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Parametric instabilities of circularly polarized small-amplitude Alfvén waves in Hall plasmas

MICHAEL S. RUDERMAN and PHILIPPE CAILLOL

Department of Applied Mathematics, University of Sheffield, Hicks Building, Hounsfield Road, Sheffield S3 7RH, UK
(m.s.ruderman@sheffield.ac.uk)

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Abstract. We study the stability of circularly polarized Alfvén waves (pump waves) in Hall plasmas. First we re-derive the dispersion equation governing the pump wave stability without making an *ad hoc* assumption about the dependences of perturbations on time and the spatial variable. Then we study the stability of pump waves with small non-dimensional amplitude a ($a \ll 1$) analytically, restricting our analysis to $b < 1$, where b is the ratio of the sound and Alfvén speed. Our main results are the following. The stability properties of right-hand polarized waves are qualitatively the same as in ideal MHD. For any values of b and the dispersion parameter τ they are subject to decay instability that occurs for wave numbers from a band with width of order a. The instability increment is also of order a. The left-hand polarized waves can be subject, in general, to three different types of instabilities. The first type is the modulational instability. It only occurs when b is smaller than a limiting value that depends on τ. Only perturbations with wave numbers smaller than a limiting value of order a are unstable. The instability increment is proportional to a^2. The second type is the decay instability. It has the same properties as in the case of right-hand polarized waves; however, it occurs only when $b < 1/\tau$. The third type is the beat instability. It occurs for any values of b and τ, and only perturbations with the wave numbers from a narrow band with the width of order a^2 are unstable. The increment of this instability is proportional to a^2, except for τ close to τ_c when it is proportional to a, where τ_c is a function of b.

1. Introduction

Parametric instabilities of finite-amplitude circularly polarized Alfvén waves have been studied for more than four decades. Galeev and Oraevskii (1963) were the first who studied the parametric instability of a circularly polarized Alfvén wave (pump wave in what follows) with a small amplitude in a low-β plasma using the ideal MHD approximation (see also Sagdeev and Galeev 1969). Derby (1978) and Goldstein (1978) derived the dispersion equation determining the stability of pump waves with arbitrary amplitudes and in finite β plasmas once again using the ideal MHD approximation. Sakai and Sonnerup (1983) and Longtin and Sonnerup (1986) studied the stability of pump waves using the two-fluid plasma description, which takes both ion and electron inertia into account (see also Brodin and Stenflo 1988). Wong and Goldstein (1986) investigated the pump wave stability in Hall plasmas.

Including the Hall term in Ohm's equation is equivalent to taking into account the ion inertia. Hence, the equations used by Wong and Goldstein (1986) are obtained from equations used by Sakai and Sonnerup (1983) and Longtin and Sonnerup (1986) in the limit $k_0\ell_e \ll 1$, where k_0 is the wave number of a pump wave and ℓ_e is the electron inertia length. Brodin and Stenflo (1990) suggested a new approach to studying the stability of MHD waves in Hall plasmas. These authors considered the resonant interaction of three waves and calculated the coupling coefficients of this interaction. Using this result they studied the decay and modulational instabilities of a large-amplitude pump wave.

The stability analysis of circularly polarized Alfvén waves was then extended in different directions. Stenflo (1976) considered relativistic multi-component plasmas and described the general method for deriving the dispersion relation determining the pump wave stability. Then he studied in detail the case of electronic plasmas with immovable ions. Lashmore-Davies and Stenflo (1979) studied the stability of a helical magnetic field similar to the one in an Alfvén wave, but created by an external current. Shukla and Stenflo (1985) considered the nonlinear behaviour of ion-cyclotron Alfvén waves. Brodin and Lundberg (1990) analyzed the stability of an electromagnetic circularly polarized wave in a plasma with anisotropic pressure. Viñas and Goldstein (1991) investigated the linear stability of pump waves with respect to obliquely propagating perturbations. Ghosh et al. (1993, 1994) and Ghosh and Goldstein (1994) analysed the linear stability and nonlinear evolution of circularly polarized Alfvén waves in two dimensions numerically. Hollweg et al. (1993) and Jayanti and Hollweg (1994) studied the stability of circularly polarized Alfvén waves in a plasma with streaming He^{++} ions. Ling and Abraham-Shrauner (1979), Spangler (1989, 1990) and Inhester (1990) used the kinetic description. A comparison of theory and observations near the Earth's bow shock was given by Spangler (1997).

Among recent publications, it is worth noting the following. Del Zanna et al. (2001) and Del Zanna and Velli (2002) studied the stability and nonlinear evolution of pump waves in three dimensions numerically. Hertzberg et al. (2003, 2004a,b) and Cramer et al. (2003) investigated the parametric instabilities of pump waves in multi-component and dusty plasmas. Matsukiyo and Hada (2003) considered the instabilities of pump waves in a relativistic electron–positron plasma. Ruderman and Simpson (2004b, 2005), Simpson and Ruderman (2005) and Simpson et al. (2006) investigated absolute and convective instabilities of pump waves in the ideal MHD approximation.

Sakai and Sonnerup (1983) and Longtin and Sonnerup (1986) restricted their analysis to the case where the perturbation wave number is much smaller than the wave number of a pump wave, and studied only the modulational instability. Wong and Goldstein (1986) did not impose any restriction either on the pump wave or on perturbations. As a result, they had to deal with such a complicated dispersion equation determining the stability that only numerical analysis was possible. To derive the dispersion equation, Sakai and Sonnerup (1983), Longtin and Sonnerup (1986) and Wong and Goldstein (1986) prescribed the dependence of the density perturbation on the spatial variable and time. So, from their analysis it was not clear if they studied the stability with respect to arbitrary perturbation, or only with respect to perturbations of a particular form. Jayanti and Hollweg (1993a) used Floquet's theorem to derive the dispersion equation without any *ad hoc* assumptions about the density perturbation. In this paper we also derive the dispersion equation without any *ad hoc* assumptions using a much simpler technique than that used by Jayanti and Hollweg (1993a). Then we use this dispersion equation to study the pump wave stability analytically assuming that the pump wave amplitude is small.

The paper is organized as follows. In the next section we write down the system of equations of Hall MHD and discuss its relevance for applications to space plasmas. In Sec. 3 we derive the dispersion equation determining the stability of pump waves. In Sec. 4 we study the stability of a pump wave with small amplitude. Sec. 5 contains a summary of the obtained results and our conclusions.

2. Equations of Hall MHD

We consider an ideal (i.e. non-dissipative) plasma with isotropic electron and ion pressure consisting of electrons and one sort of ions, and use the one-fluid approximation to describe its motion. The only difference between the Hall MHD equations that we use and the ideal MHD equations is that we take the Hall term in Ohm's law into account. The system of Hall MHD equations can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{2.1a}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}, \tag{2.1b}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}, \tag{2.1c}$$

$$\mathbf{E} = -\mathbf{v}\times\mathbf{B} + \frac{m_{\mathrm{i}}}{\rho e}\left(\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \nabla p_{\mathrm{e}}\right), \tag{2.1d}$$

$$p = p_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}. \tag{2.1e}$$

Here ρ is the plasma density, p_{e} the electron pressure and p the total pressure (electron plus ion); $\mathbf{v}$, $\mathbf{B}$ and $\mathbf{E}$ are the velocity, magnetic field and electric field respectively; m_{i} is the ion mass, e the elementary charge, γ the ratio of specific heats and μ_0 the magnetic permeability of free space. Here and in what follows, the subscript '0' indicates an equilibrium quantity.

Now we substitute (2.1d) in (2.1c) to eliminate the electric field. As a result, we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \frac{m_{\mathrm{i}}}{e\mu_0}\nabla\times\left(\frac{1}{\rho}(\nabla\times\mathbf{B})\times\mathbf{B}\right). \tag{2.2}$$

When deriving this equation, we have neglected the term proportional to $\nabla\times(\nabla p_{\mathrm{e}}/\rho)$. In the case when not only p but also p_{e} is a function of ρ this term is identically zero. This term can also be neglected without any assumption about p_{e} in low-β plasmas.

Equations (2.1a), (2.1b), (2.1e) and (2.2) constitute a closed system of equations. This system will be used in what follows to study the parametric instabilities of circularly polarized Alfvén waves.

