Sigma models with a Wess-Zumino term in twistor spaces

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Abstract.

We characterize the Riemannian manifolds whose the twistor space satisfies the geometric properties necessary to the existence of some sigma model with a Wess-Zumino term on this twistor space. We prove that these manifolds are space forms. Then we study the Riemannian manifolds for which there exists a subbundle of the twistor space which satisfies these geometric properties and prove that in most cases these manifolds are locally homogeneous. In our study, we are led to prove some theorems about metric connections with parallel curvature: we prove for example that a metric connection with parallel curvature and with restricted holonomy group $SO(n)$ must be the Levi-Civita connection and therefore the Riemannian manifold is a space form. We also propose a general method to study metric connections with parallel curvature.

Contents

1 Introduction.

1.1 The sigma models.

The classical non-linear sigma model describes harmonic maps between two (pseudo)Riemannian manifolds. More generally non-linear sigma models are a class of variational problems generalising harmonic maps. In theoretical physics, non-linear sigma-models provide a much studied class of field theories of both phenomenological and theoretical interest.
Harmonics maps $f\colon M \to N$ have some universal character: because of the simplicity of their definition, as critical points of the energy or Dirichlet functional

$$\frac{1}{2}\int_M |df|^2 d\mathrm{vol},$$

we can meet examples of harmonic maps in various situations in geometry as well in physics. In condensed matter physics, for example, harmonic maps between a 3-dimensional domain and a sphere have been used as a simplified model for nematic liquid crystals. Other models used in physics, such as the Skyrme model, Higgs models or Ginzburg-Landau models, show strong connections with the theory of harmonic maps. In theoretical physics, harmonic maps from surfaces into a Lie group (or more generally a Pseudo-Riemannian manifold) are extensively studied, in particular since they have strong analogies with 4-dimensional (self-dual) Yang-Mills equations but are simpler to handle. The chiral model for example summarizes many low energy QCD interactions while more generally 2-dimensional sigma-models may possess non trivial classical field configurations.
Some time ago, the interest of physicists in sigma-models have been reinforced since their quantization leads to examples of conformal quantum field theories.
The study of Wess-Zumino (WZ) terms has received considerable attention since they were introduced in four-dimensional chiral field theories as effective Lagrangians describing the low energy consequences of the anomalous Ward identities of the theory. Later it was realised that one could have WZ terms also in two spacetime dimensions [5].
Indeed, the classical sigma-model has been generalized by introducing a Wess-Zumino term into the Lagrangian. This term may be interpreted as adding torsion to the canonical Levi-Civita connection of the earlier models. The addition of such torsion imposes constraints on the possible geometries of the target [5]. Therefore, the study of the properties of non-linear sigma models involves often the geometry of the target space on which these theories are defined.
Furthermore, the two-dimensional nonlinear sigma model plays an important role in the context of string theory. Two dimensional sigma-models and their quantum mechanical reductions have already proven a fertile arena for the interplay of topology, geometry, and physics [8].
Moreover, non-linear sigma-models have interesting connections with geometry and topology [30, 31, 32, 33, 34, 16].

Now, let us present the particular family of non-linear sigma models that will concern us in our paper.

1.2 Stringy harmonic maps.

In [20], we have introduced new classes of non-linear second elliptic equations. The so-called *Stringy harmonic maps* generalize harmonic maps. These take values into two different general classes of manifolds: almost complex manifolds and f-manifolds (i.e. endowed with a f-structure). We proved that in a good geometric context these maps are exactly the solutions of the Euler Lagrange equation of a sigma model with a Wess-Zumino term.
Here, we consider the stringy harmonic maps into f-manifolds (N, F). More precisely, we consider the particular class of metric f-manifolds (N, F, h) defined by the twistor spaces of Riemannian manifolds (M, g). Indeed, for any Riemannian manifold (M, g), endowed with any metric connection ∇, the twistor space of orthogonal complex structures on (M, g) can be endowed canonically with a structure of metric f-manifold. We want to know if stringy harmonic maps into this twistor space (endowed with its canonical f-structure) admit a variational formulation as the the Euler Lagrange equation of a sigma model with a Wess-Zumino term.
We will see that this condition implies very strong constraints on the Riemannian manifold (M, g). Indeed we will prove that the Riemannian manifolds whose the twistor space satisfies the geometric properties necessary to the existence of a variational formulation for stringy harmonicity (into this twistor space) are space forms. Then we will study the Riemannian manifolds for which there exists a subbundle of the twistor space which satisfies these geometric properties and prove that in most cases these manifolds are locally homogeneous.
Stringy harmonic maps have been introduced in the context of the geometric interpretation of the elliptic integrable systems in homogeneous geometries (in the sense of C.-L. Terng [29], see [20] for more details). As a corollary of this study [20], these maps provide new examples of integrable two-dimensional non linear sigma models, taking values in some homogeneous spaces, namely in k-symmetric spaces, which are not symmetric spaces.
In the present subsection of the introduction, we recall the definitions and main properties about stringy harmonic maps. We refer to [20, § 5, 6] for details and proofs.

1.2.1 Stringy harmonic maps into almost complex manifolds.

A map $f\colon L \to (N, J, \nabla)$ from a Riemann surface into an almost complex manifold (N, J) endowed with a linear connection ∇ is stringy harmonic if

$$-\tau_g(f) + (J \cdot T)_g(f) = 0. \tag{1.1}$$

Here $\tau_g(f) = \mathrm{Tr}_g(\nabla df)$ is the tension field of f w.r.t. ∇, g is an Hermitian metric on L, T is the torsion of ∇ and $J \cdot T = -JT(J\cdot, J\cdot)$.
Remark that if ∇ is torsion free, $T = 0$, then the stringy harmonicity coincides with the (affine) harmonicity. In particular, if ∇ is the Levi-Civita connection defined by a metric on (M, g), then we recover the usual harmonicity. Therefore to obtain a new equation, one needs to work with connection with torsion.
Furthermore, in [20], we looked for a general geometric setting in which the stringy harmonicity has an interesting interpretation. First of all, let us remark that we have to choose the connection with respect to which the stringy harmonicity will be considered. But in general we do not have a "special" connection with respect to which one can consider the stringy harmonicity. Therefore, a first problem - that we solved - was to find a general class of (almost complex) manifold in which there exists some unique "canonical" connection, with respect to which we then could consider the stringy harmonicity. Secondly it turned out also that the stringy harmonicity with respect to this new connection admits a variationnal interpretation of a sigma model with a Wess-Zumino term.

It turns out that the more rich geometric context in which stringy harmonicity admits interesting properties is the one of $\mathcal{G}_1$-manifolds, more precisely $\mathcal{G}_1$-manifolds whose the characteristic connection has a parallel torsion. Making systematic use of the covariant derivative of the Kähler form, A. Gray and L. M. Hervella, in the late seventies, classified almost Hermitian structures into sixteen classes [14]. Denote by $\mathcal{W}$ the space of all trilinear forms (on some Hermitian vector space, say $T_{y_0}N$ for some reference point $y_0 \in N$) having the same algebraic properties as $\nabla^h\Omega_J$, where $\Omega_J = \langle J\cdot, \cdot\rangle$ is the Kähler form and ∇^h is the Levi-Civita connection. Then they proved that we have a $U(n)$-irreducible decomposition $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. The sixteen classes are then respectively the classes of almost Hermitian manifolds for which $\nabla^h\Omega_J$ 'lies in' the $U(n)$-invariant subspaces $\mathcal{W}_I = \oplus_{i\in I}\mathcal{W}_i$, $I \subset \{1, \dots, 4\}$, respectively. In particular, if we take as invariant subspace $\{0\}$, we obtain the Kähler manifolds, if we take $\mathcal{W}_1$, we obtain the class of nearly Kähler manifolds. Moreover the class of $\mathcal{G}_1$-manifolds is the one defined by $\mathcal{G}_1 = \mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$. It is characterised by : (N, J, h) is of type $\mathcal{G}_1$ if and only if the Nijenhuis tensor N_J is totally skew-symmetric (i.e. a 3-form).
In [20], we are naturally led to reprove the following theorem due to Friedrich-Ivanov[11] (see also [20, Th. 5.3.1] for a little bit different and completely written proof).

Theorem 1.1 *An almost Hermitian manifold (N, J, h) admits a Hermitian connection with totally skew-symmetric torsion if and only if the Nijenhuis tensor N_J is itself totally skew-symmetric. In this case, the connection is unique and determined by its torsion which is given by*

$$T = -d^c\Omega_J + N_J.$$

The characteristic connection is then given by $\nabla = \nabla^h - \frac{1}{2}T$.

Then we proved the following key result:

Proposition 1.1 [20, §5.3.3] *Let us suppose that the almost Hermitian manifold (N, J, h) is a $\mathcal{G}_1$-manifold. Let us suppose that its characteristic connection ∇ has a parallel torsion $\nabla T = 0$. Then the 3-form*

$$H(X, Y, Z) = T(JX, JY, JZ) = \langle (J\cdot T)(X, Y), Z\rangle$$

is closed $dH = 0$.

