A population model based on a Poisson line tessellation

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***Abstract*—In this paper, we introduce a new population model. Taking the geometry of cities into account by adding roads, we build a Cox process driven by a Poisson line tessellation. We perform several shot-noise computations according to various generalizations of our original process. This allows us to derive analytical formulas for the uplink coverage probability in each case.**

***Index Terms*—uplink channel, coverage probability, Poisson line process, Cox process.**

I. Introduction

When a mobile operator wants to dimension its cellular network, it can either use large-scale realistic simulations or make some simple model assumptions that lead to analytical results on the coverage probability. In that case stochastic geometry can be most helpful, since it provides tools such as Palm calculus that give a strong sense to the intuitive idea of "typical user" or "typical antenna" (see [Bac10]).

For downlink channel studies, one usually considers a typical user at the origin and one performs shot-noise computations (see [Bac10]) on antennas' process, that is considered independent stationary and often Poisson. So no matter the exact point process users follow, provided it is stationary (see [And10] for instance). But in this paper we are interested in the uplink channel, so we want to calculate the law of interference users create with a given antenna. For that purpose we will need a population model that is realistic enough, but also analytically tractable in shot-noise computations.

In parts II and III, we gradually introduce our model that consists in throwing roads on a map according to a Poisson line process, and then throwing users on each road according to stationary Poisson point processes. In cities such as Manhattan, choosing a (restricted) Poisson line process is totally relevant. In cities that show more irregular patterns, this choice can be discussed in opposition with a Poisson Voronoi tessellation for instance (see [Glo06]). However Poisson Voronoi tessellations could not lead to analytical results such as those we present, so that we choose to keep a slightly less but still realistic model. In part IV we present our main result, which yields the coverage probability in basic cases. At last, in parts V andVI we propose some generalizations and improvements.

II. Cox process over a tessellation

A. Definition

Let us first explain how we build a Poisson line process over $\mathbb{R}^2$ (for further details, see [Sto95], p. 244-245). If D is a line of $\mathbb{R}^2$, we consider the orthogonal projection of the origin O on D, and we denote by (θ, r) its polar coordinates. Assuming $\theta \in [0;\pi[$ and $r \in \mathbb{R}$ makes (θ, r) unique. In the following, we denote by d the application: $(\theta, r) \mapsto D$. Then, a Poisson line process with intensity λ is the image by d of a Poisson point process ξ with intensity λ on half-cylinder $[0;\pi[\times\mathbb{R}$.

B. Associated random measure

If μ is a positive parameter, we associate to ξ a random measure Λ that is the total length of roads intersecting a given area:

$$\forall A \subset \mathbb{R}^2, \quad \Lambda(A) = \sum_{(\theta,r)\in\xi} \mu l(A \cap d(\theta, r)),$$

where l is the 1-dimensional Lebesgue measure. This will allow us next to introduce the population process in a very simple way, namely a Cox process with underlying measure Λ. One verifies that the equation is well-defined when A is a Borel set, that every realization of Λ is locally finite, and that Λ viewed as a random process is measurable. Note that the whole mass of Λ is concentrated on $d(\theta, r)$ lines. Then one can prove following statements (see appendix A and B):

Proposition II.1. *Λ is isotropic stationary with intensity $\pi\lambda\mu$.*

Theorem II.2. *Under Palm distribution, Λ is the sum stationary Λ and a random line through O, with independent uniform direction over $[0, \pi[$.*

C. Cox Process

A Cox process driven by a stochastic intensity Λ is a process Φ such that given Λ, Φ is a Poisson point process with intensity Λ. One can verify that Φ is stationary and isotropic as soon as Λ is. Furthermore, using the void-probabilities characterization, one obtains easily

Theorem II.3. *Under Palm distribution, Φ is the sum of an atom at O and Φ driven by Λ, generated itself under Palm.*

III. Population Model

A. Construction

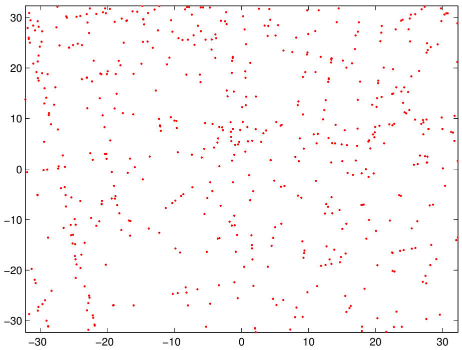
The model first consists in simulating a Poisson line process with intensity λ that represents roads on the map. Then on each road independently, we throw users with respect to a stationary Poisson point process with intensity μ (by stationarity, it is useless to introduce origins on roads). As can be seen through void probabilities, this is equivalent to a Cox process driven by Λ introduced in II-B. Grouping previous results, we obtain

Theorem III.1. *Φ is stationary, isotropic, with intensity $\pi\lambda\mu$. Under Palm, it is the sum of stationary Φ, of an independent μ-Poisson point process on a line through O with a uniform independent angle, and of an atom at O.*

B. Laplace functional of Φ

For a non-negative measurable function, let us recall that:

$$\mathcal{L}_\Phi(f) = \mathbb{E}\left[\mathrm{e}^{-\int f(\mathbf{x})\Phi(d\mathbf{x})}\right].$$



For our further needs, we mainly study functions with radial symmetry, writing abusively: $f(\mathbf{x}) = f(\|\mathbf{x}\|)$.