3. Derivation of the dispersion equation

In what follows we use Cartesian coordinates x, y, z, and assume that both equilibrium quantities and perturbations only depend on x. The system of equations (2.1a), (2.1b), (2.1e) and (2.2) has exact solutions in the form of circularly polarized Alfvén waves (pump waves) propagating along a constant magnetic field $B_x\mathbf{e}_x$, where $\mathbf{e}_x$ is the unit vector along the x-axis (Barnes and Hollweg 1974; Ovenden et al. 2004; Sakai and Sonnerup 1983). Introducing the components of the velocity and magnetic field, $\mathbf{v} = (u, v, w)$ and $\mathbf{B} = (B_x, B_y, B_z)$, we write these solutions in the form

$$
\begin{gathered}
u_0 = 0, \quad \rho_0 = \text{constant}, \quad p_0 = \text{constant}, \quad B_{0x} = B_x = \text{constant}, \\
v_{0y} = V_0 \cos\phi, \quad v_{0z} = V_0 \sin\phi, \\
B_{0y} = A_0 \cos\phi, \quad B_{0z} = A_0 \sin\phi.
\end{gathered} \tag{3.1}
$$

Here $\phi = k_0 x - \omega_0 t$, and the quantities k_0, ω_0, A_0 and V_0 are related by

$$
V_0 = -\frac{A_0 v_{\mathrm{A}}^2 k_0}{B_x \omega_0}, \qquad \frac{\omega_0}{k_0 v_{\mathrm{A}}} = \left(1 + \frac{1}{4}\ell^2 k_0^2\right)^{1/2} - \frac{1}{2}\ell k_0, \tag{3.2}
$$

where the square of the Alfvén speed, the ion cyclotron frequency and the ion inertia length are given by

$$
v_{\mathrm{A}}^2 = \frac{B_x^2}{\mu_0 \rho_0}, \quad \Omega_{\mathrm{i}} = \frac{e B_x}{m_{\mathrm{i}}}, \quad \ell = \frac{v_{\mathrm{A}}}{\Omega_{\mathrm{i}}} = \frac{m_{\mathrm{i}}}{e(\mu_0 \rho_0)^{1/2}}, \tag{3.3}
$$

respectively. The positive (negative) values of k_0 in the second equation in (3.2) correspond to left-hand (right-hand) polarized pump waves. When $|\ell k_0| \ll 1$, the second equation in (3.2) reduces to $\omega_0 = k_0 v_{\mathrm{A}}$, which is the dispersion relation for Alfvén waves in ideal MHD. We see that these waves propagate without dispersion. The dispersion effect becomes important when $|\ell k_0| \sim 1$, so that ℓ can also be called the dispersion length. The frequencies of both left-hand and right-hand polarized waves grow monotonically when $|k_0|$ increases. However, while the frequency of a right-hand polarized wave tends to infinity when $|k_0| \to \infty$, the frequency of a left-hand polarized wave tends to Ω_{i} in this limit. The right-hand polarized waves become whistler waves when their frequency is above the ion cyclotron frequency ($|\omega_0| > \Omega_{\mathrm{i}}$). The left-hand polarized waves are ion cyclotron waves for $\omega_0 \sim \Omega_{\mathrm{i}}$.

Now we represent all the dependent variables in the form $f = f_0 + f'$, where the prime indicates a perturbation, substitute in (2.1a), (2.1b), (2.1e) and (2.2), and linearize with respect to perturbations. As a result we obtain

$$
\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0, \tag{3.4a}
$$

$$
\frac{\partial u'}{\partial t} = -\frac{c_{\mathrm{S}}^2}{\rho_0}\frac{\partial \rho'}{\partial x} - \frac{A_0}{\mu_0 \rho_0}\frac{\partial}{\partial x}(B_y' \cos\phi + B_z' \sin\phi), \tag{3.4b}
$$

$$
\frac{\partial \mathbf{v}_\perp'}{\partial t} + u\frac{\partial \mathbf{v}_0}{\partial x} = \frac{B_x}{\mu_0 \rho_0}\frac{\partial \mathbf{B}_\perp'}{\partial x} - \frac{B_x \rho'}{\mu_0 \rho_0^2}\frac{\partial \mathbf{B}_0}{\partial x}, \tag{3.4c}
$$

$$
\frac{\partial \mathbf{B}_\perp'}{\partial t} = B_x \frac{\partial \mathbf{v}_\perp'}{\partial x} - \frac{\partial (u' \mathbf{B}_0)}{\partial x} - \ell v_{\mathrm{A}} \mathbf{e}_x \times \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{B}_\perp'}{\partial x} - \frac{\rho'}{\rho_0}\frac{\partial \mathbf{B}_0}{\partial x}\right), \tag{3.4d}
$$

where $\mathbf{v}_\perp = (0, v_y, v_z)$, $\mathbf{B}_\perp = (0, B_y, B_z)$ and $c_{\rm S}^2 = \gamma p_0/\rho_0$ is the square of the sound speed. When deriving (3.4), we have eliminated p from (2.1b) using (2.1e).

The traditional approach for solving the system of equations (3.4) is to assume that ρ' varies as $\exp[i(kx - \omega t)]$ (see, e.g., Galeev and Oraevskii 1963; Sagdeev and Galeev 1969; Derby 1978; Goldstein 1978; Wong and Goldstein 1986). To avoid this *ad hoc* assumption, Jayanti and Hollweg (1993a) noticed that, if the variables t and ϕ are used instead of t and x, then the coefficients of the linear system of (3.4) depend on ϕ only. Then, looking for solutions proportional to $\exp(-i\omega t)$, they obtained a system of ordinary differential equations with periodic coefficients, and applied Floquet's theory.

Jayanti and Hollweg (1993a) have used the first part of Floquet's theorem which prescribes the form of solutions to a linear system of ordinary differential equations with periodic coefficients. Ruderman and Simpson (2004a) have improved their approach and used the second part of Floquet's theorem. This second part states that, for any system of ordinary differential equations with periodic coefficients, there exists a linear transformation of dependent variables reducing this system to a system with constant coefficients. Ruderman and Simpson (2004a) have found that, in the case of ideal MHD, this transformation is very simple and is given by

$$B_+ = B'_y \cos\phi + B'_z \sin\phi, \quad B_- = B'_y \sin\phi - B'_z \cos\phi, \tag{3.5a}$$

$$v_+ = v'_y \cos\phi + v'_z \sin\phi, \quad v_- = v'_y \sin\phi - v'_z \cos\phi. \tag{3.5b}$$

It turns out that this variable transformation also works in the case of Hall MHD. In the new variables, (3.4) is rewritten as

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0, \tag{3.6a}$$

$$\frac{\partial u'}{\partial t} = -\frac{c_{\rm S}^2}{\rho_0}\frac{\partial \rho'}{\partial x} - \frac{A_0}{\mu_0\rho_0}\frac{\partial B_+}{\partial x}, \tag{3.6b}$$

$$\frac{\partial v_+}{\partial t} - \omega_0 v_- = \frac{B_x}{\mu_0\rho_0}\left(\frac{\partial B_+}{\partial x} + k_0 B_-\right), \tag{3.6c}$$

$$\frac{\partial v_-}{\partial t} + \omega_0 v_+ - V_0 k_0 u' = \frac{B_x}{\mu_0\rho_0}\left(\frac{\partial B_-}{\partial x} - k_0 B_+ + \frac{A_0 k_0 \rho'}{\rho_0}\right), \tag{3.6d}$$

$$\begin{aligned}\frac{\partial B_+}{\partial t} - \omega_0 B_- = {} & B_x\left(\frac{\partial v_+}{\partial x} + k_0 v_-\right) - A_0\frac{\partial u'}{\partial x} \\ & - \ell v_{\rm A}\left(\frac{\partial^2 B_-}{\partial x^2} - 2k_0\frac{\partial B_+}{\partial x} - k_0^2 B_- + \frac{A_0 k_0}{\rho_0}\frac{\partial \rho'}{\partial x}\right),\end{aligned} \tag{3.6e}$$

$$\begin{aligned}\frac{\partial B_-}{\partial t} + \omega_0 B_+ = {} & B_x\left(\frac{\partial v_-}{\partial x} - k_0 v_+\right) + A_0 k_0 u' \\ & + \ell v_{\rm A}\left(\frac{\partial^2 B_+}{\partial x^2} + 2k_0\frac{\partial B_-}{\partial x} - k_0^2 B_+ + \frac{A_0 k_0^2 \rho'}{\rho_0}\right).\end{aligned} \tag{3.6f}$$

This is the system of equations with constant coefficients, so that we can look for solutions proportional to $\exp[i(Kx - \Omega t)]$. As a result we reduce (3.6) to

$$\Omega\rho' - \rho_0 K u' = 0, \tag{3.7a}$$

$$\rho_0\Omega u' - K\left(c_S^2\rho' + \frac{A_0}{\mu_0}B_+\right) = 0, \tag{3.7b}$$

$$\rho_0(\Omega v_+ - i\omega_0 v_-) + \frac{B_x}{\mu_0}(KB_+ - ik_0B_-) = 0, \tag{3.7c}$$

$$\rho_0(\Omega v_- + i\omega_0 v_+ - iV_0k_0u') + \frac{B_x}{\mu_0}\left(KB_- + ik_0B_+ - \frac{iA_0k_0\rho'}{\rho_0}\right) = 0, \tag{3.7d}$$

$$\Omega B_+ - i\omega_0 B_- + B_x(Kv_+ - ik_0v_-) - A_0Ku' - \ell v_A\left(iK^2B_- - 2k_0KB_+ + ik_0^2B_- + \frac{A_0k_0K\rho'}{\rho_0}\right) = 0, \tag{3.7e}$$

$$\Omega B_- + i\omega_0 B_+ + B_x(Kv_- + ik_0v_+) - iA_0k_0u' + \ell v_A\left(iK^2B_+ + 2k_0KB_- + ik_0^2B_+ - \frac{iA_0k_0^2\rho'}{\rho_0}\right) = 0. \tag{3.7f}$$