Which then gives us the following variational interpretation

Theorem 1.2 [20, §5.3.3] *Let us suppose that the almost Hermitian manifold (N, J, h) is a $\mathcal{G}_1$-manifold. Let us suppose that its characteristic connection ∇ has a parallel torsion $\nabla T = 0$. Then the equation for stringy harmonic maps $f: L \to N$ is exactly the Euler-Lagrange equation for the sigma model in N with a Wess-Zumino term defined by the closed 3-form*

$$H = -d\Omega_J + JN_J.$$

The action functional of the sigma model in (N, h) with a Wess-Zumino term defined by a closed 3-form H is given by

$$S(f) = E(f) + S^{WZ}(f) = \frac{1}{2}\int_L |df|^2 d\text{vol}_g + \int_B H,$$

where B is 3-submanifold (or indeed a 3-chain) in N whose boundary is $f(L)$.
Then since $dH = 0$, the variation of the Wess-Zumino term is a boundary term

$$\delta S^{WZ} = \int_B L_{\delta f} H = \int_B d\imath_{\delta f} H = \int_{f(L)} \imath_{\delta f} H,$$

whence its contribution to the Euler-Lagrange equation involves only the original map $f\colon L \to N$. In our case it gives the torsion terms $(J \cdot T)_g(f)$, in (1.1).

Remark 1.1 It is important to mention that in fact there is two notions of stringy harmonicity: the stringy harmonicity defined above and the $\star$-stringy harmonicity. The $\star$-stringy harmonicity is obtained when in (1.1), we replace the terms $J \cdot T$ by the terms $J \star T$, where $J \star T$ is another linear action of J on T defined by

$$J \star B = \frac{1}{2}\left(B(J\cdot, J\cdot, J\cdot) + B(J\cdot, \cdot, \cdot) + B(\cdot, J\cdot, \cdot) + B(\cdot, \cdot, J\cdot)\right) = \frac{1}{2}\left(J \cdot B + J \circlearrowleft B\right),$$

for any $B \in \mathcal{C}(\Lambda^2 T^*N \otimes TN)$. In [20, cor. 5.3.1], we prove that the $\star$-stringy harmonicity is equivalent to stringy harmonicity w.r.t. a new almost complex structure. Moreover, the variational interpretation above holds identically for $\star$-stringy harmonicity (just replace H by $H^\star = -d\Omega_J + \frac{1}{2} J N_J$, see [20, §5.3.3]).

Remark 1.2 More generally, the variational formulation of stringy harmonicity holds if and only if the 3-form H is closed and we have also the following equivalences: $dH = 0 \Leftrightarrow dH^* = 0 \Leftrightarrow d(JN_J) = 0$. Therefore an important question would be to study almost Hermitian $\mathcal{G}_1$-manifolds which satifies $d(JN_J) = 0$. The proposition 1.1 says that almost Hermitian $\mathcal{G}_1$-manifolds whose the characteristic connection has a parallel torsion satisfies the condition $d(JN_J) = 0$.

Example 1 Let (N, J, h) be a Nearly Kähler manifold, i.e. $(\nabla^h_X J)X = 0$, for all $X \in TN$, where ∇^h is the Levi-Civita connection. Then it is also a $\mathcal{G}_1$-manifold and its caracteristic connection is nothing but its canonical Hermitian connection: $\nabla = \nabla^h - \frac{1}{2} J \nabla^h J$. Moreover according to Kirichenko, [18, 3], in a nearly Kähler manifold the canonical Hermitian connection has a parallel torsion: $\nabla T = 0$. Therefore, Nearly Kähler manifolds satisfies all the hypothesis of theorem 1.2 above. Moreover the closed 3-form is then given by $H = \frac{1}{3} d\Omega_J$ which is then exact. Therefore the action functional of the sigma model is then

$$S(f) = \frac{1}{2}\int_L |df|^2 d\mathrm{vol}_g + \int_L f^*\Omega_J.$$

Example 2 Any $(2k+1)$-symmetric space $(G/G_0, \underline{\mathrm{J}}, h)$ endowed with its canonical almost complex structure $\underline{\mathrm{J}}$ and a naturally reductive G-invariant metric h (for which $\underline{\mathrm{J}}$ is orthogonal) is a $\mathcal{G}_1$-manifold and moreover its characteristic connection coincides with its canonical connection ∇^0. Finally, the torsion of the canonical connection is obviously parallel. Therefore we obtain an interpretation of the determined ellitpic integrable system associated to a $(2k+1)$-symmetric space in terms of a sigma model with a Wess-Zumino term (see [20]).

1.2.2 Stringy harmonic maps into f-manifolds.

Let (N, F) be an f-manifold with ∇ a linear connection. Then F defines a splitting $TN = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \ker F$ and $\mathcal{H} = \operatorname{Im} F$. Let us denote by $\bar{J} = F_{|\mathcal{H}}$ the complex structure in $\mathcal{H}$ induced by F. Then we will say that a map $f\colon L \to N$ from a Riemann surface into N is *stringy harmonic* if it is solution of *the stringy harmonic maps equation*:

$$-\tau_g(f) + (F \bullet T)_g(f) = 0.$$

where $F \bullet B$, for $B \in \mathcal{C}(\Lambda^2 T^*N \otimes TN)$, denotes some natural (linear) action of F on $\mathcal{C}(\Lambda^2 T^*N \otimes TN)$. For more simplicity, let us write it in the case where (N, F) is endowed with a compatible metric h (i.e. $\mathcal{V} \perp \mathcal{H}$ and $\bar{J} = F_{|\mathcal{H}}$ is orthogonal with respect to $h_{|\mathcal{H}\times\mathcal{H}}$; in other words F is skew-symmetric w.r.t. h):

$$\begin{aligned} F \bullet B &= B(F\cdot, F\cdot, F\cdot) + \frac{1}{2} F \circlearrowleft (B - B_{\mathcal{H}^3}) \\ F \circlearrowleft A &= A(F\cdot, \cdot, \cdot) + A(\cdot, F\cdot, \cdot) + A(\cdot, \cdot, F\cdot) \end{aligned}$$

for all $B, A \in \mathcal{C}(\Lambda^2 T^*N \otimes TN)$. Let us remark that the splitting $TN = \mathcal{H} \oplus \mathcal{V}$ gives rise to some decomposition $\otimes^3 T^*N = (\otimes^3 \mathcal{H}^*) \oplus \mathcal{S}(\mathcal{H}, \mathcal{V})^* \oplus (\otimes^3 \mathcal{V}^*)$. Then any $B \in \mathcal{C}(\Lambda^2 T^*N \otimes TN) \subset \mathcal{C}(\otimes^3 T^*N)$ admits a decomposition of the form: $B = \bar{B} + \mathring{B} + B_{|\mathcal{V}^3}$, where $\bar{B} = B_{|\mathcal{H}^3}$, $\mathring{B} = B_{|\mathcal{S}(\mathcal{H},\mathcal{V})}$.
Therefore, we see that $F \bullet B$ is a sum of an horizontal term $F \cdot B = B(F\cdot, F\cdot, F\cdot) = \bar{J} \cdot \bar{B}$ and a coupling term in $\mathcal{S}(\mathcal{H}, \mathcal{V})^*$ which is $F \circlearrowleft (B - \bar{B}) = F \circlearrowleft \mathring{B}$.
In [20], we tried to find a class of f-manifolds for which there exists some unique characteristic connection which preserves the structure and then we looked for a variational interpretation of the stringy harmonicity with respect to this connection.

Best Geometric context. We looked for metric f-manifolds (N, F, h) for which there exists a metric f-connection ∇ (i.e. $\nabla F = 0$ and $\nabla h = 0$) with skew-symmetric torsion T. In a first step, we considered metric connections which preserve the splitting $TN = \mathcal{V} \oplus \mathcal{H}$ (i.e. $\nabla q = 0$, where q is the projection on $\mathcal{V}$) and we characterized the manifolds (N, h, q) for which there exists such a connection with skew-symmetric torsion, and called these *reductive* metric f-manifolds.
Then, saying about a metric f-manifold (N, F, h) that it is of global type $\mathcal{G}_1$ if its extended Nijenhuis tensor $\widetilde{N}_F$ (Def. 2.15) is skew-symmetric, we proved the following theorem:

Theorem 1.3 [20, Th. 6.2.3] *A metric f-manifold (N, F, h) admits a metric f-connection ∇ with skew-symmetric torsion if and only if it is reductive and of global type $\mathcal{G}_1$. Moreover, in this case, for any $\alpha \in \mathcal{C}(\Lambda^3 \mathcal{V}^*)$, there exists a unique metric connection ∇ with skew-symmetric torsion such that $T_{|\Lambda^3\mathcal{V}} = \alpha$. This unique connection is given by*

$$T = (-d^c \Omega_F + N_{F|\mathcal{H}^3}) + \mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_\mathcal{V}) + \alpha.$$

where $\Omega_F = \langle F\cdot, \cdot\rangle$, N_F is the Nijenhuis tensor of F, Φ and $\mathrm{R}_\mathcal{V}$ are resp. the curvature of $\mathcal{H}$ and $\mathcal{V}$ resp., and Skew *is the sum of all the circular permutations on the three variables.*

On a metric f-manifold (N, F, h), a metric f-connection ∇ with skew-symmetric torsion is called a *characteristic connection*.

