired from newly built satellites[1] can actually reach a size of 30000×30000 pixels. Dealing with such a volume of data has important consequences on embedded resources, which require more memory, more computing capacity and therefore more powerful electrical sources.

In a classical satellite imaging system, the observed image is sampled at the Shannon frequency to give N pixels and then compressed by some coding algorithms, like the JPEG standards. The purpose of the coding step is to represent the image on a limited number $K << N$ of coefficients to match the low capacity of the on-board mass storage. However, using an expensive scheme to sample the whole image for, finally, retaining only K coefficients may appear to be wasteful. Many on-board resources could then be saved up if the compressed coefficients were directly acquired out of the sensor.

Recently, a new theory of sampling has been emerged in the signal processing community. This theory, introduced as the Compressive Sampling or Compressed Sensing (CS) [1], suggests that one can reconstruct perfectly a signal, supposed to be sparse in some basis, from a limited (i.e. fewer than Shannon) number of incoherent measurements. Although the design of a sensor able to produce these measurements is difficult and beyond the scope of this paper, the CS technique clearly appears to be adapted to the satellite imaging chain. It could indeed drastically simplify the process of image acquisition by providing a reduced number of measurements, directly outcomed from the sensor, therefore saving an important quantity of resources. It is also valuable to point out that the CS framework provides an acquisition technique whose performances depend mainly on the reconstruction algorithm done on the ground. In comparison, the current acquisition imaging chain is bounded by the efficiency of the compression scheme embedded on-board. If one wants to increase the quality of the final image, one has to design a new satellite imaging chain. This "universal" coding feature [2] of the CS is thus very attractive.

Though it is not a general result (see [3] for example), previous works [4] have shown that the CS technique may be competitive regarding to a wavelet-based compression scheme on smoothed classical test images. But to the best of our knowledge, no works have been dedicated to this comparison for high-resolution satellite imaging, taking into account the degradations of the satellite imaging acquisition chain (blur, instrumental and quantizing noises). Our contribution is then to present a numerical exploration of the Compressed Sensing technique applied to satellite imaging. We present a satellite imaging chain

[1]such as the european PLEIADES-HR satellite.

based on this technique and we evaluate its efficiency with respect to the classical imaging chain composed of a Nyquist sampling followed by a wavelet-based compression scheme. Note that this work only focusses on the problem of CS image reconstruction in application to satellite imaging and does not deal with the problem of designing a sensor able to produce the measurements. The conception of such sensor is a very difficult task. Some results have been obtained on this aspect [5] but much work still needs to be addressed to provide efficient high-resolution CS sensors.

The rest of the paper is organized as follows. In section 2, we present the Compressed Sensing theory. We propose, in section 3, an algorithm adapted to the satellite imaging which reconstructs the image from the CS measurements. The originality of the proposed algorithm is that it takes into account the main degradations of the imaging chain such as blur, instrumental and quantizing noise. We compare, in section 4, the reconstruction results of this method to the ones obtained using the classical acquisition chain based on a wavelet transform. We show global rate-distortion and visual results on a real satellite data. We conclude in section 5 and present perspectives for future works.

2. COMPRESSIVE SAMPLING FRAMEWORK

Compressive Sampling (CS) is a new emergent sampling theory introduced independently by E.J Candès [6] and D.L. Donoho [7]. The motivation behind the CS technique is to perform in the same time the acquisition and the compression of the signal. We give a quick overview of this technique in this section but more information can be found in the referred works.

In a classical satellite imaging system, the image is firstly sampled at the Nyquist frequency to give N pixels. However, it is well-known that natural images have a sparse representation in wavelet basis [8] such that almost all the information of the image holds on a limited number $K << N$ of coefficients. This property of sparsity is strongly used by image compression systems like the JPEG-2000 standard which projects the image on a wavelet basis such that only K coefficients become significant. But this technique does not seem to be adapted to low-resources applications as all the image is usually required to perform the compression. One needs then to design an acquisition chain which has sufficient resources to sample and store the N pixels of the image which, considering its size, is very expensive to achieve.