This is the system of six linear homogeneous algebraic equations for six variables. It has a non-trivial solution only when its determinant is zero. This condition gives the dispersion equation that can be written as

$$D(\omega, k) \equiv \sum_{j=0}^{6} q_j(k)\omega^{6-j} = 0. \tag{3.8}$$

In this equation we use the dimensionless quantities

$$\tau = \frac{k_0 v_A}{\omega_0}, \quad k = \frac{K}{k_0}, \quad \omega = \frac{\Omega}{\omega_0}, \quad a = \frac{A_0}{B_x}, \quad b = \frac{c_S}{v_A}. \tag{3.9}$$

Note that, in accordance with (3.2), the dispersion length ℓ is related to τ by

$$\ell k_0 = (\tau^2 - 1)/\tau, \tag{3.10}$$

so that $\tau > 1$ ($0 < \tau < 1$) corresponds to left-hand (right-hand) polarized pump waves. The coefficient functions $q_0(k), \dots, q_6(k)$ are given in Appendix A. Note that, with the accuracy up to the notation, the dispersion equation (3.8) coincides with the one derived by Longtin and Sonnerup (1986) if we take the electron mass, m_e, equal to zero in the latter. It also coincides with the dispersion equation obtained by Wong and Goldstein (1986).

In what follows, we restrict our analysis to the case of small plasma beta, $b < 1$. The reason for this is that the basic system of Hall MHD equations (2.1) is only valid for low-beta plasmas. For high-beta plasmas, kinetic effects and effects of pressure anisotropy become very important (e.g. Khanna and Rajaram 1982; Mjølhus and Wyller 1988).

Using (A 1) it is straightforward to see that, if the pair (k, ω) satisfies the dispersion equation, then the pair $(-k, -\omega)$ also satisfies it. This observation enables us to restrict the analysis to $k > 0$.

4. Stability analysis

In this section we study the stability of pump waves with small amplitudes, $a \ll 1$. Our analysis is very similar to that made by Jayanti and Hollweg (1993b) for non-dispersive waves using the ideal MHD description. Using (A 1) we write the dispersion function in the form $D(\omega, k) = D_0(\omega, k) + a^2 D_1(\omega, k)$, where

$$D_0 = (\omega^2 - b^2\tau^2k^2)\{(\omega - 1)^2 - (k-1)^2[1 + \omega(\tau^2 - 1)]\}$$
$$\times \{(\omega + 1)^2 - (k+1)^2[1 - \omega(\tau^2 - 1)]\}, \tag{4.1}$$

$$D_1 = \tau^2 k^2 (k - \omega)\{k[1 + (2\tau^2 - 1)\omega^2] + \omega[\omega^2 - (2\tau^2 + 1)]\}. \tag{4.2}$$

When $a = 0$ the dispersion equation (3.8) reduces to $D_0(\omega, k) = 0$. The roots of this equation considered as an equation for k are

$$k_{1,2} = \pm \frac{\omega}{b\tau}, \tag{4.3a}$$

$$k_{3,4} = 1 \pm \frac{\omega - 1}{[1 + \omega(\tau^2 - 1)]^{1/2}}, \tag{4.3b}$$

$$k_{5,6} = -1 \pm \frac{\omega + 1}{[1 - \omega(\tau^2 - 1)]^{1/2}}. \tag{4.3c}$$

In (4.3a) the upper sign corresponds to k_1 and the lower to k_2. The same convention is applied to $k_{3,4}$ and $k_{5,6}$. It is straightforward to show that $k_j(\omega)$ is a monotonic function with the range $(-\infty, \infty)$ $(j = 1, \ldots, 6)$. This implies that equation $D_0(\omega, k) = 0$ considered as an equation for ω has six real roots for any $k > 0$, and the pump wave is stable in the limit $a \to 0$.

The curves defined by (4.3) in the ωk-plane are shown in Fig. 1. It is easy to see that k_1 and k_2 correspond to forward- and backward-propagating sound waves, respectively. Following Jayanti and Hollweg (1993b), we denote them as fs and bs (i.e. forward sound and backward sound) in Fig. 1. The remaining four roots are Alfvén waves. Ω and K are the frequency and wave number of density fluctuations. Then it follows from (3.5) that the frequency and wave number of fluctuations of $\mathbf{v}'_\perp$ and $\mathbf{B}'_\perp$ are $\Omega \pm \omega_0$ and $K \pm k_0$, which correspond to $\omega_\pm = \omega \pm 1$ and $k_\pm = k \pm 1$ in the dimensionless variables. This observation implies that k_3 and k_5 are the roots corresponding to forward-propagating Alfvén waves involving (k_-, ω_-) and (k_+, ω_+), respectively, and k_4 and k_6 are the roots corresponding to backward-propagating Alfvén waves involving (k_-, ω_-) and (k_+, ω_+), respectively. These roots are denoted as $\mathrm{fA}_\pm$ and $\mathrm{bA}_\pm$ (i.e. forward Alfvén and backward Alfvén) in Fig. 1.

The intersection points of the different curves in Fig. 1 correspond to the double roots of equation $D_0(\omega, k) = 0$ considered as an equation for ω. Taking $a \neq 0$, $a \ll 1$ corresponds to a small perturbation of this dispersion equation. This perturbation causes small perturbations of simple roots of the dispersion equation $D_0(\omega, k) = 0$, but these simple roots remain real. As for a double root, the perturbation causes its split either into two real roots or into a pair of complex conjugate roots. The latter case corresponds to instability. Hence, only a perturbation with k close to one of the values corresponding to double roots can be unstable when $a \ll 1$. This discussion shows that the intersection points in Fig. 1 are of crucial importance for the stability analysis. Fig. 1 clearly shows that the number of intersection points

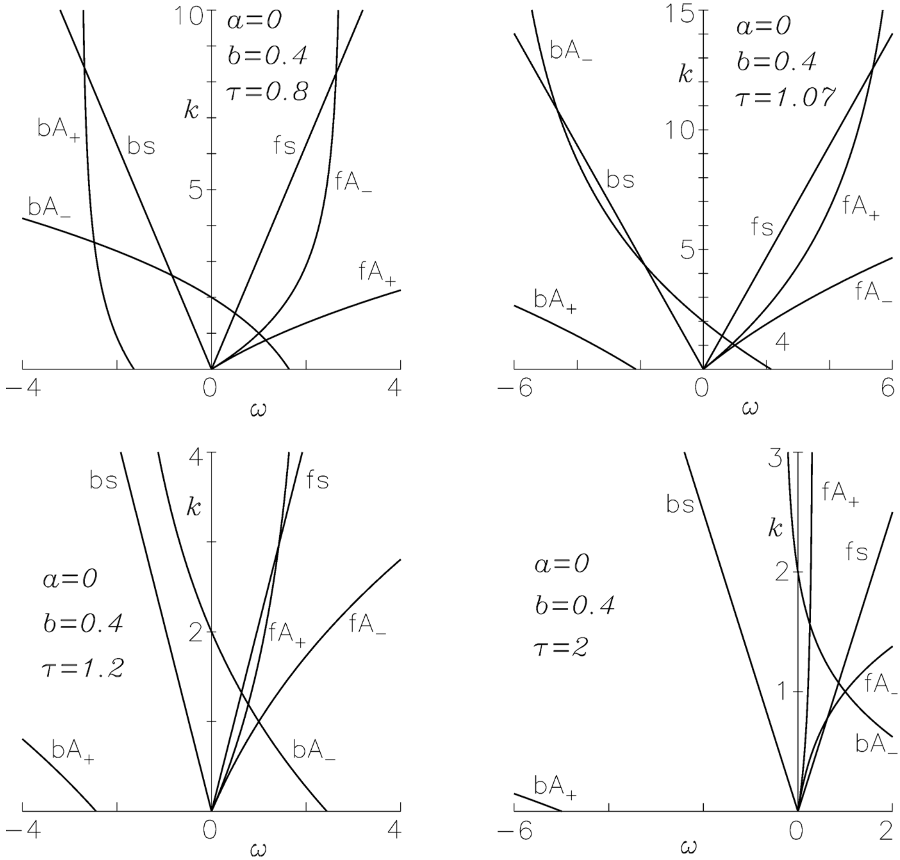

Figure 1. The dimensionless wave number k as a function of the dimensionless frequency ω for $a = 0$ and $b = 0.4$. The upper left panel corresponds to $\tau = 0.8$ (right-hand polarized pump wave). The three other panels correspond to left-hand polarized pump waves. The upper right, lower left and lower right panels correspond to $\tau = 1.07$ ($b < b_1(\tau)$), $\tau = 1.2$ ($b_1(\tau) < b < b_2(\tau)$) and $\tau = 2$ ($b > b_2(\tau)$), respectively. The notation fs and bs is used for forward and backward sound waves, respectively. The notation fA_- and bA_- is used for forward and backward Alfvén waves involving k_- and ω_-. The notation fA_+ and bA_+ is used for forward and backward Alfvén waves involving k_+ and ω_+.

depends on the value of τ. This observation inspires us to carry out the stability analysis for the right-hand polarized and left-hand polarized waves separately.