Theorem III.2. *If f has a radial symmetry, then*

$$\mathcal{L}_\Phi(f) = exp\Big[-2\pi\lambda \int_{r\geq 0} \Big(1 - exp\Big(-2\mu \int_{t\geq 0} \Big(1 - e^{-f(\sqrt{r^2+t^2})}\Big)\, dt\Big)\Big) dr\Big].$$

For a proof, see appendix C. In the case where f has no radial symmetry, the same proof as in appendix C leads to

Theorem III.3. *In the general case we have:*

$$\mathcal{L}_\Phi(f) = exp\Big[-2\lambda \int_{\theta=0}^{\pi} \int_{r\in\mathbb{R}} \Big(1 - exp\Big(-\mu \int_{t\in\mathbb{R}} \Big(1- e^{-f(r\cos(\theta)-t\sin(\theta), r\sin(\theta)+t\cos(\theta))}\Big) dt\Big)\Big) d\theta\ dr\Big].$$

IV. Network model and Uplink coverage

Antennas are assumed to follow an independent Poisson point process Φ_a with intensity λ_a. We also suppose that users always connect to their closest antenna. Our question is: given a typical user (in the sense of Palm), what is the probability that its antenna receives the target SINR on the uplink channel?

A. Coverage

From typical user's point of view, we are at O and Φ is denoted by Φ^0. Since Φ_a is independent of Φ, it still can be regarded stationary. Conditioning on the closest antenna, we just have to calculate the probability for the SINR to be high enough. Following [And10], let us recall the standard propagation model they use:

- the power emission of the mobiles is constant equal to $1/\epsilon$;
- at distance r, the path loss is r^α ($\alpha > 2$);
- we include some Rayleigh fading (according to an exponential law with parameter 1); hence, the received power is $hr^{-\alpha}$, where $h \sim \exp(\epsilon)$;
- the noise power is constant, equal to σ^2;
- the SINR target threshold is denoted by T;
- SNR=$1/\epsilon\sigma^2$ is the mean signal-noise ratio at distance 1.

Thanks to Radio Resource Management (RRM), users do not interfere systematically one with another. They use orthogonal resources, being through time division (HSDPA), time-frequency division (GSM, LTE), or code division (UMTS). So there is only a fraction η of users interfering, that we must take into account in the choice of μ. The easiest way consists in making a η-thinning of each Poisson point process on roads. The new Φ follows the same law as the old one, except that its intensity is multiplied by η. All computations still hold, provided we take $\mu_{\text{radio}} = \eta\mu_{\text{road}}$.

Theorem IV.1. *The uplink coverage probability is given by:*

$$\mathbb{P}(cov) = 2\pi\lambda_a \int_{\rho=0}^{\infty} e^{-\pi\lambda_a\rho^2} e^{-\epsilon T\rho^\alpha\sigma^2} p(T,\rho) q(T,\rho) \rho d\rho,$$

with

$$p(T,\rho) = exp\Big[-2\pi\lambda\rho \int_{r=0}^{\infty} \Big(1 - exp\Big(-2\mu\rho \int_{t=0}^{\infty} \frac{dt}{1 + \frac{(r^2+t^2)^{\alpha/2}}{T}}\Big)\Big) dr\Big]$$

$$q(T,\rho) = \frac{1}{\pi} \int_{\theta=0}^{\pi} exp\left[-2\mu\rho \int_{t=0}^{\infty} \frac{dt}{1 + \frac{(\sin^2\theta + t^2)^{\alpha/2}}{T}}\right] d\theta.$$

Proof - Let ρ be the distance to O of the closest antenna. By isotropy, suppose it is at $(\rho, 0)$. Received SINR is $\frac{h\rho^{-\alpha}}{\sigma^2 + I_\rho}$, where $I_\rho = \sum_{\mathbf{x}_j \in \Phi^0 \setminus \{0\}} h_j R_j^{-\alpha}$ is the interference and R_j is the distance between $\mathbf{x}_j$ and $(\rho, 0)$. Until equation (10) of [And10], nothing changes. Conditioning on ρ we have:

$$\mathbb{P}(\text{SINR} > T) = \int_{\rho>0} \mathrm{e}^{-\pi\lambda_a\rho^2} \mathrm{e}^{-\epsilon T\rho^\alpha\sigma^2} \mathcal{L}_{I_\rho}(\epsilon T\rho^\alpha) 2\pi\lambda_a\rho d\rho.$$

Obtaining $\mathcal{L}_{I_\rho}(s)$ leads to a classical shot-noise computation. First:

$$\mathcal{L}_{I_\rho}(s) = \mathbb{E}^0\left[\prod_{\mathbf{x}_j \in \Phi^0\setminus\{0\}} \frac{1}{1 + \frac{s}{\epsilon} R_j^{-\alpha}}\right].$$

According to theorem III.1, we split $\mathcal{L}_{I_\rho}(s)$ into two terms:

- the expectation on stationary Φ, which allows to suppose $\rho = 0$. We denote by p the corresponding term;

  $$p = \mathcal{L}_\Phi(f) \quad \text{with} \quad f(\mathbf{x}) = \log\left(1 + \frac{s}{\epsilon}\|\mathbf{x}\|^{-\alpha}\right),$$

  which has a radial symmetry. Applying theorem III.2, putting $s = \epsilon T\rho^\alpha$, $r \mapsto r/\rho$ and $t \mapsto t/\rho$ we obtain p.
- the expectation on Φ_i over a line $d(\theta, 0)$ through O, denoted by q. Conditioning on θ we get:

  $$q = \frac{1}{\pi}\int_{\theta=0}^{\pi} \mathbb{E}\left[\prod_{\mathbf{x}_i \in d(\theta,0)} \frac{1}{1 + \frac{s}{\epsilon} R_i^{-\alpha}}\right] d\theta = \mathcal{L}_{\Phi_i}(f)$$

  where $f(t) = \log\left(1 + \frac{s}{\epsilon} R_i^{-\alpha}\right)$ and t is the abscissa of the points on $d(\theta, 0)$ (for example starting at O). Using:

  $$f(t) = \log\left(1 + \frac{s}{\epsilon}(\rho^2 + t^2 - 2t\rho\cos\theta)^{-\alpha/2}\right),$$

  using the well-known formula for the Laplace transform of a Poisson point process (see [Bac10]), then putting $s = \epsilon T\rho^\alpha$, $r \mapsto r/\rho$, $t \mapsto t/\rho - \cos(\theta)$ and using parity, we obtain the result. ■

Remark - p and q are independent of the power parameter ϵ.

B. Comparison with an ordinary Poisson point process

Φ has intensity $\pi\lambda\mu$. If we replace it by a Poisson point process with the same intensity, the coverage probability is still :

$$\mathbb{P}(\text{SINR} > T) = 2\pi\lambda_a \int_{\rho>0} \mathrm{e}^{-\pi\lambda_a\rho^2}\mathrm{e}^{-\epsilon T\rho^\alpha\sigma^2}\mathcal{L}_{J_\rho}(\epsilon T\rho^\alpha)\rho d\rho,$$

where J_ρ is the new interference. So it is enough to compare $\mathcal{L}_{I_\rho}$ and $\mathcal{L}_{J_\rho}$. This time $\Phi^0\backslash\{0\}$ is equivalent to stationary Φ according to Slivnyak's theorem (see [Bac10]), and the same computation leads to :

$$\mathcal{L}_{J_\rho}(\epsilon T\rho^\alpha) = \exp\left[-2\pi^2\lambda\mu\rho^2\int_{r=0}^{\infty}\frac{rdr}{1+\frac{r^\alpha}{T}}\right].$$

After using generic inequality $1-\mathrm{e}^{-A} \leq A$ and changing to polar coordinates, we obtain $\mathcal{L}_{J_\rho}(\epsilon T\rho^\alpha) \leq p(T,\rho)$. But on the other hand, multiplying p by $q < 1$ produces the opposite effect. Which effect is greater than the other ? Our computations show that it depends on T (fig. 1), even if the difference happens to be very close to 0.

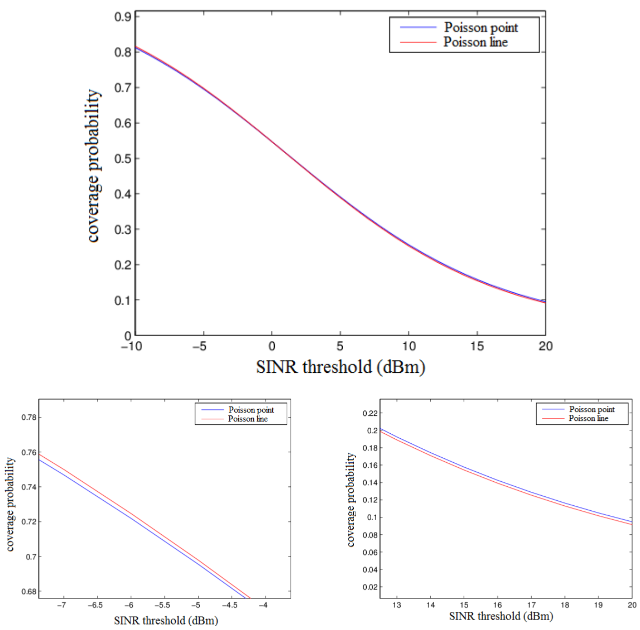


Fig. 1. Comparison of uplink coverage probabilities between Poisson line model and Poisson point model. $\lambda_a = 10$ km^{-2}, SNR = 19 dB, $\alpha = 3,57$, $\lambda = 15$ km^{-1}, $\mu = 0,1$ km^{-1}.

C. Fitting parameters to reality

The most common path-loss model is the COST-Hata model : $(P_{\text{received}})_{\text{dBm}} = (P_{\text{max}})_{\text{dBm}} - A - B\log_{10}(r)$, with typically P_{max} = 34 dBm, A = 128 dB, B = 35,7 dB and r in km. By identification, we have $\alpha = B/10 = 3.57$ and $1/\epsilon = 3.98\times10^{-13}$ W. For GSM systems, one usually takes $\sigma^2 = -174$ dBm / Hz, multiplied by a noise factor around $10^{0,8}$. Over a 200 kHz spectrum, we get $\sigma^2 = 5.02\times10^{-15}$ W, which leads to SNR = 19 dB.