The main result of the Compressive Sampling theory states that the original signal can be recovered from a small number of measurements [6] directly outcomed from the sensor. The key of the CS theory relies on the supposed sparsity of the original signal, meaning that it can be perfectly represented in some basis Ψ with only K non-null coefficients. Based on this property of sparsity, the authors of [6] showed that only M (with $M << N$) measurements are required to perfectly reconstruct the original signal with a high probability. These M observations are obtained by the projection of the true image $x_0 \in \mathbb{R}^N$ on a measurement matrix $\Phi : \mathbb{R}^N \to \mathbb{R}^M$

$$y = \Phi x_0. \tag{1}$$

This formulation is actually very common and the particularity of the CS lies in the design of the measurement matrix. As mentioned by [6], the matrix Φ should spread the information of the image and most of random matrices perfectly meet this condition. More generally, a property named the Restricted Isometry Property (RIP) has been introduced in [2] to ensure that the image may be recovered form the incoherent measurements. To simplify, the RIP requires the sub-matrices of the measurement matrix Φ to be orthogonal enough to be able to recover the K-sparse components of the image x_0.

If the RIP condition is satisfied, the image x_0 can be recovered by minimizing the l^1-norm of its coefficients under the constraint that its projection on Φ is equal to the observed vector y [9]

$$\text{Find } \tilde{x} = \begin{array}{rl} \arg\min & \|\Psi x\|_1 \\ \text{subject to} & x \in \mathbb{R}^N \\ & y = \Phi x \end{array} . \tag{2}$$

The optimization problem (2) is a particular instance of the Basis Pursuit (BP) problem [10] which can be solved using classical algorithms from the linear programming literature. It can be interpreted as follows. The measurement matrix Φ is a matrix which spreads the image in the measurement vector y. Although it is not orthogonal, if Φ satisfies the RIP, then the inverse solution $\Phi^\dagger y$ contains all the information of the image x but in disorder. Also remind that the representation of the image x in the basis Ψ is sparse or, in other words, strongly compact. Minimizing the l^1-norm of its coefficients will then put the non-null coefficients back at the correct position, recovering therefore the whole image.

The more sparse is x_0, the easier it will be for the algorithm (2) to place these non-null coefficients. Clearly, recovering the image x_0 highly depends on the link between the compactness of the decomposition basis Ψ and the diffusion of the measurement matrix Φ. More generally, the algorithm (2) efficiently recovers the original image only if matrices Φ and Ψ are completely uncorrelated.

A mutual coherence (MC) μ has been introduced in [11] to measure this correlation and more precisely, to measure the correlation between each vector basis ϕ_i and ψ_i of Φ and Ψ. This coherence belongs to $[1, \sqrt{N}]$ [1]; a small value of μ meaning that the matrices Ψ and Φ are completely uncorrelated. For example, if Φ is the Fourier basis, then the minimal coherence is obtained with $\Psi = I$ (the sampling operator) and is equal to 1. For our application, we will detail the choice of matrices Φ and Ψ in the numerical section of the paper. More generally, solving (2) recovers x_0 exactly if

$$M \geq C\mu^2(\Phi, \Psi)K \log(N), \quad C < 1 \text{ is a constant.} \tag{3}$$

In classical imaging systems, acquired images are usually degraded by both blur and instrumental noise. As shown in [12], the CS technique can be extended to this scenario. The acquisition model becomes [12]

$$y = \Phi H x_0 + z, \tag{4}$$

where $H : \mathbb{R}^N \to \mathbb{R}^N$ is the blur matrix and $z \in \mathbb{R}^M$ is an additive noise. In the classical case of an additive white Gaussian noise of variance σ_z^2, the reconstruction algorithm may write [12]

$$\text{Find } \tilde{x} = \begin{array}{rl} \arg\min & \|\Psi x\|_1 \\ \text{subject to} & x \in \mathbb{R}^N \\ & \|y - \Phi H x\|_2^2 \leq M\sigma_z^2 \end{array}. \tag{5}$$

Similarly to (2), the optimization problem (5) is a particular instance of Basis Pursuit Denoising (BPDN) which can also be solved using linear programming techniques [10]. Based on the CS framework, we formulate in the next part the acquisition model in the case of the satellite chain and we propose an algorithm to reconstruct the image from the measurements vector.

3. APPLICATION TO SATELLITE IMAGING

As said previously, we assume that we have at our disposal a sensor able to produce incoherent measurements, in the sense of the CS framework. We are interested in evaluating the quality of the reconstructed image in comparison to the image obtained using the current acquisition chain based on wavelet compression [8].