Contrary to the case of stringy harmonic maps into an almost Hermitian $\mathcal{G}_1$-manifolds, in the present case, the hypothesis that the torsion of one characteristic connection is parallel $\nabla T = 0$ does not imply the closure of the 3-form $H = F \bullet T$. However, we characterize this closure under the hypothesis $\nabla T = 0$ and $\mathrm{R}_\mathcal{V} = 0$, by 2 purely algebraic conditions on the horizontal curvature Φ and the Nijenhuis tensor (see § 2.3.9).

Moreover, we can prove ([20, §6]) that for any reductive metric f-manifold of global type $\mathcal{G}_1$, we have

$$H = F \bullet T = F \cdot N_F - \frac{1}{2} F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_\mathcal{V})) - d\Omega_F.$$

Therefore the closure of $H = F \bullet T$ is equivalent to the closure of the 3-form $F \cdot N_F - \frac{1}{2} F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_\mathcal{V}))$. In particular, contrary to what happens in the case of almost

complex manifolds, we do not have "$dH = 0 \Leftrightarrow d(F\cdot N_F)$", which is not surprising since $F \cdot N_F$ is only an horizontal 3-form (a term in $\mathcal{S}(\mathcal{H},\mathcal{V})^*$ is missing).
This leads us to make the following remark: in the definition of $F \bullet B$, the coefficient[1] $\frac{1}{2}$ could appear artificial and one could define $F \bullet_t B = F\cdot B + t\, F \circlearrowleft \mathring{B}$, for any $t \in]0,+\infty[$ (i.e. take an arbitrary coefficient $t > 0$ of coupling between the two terms). This would give rise to a family of 3-form $H_t = F \bullet_t T$ and then H_t is closed for all t (or only for two different values of t) if and only if $d(F\cdot N_F) = 0$ and $d\left(F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))\right) = 0$.
Another way to say that, is to mention that like for almost complex manifolds (remark 1.1), here also it is useful and important to consider a second linear action $F \star B$ of F on TN-valued 2-forms B:

$$F \star B = \frac{1}{2}\left(F\cdot B + F \circlearrowleft B\right) = \bar{J} \star \bar{B} + \frac{1}{2} F \circlearrowleft (B - \bar{B}).$$

This allows to define a second notion: the $\star$-stringy harmonicity. The corresponding 3-form is then $H^\star = F \star T$, and we compute that

$$H^\star = F \bullet T - \frac{1}{2} F\cdot N_F = \frac{1}{2} F \cdot N_F - \frac{1}{2} F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}})) - d\Omega_F.$$

Therefore $H = F \bullet T$ and $H^\star = F \star T$ are closed if and only if $F\cdot N_F$ and $F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))$ are simultaneously closed. In a word, we are led to the following definition.

Definition 1.1 *Let (N,F,h) be a reductive metric f-manifold of global type $\mathcal{G}_1$. We will say that (N,F,h) **has a closed stringy structure** if the 3-forms $F\cdot N_F$ and $F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))$ are closed. This is equivalent to say that the two 3-forms H and $H^\star$ are closed.*

Then we have

Theorem 1.4 [20, §6.4.3] *Let (N,F,h) be a reductive metric f-manifold of global type $\mathcal{G}_1$. Let us suppose that (N,F,h) has a closed stringy structure. Let ∇ be one characteristic connection.*
• Then the equation for stringy harmonic maps (w.r.t. ∇) $f\colon L \to N$ is exactly the Euler-Lagrange equation for the sigma model in N with a Wess-Zumino term defined by the closed 3-form

$$H = -d\Omega_F + F\cdot N_F - \frac{1}{2} F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))\,.$$

• Moreover the equation for $\star$-stringy harmonic maps $f\colon L \to N$ is exactly the Euler-Lagrange equation for the sigma model in N with a Wess-Zumino term defined by the closed 3-form

$$H^\star = -d\Omega_F + \frac{1}{2} F\cdot N_F - \frac{1}{2} F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))\,.$$

Moreover, we proved the following characterisation of closed stringy structures, under the hypothesis $\nabla T = 0$:

Theorem 1.5 [20, §6.4.3] *Let (N,F,h) be a reductive metric f-manifold of global type $\mathcal{G}_1$. Let us suppose that one of its characteristic connections, ∇, has a parallel torsion $\nabla T = 0$. Let us supppose that $\mathrm{R}_{\mathcal{V}} = 0$ and that the horizontal curvature Φ is pure. The following statement are equivalent:*

- *The horizontal 3-form $F\cdot N_F$ is closed.*
- *(N,F,h) has a closed stringy structure.*
- *The horizontal complex structure $\bar{J}$ is a cyclic permutation of the horizontal curvature, and the 2-forms $N_{\bar{J}}$ and Φ have orthogonal supports (§ 2.3.9).*

[1] Dictated by the geometric interpretation of elliptic integrable systems.

Example 3 Any $2k$-symmetric space $(G/G_0, F, h)$ endowed with its canonical f-structure and a naturally reductive G-invariant metric h (compatible with F) is reductive and of global type $\mathcal{G}_1$, and moreover its canonical connection ∇^0 is a characteristic connection. Furthermore, the torsion of the canonical connection is obviously parallel. Finally, we proved in [20, §6.4.5] that any $2k$-symmetric space $(G/G_0, F, h)$ satisfies the two algebraic hypothesis above. Therefore we obtained an interpretation of the determined elliptic integrable system associated to a $2k$-symmetric space in terms of a sigma model with a Wess-Zumino term.

1.3 Metric connections with skew-symmetric torsion, integrable non linear sigma models and twistor spaces.

1.3.1 Metric connections with skew-symmetric torsion.

In mathematical physics. We just saw how metric connections with parallel torsion play an important role in the variational interpretation of stringy harmonicity. More generally, these recently became a subject of interest in theoretical and mathematical physics [28]. Let us give here some examples (taken from [11]).

• The target space of supersymmetric sigma models with a Wess-Zumino term carries a geometry of a metric connection with skew-symmetric torsion.
• In supergravity theories, the geometry of the moduli space of a class of black holes is carried out by a metric connection with skew-symmetric torsion.
• The geometry of NS-5 brane solutions of type II supergravity theories is generated by a metric connection with skew-symmetric torsion.
• The existence of parallel spinors with respect to a metric connection with skew-symmetric torsion on a Riemannian spin manifold is of importance in string theory, since they are associated with some string solitons (BPS solitons).

In differential geometry. This kind of connection was used by Bismut to prove a local index theorem for non-Kähler Hermitian manifolds. Such a connection is known as a KT-connection (Kähler with torsion) or a Bismut connection on an almost Hermitian manifold. The KT-geometry is a natural generalization of the Kähler geometry, since when the torsion is zero the KT-connection coincides with the Levi-Civita connection. According to Gauduchon [13], on any Hermitian manifold, there exists a unique Hermitian connection with totally skew-symmetric torsion.
Furthermore, Friedrich and Ivanov [11] describe all almost contact metric, almost hermitian and G_2-structures admitting a connection with totally skew-symmetric torsion tensor, and prove that there exists at most one such connection. They investigate its torsion form, its Ricci tensor, the Dirac operator and the ∇- parallel spinors. In particular, they obtain partial solutions of the type II string equations in dimension $n = 5$, 6 and 7.
Moreover, a theorem of Kirichenko says that the characteristic connection in a Nearly Kähler manifold has a parallel torsion. Riemannian manifolds endowed with a G-structure admitting a G-connection with parallel skew-symmetric torsion became a subject of great interest [3, 2, 9, 10, 24].

1.3.2 Integrable non linear sigma models

It could turn out to be very important to study the integrability of two-dimensional non linear sigma models. Indeed, there are only a few known examples of integrable non linear sigma

models. Heuristically an integrable sigma model possesses an infinite number of conserved observables which allows to make the system "solvable". Consequently the problem of the existence of new integrable sigma models is very important.
More generally, an integrable system is a non linear partial differential equation (PDE) with "symmetries" and exceptional properties (as existence of solitons, of an Hamiltonian structure, infinitely many symmetries,...) and solutions of which can be constructed by algorithms using various techniques related to algebraic geometry and loops group factorisations. During the last decades, the list of integrable systems has been considerably enriched by examples coming from differential geometry: constant mean curvature surfaces, harmonic maps into symmetric spaces, Hamiltonian stationary Lagrangian surfaces into Hermitian symmetric spaces, etc. for two variables systems; Yang-Mills self dual connections and self dual Einstein metric, or Hyperkählerian metric, for four variables systems. Concurrently, the important role that these integrable systems seem to play in the quantum fields theories has been confirmed, owing notably to the important progress which has been realized in the comprehension of the various dualities between "quantum" and "solitons", from the duality between sin-Gordon equation and Thiring model (in dimension 2) until the superstrings theories in dimension 10, passing through the monopoles of 't Hooft-Polyakov in dimension 4. All these results constitute an encouragement to search new integrable systems, especially those having a geometrical and/or physical meaning.