We still have to choose spatial parameters λ, λ_a and μ. The average number of roads intersecting a disk with radius 1 km is $2\pi\lambda$. In Paris there are more than 6000 streets, many of which are alleys or dead-ends, and its surface is slightly more than 100 km^2. So we can legitimately take $\lambda = 15$km^{-1}. Considering that each antenna covers a disk with radius about 300m and that most antenna are 3-sector, we shall take $\lambda_a = 10$km^{-2}. Each antenna can manage up to $1/\eta$ simultaneous connections. Over a 1 km^2 square with maximum load, we would have $\pi\lambda\mu_{\text{road}} = 1/\eta\times\lambda_a$, which gives $\mu_{\text{radio}} = \lambda_a/\pi\lambda = 0,21$ km^{-1}. So a reasonable value of μ would be 0.1 km^{-1}.

Results are represented on fig. 1. The difference between both processes is always very small, which comes from the fact that μ is very small too, so that alignements on the roads almost disappear. With more critical parameters or with very rough RRM techniques causing multiple interferences, one can expect the difference to increase.

D. Walkers vs. drivers

Suppose now you have two types of users : drivers who stay on roads, and walkers who can be anywhere. If walkers are supposed to follow a Poisson point process, from their point of view, interference corresponds to the sum of a stationary Poisson point process (according to Slivnyak's theorem) and a stationary Poisson line process. On the other hand, from a driver's point of view, interference corresponds to the same sum, but also with a Poisson point process on a road through the driver. It is always greater, and the coverage probability is lower, so that it is better to be a walker than a driver !

V. Two generalizations

A. First generalization : with μ non-constant

To be more realistic, we wish to model a network where roads do not all experience the same traffic, so we make μ random. Let us introduce a family $\mu_{(\theta,r)}$ that is iid and independent of ξ, and write :

$$\Lambda(A) = \sum_{(\theta,r)\in\xi}\mu_{(\theta,r)}l(A\cap d(\theta,r)).$$

Of course Λ remains stationary and isotropic. Same computations as before lead then to

Theorem V.1. *Under Palm distribution, Λ is the sum of stationary Λ and an independent line through O. The angle of the line is uniform, but its intensity follows density*

$$\mathbb{P}'(d\mu) = \frac{\mu}{\mathbb{E}[\mu]}\mathbb{P}(d\mu).$$

Theorem V.2. *With μ random, the coverage probability is :*

$$\mathbb{P}(cov) = 2\pi\lambda_a\int_{\rho=0}^{\infty}e^{-\pi\lambda_a\rho^2}e^{-\epsilon T\rho^\alpha\sigma^2}p(T,\rho)q(T,\rho)\rho d\rho,$$

with

$$p(T,\rho) = exp\Bigg[-2\pi\lambda\rho\int_{r=0}^{\infty}\Bigg(1- \mathcal{L}_\mu\left(2\rho\int_{t=0}^{\infty}\frac{dt}{1+\frac{(r^2+t^2)^{\alpha/2}}{T}}\right)\Bigg)dr\Bigg]$$

$$q(T,\rho) = \int_{\theta=0}^{\pi}\int_\mu exp\Bigg(-2\mu\rho \int_{t=0}^{\infty}\frac{dt}{1+\frac{(\sin^2\theta+t^2)^{\alpha/2}}{T}}\Bigg)\frac{d\theta}{\pi}\,\frac{\mu}{\mathbb{E}[\mu]}\,\mathbb{P}(d\mu).$$

Remark - We see that high-loaded roads contribute more to interference than before, resulting in a bias of coverage towards low values. If we come back to the comparison of section IV-B, letting Var(μ) grow while $\mathbb{E}[\mu]$ remains constant, one can expect a downward trend to appear. This is exactly what show our computations (fig. 2). Of course, the difference is quite small (since μ is small too), but if we zoom in the figure, the effect is visible. Furthermore, with more extreme parameters, one can expect the difference to become obvious.

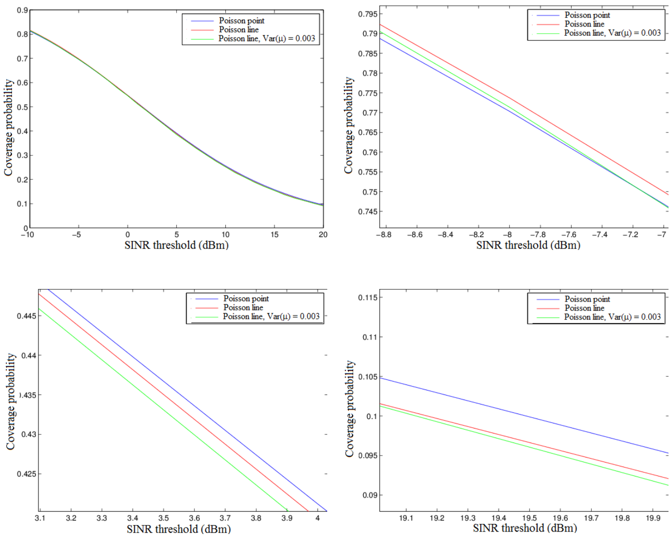


Fig. 2. Comparison of different coverage probabilities. Radio and spatial parameters are the same as in fig. 1. In blue, this is Poisson Point. In red, Poisson line with constant $\mu = 0,1$ km^{-1}. In green, Poisson line with random μ (uniform law with mean 0.1 km^{-1} and variance 0.003 km^{-2}).