4.1. Stability of right-hand polarized waves ($\tau < 1$)

In what follows, we use the notation (ω_{ij}, k_{ij}) for the coordinates of the intersection point of the curves $k_i(\omega)$ and $k_j(\omega)$. When $\tau < 1$, there are seven intersection points: (ω_{13}, k_{13}), (ω_{14}, k_{14}), (ω_{24}, k_{24}), (ω_{26}, k_{26}), (ω_{34}, k_{34}), (ω_{45}, k_{45}) and (ω_{46}, k_{46}), where

$$\omega_{13} = b\tau + \frac{(1-b\tau)[1+b\tau+\sqrt{(1-b\tau)^2+4b\tau^3}]}{2(1-\tau^2)}, \quad k_{13} = \frac{\omega_{13}}{b\tau}, \tag{4.4a}$$

$$\omega_{14} = b\tau + \frac{(1-b\tau)[1+b\tau-\sqrt{(1-b\tau)^2+4b\tau^3}]}{2(1-\tau^2)}, \quad k_{14} = \frac{\omega_{14}}{b\tau}, \tag{4.4b}$$

$$\omega_{24} = -b\tau + \frac{(1+b\tau)[1-b\tau-\sqrt{(1+b\tau)^2-4b\tau^3}]}{2(1-\tau^2)}, \quad k_{24} = -\frac{\omega_{24}}{b\tau}, \tag{4.4c}$$

$$\omega_{26} = b\tau - \frac{(1+b\tau)[1-b\tau+\sqrt{(1+b\tau)^2-4b\tau^3}]}{2(1-\tau^2)}, \quad k_{26} = -\frac{\omega_{26}}{b\tau}, \tag{4.4d}$$

$$\omega_{34} = 1, \quad k_{34} = 1, \tag{4.4e}$$

$$\omega_{45} = \sqrt{1-2\tau^4+\frac{2\tau^2(1-\chi\tau^2)}{1-\tau^2}}, \quad k_{45} = 1 - \frac{\omega_{45}-1}{[1+\omega_{45}(\tau^2-1)]^{1/2}}, \tag{4.4f}$$

$$\omega_{46} = -\sqrt{1-2\tau^4+\frac{2\tau^2(1+\chi\tau^2)}{1-\tau^2}}, \quad k_{46} = 1 - \frac{\omega_{46}-1}{[1+\omega_{46}(\tau^2-1)]^{1/2}}, \tag{4.4g}$$

with $\chi = \sqrt{1+(1-\tau^2)^2}$. The characteristic picture of curves $k_j(\omega)$ $(j = 1,\ldots,6)$ for $\tau < 1$ is shown in the upper left panel of Fig. 1.

Note that there is one additional intersection point, $\omega = k = 0$. In this point four curves, fs, bs, fA_- and fA_+, intersect, so that $\omega = 0$ is the four-fold root of $D_0(\omega,k) = 0$ at $k = 0$. When $0 < k \ll 1$, this four-fold root splits into two single real roots, $\omega_{1,2} = \pm b\tau k$, and one real double root $\omega_3 = 2k/(1+\tau^2)$. It is straightforward to show that the roots ω_1 and ω_2 remain real when $0 < a \ll 1$. The double root ω_3 splits into two roots, $\omega_3^{\pm}$. To calculate $\omega_3^{\pm}$ we introduce $\omega = a\tilde{\omega}$ and $k = a\tilde{k}$, and then look for $\tilde{\omega}$ in the form

$$\tilde{\omega} = \tilde{k}\left(\frac{2}{1+\tau^2} + a\kappa\right). \tag{4.5}$$

Substituting (4.5) into the dispersion equation $D(\omega,k) = 0$ and collecting terms of the lowest order with respect to a, we easily obtain

$$\kappa^2 = \frac{(1-\tau^2)(1+3\tau^2)}{(1+\tau^2)^2}\left\{\frac{\tau^2}{4-b^2\tau^2(1+\tau^2)^2} + \tilde{k}^2\frac{(1-\tau^2)(1+3\tau^2)}{(1+\tau^2)^4}\right\}. \tag{4.6}$$

We see that $\kappa^2 > 0$ when $\tau < 1$. Hence, both ω_3^- and ω_3^+ are real, and there is no instability related to the intersection point $\omega = k = 0$.

Let us calculate the expansion of $D_0(k,\omega)$ in the Taylor series in the vicinity of the intersection point (ω_{ij}, k_{ij}). Since the equation $D_0(k,\omega) = 0$ considered as an equation for k has a double root at $\omega = \omega_{ij}$, then $\partial D_0/\partial k = 0$ at $k = k_{ij}$, $\omega = \omega_{ij}$. Similarly, since the equation $D_0(k,\omega)$ considered as an equation for ω has a double root at $k = k_{ij}$, then $\partial D_0/\partial\omega = 0$ at $k = k_{ij}$, $\omega = \omega_{ij}$. Hence, the Taylor expansion of $D_0(k,\omega)$ in the vicinity of (k_{ij}, ω_{ij}) is

$$D_0(k,\omega) = \tfrac{1}{2}D_{kk}(k-k_{ij})^2 + D_{k\omega}(k-k_{ij})(\omega-\omega_{ij}) + \tfrac{1}{2}D_{\omega\omega}(\omega-\omega_{ij})^2 + \cdots, \tag{4.7}$$

where

$$D_{kk} = \left.\frac{\partial^2 D_0}{\partial k^2}\right|_{\substack{k=k_{ij}\\ \omega=\omega_{ij}}}, \quad D_{k\omega} = \left.\frac{\partial^2 D_0}{\partial k\partial\omega}\right|_{\substack{k=k_{ij}\\ \omega=\omega_{ij}}}, \quad D_{\omega\omega} = \left.\frac{\partial^2 D_0}{\partial\omega^2}\right|_{\substack{k=k_{ij}\\ \omega=\omega_{ij}}}.$$

Since the equation $D_0(k,\omega) = 0$ has two solutions in the vicinity of (ω_{ij}, k_{ij}), $k = k_i(\omega)$ and $k = k_j(\omega)$, the quadratic equation for $k - k_{ij}$,

$$D_{kk}(k-k_{ij})^2 + 2D_{k\omega}(k-k_{ij})(\omega-\omega_{ij}) + D_{\omega\omega}(\omega-\omega_{ij})^2 = 0, \tag{4.8}$$

has two different real roots, which implies that

$$D_{k\omega}^2 - D_{kk}D_{\omega\omega} > 0. \tag{4.9}$$

Let us now put $k = k_{ij} + a\tilde{k}$ and $\omega = \omega_{ij} + a\tilde{\omega}$, and use the expansion (4.7). Then, in the vicinity of (ω_{ij}, k_{ij}), the dispersion equation $D(\omega, k) = 0$ takes the approximate form

$$D_{\omega\omega}\tilde{\omega}^2 + 2D_{k\omega}\tilde{\omega}\tilde{k} + D_{kk}\tilde{k}^2 + 2D_1(\omega_{ij}, k_{ij}) = 0. \tag{4.10}$$

This is the quadratic equation for $\tilde{\omega}$. Its discriminant is

$$\Delta = (D_{k\omega}^2 - D_{kk}D_{\omega\omega})\tilde{k}^2 - 2D_{\omega\omega}D_1(\omega_{ij}, k_{ij}). \tag{4.11}$$

If $D_{\omega\omega}D_1(\omega_{ij}, k_{ij}) < 0$, then $\Delta > 0$ for any value of $\tilde{k}$, and the two roots of (4.10) are real. On the other hand, when

$$D_{\omega\omega}D_1(\omega_{ij}, k_{ij}) > 0, \tag{4.12}$$

then $\Delta < 0$ if $\tilde{k}$ satisfies

$$\tilde{k}^2 < \eta_{ij}^2 \equiv 2D_{\omega\omega}D_1(\omega_{ij}, k_{ij})(D_{k\omega}^2 - D_{kk}D_{\omega\omega})^{-1}, \tag{4.13}$$

and (4.10) has two complex conjugate roots. Hence, we conclude that the pump wave is unstable with respect to perturbation with the wave number $k = k_{ij} + a\tilde{k}$ when $|\tilde{k}|$ is sufficiently small if and only if the inequality (4.12) is satisfied. Note that this inequality can be rewritten as $\eta_{ij}^2 > 0$.