In our study [20] of elliptic integrable systems in homogeneous geometries, we obtain an interpretation of these systems in terms of stringy harmonic maps in some homogeneous spaces (namely k-symmetric spaces [23]) which satisfies the geometric properties giving rise to the variational interpretation of stringy harmonicity. This result provides a new contribution to the field of (integrable) non linear sigma models. Indeed this gives new examples of integrable two-dimensional non linear sigma models. These new examples take place in some homogeneous spaces, namely in k-symmetric spaces, which are not symmetric spaces. At our knowledge, all the already known integrable two-dimensional non linear sigma models take place in symmetric spaces or (equivalently) in Lie groups. Let us precise that the k-symmetric spaces with k odd can be endowed canonically with a structure of almost Hermitian manifold of type $\mathcal{G}_1$, whereas the k-symmetric spaces with k even can be endowed canonically with a structure of reductive metric f-manifold of global type $\mathcal{G}_1$. Therefore these 2 different family of homogeneous spaces give rise to 2 different families of new integrable two-dimensional non linear sigma models.
Moreover, these previous homogeneous spaces G/G_0 can be embedded canonically in some twistor space over another homogeneous spaces G/H. This leads us naturally to study stringy harmonic maps into twistor spaces (and the possibility of their variational formulation). In particular a natural question is: the Riemannian manifolds (M, g) such that the stringy harmonic maps into the corresponding twistor space admits a variational formulation, are they only the homogeneous spaces? Therefore does the existence of the variational formulation (of stringy harmonic maps into the twistor space) implies the integrability (of the stringy harmonicity)?

1.3.3 The twistor spaces and position of the problem.

In the study [20] of elliptic integrable systems in homogeneous geometries, for each geometric interpretation in the homogeneous target space $N = G/G_0$ under consideration, there is a corresponding geometric interpretation in the twistor space:

$$\mathcal{Z}_k(M) = \{J \in SO(TM)|\, J^k = \mathrm{Id}, J^p \neq \mathrm{Id} \text{ if } p < k, \ker(J \pm \mathrm{Id}) = \{0\}\} \tag{1.2}$$

which is the bundle of isometric endomorphisms of TM with finite order k and with no eigenvalues $= \pm 1$. In particular $\mathcal{Z}_4(M)$ is the familiar twistor bundle $\Sigma(M)$ of orthogonal almost complex structures on M

$$\Sigma(M) = \{J \in \mathfrak{so}(TM) |\, J^2 = -\mathrm{Id}\}.$$

It is proven in [20] that any k-symmetric space $N = G/G_0$ can be embedded into the twistor space $\mathcal{Z}_k(M)$. More precisely in the even case ($k = 2p$) we have an injective morphism of bundle over the associated p-symmetric space $M = G/H$ (defined by the square of the order k automorphism τ defining N, i.e. $G_0 = G^\tau$ and $H = G^{\tau^2}$). This morphism of bundle is moreover an embedding

$$G/G_0 \hookrightarrow \mathcal{Z}_{2p}(G/H).$$

In the odd case we have a section defining then an embedding

$$G/G_0 \hookrightarrow \mathcal{Z}_{2p+1}(G/G_0).$$

Under these embeddings, stringy harmonic maps f taking values in the homogeneous space N are sent into stringy harmonic maps $\mathfrak{J} \circ f$ taking values in the twistor space $\mathcal{Z}_k(M)$, where $\mathfrak{J}\colon N \to \mathcal{Z}_k(M)$ is one of the embedding decribed above. More precisely f is stringy harmonic if and only if $\mathfrak{J} \circ f$ is so.

The geometric variational formulation of stringy harmonicity take place in some manifolds endowed with some particular geometric structure. This could simply be, for example, the almost Hermitian $\mathcal{G}_1$-manifolds whose characteristic connection has a parallel torsion, or the reductive metric f-manifolds of global type $\mathcal{G}_1$. For example, the k-symmetric spaces are very particular examples of this kind of manifolds. Moreover, it is natural to try to make these variational formulation more universal by writting them in a more general setting. More precisely we would like to ask the question of the possibility to find some universal prototype of these particular manifolds (i.e. endowed with some particular structure), which can be endowed canonically with the needed geometric structure and such that 'many' of our particular manifolds, can be embedded in this prototype. The role of this universal prototype will be played of course by the twistor spaces.

The geometric formulation of stringy harmonicity in the twistor spaces could a priori be considered as enough "universal" since these twistor spaces are defined for any Riemannian manifold endowed with some metric connection, and are endowed canonically with (most of) the different geometric structures that we need (§2.3.8). That is to say the geometric structures we need to endow the target space N with, in our geometric formulations. For example, these twistor space admit a canonical metric f-structure and a connection preserving their structure with a torsion whose (almost) all the components are skew-symmetric. This connection is called the paracharacteristic connection in [20, §6.2.4] and the stringy harmonicity can be considered naturally w.r.t. this connection (§2.3.5).

For example, as concerns k-symmetric spaces, we already know that they can be embedded canonically into some twistor bundles. In the even case, the fibration $\pi\colon G/G_0 \to G/H$ imposes to view canonically any $2p$-symmetric space as a subbundle of $\mathcal{Z}_{2p}(G/H)$ so that the twistorial interpretation is in some sense dictated by the structure of the $2p$-symmetric space.

More generally, suppose that we want to study stringy harmonicity in metric f-manifolds (N, F, h). For example, we have seen that among the list of hypothesis which together give a sufficient condition for our variationnal interpretation of stringy harmonicity there is the hypothesis $\mathrm{R}_{\mathcal{V}} = 0$. It is then natural to consider the particular case where the vertical subbundle $\mathcal{V}$ is the tangent space to the fibre of a Riemannian submersion $\pi\colon (N, h) \to (M, g)$, i.e. $\mathcal{V} = \ker d\pi$. In this particular case of a Riemannian submersion, the f-structure F defines a complex structure $\bar{J}$ on $\pi^*TM = \mathcal{H}$ which itself gives rise to a morphism of submersion $\mathcal{I}\colon N \to \Sigma(M)$, $y \mapsto (\pi(y), \bar{J}(y))$.

This shows that the twistor bundle $\Sigma(M)$ appears naturally in the general context - even though the morphism $\mathcal{I}$ is not injective in general.
In the present paper we will consider a particular class of Riemannian submersion $\pi\colon (N,h)\to (M,g)$: the homogeneous fibre bundles, of which the twistor bundles $\mathcal{Z}_p(M)$ are particular examples ($p\in\mathbb{N}^*$).

Now we are lead to the following questions. Let (M,g,∇) be a Riemannian manifold endowed with a metric connection, and let us consider the different twistor spaces $\mathcal{Z}_{2k}(M)$ over M (and in particular the usual twistor bundle of almost complex structures). These twistor spaces endowed with their canonical structures, are they enough general so that (for example) an enough large class of metric f-manifolds of global type $\mathcal{G}_1$ can be embedded in one of these twistor spaces? And in this case, this twistor space is it itself of global type $\mathcal{G}_1$? Or on the contrary, does the fact to impose to a twistor space to be of global type $\mathcal{G}_1$ or more weakly, to contain a twistor subbundle of global type $\mathcal{G}_1$, be a strong condition which implies strong constraints on (M,g,∇) (the Riemannian manifold endowed with a metric connection)? In this case, which are these strong constraints and which are the metric f-manifold of global type $\mathcal{G}_1$ isomorphic to these twistor subbundle of global type $\mathcal{G}_1$?

Precise statement of the Problem studied in the present paper. As we will see in §2.3, the twistor space $\mathcal{Z}_{2k}(M)$ can be endowed with some canonical *paracharacteristic* connection preserving the structure which admits a torsion T such that the components $T_{|\mathcal{H}^3}$, $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{V}\times\mathcal{V})}$, and $T_{|\mathcal{V}^3}$ are skew-symmetric. But we *prove here* that in general, it not possible to have also $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ skew-symmetric, i.e. in general the twistor space endowed with its canonical metric f-structure is not of global type $\mathcal{G}_1$: it is reductive and only *horizontally of type* $\mathcal{G}_1$ [2]. So the questions that we want to solve here are the following:

1. Characterize the Riemannian manifold (M,g) for which there exists a metric connection ∇ such that the twistor space over (M,g) endowed with the canonical metric f-structure defined by ∇ is of global type $\mathcal{G}_1$ (i.e. their exists a metric connection on the twistor space which preserves the structure and with skew-symmetric torsion).

2. Characterize the Riemannian manifold (M,g) for which there exists a metric connection ∇ such that the twistor space over (M,g) endowed with the canonical metric f-structure defined by ∇ admits a subbundle which is of global type $\mathcal{G}_1$.

3. In this case, find those Riemannian manifolds (M,g) such that the previous twistor subbundle of global type $\mathcal{G}_1$ has a closed stringy structure: i.e. stringy harmonicity into this subbundle admits a variational interpretation.

Prospects The discussion that precedes was concerning metric f-manifolds. Now, what about almost Hermitian manifolds? For example, let us consider the case of $(2k+1)$-symmetric spaces. In this odd case, the use of the twistor space $\mathcal{Z}_{2k+1}(N)$ is less pertinent than in the even case. Indeed in the even case, we had some particular fibration that the twistor space allows to realise more universally as some subbundle of endomorphisms over M. Here we do not have this problem of fibration and therefore do not need a priori the twistor space. In the odd case, we have a canonical section $J_1\colon G/G_0\to\mathcal{Z}_{2k+1}(G/G_0)$, which allows to duplicate each geometric property satisfied by the geometric map $f\colon L\to N$ into 2 "identical" properties in each subbundle $\mathcal{H}$ and $\mathcal{V}$ of the tangent bundle of the twistor space.
However one can try to obtain analogous results for stringy harmonic maps into almost Hermitian

[2] The precise definitions of these notions are recalled in §2.3.

manifolds, as those concerning the twistorial formulation of harmonic maps, for example: a map $f\colon L \to (M,g)$ of a Riemann surface has a holomorphic lift into $\Sigma^+(M)$ if and only if it is weakly conformal, harmonic and $f^*w_1(M) = 0$ (see [26, Th. 9.10] or [6, Th. 2.5]).