B. Second generalisation : Manhattan model

This time, we suppose again μ constant, but roads are orthogonal, ie $\theta = 0$ or $\pi/2$. To perform that, we concentrate the mass of ξ on both axes $\{0\} \times \mathbb{R}$ and $\{\pi/2\} \times \mathbb{R}$. It is immediate that Λ remains stationary (but not isotropic!), and the same arguments as before lead to

Theorem V.3. *Under Palm distribution, Λ is the sum of stationary Λ and a line through O, whose angle is 0 or $\pi/2$ with probability $1/2$.*

Theorem V.4. *With equal intensities, Manhattan model and Poisson line model have the same coverage probability.*

VI. How to take RRM better into account

A. Motivation

So far, our model has taken orthogonal multiple access into account, but users of the same time-frequency pattern have been uniformly spread over territory. Now it would be more realistic to spread interferers only outside of current user's cell. Let us denote by $\mathcal{C}(O)$ the Voronoï cell of Φ_a that covers O. In the coverage computation, we should first condition on Φ_a, and then compute our shot-noise on same f as in theorem IV.1, except that we would make it zero inside $\mathcal{C}(O)$. But making that, we lose radial symmetry[1], and worse, there is no analytical simple result on $\mathcal{C}(O)$ geometry.

[1] which is not a crucial issue, since we know general formula for Laplace transform although it is quite uncomfortable, see theorem III.3.

B. Approximation

Then, a good approximation consists in replacing $\mathcal{C}(O)$ by a disk of same (mean) surface. More accurately, we make f zero on a O-centered disk, whose surface is the mean surface of $\mathcal{C}(O)$ conditionally to ρ. If we denote by $R(\lambda_a, \rho)$ the radius of the disk and if we write $x^+ = \max(0, x)$, we obtain

Theorem VI.1. *Taking RRM into account, the new coverage is well-approximated by :*

$$\mathbb{P}(cov) \approx 2\pi\lambda_a \int_{\rho=0}^{\infty} e^{-\pi\lambda_a\rho^2} e^{-\epsilon T\rho^\alpha\sigma^2} p(T,\rho) q(T,\rho) \rho d\rho,$$

with

$$p(T,\rho) = exp\Bigg[-2\pi\lambda\rho \int_{r=0}^{\infty}\Bigg(1 - exp\Bigg(-2\mu\rho \int_{t=\sqrt{\left(\frac{R(\lambda_a,\rho)^2 - r^2}{\rho^2}\right)^+}}^{\infty} \frac{dt}{1 + \frac{(r^2+t^2)^{\alpha/2}}{T}}\Bigg)\Bigg)dr\Bigg],$$

$$q(T,\rho) = \frac{1}{\pi}\int_{\theta=0}^{\pi} exp\Bigg[-2\mu\rho \int_{t=\sqrt{\left(\frac{R(\lambda_a,\rho)^2}{\rho^2} - \sin^2\theta\right)^+}}^{\infty} \frac{dt}{1 + \frac{(\sin^2\theta + t^2)^{\alpha/2}}{T}}\Bigg] d\theta.$$

C. Equivalent radius issue

It is quite probable that no analytical formula exists for $R(\lambda_a, \rho)$. So the easiest way of finding it is to compute a table. We show the result in fig. 3 for $\lambda_a = 1$ km^{-2}. For practical applications, it is useless to compute values corresponding to $\rho \geq 2$, since $\mathbb{P}(\rho \geq 2) = \mathrm{e}^{-4\pi} \approx 3,5 \times 10^{-6}$.

ρ	eq. surf.	ρ	eq. surf.	ρ	eq. surf.
0,05	1,0060	0,70	1,4317	1,35	2,1630
0,10	1,0158	0,75	1,4809	1,40	2,2312
0,15	1,0279	0,80	1,5281	1,45	2,2922
0,20	1,0471	0,85	1,5832	1,50	2,3580
0,25	1,0778	0,90	1,6354	1,55	2,4227
0,30	1,1024	0,95	1,6899	1,60	2,4919
0,35	1,1348	1,00	1,7486	1,65	2,5599
0,40	1,1745	1,05	1,8030	1,70	2,6284
0,45	1,2140	1,10	1,8617	1,75	2,6832
0,50	1,2532	1,15	1,9219	1,80	2,7556
0,55	1,2936	1,20	1,9801	1,85	2,8272
0,60	1,3380	1,25	2,0432	1,90	2,8999
0,65	1,3850	1,30	2,1059	1,95	2,9679

Fig. 3. mean surface of $\mathcal{C}(O)$ conditionally to ρ for $\lambda_a = 1$ km^{-2}.