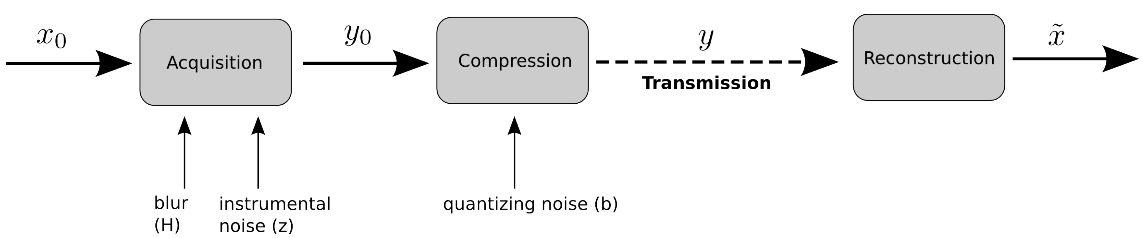

Fig. 1. Considered acquisition imaging chain.

A classical satellite imaging chain is represented figure 1. It is mainly composed by three parts: The acquisition which captures the scene, the compression which reduces the weight of the measured image and the restoration which reconstructs the image from the measurements and reduces the degradations collected during the processing chain. We assume that the transmission does not introduce any degradations on the measured coefficients.

The acquisition process affects the quality of the true image by adding blur and instrumental noise. The blur H is mainly caused by the natural environment and the imperfection of the acquisition components. It is supposed to be known. The instrumental noise z is also the composition of several noise sources such as a photon noise and an electronic noise. It can be globally modeled as a zero-mean Gaussian distribution with a known variance σ_z^2.

In addition to blur and instrumental noise, the measurements are also degraded by quantizing noise. A satellite is not able to continuously transmit the acquired images as it sometimes flies over "shadow zones" where a transmission is not possible. It has to save the acquired images on its on-board mass storage to transmit them later. But the on-board storage capacity of a satellite is highly limited. The measurements are thus always coded to fit a target compression rate, which is also beneficial for fast transmission. This compression introduces an irreversible error known as the coding or quantizing noise.

This quantization Q can be modeled as a scalar uniform quantization which quantizing step Δ_i depends on the coefficient ${y_0}_i, i \in \{1, \ldots, M\}$ regarded

$$Q\left[{y_0}_i\right] = \Delta_i \left\lfloor \frac{{y_0}_i}{\Delta_i} + \frac{1}{2} \right\rfloor, \tag{6}$$

where $\lfloor . \rfloor$ is the floor function which returns the greatest integer less than or equal to its argument. The quantizing step Δ_i can be transmitted with the image as in the JPEG standard [13] or can be deduced during the decoding algorithm for more recent methods [14, 15]. We assume in the following that the quantizing steps Δ_i are known. Let $b = Q[y_0] - y_0$ be the quantizing error. From (6), we have for each coordinate b_i of b

$$-\frac{\Delta_i}{2} \leq b_i < \frac{\Delta_i}{2}, \quad \forall i \in \{1, \ldots, M\} \tag{7}$$

or equivalently

$$b \in B, \quad \text{with } B := \left\{ b \in \mathbb{R}^M, -\frac{\Delta_i}{2} \leq b_i < \frac{\Delta_i}{2} \right\}. \tag{8}$$

Using the previous definition of b, we propose to modelize the the observed measurements as

$$y = Q\left[\Phi H x_0 + z\right] = \Phi H x_0 + z + b, \tag{9}$$

where y is the measurements vector. The extension of the reconstruction algorithm (5) to the acquistion model (9) is simple. First, simply remark that the problem (5) can also be written

$$\begin{array}{rrl} \text{Find } \tilde{x} = & \arg\min & \|\Psi x\|_1 \\ & \text{subject to} & x \in \mathbb{R}^N, z \in \mathbb{R}^M \\ & & \|z\|_2^2 \leq M\sigma_z^2 \\ & & y = \Phi H x + z \end{array}. \tag{10}$$

The variable b needs to be add to the problem (10) to take into account the presence of the coding noise. Using (8) and (9), the reconstruction problem writes

$$\begin{array}{rrl} \text{Find } \tilde{x} = & \arg\min & \|\Psi x\|_1 \\ & \text{subject to} & x \in \mathbb{R}^N,\ z \in \mathbb{R}^M,\ b \in \mathbb{R}^M \\ & & \|z\|_2^2 \leq M\sigma_z^2, \\ & & b \in B, \\ & & y = \Phi H x + z + b \end{array}. \tag{11}$$