1.4 Organisation of the paper and list of the main results.

Section 2. In this section, we present the materials that will be used in the all the following of paper. In particular, we present several geometric structures the twistor space will be endowed with: Homogeneous fibre bundle structures and metric f-structures. An important part of the results recalled in this section is taken from [20]. This section contains also some new (technical) results contained in § 2.2.5.

Section 3. We study metric connection with parallel curvature.

First, we prove a very useful result (Lemma 3.1). *Let (M,g) a Riemannian manifold,* $\dim M \geq 4$, *endowed with a metric connection ∇. Suppose that the curvature operator R of ∇ is given by $R(X,Y)Z = k\left(\langle X,Z\rangle Y - \langle Y,Z\rangle X\right)$, for some $k \in \mathcal{C}^\infty(M)$. Then its torsion satisfies $T(X,Y) = \dfrac{1}{2k}\left((Y\cdot k)X - (X\cdot k)Y\right)$ on $\Omega = \{k \neq 0\}$ (and $dk = 0$ on Ω^c). In particular, k is constant if and only if ∇ coincides with the Levi-Civita connection, and therefore (M,g) has a constant sectional curvature. Furthermore, ∇ is geodesically equivalent to Levi-Civita if and only if it coincides with Levi-Civita.*

Then, in § 3.2, we remark that the second Bianchi identity for a connection with parallel curvature, reduces to an algebraic equation in terms of the curvature and the torsion: $R\circ\mathcal{S}(T) = 0$, i.e. $\operatorname{Im}\mathcal{S}(T) \subset \operatorname{Ker} R$. Here $\mathcal{S}\colon T \mapsto \mathcal{S}(T)$ is a natural linear map which to each "torsion" $T \in \Lambda^2E^*\otimes E$ associates a natural linear map $\mathcal{S}(T)\colon \Lambda^3E \to \Lambda^2E$ (where E is any vector space of finite dimension, and of course $E = T_xM$ in our case). We prove that $\mathcal{S}$ is injective (Lemma 3.3). This allows to prove (still under the hypothesis $\nabla R = 0$) that:
if the curvature operator $R\colon \Lambda^2TM \to \operatorname{End}(TM)$ is injective then the torsion vanishes $T = 0$. In particular if ∇ is a metric connection with respect to some Riemannian metric g on M, then the following statements are equivalent: **(i)** *R is injective,* **(ii)** $\operatorname{Hol}^0(\nabla) = SO(n)$, **(iii)** *this metric connection is the Levi-Civita connection, $\nabla = \nabla^g$, and $\operatorname{Hol}^0(\nabla^g) = SO(n)$,* **(iv)** *$\nabla = \nabla^g$ and M has a constant sectional curvature.* (See Theorem 3.1).

In § 3.3, we study the $GL(E)$-invariant injective linear maps $\mathcal{S}\colon \Lambda^2E^*\otimes E \mapsto \Lambda^3E^*\otimes\Lambda^2E$. This allows us to prove the following (Theorem 3.2). *Let E be an Euclidean space of dimension $n \geq 5$. Let $\mathrm{L}^*(\Lambda^3E,\Lambda^2E)$ be the set of surjective linear maps $\Lambda^3E \to \Lambda^2E$. Then $\mathcal{U}(E) := \mathcal{S}^{-1}\left(\mathrm{L}^*(\Lambda^3E,\Lambda^2E)\right)$ is an open dense set in $\Lambda^2E^*\otimes E$.*
This yields the following result (corollary 3.1). *Let M be a manifold of dimension $n \geq 5$, ∇ a linear connection on M with parallel curvature $\nabla R = 0$, and we still denote by T its torsion. The set $\{x \in M | T_x \in \mathcal{U}(T_xM)\}$ is an open set in M. In particular if there exists $x_0 \in M$ such that $T_{x_0} \in \mathcal{U}(T_{x_0}M)$ then we have $R = 0$ in all a neighbourhood of $x_0 \in M$.*

Finally, we conclude the section by a study of metric connection with vectorial torsion and parallel curvature (§ 3.4).

Section 4. This section is devoted to the proof of the theorem 4.1 which says the following.
Let (M,g) be a Riemannian manifold of dimension $2n \geq 6$, endowed with a metric connection ∇. Let us consider the twistor bundle $(\Sigma^+(M),\mathcal{F},h)$ endowed with its canonical f-structure and its Kaluza-Klein metric (see §2.3.8). Then $(\Sigma^+(M),\mathcal{F},h)$ is globally of type $\mathcal{G}_1$ if and only if $\nabla = \nabla^g$ and (M,g) has a constant non vanishing sectional curvature. In this case, $\Sigma^+(M)$ is a locally 4-symmetric space and the corresponding 4-symmetric fibration is $\Sigma^+(M) \to M$.
After using the tools, techniques and results presented in the two previous sections, we reduce

the proof of the theorem to an algebraic problem on the curvature R which consists itself in proving the following algebraic result (Theorem 4.2). *Let $n \geq 3$, if $R \in \mathfrak{so}(2n) \otimes \mathfrak{so}(2n)$ satisfies $\forall J \in \Sigma^+(\mathbb{R}^{2n})$, $R_J^{--} = 0$, then we have $R = 0$. In other words,*

$$\sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathfrak{so}_-(J) \otimes \mathfrak{so}_-(J) = \mathfrak{so}(2n) \otimes \mathfrak{so}(2n),$$

where we have set $\mathfrak{so}_-(J) = \{A \in \mathfrak{so}(2n) | \, AJ + JA = 0\}$. We prove this algebraic results by using basic representation theory: more precisely we use the $SO(2n)$-invariant decomposition of $\mathfrak{so}(2n) \otimes \mathfrak{so}(2n)$.

Section 5. In this subsection we want to characterize the manifolds (M, g, ∇) such that the associated twistor bundle has some particular properties like: respectively the horizontal type $\mathcal{G}_1$, the parallelness of the torsion of the paracharacteristic connection, and finally the pureness of the horizontal curvature. The aim of the present study is to begin to understand what happens for admissible subbundles of $\Sigma^+(M)$. Indeed, we want to understand what each particular property (among those listed above) implies on $\Sigma^+(M)$, with the aim to generalise the obtained results for admissible subbundles of $\Sigma^+(M)$.

Section 6. We study the Riemannian manifolds for which there exists an admissible subbundle of the twistor space which is of global type $\mathcal{G}_1$ and has a closed stringy structure and prove that in most cases these manifolds are locally homogeneous.

2 Preliminaries.

2.1 Some notations, definitions, conventions and basic recalls.

2.1.1 Canonical identifications using a metric, and multilinear algebra.

In the following, when a metric h is given, on a manifold N, we use the following convention: each TN-valued bilinear form on N, $B \in \mathcal{C}(T^*N \otimes T^*N \otimes TN)$, will be identified (via the metric h) with the corresponding trilinear form:

$$B(X, Y, Z) := \langle B(X, Y), Z \rangle.$$

Moreover, we denote by Ω_A the bilinear form associated (via the metric h) to an endomorphism $A \in \mathcal{C}(\mathrm{End}(TN))$:

$$\Omega_A(X, Y) = \langle A(X), Y \rangle, \quad \forall X, Y \in TN.$$

Then, under our convention, for any endomorphism $A \in \mathcal{C}(\mathrm{End}(TN))$, $D^h A$ is identified to $D^h \Omega_A$, where D^h is the Levi-Civita connection.

More generally, everywhere it will be relevant, we will implicitely identify a Euclidean space E with its dual (via the metric h). For example, $\mathfrak{so}(E)$ could be identified to $\Lambda^2 E^*$ once it will be pertinent. Moreover:

Convention: When a metric is given on some vector space (or bundle) E, we consider that any tensorial product of E and its dual E^* is endowed canonically with the induced tensorial metric.

Let us consider the following natural map $R_0 \colon \Lambda^2 E \to \mathfrak{so}(E)$,

$$R_0(X \wedge Y) = \langle X, \cdot \rangle Y - \langle Y, \cdot \rangle X.$$

This is nothing but the natural identification between $\Lambda^2 E$ and $\mathfrak{so}(E)$. Indeed, the isomorphism $E \otimes E \cong E^* \otimes E = \mathrm{End}(E)$ is given by $x \otimes y \mapsto \langle x, \cdot \rangle \otimes y$ and the inclusion $\Lambda^2 E \to E \otimes E$ is

given by $x \wedge y = x \otimes y - y \otimes x$.
It is useful to remark that

$$R_0(X, Y, Z, W) := \langle R_0(X \wedge Y)Z, W\rangle = \langle X \wedge Y, Z \wedge W\rangle.$$

This allows to recover that $R_0 = \mathrm{Id}_{\Lambda^2 E}$ through the identification $(\Lambda^2 E)^* \otimes \mathfrak{so}(E) \cong (\Lambda^2 E)^* \otimes \Lambda^2 E$.
Let us note that we have $\forall V \in \mathfrak{so}(E)$

$$\langle R_0(X, Y), V\rangle = 2\langle VX, Y\rangle \tag{2.1}$$

where in the left hand side we use the tensorial inner product on $\mathrm{End}(E) = E^* \otimes E$ that is to say

$$\langle A, B\rangle = \mathrm{Tr}(AB^t), \quad \forall A, B \in \mathrm{End}(E). \tag{2.2}$$

Remark that the induced inner product on $\mathfrak{so}(E)$ coincides with (the opposite of) its Killing form.