Note that you can always come down to $\lambda_a = 1$ km^{-2}. Indeed, a Poisson point process with intensity λ_a can be seen as a process with intensity 1 dilated by $\sqrt{\lambda_a}$. This gives :

$$R(\lambda_a,\rho) = R\left(1, \rho\sqrt{\lambda_a}\right)/\sqrt{\lambda_a}.$$

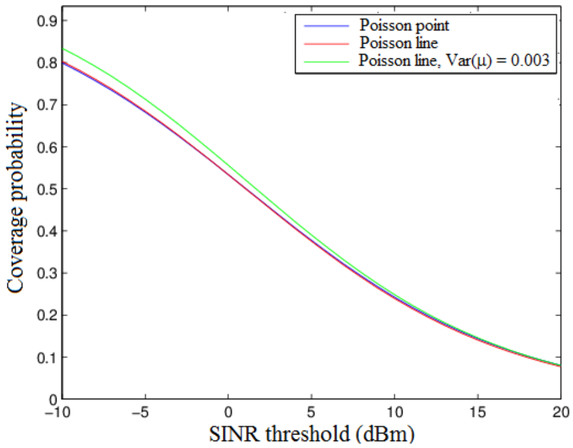


Fig. 4. Computation of the coverage probability for a Poisson line process taking RRM into account. Radio and spatial parameters are the same as fig. 1. As expected, the new coverage is slightly better, since we do not take into account intra-cellular interference.

VII. Conclusion

We have built a robust model, generic enough to cover a wide range of situations provided we add some generalizations. Of course there are far more than the both we propose : one could also take another law for the fading, or the noise, or mix several classes of users with several classes of roads, etc. Each time, we have shown how to perform computations for the coverage probability that lead to easy-to-implement analytical results.

Another feature of the model is that it allows backwards engineering : if a telecommunication operator can measure the coverage probability in his network through call dropping rates, then he can deduce μ and determine how many users are in the zone.

It would be also interesting to add some time-variation in our model, making users move along the roads to be even more realistic. This would probably lead to ergodic results about the time variation of coverage.

Appendix

A. Proof of proposition II.1

The law of Λ is characterized by finite distributions $(\Lambda(A_1), \ldots, \Lambda(A_k))$ when A_js run through Borel sets of $\mathbb{R}^2$, and these distributions are themselves characterized by their Laplace transform :

$$\mathbb{E}\left[\mathrm{e}^{-\sum_j s_j \Lambda(A_j)}\right] = \mathbb{E}\left[\mathrm{e}^{-\sum_j s_j \sum_{(\theta,r)\in\xi} \mu l(A_j \cap d(\theta,r))}\right].$$

After rotating the plane of θ_0 around origin, the new Laplace transform is given by :

$$\mathbb{E}\left[\exp\left(-\sum_j s_j \sum_{(\theta,r)\in\xi} \mu l(\mathrm{rot}_{\theta_0}(A_j) \cap d(\theta,r))\right)\right]$$

$$= \mathbb{E}\left[\exp\left(-\sum_j s_j \sum_{(\theta,r)\in\xi} \mu l(A_j \cap d(\theta-\theta_0,r))\right)\right]$$

(since l is invariant under rotations). Hence, we just have applied a horizontal translation (and also possibly a horizontal symmetry[2]) along $-\theta_0$ over the half-cylinder $[0;\pi[\times\mathbb{R}$. ξ is invariant by such a transformation, since it is characterized by void probabilities $\mathbb{P}(\xi(K) = 0) = \mathrm{e}^{-\lambda m(K)}$, when K runs through compact sets of $[0;\pi[\times\mathbb{R}$ and where m is the 2-dimensional Lebesgue measure. Hence Λ is isotropic.

Now, to show that it is stationary, it is enough to show that it is invariant under translations along horizontal vectors $\rho\mathbf{i}$. Also working on Laplace transforms, the only difference is that (θ, r) becomes $(\theta, r - \rho\cos(\theta))$. But this transformation has jacobian 1, hence $m(K)$ does not change.

Now that we know that Λ is stationary, one can compute its intensity. If A is a Borel set of $\mathbb{R}^2$, we denote by $d^{-1}(A)$ the region of $[0;\pi[\times\mathbb{R}$ that corresponds to lines intersecting A. Let B be the O-centered unit disk. $d^{-1}(B) = [0,\pi[\times[-1,1]$. On this strip, ξ law consists in generating an integer N with respect to a $2\pi\lambda$-paramater Poisson law, and then throwing N uniform iid points (θ_i, r_i) in the strip. Using Wald's identity, one finds that the intensity of Λ is $\pi\lambda\mu$. ■

B. Proof of theorem II.2

We characterize Λ under Palm distribution by the Laplace transform of its finite distributions. Without loss of generality, we can compute them for one Borel set A, since calculus immediately extends to the general case.