However, the criterion (4.12) does not work in the vicinity of $(1, 1)$, (ω_{45}, k_{45}) and (ω_{46}, k_{46}) because $D_1(1, 1) = D_1(\omega_{45}, k_{45}) = D_1(\omega_{46}, k_{46}) = 0$. Simple calculation shows that $\partial D_1/\partial k = \partial D_1/\partial \omega = 0$ when $k = 1$ and $\omega = 1$. This implies that the approximate solutions of the dispersion equation in the vicinity of (1,1) are given by (4.8), i.e. they are real. Hence, there are no unstable modes with k and ω close to unity.

In order to have an equation similar to (4.11) with the contributions from D_0 and D_1 of the same order when $(\omega, k) = (\omega_{45}, k_{45})$ or $(\omega, k) = (\omega_{46}, k_{46})$, we have to take $k = k_{4j} + a^2\hat{k}$ and $\omega = \omega_{4j} + a^2\hat{\omega}$, where $j = 5$ or $j = 6$. Then in the vicinity of (ω_{4j}, k_{4j}) $(j = 5, 6)$, the dispersion equation $D(\omega, k) = 0$ takes the approximate form

$$D_{\omega\omega}\hat{\omega}^2 + 2D_{k\omega}\hat{\omega}\hat{k} + D_{kk}\hat{k}^2 + 2D_{1\omega}\hat{\omega} + 2D_{1k}\hat{k} = 0, \tag{4.14}$$

where

$$D_{1\omega} = \left.\frac{\partial D_1}{\partial \omega}\right|_{\substack{k=k_{4j}\\ \omega=\omega_{4j}}}, \quad D_{1k} = \left.\frac{\partial D_1}{\partial k}\right|_{\substack{k=k_{4j}\\ \omega=\omega_{4j}}} \qquad (j = 5, 6).$$

The discriminant of this quadratic equation is

$$\widetilde{\Delta} = (D_{k\omega}^2 - D_{kk}D_{\omega\omega})\hat{k}^2 + 2(D_{k\omega}D_{1\omega} - D_{\omega\omega}D_{1k})\hat{k} + D_{1\omega}^2. \tag{4.15}$$

We see that $\widetilde{\Delta}$ is a quadratic trinomial with respect to $\hat{k}$. Its discriminant is equal to

$$\begin{aligned}\widehat{\Delta} &= D_{\omega\omega}\left(D_{\omega\omega}D_{1k}^2 + D_{kk}D_{1\omega}^2 - 2D_{k\omega}D_{1k}D_{1\omega}\right)\\ &= \frac{4\tau^4k^4(k-\omega)^2(\omega^2-1)^2(\omega^2-b^2\tau^2k^2)^2[(1+\tau^2)^2-(1-\tau^2)^2\omega^2]}{1-(1-\tau^2)^2\omega^2}UW,\end{aligned} \tag{4.16}$$

where

$$U = \frac{[1+\tau^2-(1-\tau^2)\omega][1+(2\tau^2-1)\omega^2]}{1-(1-\tau^2)\omega} + 4\omega(\omega-1)(2\tau^2-1)$$

$$- 2[3\omega^2 + 2\omega(2\tau^2-1) - (2\tau^2+1)]\sqrt{1-(1-\tau^2)\omega}, \tag{4.17a}$$

$$W = \frac{[1+\tau^2-(1-\tau^2)\omega][1+(2\tau^2-1)\omega^2]}{1+(1-\tau^2)\omega} \mp 4\omega(\omega-1)(2\tau^2-1)$$

$$\pm 2[3\omega^2 + 2\omega(2\tau^2-1) - (2\tau^2+1)]\sqrt{1+(1-\tau^2)\omega}. \tag{4.17b}$$

In (4.16) and (4.17) either $(\omega, k) = (\omega_{45}, k_{45})$ or $(\omega, k) = (\omega_{46}, k_{46})$. In (4.17b) the upper sign corresponds to $\omega = \omega_{45}$ and the lower sign to $\omega = \omega_{46}$.

It is straightforward to see that the sign of $\widehat{\Delta}$ coincides with the sign of UW. We verified analytically that $U_{4j} < 0$ and $W_{4j} > 0$ for $\tau \ll 1$ and $1-\tau \ll 1$, where U_{4j} and W_{4j} are given by (4.17) with $\omega = \omega_{4j}$ and $j = 5, 6$. We also calculated U_{4j} and W_{4j} numerically for τ varying from 0.01 to 0.99 and found that $U_{4j} < 0$ and $W_{4j} > 0$. Hence, $\widehat{\Delta} < 0$ both for $(\omega, k) = (\omega_{45}, k_{45})$ and $(\omega, k) = (\omega_{46}, k_{46})$. This implies that $\widetilde{\Delta} > 0$ for any $\hat{k} \neq 0$. Then it follows that the two roots of equation (4.14) are real for any $\hat{k}$, so that there are no unstable modes with (ω, k) close to either (ω_{45}, k_{45}) or (ω_{46}, k_{46}).

We proved analytically that the inequality (4.12) is not satisfied when $(i, j) = (1, 3)$, $(i, j) = (2, 4)$ or $(i, j) = (2, 6)$. This implies that there are no unstable modes with (ω, k) close to (ω_{13}, k_{13}), (ω_{24}, k_{24}) or (ω_{26}, k_{26}).

The expression for η_{14}^2 is given by

$$\eta_{14}^2 = \frac{4\tau^8(1-b\tau)[1+\tau^2-(1-\tau^2)\omega_{14}]}{\omega_{14}[1-(1-\tau^2)\omega_{14}][(1-b\tau)^2+4b\tau^3][\sqrt{(1-b\tau)^2+4b\tau^3}-(1-b\tau)+2\tau^2]}$$

$$\times \{(1-b\tau)^2 + 4b\tau^3 - 2\tau^2(1-\tau^2) - (1-b\tau-2\tau^2)\sqrt{(1-b\tau)^2+4b\tau^3}\}^{-1}. \tag{4.18}$$

It is easy to prove that the last multiplier in this expression is positive for any values of b and τ. Taking into account that, in accordance with (4.3b) and (4.4b), $\omega_{14} > 0$ and $1-(1-\tau^2)\omega_{14} > 0$, we conclude that the sign of η_{14}^2 coincides with the sign of $1-b\tau$. Since $b < 1$, it follows that $\eta_{14}^2 > 0$ when $\tau < 1$. This implies that there are unstable modes with (ω, k) close to (ω_{14}, k_{14}). The interval of wave numbers corresponding to unstable modes is determined by $|k - k_{14}| < a\eta_{14}$. It follows from (4.11) that the instability increment takes its maximum value when $\tilde{k} = 0$, i.e. when $k = k_{14}$. It is equal to $\gamma_{14} = a\tilde{\gamma}_{14}$, where $\tilde{\gamma}_{14}$ is given by

$$\tilde{\gamma}_{14} = \sqrt{\frac{2D_1(\omega_{14}, k_{14})}{D_{\omega\omega}}} = \frac{\tau}{2b}\left\{\frac{\omega_{14}(1-b\tau)[1-(1-\tau^2)\omega_{14}]}{1+\tau^2-(1-\tau^2)\omega_{14}}\right.$$

$$\left.\times \frac{\sqrt{(1-b\tau)^2+4b\tau^3}-(1-b\tau)+2\tau^2}{(1-b\tau)^2+4b\tau^3-2\tau^2(1-\tau^2)-(1-b\tau-2\tau^2)\sqrt{(1-b\tau)^2+4b\tau^3}}\right\}^{1/2}. \tag{4.19}$$

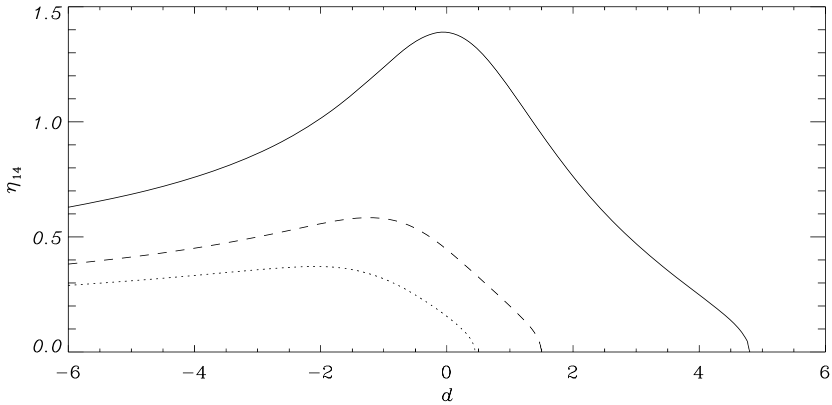

Figure 2. The dependences of η_{14} on the dispersion parameter d. The solid, dashed and dotted curves correspond to $b = 0.2$, 0.5 and 0.8, respectively. Negative (positive) values of d correspond to the right-hand (left-hand) polarized waves.