Remark 2.1 If (e_i) is an orthonormal basis of E, then $(e_i \otimes e_j)$ is an orthonormal basis of $\otimes^2 E$, but $(e_i \wedge e_j)$ is an orthogonal basis of $\Lambda^2 E$ such that $|e_i \wedge e_j| = \sqrt{2}$.

We denote by $\mathcal{S}(E)$ the set of symmetric endomorphisms of E.
Moreover, we set for all $B \in \mathcal{C}(E^* \otimes E^* \otimes E)$

$$\begin{aligned} \mathrm{Sym}(B)(X, Y) &= B(X, Y) + B(Y, X), \\ \mathrm{Alt}_{12}(B)(X, Y) &= B(X, Y) - B(Y, X), \quad \forall X, Y \in E. \end{aligned}$$

We denote by Skew the following linear endomorphism of $\otimes^3 E^*$:

$$\mathrm{Skew}(B)(X, Y, Z) = B(X, Y, Z) + B(Y, Z, X) + B(Z, X, Y).$$

Then the *Bianchi projector* is defined by $b = \dfrac{1}{3}\mathrm{Skew}$. In particular, when restricted to $\Lambda^2 E^* \otimes E$, it defines a projector onto $\Lambda^3 E^*$.

We will use the notation $S^k(E) = \odot^k E$ (for any vector space E).
Furthermore, let E_1, E_2, E_3 be vector bundles, then we set also $\mathcal{S}(E_1 \times E_2 \times E_3) = \underset{i,j,k}{\mathfrak{S}} E_i \otimes E_j \otimes E_k$, where $\underset{i,j,k}{\mathfrak{S}}$ means that we make a direct sum on the circular permutations of 1,2,3.
Let E, F be two vector spaces, we denote by $\mathrm{L}(E, F)$ the vector space of linear maps from E to F, and by $\mathrm{L}^*(E, F)$ the open subset of surjective linear maps. Recall that $\mathrm{L}^*(E, F)$ is dense in $\mathrm{L}(E, F)$ if $\dim F \leq \dim E \leq +\infty$.
Let us conclude this § 2.1.1 by the following lemma which will be used in § 3.

Lemma 2.1 *Let E and F be two real vector spaces such that $\dim F \leq \dim E \leq +\infty$. Let $\mathcal{B}$ be a vector subspace of $\mathrm{L}(E, F)$ such that $\mathcal{B} \cap \mathrm{L}^*(E, F) \neq \varnothing$. Then $\mathcal{B} \cap \mathrm{L}^*(E, F)$ is open and dense in $\mathcal{B}$.*

2.1.2 Metric connection and torsion.

Let us recall some useful properties about connections. Our references are [20, § 5.2.1] and [1, 2.1].

Definition 2.1 *Two linear connection ∇ and ∇' in a manifold N are said to be geodesically equivalent if they have the same geodesics. A connection ∇ on a Riemannian manifold (N, h) is said to be geodesic-preserving if it is geodesically equivalent to the Levi-Civita connection ∇^h of h.*

Definition 2.2 *Let (M, g) be a Riemannian manifold and (N, ∇) be a manifold endowed with a linear connection ∇. The tension field of a map $f\colon (M, g) \to (N, \nabla)$ is given by $\tau_g(f) = \mathrm{Tr}_g(\nabla df)$.*

Proposition 2.1 *Let ∇ be a connection on a manifold N and $A \in \mathcal{C}(T^*N \otimes \mathrm{End}(TN))$. Then the connection*

$$\nabla' = \nabla + A$$

has the same geodesic as ∇ if and only if $A(\cdot, \cdot)$ is skew-symmetric (as a bilinear map). In this case for any map $f\colon (M, g) \to N$, from a Riemannian manifold into N, we have $\tau'_g(f) = \tau_g(f)$, where $\tau'_g(f)$ and $\tau_g(f)$ are the tension fields w.r.t. ∇ and ∇' respectively. Moreover (still in this case), we have

$$T^{\nabla'} = T^{\nabla} + 2A.$$

Now, let us suppose that ∇ is metric w.r.t. some metric h in N. Then ∇' is metric if and only if A takes values (as a 1-form) in the skew-symmetric endomorphisms of TN: $A \in \mathcal{C}(T^*N \otimes \mathfrak{so}(TN))$. Therefore ∇' is metric and geodesically equivalent to ∇ if and only if A is totally skew-symmetric *which means that the associated 3-linear map defined by $A(X, Y, Z) = \langle A(X, Y), Z\rangle$ is a 3-form on N.*

Proposition 2.2 *Let (N, h) be a Riemannian manifold, and let us denote by D^h its Levi-Civita connection. Then a metric connection ∇ on N is entirely determined by its torsion T. Moreover a metric connection ∇ on N is geodesic-preserving if and only if its torsion T is totally skew-symmetric. Then in this case we have*

$$\nabla = D^h + \frac{1}{2}T.$$

Proof. For any metric connection $\nabla = D^h + A$, we have

$$T(X, Y) = A(X, Y) - A(Y, X) \tag{2.3}$$
$$2A(X, Y, Z) = T(X, Y, Z) + T(Z, X, Y) + T(Z, Y, X), \tag{2.4}$$

where the second equation (2.4) is derived directly from the first one (2.3) (compute the right hand side of the second equation using the first equation and that $A(X, Y, Z) = -A(X, Z, Y)$). This proves the first assertion. Concerning the second assertion, we see (according to (2.3-2.4)) that A is totally skew-symmetric if and only if T is so, i.e., according to proposition 2.1, ∇ is geodesic preserving if and only if T is totally skew-symmetric. Then in this case $T = 2A$ i.e. $\nabla = D^h + \frac{1}{2}T$. This completes the proof. □

We are led to the following

Definition 2.3 *Let E be an Euclidean space. We define the map* $\mathrm{A}\colon \Lambda^2 E^* \otimes E \longrightarrow E^* \otimes \mathfrak{so}(E)$ *as the $O(E)$-equivariant linear isomorphism defined by equation* (2.4)*, i.e.*

$$\mathrm{A}(T) = \frac{1}{2}(T(X,Y,Z) + T(Z,X,Y) + T(Z,Y,X)).$$

Then A^{-1} *is given by* (2.3)*. Moreover* A *induces, by restriction, an automorphism of* $\Lambda^3 E^*$ *which is nothing but* $\frac{1}{2}\,\mathrm{Id}$*. Furthermore, if now E is a Riemannian vector bundle, we still denote by* A *the corresponding isomorphism.*

Now, proposition 2.2 means the following: *let (N,h) be a Riemannian manifold, then $A := \nabla - D^h$ is given by $A = \mathrm{A}(T)$, where T is the torsion of ∇.*

Remark 2.2 There is another way to interpret the equation (2.4). Let us set $U(X,Y,Z) = \langle U(X,Y),Z\rangle = T(Z,X,Y) + T(Z,Y,X)$, and $A = \frac{1}{2}(T+U)$. We remark that U is symmetric w.r.t. to the variables X,Y, so that the connection $\nabla - A = \nabla - \frac{1}{2}(T+U)$ is torsion free. Moreover we see that $A(X,Y,Z) = \frac{1}{2}(T(X,Y,Z) + T(Z,X,Y) + T(Z,Y,X))$ is skew symmetric w.r.t. the two last variables Y,Z. Therefore $\nabla - A$ is metric and thus this is the Levi-Civita connection D^h:

$$D^h = \nabla - \frac{1}{2}(T+U).$$

Moreover, T is totally skew-symmetric if and only if the "natural reductivity term" $U = 0$.

2.1.3 Irreducible decomposition of the space of torsions $\mathcal{T}(\mathbb{R}^n) = \mathfrak{so}(n) \otimes \mathbb{R}^n$.

Let E be an Euclidean vector space (sometimes identified to $\mathbb{R}^n$ if it is relevant). Let us set

$$\mathcal{T}(E) = (\Lambda^2 E^*) \otimes E \quad \text{and} \quad \mathcal{A}(E) = E^* \otimes \mathfrak{so}(E).$$

Let us consider the Bianchi projector $b\colon \mathcal{T}(E) \cong \Lambda^2 E^* \otimes E^* \to \Lambda^3 E^*$,

$$b(T) = \frac{1}{3}\mathrm{Skew}(T).$$

Moreover the trace on $\mathcal{T}(E)$ is defined by $\mathrm{Tr}(T)(X) = \sum_{i=1}^n T(X,e_i,e_i)$. It can be realized as a projector. Indeed, we can define an injection $E^* \to \mathcal{T}(E)$: for each $\alpha \in E^*$ one sets

$$\tilde{\alpha}(X,Y,Z) = \frac{1}{n-1}\left(\alpha(X)\langle Y,Z\rangle - \alpha(Y)\langle X,Z\rangle\right)$$

or equivalently (via the identification $E^* = E$) to each $\xi \in E$ we associate $\frac{1}{n-1} T_\xi \in \mathcal{T}(E)$ defined by

$$T_\xi(X,Y) = R_0(X \wedge Y)\xi.$$

Then we set

$$(\ker b)_0 = \ker b \cap \ker \mathrm{Tr}.$$

We could also choose to work with $\mathcal{A}(E) = E^* \otimes \mathfrak{so}(E)$ as well. It suffices to use the isomorphism $\mathrm{A}\colon \mathcal{T}(E) \to \mathcal{A}(E)$ to translate everything. Let us set $A_\xi = \mathrm{A}(T_\xi)$. Then we compute that

$$A_\xi(X,Y) = R_0(X \wedge \xi)Y.$$

Moreover, the corresponding injection $E^* \to \mathcal{A}(E)$ is then $\hat{\alpha}(X,Y,Z) = \dfrac{1}{n-1}\left(\alpha(Z)\langle X,Y\rangle - \alpha(Y)\langle Z,X\rangle\right)$.
This injection allows to realize the trace on $\mathcal{A}(E)$ as a projector.
The following lemma will be very useful.