Let us find $\mathbb{E}^0\left[\mathrm{e}^{-s\Lambda(A)}\right]$ applying Palm formula on unit disk B. First we generate number N of points falling in $[0,\pi[\times[-1,1]$, and then we throw N uniform iid couples (θ_i, r_i) in this strip. For each point, we introduce a unit vector $\mathbf{u}_i$ directing $d(\theta_i, r_i)$ (no matter its orientation). Then $d(\theta_i, r_i)\cap B$ is the segment with middle $(r_i\cos(\theta_i), r_i\sin(\theta_i))$ (that we shall denote by $\mathbf{y}_i$ for more simplicity), direction $\mathbf{u}_i$ and length $2\sqrt{1-r_i^2}$. We have :

$$\mathbb{E}^0\left[\mathrm{e}^{-s\Lambda(A)}\right] = \frac{1}{\pi\lambda\mu\ m(B)}\ \mathbb{E}\left[\int_{\mathbf{y}\in B} \mathrm{e}^{-s\Lambda(A+\mathbf{y})}\Lambda(d\mathbf{y})\right]$$

$$= \frac{1}{\pi^2\lambda\mu}\ \mathrm{e}^{-2\pi\lambda}\sum_{n\geq 0}\frac{(2\pi\lambda)^n}{n!}\frac{1}{(2\pi)^n}\int_{\substack{\theta_1\in[0,\pi[\\ r_1\in[-1,1]}}\cdots\int_{\substack{\theta_n\in[0,\pi[\\ r_n\in[-1,1]}}$$

$$\sum_{i=1}^n \int_{t=-\sqrt{1-r_i^2}}^{\sqrt{1-r_i^2}} \mathbb{E}\left[\mathrm{e}^{-s\Lambda(A+\mathbf{y}_i+t\mathbf{u}_i)}\middle|\xi \text{ on } d^{-1}(B)\right]$$

$$\mu\ dt\ d\theta_1\ dr_1\ \ldots d\theta_n\ dr_n.$$

By symmetry :

$$\mathbb{E}^0\left[\mathrm{e}^{-s\Lambda(A)}\right] = \frac{\mathrm{e}^{-2\pi\lambda}}{\pi^2\lambda\mu}\sum_{n\geq 1}\frac{\lambda^n}{n!}n\int_{\substack{\theta_1\in[0,\pi[\\ r_1\in[-1,1]}}\cdots\int_{\substack{\theta_n\in[0,\pi[\\ r_n\in[-1,1]}}$$

$$\int_{t=-\sqrt{1-r_1^2}}^{\sqrt{1-r_1^2}} \mathbb{E}\left[\mathrm{e}^{-s\Lambda(A+\mathbf{y}_1+t\mathbf{u}_1)}\middle|\xi \text{ on } d^{-1}(B)\right]$$

$$\mu\ dt\ d\theta_1\ dr_1\ \ldots d\theta_n\ dr_n.$$

[2] if $\theta - \theta_0$ runs out of $[0,\pi[$, we have to bring it back modulo π, possibly turning r to $-r$: the half-cylinder is in fact a Möbius strip, since $(\theta - \pi, r)$ identifies itself to $(\theta, -r)$.

Then we separate (θ, r) regarding $|r| \leq 1$ or $|r| > 1$, and we use ξ's complete independence property :

$$\mathbb{E}^0\left[\mathrm{e}^{-s\Lambda(A)}\right] = \frac{\mathrm{e}^{-2\pi\lambda}}{\pi^2}\sum_{n\geq 1}\frac{\lambda^{n-1}}{(n-1)!}\int_{\substack{\theta_1\in[0,\pi[\\ r_1\in[-1,1]}}\cdots\int_{\substack{\theta_n\in[0,\pi[\\ r_n\in[-1,1]}}$$
$$\int_{t=-\sqrt{1-r_1^2}}^{\sqrt{1-r_1^2}}\exp\left(-s\mu\sum_{i=1}^{n} l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta_i, r_i)\right)\right)$$
$$\mathbb{E}\left[\exp\left(-s\mu\sum_{(\theta,r)\in\xi}^{|r|>1} l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta, r)\right)\right)\right]$$
$$dt\ d\theta_1\ dr_1\ \ldots d\theta_n\ dr_n.$$

We split $\sum_i l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta_i, r_i)\right)$ into two parts :

- for $i = 1$, translating by $-\mathbf{y}_1 - t\mathbf{u}_1$ yields :
$$l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta_1, r_1)\right) = l\left(A\cap d(\theta_1, 0)\right).$$
- for $i > 1$, all terms are symmetric.

We obtain :

$$\mathbb{E}^0\left[\mathrm{e}^{-s\Lambda(A)}\right] = \frac{\mathrm{e}^{-2\pi\lambda}}{\pi^2}\sum_{n\geq 1}\frac{\lambda^{n-1}}{(n-1)!}\int_{\theta_1=0}^{\pi}\mathrm{e}^{-s\mu l(A\cap d(\theta_1,0))}$$
$$\int_{r_1}\int_{t=-\sqrt{1-r_1^2}}^{\sqrt{1-r_1^2}}\left(\int_{\theta,r}\mathrm{e}^{-s\mu l((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta,r))}dr\ d\theta\right)^{n-1}$$
$$\mathbb{E}\left[\exp\left(-s\mu\sum_{(\theta,r)\in\xi}^{|r|>1} l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta, r)\right)\right)\right] dtdr_1d\theta_1$$
$$= \frac{1}{\pi^2}\int_{\theta_1=0}^{\pi}\mathrm{e}^{-s\mu l(A\cap d(\theta_1,0))}\int_{r_1}\int_{t=-\sqrt{1-r_1^2}}^{\sqrt{1-r_1^2}}$$
$$\exp\left(-2\pi\lambda + \lambda\int_{\theta,r}\mathrm{e}^{-s\mu l((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta,r))}dr\ d\theta\right)$$
$$\mathbb{E}\left[\exp\left(-s\mu\sum_{(\theta,r)\in\xi}^{|r|>1} l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta, r)\right)\right)\right] dtdr_1d\theta_1.$$