The problem (11) can be further simplified by noting that the variable b can be replaced by $y - (\Phi H x + z)$. We finally propose to formulate the reconstruction problem as

$$\begin{array}{rrl} \text{Find } \tilde{x} = & \arg\min & \|\Psi x\|_1 \\ & \text{subject to} & x \in \mathbb{R}^N,\ z \in \mathbb{R}^M \\ & & \|z\|_2^2 \leq M\sigma_z^2, \\ & & y - (\Phi H x + z) \in B \end{array}. \tag{12}$$

The optimization problem (12) is a convex problem constrained on convex sets and thus admits a unique (convex) set of solutions [16]. However, the presence of the linear operators Ψ, Φ and H make it difficult to solve. We propose here to use the Alternating Direction Method of Multipliers (ADMM) proposed in [17]. The idea of this algorithm is to insert split variables such that the linear operators in (12) are moved into a linear constraint. Its presentation is however beyond the scope of this paper, and we refer the interested reader to [17] (and references therein) for futher information on the numerical implementation of this algorithm.

4. NUMERICAL RESULTS

In this part, we evaluate the performances of the CS technique for satellite imaging in comparison to the chain based on a wavelet compression scheme. For the numerical experiments, we choose the measurement matrix Φ to be the noiselet transform [18] and set Ψ to be the gradient operator such that $||\Psi x|||_1$ is the Total Variation (TV) [19]. We made the choice of the TV as it is almost equivalent to a Haar basis which, as required by the CS framework, shares a small mutual coherence with the noiselet transform [1]. Note that a TV prior is also well adapted to the regularity of high-resolution satellite images.

As mentioned previously, we compare the CS acquisition technique to the classical acquisition chain which consists in sampling the real image at the Shannon frequency followed by a compression scheme. The considered compression algorithm is close to the JPEG-2000 standard and uses the bi-orthogonal Cohen-Daubechies-Feauveau (CDF) 9/7 wavelet transform described in [20] followed by the same quantization process as the one defined in (6). As in the CS technique, we can design an algorithm to reconstruct the image from the noisy observed wavelet coefficients y

$$\text{Find } \tilde{x} = \begin{array}{rl} \arg\min & \|\Psi x\|_1 \\ \text{subject to} & x \in \mathbb{R}^N, \ z \in \mathbb{R}^N \\ & y - W(Hx + z) \in B \\ & \|z\|_2^2 \leq N\sigma_z^2 \end{array}, \tag{13}$$

where W is the CDF 9/7 wavelet transform and Ψ is the gradient operator. Note that the formulation (13) is not expressed using any matrix Φ as, in this case, the measurement matrix is the sampling operator ($\Phi = I$). We will compare the results of techniques (12) and (13) visually but also in a rate-distortion sense. As both techniques offer different ways to control the target coding rate, we now detail the choice of the chain in each case.

For the CS technique, we take benefit from the fact that the image can ideally be reconstructed from less measurements than Shannon. More precisely, for a low target rate, we will restrict the number of measurements M to be small and when the target rate is high, we will increase this number, the maximum number of measurements being equal to the number of pixels N.
This particular choice comes from the fact that the distribution of the CS coefficients is quite large and that a high quantization has to be applied on these coefficients to reach low target rates [21]. It seems then more appropriate to tune the number of measurements M instead of tuning the quantizing steps, for a given coding rate. Consequently, we will always take $\Delta_i = 1, \forall i \in \{1, \ldots, M\}$ for all coding rates. Note that these measurements will be taken randomly and that the position of the retained coefficients can be known at each side of the chain by transmitting the seed of the random generator.

The imaging chain based on a wavelet scheme does not however offer such feature. More precisely, all the coefficients have to be retained to be able to reconstruct the image. Consequently, for this technique, we will keep all the coefficients and we will tune the quantizing steps to reach the target coding rate. For more simplicity, we will take the same quantizing step for all coefficients $\Delta_i = \Delta, \forall i \in \{1, \ldots, N\}$.

The reconstruction error will be evaluated using the Peak Signal-To-Noise Ratio (PSNR) defined, for 12 bits images, as

$$PSNR(x_0, \tilde{x}) = 10 \log_{10} \left(\frac{4095^2}{\frac{1}{N}\|x_0 - \tilde{x}\|_2^2} \right). \tag{14}$$

where x_0 is the reference image and $\tilde{x}$ is the reconstructed one. As mentioned previously, we will evaluate the results in a rate-distortion sense. The distortion will be evaluated using the reconstruction error (14). For the evaluation of the coding rate, we assume that the quantized coefficients will be encoded using an entropy encoder. The coding rate can then be measured using the entropy (expressed in bits/symbol) of the coefficients y [22]. It is defined as

$$H(y) = -\sum_{m=-\infty}^{\infty} p_y(m) \log_2(p_y(m)), \tag{15}$$

where $p_y(m)$ is the probability for a quantized coefficient to get the symbol m. Note that for the imaging chain based on the CS technique, we only retain M coefficients. The entropy of the quantized coefficients will be thus multiplied by the ratio between the number of measurements and the number of pixels for that case.