Lemma 2.2 *If $n \geq 2$, the $SO(n)$-irreducible decomposition of $\mathcal{T}(\mathbb{R}^n) = \mathfrak{so}(n) \otimes \mathbb{R}^n$ is given by*

$$\begin{aligned}\mathcal{T}(\mathbb{R}^n) &= \ker b \oplus \operatorname{Im} b = \mathcal{T}_1 \oplus (\ker b)_0 \oplus \operatorname{Im} b \\ &= \mathbb{R}^n \oplus (\ker b)_0 \oplus \Lambda^3(\mathbb{R}^n)^*.\end{aligned}$$

where

$$\mathcal{T}_1 = \{T_\xi |\, \xi \in \mathbb{R}^n\}.$$

2.1.4 Symmetric curvature and vectorial torsion.

Lemma 2.3 *Let (M, g) be a Riemannian manifold and ∇ a metric connection on (M, g). Let us decompose $\nabla = D^g + A$, where $A = \mathrm{A}(T)$. Then the curvature operators of ∇ and of D^g are related by*

$$\begin{aligned}R(X,Y) &= R^g(X,Y) + (d^\nabla A)(X,Y) - [A(X), A(Y)] \\ R(X,Y) &= R^g(X,Y) + (d^{D^g} A)(X,Y) + [A(X), A(Y)] \\ R(X,Y) &= R^g(X,Y) + \mathrm{Alt}_{1,2}(\nabla A)(X,Y) + A(T(X,Y)) - [A(X), A(Y)]\end{aligned}$$

Definition 2.4 *Let (M, g) be a Riemannian manifold and ∇ a metric connection on (M, g). We will say that*

- *∇ has a vectorial torsion if its torsion takes values in $\mathcal{T}_1(TM)$, i.e. $T \in \mathcal{C}(\mathcal{T}_1(TM))$.*
- *∇ admits a symmetric curvature if its curvature operator R (considered as a section of $\Lambda^2 T^*M \otimes \Lambda^2 TM$) is a symmetric endomorphism, i.e. $R \in \mathcal{S}(\Lambda^2 TM)$.*

Proposition 2.3 *Let (M, g) be a Riemannian manifold and ∇ a metric connection on (M, g). Suppose that its torsion is parallel: $\nabla T = 0$. Then the following holds.*

- *If the torsion T is skew-symmetric then the curvature is symmetric.*
- *If the torsion T is vectorial then the curvature is symmetric.*

Proof. According to § 2.1.2 we can write $\nabla = D^g + A$, where $A = \mathrm{A}(T)$. Then if $\nabla T = 0$ (and therefore $\nabla A = 0$), we have according to the previous lemma,

$$R^\nabla(X,Y) = R^g + A(T(X,Y)) - [A(X), A(Y)], \quad \forall X, Y \in TM,$$

where R^g is the curvature of the Levi-Civita connection. We already know that R^g is symmetric. We have to prove that, in the two cases in concerns, $\langle (A(T(X,Y)) - [A(X), A(Y)])\, Z, W\rangle$ is symmetric w.r.t. the two variables (X, Y) and (Z, W).
If T is skew-symmetric, then $A = \mathrm{A}(T) = \frac{1}{2}T$. If $T = T_\xi$ for some $\xi \in \mathcal{C}(TM)$, then $A = \mathrm{A}(T_\xi) = A_\xi$. Therefore $A(T(X,Y)) = \frac{1}{2}T(T(X,Y), \cdot)$ if T is skew-symmetric, and $A(T(X,Y)) = A_\xi(T_\xi(X,Y))$ if $T = T_\xi$ for some $\xi \in \mathcal{C}(TM)$. Then it suffices to check that

$$\frac{1}{2}\langle T(T(X,Y), Z), W\rangle - \frac{1}{4}\langle [T(X), T(Y)]Z, W\rangle$$

with T skew-symmetric, and

$$\langle A_\xi(T_\xi(X,Y), Z), W\rangle - \langle [A_\xi(X), A_\xi(Y)]Z, W\rangle$$

are symmetric w.r.t. to the two variables (X, Y) and (Z, W), which is nothing but computation. We find in particular that the terms $\langle A_\xi(T_\xi(X, Y), Z), W\rangle$ and $\langle [A_\xi(X), A_\xi(Y)]Z, W\rangle$ are each symmetric. This completes the proof. □

In the vectorial case, we can improve the previous proposition.

Proposition 2.4 *Let (M, g) be a Riemannian manifold and ∇ a metric connection on (M, g). If the torsion T is vectorial then the curvature is symmetric.*

Proof. According to the previous lemma, it suffices to prove that $d^{D^g} A_\xi$ is symmetric, because we have seen in the proof of prop. 2.3 that $\langle [A_\xi(X), A_\xi(Y)]Z, W\rangle$ is symmetric. Let us compute:

$$\begin{aligned}\langle d^{D^g} A_\xi(X, Y)Z, W\rangle &= \langle \mathrm{Alt}_{1,2}(D^g A_\xi)(X, Y)Z, W\rangle = \langle Y \wedge D^g_X \xi - X \wedge D^g_Y \xi, Z \wedge W\rangle = \\ &\langle Y, Z\rangle\langle D^g_X \xi, W\rangle - \langle Y, W\rangle\langle D^g_X \xi, Z\rangle - \langle X, Z\rangle\langle D^g_Y \xi, W\rangle + \langle X, W\rangle\langle D^g_Y \xi, Z\rangle \\ &= \langle Y, Z\rangle\langle X, -D^g_\xi W + [\xi, W]\rangle - \langle Y, W\rangle\langle X, -D^g_\xi Z + [\xi, Z]\rangle - \langle X, Z\rangle\langle Y, -D^g_\xi W + [\xi, W]\rangle \\ &+ \langle X, W\rangle\langle Y, -D^g_\xi Z + [\xi, Z]\rangle = \langle Y, Z\rangle\langle X, -D^g_W \xi\rangle - \langle Y, W\rangle\langle X, -D^g_Z \xi\rangle - \langle X, Z\rangle\langle Y, -D^g_W \xi\rangle \\ &+ \langle X, W\rangle\langle Y, -D^g_Z \xi\rangle = \langle X \wedge Y, Z \wedge D_W \xi\rangle - \langle X \wedge Y, W \wedge D^g_Z \xi\rangle \\ &= \langle X \wedge Y, (d^{D^g} A_\xi)(Z \wedge W)\rangle.\end{aligned}$$

At the third equality, we used that D^g is torsion free and that $D^g_X g = L_X g = 0$. In the fourth equality, we used again that D^g is torsion free (in fact, in the third and fourth equalities, we only reproved that $D^g \xi$ is a skew-symmetric endomorphism). This completes the proof. □

Remark 2.3 • The prop 2.3 does not hold, in general, if $\mathcal{T} \in \mathcal{C}(\Lambda^3 T^*M) \oplus \mathcal{T}_1(TM)$, i.e. if $[T]_{(\ker b)_0} = 0$. Indeed, if $T = T_a + T_\xi$, then, in general, the term $A(T(X, Y)) - [A(X), A(Y)]$ contains the additional crossing term $\frac{1}{2}T_a(T_\xi(X, Y), \cdot) + A_\xi(T_\xi(X, Y), \cdot) + \frac{1}{2}[T_a(X), A_\xi(Y)] + \frac{1}{2}[A_\xi(X), T_a(Y)]$ which is not symmetric.
• Moreover, in general, a metric connection with skew-symmetric torsion does not admit a symmetric curvature operator. Indeed, we then have $\langle d^{D^g} A_\xi(X, Y)Z, W\rangle = \langle \mathrm{Alt}_{1,2}(D^g A_\xi)(X, Y)Z, W\rangle = (\nabla_X T)(Y, Z, W) - (\nabla_Y T)(X, Z, W)$, which is not symmetric in general.