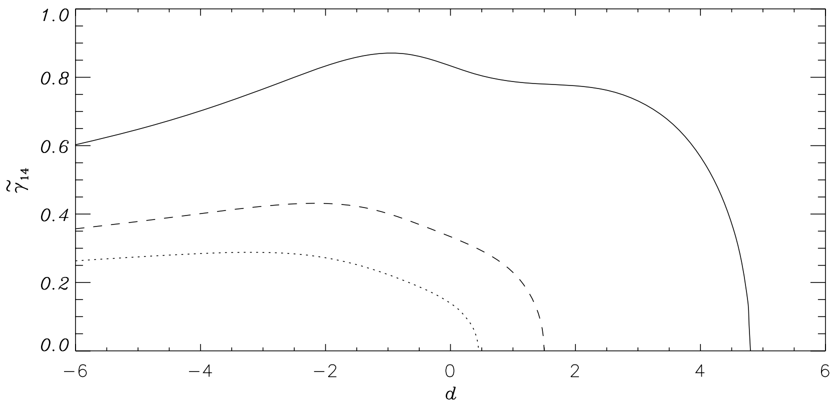

Figure 3. The same as Fig. 2, but for $\tilde{\gamma}_{14}$.

In the limit of ideal MHD ($\tau \to 1$) the interval of unstable wave numbers and the maximum instability increment are given by

$$\left|k - \frac{2}{1+b}\right| < \frac{a(1-b)^{1/2}}{b^{1/2}(1+b)^2}, \quad \gamma = \frac{a(1-b)^{1/2}}{2b^{1/2}(1+b)}. \tag{4.20}$$

These results coincide with those obtained by Jayanti and Hollweg (1993b). The dependences of η_{14} and $\tilde{\gamma}_{14}$ on the dispersion parameter $d = \ell k_0 = (\tau^2 - 1)/\tau$ for different values of b are shown in Figs 2 and 3.

This particular type of instability results in decay of the pump wave in the forward-propagating sound wave and backward-propagating Alfvén wave, so that it is called the decay instability (Galeev and Oraevskii 1963; Sagdeev and Galeev 1969; Derby 1978; Goldstein 1978; Wong and Goldstein 1986).

4.2. Stability of left-hand polarized waves ($\tau > 1$)

For any $\tau > 1$ and any $b < 1$ there are three points of intersection of two curves, (ω_{14}, k_{14}), (ω_{34}, k_{34}) and (ω_{45}, k_{45}), determined by (4.4b), (4.4e) and (4.4f), and the point $\omega = k = 0$ where four curves intersect. There are additional points of intersection of two curves with their number depending on the values of τ and b.

(i) $b < b_1(\tau) \equiv \tau^{-1}(\tau - \sqrt{\tau^2 - 1})^2$.In this case there are three additional points of intersection: one of curves $k_1(\omega)$ and $k_5(\omega)$ at point (ω_{15}, k_{15}), and two of curves $k_2(\omega)$ and $k_4(\omega)$ at points $(\omega_{24}^{\pm}, k_{24}^{\pm})$, where

$$\omega_{15} = -b\tau + \frac{(1 - b\tau)\left[b\tau + 1 + \sqrt{(b\tau - 1)^2 + 4b\tau^3}\right]}{2(\tau^2 - 1)}, \quad k_{15} = \frac{\omega_{15}}{b\tau}. \tag{4.21}$$

$$\omega_{24}^{\pm} = -b\tau + \frac{(1 + b\tau)\left[b\tau - 1 \pm \sqrt{(1 + b\tau)^2 - 4b\tau^3}\right]}{2(\tau^2 - 1)}, \quad k_{24}^{\pm} = -\frac{\omega_{24}^{\pm}}{b\tau}. \tag{4.22}$$

(ii) $b_1(\tau) < b < b_2(\tau) \equiv 2/\tau(\tau^2 + 1)$.In this case, there is only one additional point of intersection, (ω_{15}, k_{15}).

(iii) $b > b_2(\tau)$.In this case once again there is only one additional point of intersection, (ω_{13}, k_{13}), determined by (4.4a). The characteristic pictures of curves $k_j(\omega)$ ($j = 1, \ldots, 6$) for $\tau > 1$ are shown in Fig. 1. The upper right, lower left and lower right panels correspond to $b < b_1(\tau)$, $b_1(\tau) < b < b_2(\tau)$ and $b > b_2(\tau)$, respectively.

Similarly to Sec. 4.1, we start our analysis from the point $\omega = k = 0$. Once again, for $0 < k \ll 1$, there are two real simple roots of $D_0(\omega, k) = 0$ and one real double root satisfying $|\omega| \ll 1$. Once again the two simple roots remain real when $0 < a \ll 1$, while the double root splits into two roots, $\omega_3^{\pm}$, given by (4.5). In accordance with (4.6), these roots are real when $b\tau(1+\tau^2) > 2$. However, they are complex conjugate when $b\tau(1 + \tau^2) < 2$, and $\tilde{k}$ satisfies

$$\tilde{k} < \tilde{k}_{\rm lim} = \frac{\tau(1 + \tau^2)^2}{\sqrt{(3\tau^2 + 1)(\tau^2 - 1)[4 - b^2\tau^2(\tau^2 + 1)^2]}}. \tag{4.23}$$

Hence, for $b\tau(1 + \tau^2) < 2$ (which means that $c_{\rm S}$ is smaller than the pump wave group velocity; see, e.g., Longtin and Sonnerup (1986)) there is an unstable mode with $|\omega| \ll 1$ when $k < a\tilde{k}_{\rm lim}$, where $\tilde{k}_{\rm lim}$ is given by (4.23). This mode corresponds to the modulational instability studied by many authors (e.g. Sakai and Sonnerup 1983; Longtin and Sonnerup 1986; Wong and Goldstein 1986). Its increment is given by

$$\gamma_{\rm mod} = k\sqrt{\frac{(\tau^2 - 1)(3\tau^2 + 1)}{(1 + \tau^2)^2}\left\{\frac{a^2\tau^2}{4 - b^2\tau^2(\tau^2 + 1)^2} - k^2\,\frac{(\tau^2 - 1)(3\tau^2 + 1)}{(1 + \tau^2)^4}\right\}}. \tag{4.24}$$

The instability increment $\gamma_{\rm mod}(k)$ takes its maximum value,

$$\gamma_{\rm m} = \frac{a^2\tau^2(1 + \tau^2)}{2[4 - b^2\tau^2(\tau^2 + 1)^2]}, \tag{4.25}$$

when $k = k_{\rm m} = a\tilde{k}_{\rm lim}/\sqrt{2}$. Obviously, (4.23)–(4.25) are only valid when $b\tau(\tau^2 + 1)$ is not too close to 2. With the accuracy up to the notation, the expressions for $\tilde{k}_{\rm lim}$, $\gamma_{\rm mod}$, $k_{\rm m}$ and $\gamma_{\rm m}$ exactly coincide with the corresponding expressions obtained by Longtin and Sonnerup (1986).

The general theory described in Sec. 4.1, of course, remains valid for $\tau > 1$. Once again the pump wave is unstable with respect to perturbation with the wave number $k = k_{ij} + a\tilde{k}$ when $|\tilde{k}|$ is sufficiently small if and only if the inequality (4.12) is satisfied. We proved analytically that (4.12) is not satisfied for $(\omega, k) = (\omega_{15}, k_{15})$

and $(\omega, k) = (\omega_{24}^{\pm}, k_{24}^{\pm})$. Hence, there are no unstable modes with (ω, k) close to either (ω_{15}, k_{15}) or $(\omega_{24}^{\pm}, k_{24}^{\pm})$.

It follows from the analysis of Sec. 4.1 that the sign of η_{14}^2 coincides with the sign of $1-b\tau$. Hence, $\eta_{14}^2 > 0$ when $\tau < 1/b$ and $\eta_{14}^2 < 0$ when $\tau > 1/b$. This implies that the perturbations with the wave numbers satisfying $|k - k_{14}| < a\eta_{14}$ are unstable when $\tau < 1/b$. The increment of these perturbations takes its maximum value $\gamma_{14} = a\tilde{\gamma}_{14}$, with $\tilde{\gamma}_{14}$ given by (4.19), when $k = k_{14}$. If $\tau > 1/b$, then there are no unstable perturbations with (ω, k) close to (ω_{14}, k_{14}). The dependences of η_{14} and $\tilde{\gamma}_{14}$ on the dispersion parameter $d = \ell k_0 = (\tau^2 - 1)/\tau$ for different values of b are shown in Figs 2 and 3.