We simulate the two imaging chains on the reference image depicted figure 2. This image is a simulation of the PLEAIDES-HR imaging chain. The blur H used in this simulation is then the Point Spread Function (PSF) of the optics of the PLEAIDES-HR satellite which has been provided by the CNES. For the simulation, we set the standard deviation of the instrumental noise to be equal to 10, $\sigma_z = 10$.

Fig. 2. Reference image, Cannes harbour (1024×1024 pixels). The dynamic range of this image is 12 bits.

Results are shown on figures 3 and 4. From the rate-distortion function displayed on figure 3, we see that the CS technique does not give competitive reconstruction results in comparison to the wavelet-based technique, and stands 5-6 dB below this technique, for all compression rates. Visually, the reconstructed images are not very good as well. We can see on figure 4 that the CS reconstruction algorithm overregularizes the solution and creates large patterns, therefore losing the details of the image. Although it seems clear that the CS is a good acquisition technique as it better spreads the information than a wavelet transform, it also appears that high-resolution satellites images are not sparse enough, in usual basis, to be used with this technique.

Moreover, as said previously the CS coefficients have a large distribution (larger than wavelet coefficients) making their coding difficult to perform, even when one only retains a limited number of these coefficients. We have however strong thoughts that the CS could be an efficient acquisition strategy for satellite images as it has already shown interesting results in application where the image is naturally strongly sparse, such as in MRI application [23]. Following this idea, an imaging chain based on the CS technique may be interesting for galaxy observation missions which naturally give sparse images, as in astronomy where the CS exhibits great performances [24]. But in the case of earth observation missions, the approximative sparsity of the images does not seem to be sufficient to make the CS technique competitive regarding to the classical wavelet approach.

5. CONCLUSION

In this paper, we have experimentaly studied the performances of the CS acquisition technique in application to satellite imaging. We have proposed an algorithm to take into account the degradations of the satellite imaging chain (blur, instrumental and quantizing noise) and showed reconstruction results, visually and in a rate-distortion sense, on a real satellite data. We performed a comparison of this method to the classical acquisition method based on a wavelet transform and we showed that the CS acquisition technique does not give competitive results for earth observation imaging but may be interesting for observation missions which give sparse images. There is also room for improvements by considering more properly the distribution of satellite images to enhance usual priors and quantizing strategies which need to better fit the characteristics of the CS coefficients distribution. This will be the focus of future works.

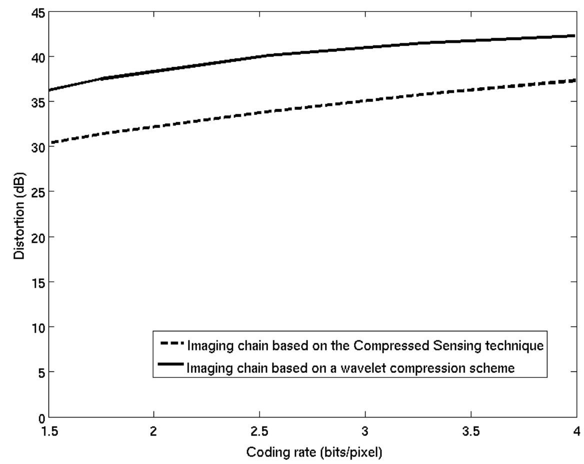

Fig. 3. Rate-distortion function for the two acquisition techniques. The dashed curve is the PSNR w.r.t. the compression rate for the CS acquisition technique while the solid curve is the PSNR w.r.t. the compression rate for the wavelet-based method.

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Fig. 4. Reconstruction results for the two acquisition techniques at a compression rate of 2.5 bits/pixel. For each line, from left to right: Zoom on the original image, zoom on the reconstructed image using the CS technique ($PSNR = 33.8\ dB$) and zoom on the reconstructed image using the wavelet-based technique ($PSNR = 40\ dB$).

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