The exponential can be rewritten as :

$$\exp\left(-\int_{\theta=0}^{\pi}\int_{r\in\mathbb{R}}\left(1-\mathrm{e}^{-f(\theta,r)}\right)\lambda\ dr\ d\theta\right),$$

with $f(\theta, r) = s\mu\ l((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta, r))\ \mathbb{1}(|r| \leq 1)$. One recognizes the Laplace transform $\mathcal{L}_\xi(f)$:

$$\mathbb{E}\left[\exp\left(-s\mu\sum_{(\theta,r)\in\xi}^{|r|\leq 1} l((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta, r))\right)\right].$$

Using complete independence again, we group both expectations in the main computation :

$$\mathbb{E}^0\left[\mathrm{e}^{-s\Lambda(A)}\right] = \frac{1}{\pi^2}\int_{\theta_1=0}^{\pi}\mathrm{e}^{-s\mu l(A\cap d(\theta_1,0))}\int_{r_1}\int_{t=-\sqrt{1-r_1^2}}^{\sqrt{1-r_1^2}}$$
$$\mathbb{E}\left[\exp\left(-s\mu\sum_{(\theta,r)} l\left((A+\mathbf{y}_1+t\mathbf{u}_1)\cap d(\theta, r)\right)\right)\right] dtdr_1d\theta_1$$
$$= \frac{1}{\pi^2}\int_{\theta_1=0}^{\pi}\mathrm{e}^{-s\mu l(A\cap d(\theta_1,0))}\int_{r_1}\int_{t=-\sqrt{1-r_1^2}}^{\sqrt{1-r_1^2}}$$
$$\mathbb{E}\left[\mathrm{e}^{-s\Lambda(A+\mathbf{y}_1+t\mathbf{u}_1)}\right]\ dt\ dr_1\ d\theta_1$$
$$= \mathbb{E}\left[\mathrm{e}^{-s\Lambda(A)}\right]\frac{1}{\pi}\int_{\theta_1=0}^{\pi}\mathrm{e}^{-s\mu l(A\cap d(\theta_1,0))}\ d\theta_1$$

since Λ is stationary. ■

C. Proof of theorem III.2

First suppose that f is zero outside of a radius $R \geq 0$. Conditioning on lines that intersect disk of radius R, using symmetry with respect to θ and independence of restrictions Φ_i on each line, we obtain :

$$\mathcal{L}_\Phi(f) = \mathrm{e}^{-2\pi R\lambda}\sum_{n\geq 0}\frac{(2\pi R\lambda)^n}{n!(2R)^n}\int_{r_1,\ldots,r_n=-R}^{R}\left(\prod_{i=1}^{n} g(r_i)\right)dr_i,$$

where $g(r_i) = \mathbb{E}\left[\mathrm{e}^{-\int f(\mathbf{x})\Phi_i(d\mathbf{x})}\right]$.

Since g is even, $\mathcal{L}_\Phi(f) = \exp\left[-2\pi\lambda\int_0^R(1-g(r))dr\right]$. To find $g(r)$, one can suppose without loss of generality that the corresponding line is vertical. Its intersection with the disk of radius R is a segment with length $2\sqrt{R^2-r^2}$. Conditioning on points falling in the segment :

$$\begin{aligned} g(r) &= \mathrm{e}^{-2\mu\sqrt{R^2-r^2}}\sum_{m\geq 0}\frac{(2\mu\sqrt{R^2-r^2})^m}{m!(2\sqrt{R^2-r^2})^m} \\ &\qquad\left(\int_{t=-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\mathrm{e}^{-f(\sqrt{r^2+t^2})}dt\right)^m \\ &= \exp\left[-2\mu\sqrt{R^2-r^2} + \mu\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\mathrm{e}^{-f(\sqrt{r^2+t^2})}dt\right] \\ &= \exp\left[-2\mu\int_0^{\sqrt{R^2-r^2}}\left(1-\mathrm{e}^{-f(\sqrt{r^2+t^2})}\right)dt\right]. \end{aligned}$$

Since $1-\mathrm{e}^{-f(\sqrt{r^2+t^2})}$ is zero for $t > \sqrt{R^2-r^2}$:

$$g(r) = \exp\left[-2\mu\int_0^{\infty}\left(1-\mathrm{e}^{-f(\sqrt{r^2+t^2})}\right)dt\right].$$

We see that $g(r) = 1$ as soon as $r > R$. Hence, this gives the result for f with bounded support. In general case, apply previous result to $\tilde{f}(\mathbf{x}) = f(\mathbf{x})\mathbb{1}_{\|\mathbf{x}\|\leq R}$ and use monotone convergence theorem. ■

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