Once again the criterion (4.12) does not work when either $(i, j) = (1, 3)$ or $(i, j) = (4, 5)$ because $D_1(1, 1) = D_1(\omega_{45}, k_{45}) = 0$. $\partial D_1/\partial k = \partial D_1/\partial\omega = 0$ when $k = 1$ and $\omega = 1$, so that the approximate solutions of the dispersion equation in the vicinity of (1,1) are given by (4.8), they are real and there are no unstable modes with k and ω close to unity.

We have seen in Sec. 4.1 that the existence of unstable modes with ω and k close to ω_{45} and k_{45}, respectively, depends on the sign of the quantity $\widehat{\Delta}$ determined by (4.16). The conclusion that the sign of $\widehat{\Delta}$ coincides with the sign of UW remains valid for $\tau > 1$. We showed analytically that $U_{45} > 0$ and $W_{45} > 0$ for $\tau - 1 \ll 1$ and for $\tau \gg 1$. We calculated U_{45} and W_{45} numerically for $1.01 \leqslant \tau \leqslant 10$ and obtained that $U_{45} > 0$ and $W_{45} > 0$ for τ in this interval. Hence, we conclude that $\widehat{\Delta} > 0$ for any $\tau > 1$ when $(\omega, k) = (\omega_{45}, k_{45})$, which implies that there are unstable modes with (ω, k) close to (ω_{45}, k_{45}). These modes correspond to the beat instability (Forslund et al. 1972; Wong and Goldstein 1986). The wave numbers of the unstable modes satisfy the condition $\widetilde{\Delta} < 0$. Using this condition we obtain that the interval of the wave numbers corresponding to the unstable modes is determined by

$$k_{45} + a^2\hat{k}_1 < k < k_{45} + a^2\hat{k}_2, \tag{4.26}$$

where $\hat{k}_1$ and $\hat{k}_2$ are given by

$$\begin{aligned}
\hat{k}_{1,2} = {} & \frac{\tau^2\omega_{45}k_{45}^2(\tau^2 - 1)[1 + (2\tau^2 - 1)\omega_{45}]}{8(1 - \omega_{45}^2)(b^2\tau^2 k_{45}^2 - \omega_{45}^2)} \\
& \times \big(\sqrt{[1 - (\tau^2 - 1)\omega_{45}][1 + \tau^2 + (\tau^2 - 1)\omega_{45}]W_{45}} \\
& \pm \sqrt{[1 + (\tau^2 - 1)\omega_{45}][1 + \tau^2 - (\tau^2 - 1)\omega_{45}]U_{45}}\big)^2 \\
& \times \big([1 + \tau^2 + (\tau^2 - 1)\omega_{45}][1 - (\tau^2 - 1)\omega_{45}]^{3/2} \\
& + [1 + \tau^2 - (\tau^2 - 1)\omega_{45}][1 + (\tau^2 - 1)\omega_{45}]^{3/2}\big)^{-2}.
\end{aligned} \tag{4.27}$$

The increment of these perturbations takes its maximum value at $k = k_{45} + a^2(\hat{k}_1 + \hat{k}_2)/2$. This maximum value is equal to $\gamma_{45} = a^2\hat{\gamma}_{45}$, where

$$\begin{aligned}
\hat{\gamma}_{45} = {} & \frac{\tau^2 k_{45}^2|k_{45} - \omega_{45}|[1 - (\tau^2 - 1)^2\omega_{45}^2]\sqrt{U_{45}W_{45}}}{|\omega_{45}^2 - b^2\tau^2 k_{45}^2|[(1 + \tau^2)^2 - (\tau^2 - 1)^2\omega_{45}^2]\sqrt{2(1 - \omega_{45}^2)}} \\
& \times \big([1 + \tau^2 + (\tau^2 - 1)\omega_{45}][1 - (\tau^2 - 1)\omega_{45}]^{3/2} \\
& + [1 + \tau^2 - (\tau^2 - 1)\omega_{45}][1 + (\tau^2 - 1)\omega_{45}]^{3/2}\big)^{-1/2}.
\end{aligned} \tag{4.28}$$

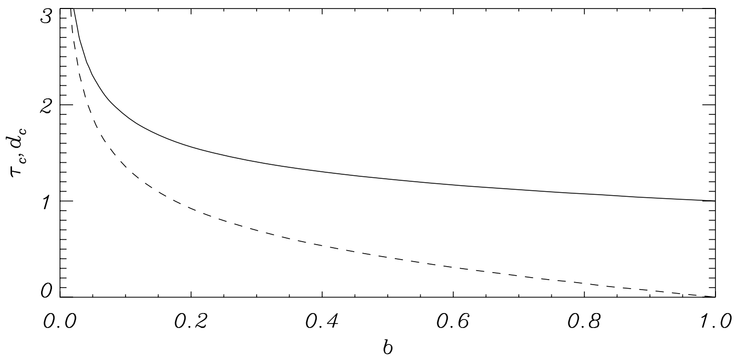

Figure 4. The dependence of τ_c (solid line) and $d_c = (\tau_c^2 - 1)/\tau_c$ (dashed line) on b.

The following asymptotic formulae are valid:

$$\hat{k}_1 \approx \hat{k}_2 \approx -\frac{1}{4(1-b^2)}, \quad \hat{\gamma}_{45} \approx \frac{d\sqrt{2}}{4(1-b^2)} \quad \text{for } d = \frac{\tau^2-1}{\tau} \ll 1, \tag{4.29}$$

$$\hat{k}_1 \approx \frac{3+\sqrt{3}}{18b^2}, \quad \hat{k}_2 \approx \frac{3+\sqrt{3}}{6b^2}, \quad \hat{\gamma}_{45} \approx \frac{3^{3/4}}{2b^2 d^3} \quad \text{for } d = \frac{\tau^2-1}{\tau} \gg 1. \tag{4.30}$$

We can observe that both expressions (4.27) for $\hat{k}_{1,2}$ and (4.28) for $\hat{\gamma}_{45}$ contain the multiplier $\omega_{45}^2 - b^2\tau^2k_{45}^2$ in the denominators. When $\tau = 1$, $\omega_{45} = k_{45} = 1$. Since $b < 1$, we conclude that $\omega_{45}^2 - b^2\tau^2k_{45}^2 > 0$ at $\tau = 1$. It is not difficult to show that ω_{45} decreases and k_{45} increases when τ increase, which implies that $\omega_{45}^2 - b^2\tau^2k_{45}^2$ is a monotonically decreasing function of τ. When $\tau \to \infty$, then $\omega_{45} \to 0$ and $k_{45} \to \sqrt{3}$, so that $\omega_{45}^2 - b^2\tau^2k_{45}^2 < 0$ for $\tau \gg 1$. This analysis shows that there is exactly one value $\tau = \tau_c$ such that $\omega_{45}^2 - b^2\tau^2k_{45}^2 = 0$ at $\tau = \tau_c$. Since $\omega_{45} > 0$ and $k_{45} > 0$, we have $\omega_{45} = b\tau k_{45}$ at $\tau = \tau_c$, which corresponds to the dispersion relation for the forward-propagating sound wave. Hence, at $\tau = \tau_c$ there is a resonance between three wave modes: the backward-propagating Alfvén wave bA$_-$, the forward-propagating Alfvén wave fA$_+$ and the forward-propagating sound wave fs. The dependence of τ_c and $d_c = (\tau_c^2 - 1)/\tau_c$ on b is shown in Fig. 4. Note that $\omega_{14} = \omega_{45}$ and $k_{14} = k_{45}$ when $\tau = \tau_c$.

The dependences of $\hat{k}_1$, $\hat{k}_2$ and $\hat{\gamma}_{45}$ on d for different values of b are shown in Figs 5 and 6, respectively. When $\tau = \tau_c$, the expressions for $\hat{k}_{1,2}$ and $\hat{\gamma}_{45}$ have singularities. This is not surprising at all because the expansions of the form $k = k_{45} + a^2\hat{k}$ and $\omega = \omega_{45} + a^2\hat{\omega}$ are not valid when τ is close to τ_c, more precisely, when $|\tau - \tau_c| \sim a$. Taking into account that, at $\tau = \tau_c$, ω_{45} is a three-fold root of D_0, it is not difficult to see that the correct expansion is $k = k_{45} + a\tilde{k}$ and $\omega = \omega_{45} + a\tilde{\omega}$. We did not study the behaviour of the unstable mode at $|\tau - \tau_c| \sim a$ because it only occurs in a very restricted domain of parameters b and τ.

5. Summary and conclusions

In this paper we considered the stability of circularly polarized Alfvén waves (pump waves). We re-derived the dispersion equation governing the pump wave stability. Then we carried out the stability analysis for small-amplitude pump waves. When

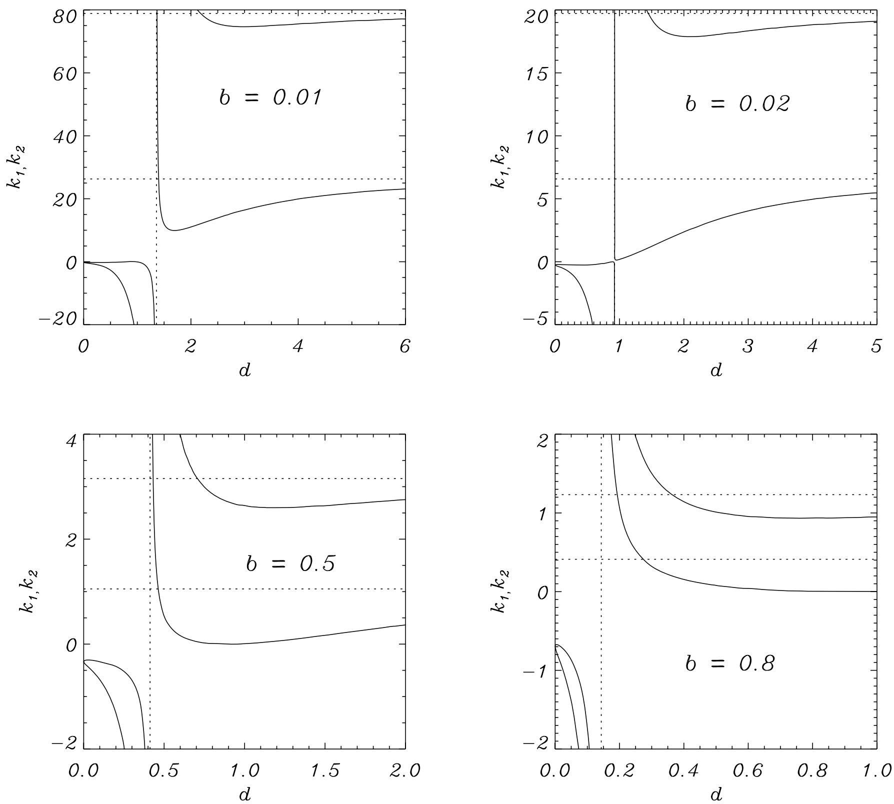

Figure 5. The dependences of $\hat{k}_1$ (upper solid lines) and $\hat{k}_2$ (lower solid lines) on d for different values of b. The horizontal dotted lines are the asymptotes for $\hat{k}_1$ and $\hat{k}_2$ as $d \to \infty$. The vertical dotted lines are the asymptotes for $\hat{k}_1$ and $\hat{k}_2$ as $d \to d_c$. Note that the gap between the line showing $\hat{k}_2$ for $d < d_c$, and the line showing $\hat{k}_1$ for $d > d_c$ is so small when $b = 0.02$ that it is almost invisible in the upper right panel.

doing so we assumed that the ratio of the sound and Alfvén speeds b is smaller than unity ($b < 1$). We found that a right-hand polarized pump wave is unstable with respect to perturbations with the dimensionless wave numbers satisfying $|k - k_{14}| < a\eta_{14}$, where a is the dimensionless pump wave amplitude, and the quantities k_{14} and η_{14} are functions of b and the dispersion parameter τ (recall that $\tau < 1$ for right-hand polarized waves and $\tau > 1$ for left-hand polarized waves). The maximum growth rate of this instability is proportional to a. This is the decay instability which leads to the decay of the pump wave into the forward-propagating sound wave and backward-propagating Alfvén wave. We see that, qualitatively, the stability properties of right-hand polarized pump waves are the same as those of pump waves in ideal MHD, where the Hall term is neglected.

The stability properties of left-hand polarized pump waves are much more complicated. In general, these waves can be subject to three different types of instabilities. The first one is the modulational instability. For a small-amplitude pump wave it occurs only if the inequality $b\tau(1 + \tau^2) < 2$ is satisfied. This inequality is equivalent to the condition that c_S is smaller than the pump wave group velocity.

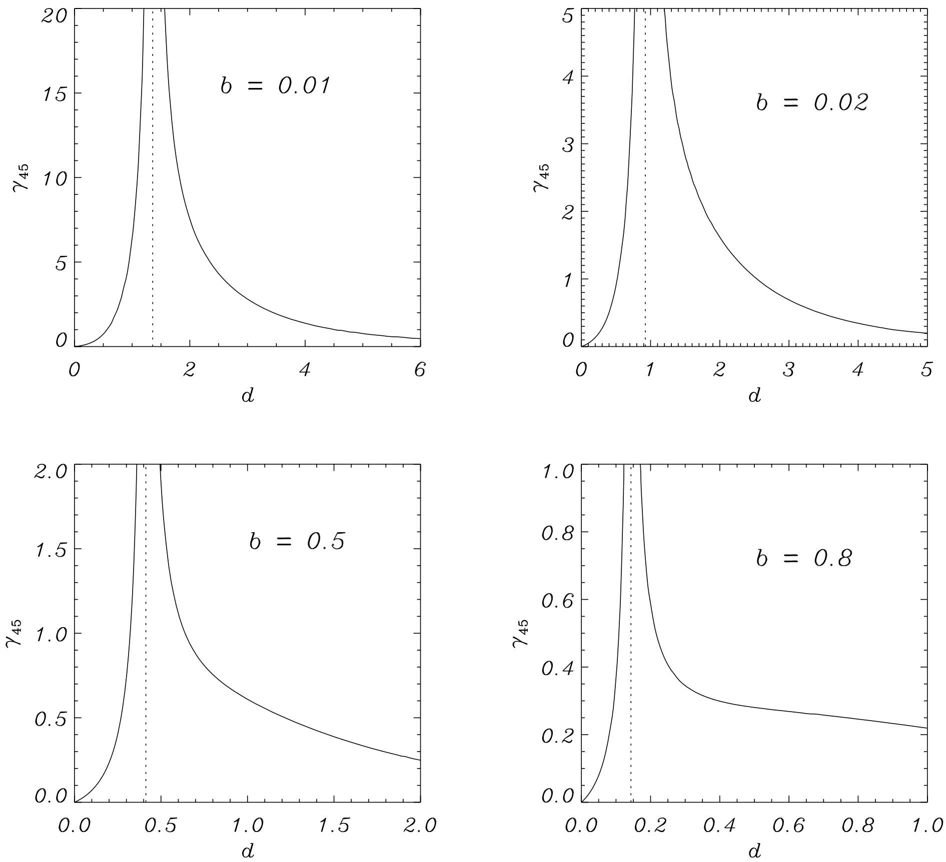

Figure 6. The dependences of $\hat{\gamma}_{45}$ on d for different values of b. The vertical dotted lines are the asymptotes for $\hat{\gamma}_{45}$ as $d \to d_c$.

A pump wave is unstable with respect to perturbations with k satisfying $k < a\tilde{k}_{\lim}$, where $\tilde{k}_{\lim}$ depends on b and τ. The maximum growth rate of this instability is proportional to a^2. A comprehensive analysis of the modulational instability for pump waves with arbitrary amplitudes can be found in Longtin and Sonnerup (1986).

The second type of instability is the same decay instability as occurs in the case of right-hand polarized pump waves. However, now it only occurs if the condition $b\tau < 1$ is satisfied. Similarly to right-hand polarized waves, the instability increment is proportional to a, and it occurs for the wave numbers satisfying $|k - k_{14}| < a\eta_{14}$.

The third type of instability occurs for any values of $\tau > 1$ and $b < 1$. Only perturbations with the wave numbers from a very narrow band $k_{45} + a^2\hat{k}_1 < k < k_{45} + a^2\hat{k}_2$ are unstable, where $\hat{k}_1$ and $\hat{k}_2$ are functions of τ only. The increment of this instability is proportional to a^2, except for τ close to τ_c when it is proportional to a. This instability results in the growth of two side-band Alfvén waves, one forward and one backward propagating. In accordance with Wong and Goldstein (1986) this is the beat instability.

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Appendix A

Here we give the coefficient functions $q_0(k), \ldots, q_6(k)$:

$$\begin{aligned}
&q_0 = 1, \quad q_1 = 4k(\tau^2 - 1),\\
&q_2 = -\{k^4(\tau^2-1)^2 + k^2[(a^2+b^2)\tau^2 - 2(\tau^2-2)^2 + 4] + (1+\tau^2)^2\},\\
&q_3 = -2k[k^2\tau^2(\tau^2-1)(a^2+2b^2) - 2(\tau^2+1)],\\
&q_4 = b^2\tau^2(\tau^2-1)^2k^6 - \{2b^2\tau^2[(\tau^2-2)^2-2] + a^2\tau^2(1-2\tau^2) - 1\}k^4\\
&\qquad + \{b^2\tau^2(\tau^2+1)^2 + a^2\tau^2(1+2\tau^2) - 4\}k^2,\\
&q_5 = -2\tau^2k^3(\tau^2+1)(a^2+2b^2), \quad q_6 = \tau^2k^4(a^2+4b^2-b^2k^2).
\end{aligned} \tag{A 1}$$

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