2.1.5 Decompositions defined by an almost complex structure.

Let (E, J) be a complex vector space and let us set

$$\mathrm{Bil}(E) = E^* \otimes E^* \otimes E.$$

and for $\varepsilon, \varepsilon' \in \mathbb{Z}_2$ we set

$$\mathrm{Bil}^{\varepsilon,\varepsilon'}(E, J) = \{A \in \mathrm{Bil}(E) | A(J\cdot, \cdot) = \varepsilon JA,\ A(\cdot, J\cdot) = \varepsilon' JA\}$$

so that we have the decomposition

$$\mathrm{Bil}(E) = \oplus_{(\varepsilon,\varepsilon')\in\mathbb{Z}_2\times\mathbb{Z}_2} \mathrm{Bil}^{\varepsilon,\varepsilon'}(E, J). \tag{2.5}$$

Let us remark that for any $A \in \mathrm{Bil}(E)$, its component $A^{\varepsilon,\varepsilon'} \in \mathrm{Bil}^{\varepsilon,\varepsilon'}(E, J)$ is given by

$$A^{\varepsilon,\varepsilon'}(X, Y) = -\frac{1}{4}\left(\varepsilon\varepsilon' A(JX, JY) + \varepsilon JA(JX, Y) + \varepsilon' JA(X, JY) - A(X, Y)\right). \tag{2.6}$$

Moreover as concerns $\mathcal{T}(E) = (\Lambda^2 E^*) \otimes E \subset \mathrm{Bil}(E)$ we also have the decomposition

$$\mathcal{T}(E) = \mathcal{T}^{2,0} \oplus \mathcal{T}^{0,2} \oplus \mathcal{T}^{1,1}, \tag{2.7}$$

where $\mathcal{T}^{2,0} = \mathrm{Bil}^{++}(E,J) \cap \mathcal{T}(E) = (\Lambda^{2,0} E^{*\mathbb{C}}) \otimes_{\mathbb{C}} E$, $\mathcal{T}^{2,0} = \mathrm{Bil}^{--}(E,J) \cap \mathcal{T}(E) = (\Lambda^{0,2} E^{*\mathbb{C}}) \otimes_{\mathbb{C}} E$ and $\mathcal{T}^{1,1} = \left(\mathrm{Bil}^{+-} + \mathrm{Bil}^{-+}\right)(E,J) \cap \mathcal{T}(E) = (\Lambda^{1,1} E^{*\mathbb{C}}) \otimes_{\mathbb{C}} E$.
Of course, these notations can be extended to the case (E,J) is a complex vector bundle. In particular, we will use these for the tangent bundle (TN,J) of an almost complex manifold, and will forget in this case the precision of the bundle in the notation and write for example simply $\mathcal{T}$ and Bil.
Furthermore, for any $B \in \mathcal{T}(E)$, we set

$$B^{**} := B^{++} + B^{+-} + B^{-+}.$$

Let us remark that (2.5) and (2.7) are orthogonal decompositions if J is orthogonal (using the conventions explained in § 2.1.1).
Let (N,J) be an almost complex manifold. To any trilinear form $\alpha \in \mathcal{C}(\otimes^3 T^*N)$ will be associated the trilinear form

$$\alpha^c = -\alpha(J\cdot, J\cdot, J\cdot).$$

In particular, if $\alpha = d\beta$, with $\beta \in \Omega^2(N) := \mathcal{C}(\Lambda^2 T^*N)$ then we set $d^c\beta := \alpha^c$.

Now, suppose that let (E,J,h) is a Hermitian vector space. We will also use the decomposition (2.5) for the space $\mathcal{A}(E) = E^* \otimes \mathfrak{so}(E)$ but w.r.t. the two last variables, that is to say, we consider the following decomposition

$$\mathrm{Bil}(E) = \oplus_{(\varepsilon,\varepsilon') \in \mathbb{Z}_2 \times \mathbb{Z}_2} \mathrm{Bil}^{*,\varepsilon,\varepsilon'}(E,J).$$

where for $\varepsilon, \varepsilon' \in \mathbb{Z}_2$ we set

$$\mathrm{Bil}^{*,\varepsilon,\varepsilon'}(E,J) = \{A \in \mathrm{Bil}(E) | A(\cdot, J\cdot) = -\varepsilon A(J\cdot,\cdot),\ JA = \varepsilon' A(J\cdot,\cdot).\}$$

This leads to the decomposition

$$\mathcal{A}(E) = \mathcal{A}^{2,0} \oplus \mathcal{A}^{0,2} \oplus \mathcal{A}^{1,1}, \tag{2.8}$$

defined as for (2.7). Then we denote by $A^{*,\varepsilon,\varepsilon'}$ and $A^{*,(2,0)}$, $A^{*,(0,2)}$, $A^{*,(1,1)}$ respectively the components of $A \in \mathcal{A}(E)$ w.r.t. the two previous decompositions.
Let us remark that $\mathrm{Bil}^{*,-,-} = \mathrm{Bil}^{-,-}$ and that $\forall B \in \mathrm{Bil}(E)$, $B^{*,--} = B^{--}$. Therefore, for any $A \in \mathcal{A}(E)$, $A^{*,(0,2)}$ will also be simply denoted by $A^{0,2}$.
Moreover, we have

$$\mathrm{A}(T^{0,2}) = \mathrm{A}(T)^{0,2}, \quad \forall T \in \mathcal{T}(E).$$

It is also useful to remark that $\mathcal{T}(E) \cap \mathcal{A}(E) = \Lambda^3 E^*$ and that when restricted to $\Lambda^3 E^*$ each of the two decompositions (2.7) and (2.8) give rise then to the usual decomposition $\Lambda^3 E^* = \Lambda^{(3,0)+(0,3)} E^* \oplus \Lambda^{(2,1)+(1,2)} E^*$, with the following correspondences $\Lambda^{(3,0)+(0,3)} E^* = \mathcal{T}^{0,2} \cap \Lambda^3 E^* = \mathcal{A}^{0,2} \cap \Lambda^3 E^*$ and $\Lambda^{(2,1)+(1,2)} E^* = \left(\mathcal{T}^{2,0} \oplus \mathcal{T}^{1,1}\right) \cap \Lambda^3 E^* = \left(\mathcal{A}^{2,0} \oplus \mathcal{A}^{1,1}\right) \cap \Lambda^3 E^*$.
One proves easily the following.

Lemma 2.4 [13, Lemme 1] *Let E be an even dimensional Euclidean vector space. The following holds: $\forall J \in \Sigma^+(E)$, $\forall \xi \in E$, $(T_\xi)_J^{0,2} = 0$ and $(A_\xi)_J^{0,2} = 0$.*

Remark 2.4 The irreducible decomposition given by lemma 2.2 is orthogonal. Hence according to lemma 2.4, $\forall J \in \Sigma^+(E)$, $\mathcal{T}_J^{0,2} \perp \mathcal{T}'$, so that $\forall J \in \Sigma^+(E)$, $\mathcal{T}_J^{0,2} \subset (\ker b)_0 \oplus \Lambda^3(\mathbb{R}^n)^*$.

Again (E, J, h) is a Hermitian vector space, then we set

$$\mathfrak{so}_+(J) = \{A \in \mathfrak{so}(E,h) | [A, J] = 0\}, \quad \mathfrak{so}_-(J) = \{A \in \mathfrak{so}(E,h) | AJ + JA = 0\}.$$

where $\mathfrak{so}(E, h)$ is of course the Lie algebra of skew-symmetric endomorphisms of (E, h). We then have the following decomposition $\mathfrak{so}(E) = \mathfrak{so}_+(J) \oplus \mathfrak{so}_-(J)$. We will denote $V^\pm$ (or $V_J^\pm$ if the almost complex structure needs to be made precise) the component of $V \in \mathfrak{so}(E)$ with respect to this decomposition. In the same way, if $R \in \mathfrak{so}(E) \otimes \mathfrak{so}(E)$, we denote by $R_J^{\varepsilon,\varepsilon'}$ (or simply $R^{\varepsilon,\varepsilon'}$) its components following the decomposition $\mathfrak{so}(E) \otimes \mathfrak{so}(E) = \bigoplus_{(\varepsilon,\varepsilon')\in\mathbb{Z}_2\times\mathbb{Z}_2} \mathfrak{so}_\varepsilon(J) \otimes \mathfrak{so}_{\varepsilon'}(J)$.

Remark that since $\mathfrak{so}_+(J) \perp \mathfrak{so}_-(J)$, we have $R_0(\mathfrak{so}_\varepsilon(J)) = \mathfrak{so}_\varepsilon(J)$, in other words we have

$$\langle R_0(X \wedge Y), V_\varepsilon\rangle = \langle R_0^{\varepsilon,\varepsilon}(X \wedge Y), V_\varepsilon\rangle = \langle R_0((X \wedge Y)_-), V_\varepsilon\rangle, \quad \forall V_\varepsilon \in \mathfrak{so}_\varepsilon(J), \forall X, Y \in E \quad (2.9)$$

Below, some additional definitions. We will use the following notations already defined in § 1.3.3. Let E be a Euclidean vector space (or bundle) of even dimension then

$$\begin{aligned} \Sigma(E) &= \{J \in \mathfrak{so}(E) |\, J^2 = -\mathrm{Id}\} \\ \mathcal{Z}_k(E) &= \{J \in SO(E) |\, J^k = \mathrm{Id}, J^p \neq \mathrm{Id} \text{ if } p < k, \ker(J \pm \mathrm{Id}) = \{0\}\}. \end{aligned}$$

Definition 2.5 *Let E be a Euclidean vector space. An isometry $A \in SO(E)$ will be called a $2k$-structure if $A \in \mathcal{Z}_{2k}(E)$.*

Definition 2.6 *Let $(E, h) \to M$ be a Riemannian vector bundle over a manifold M. Then for each $2k$-structure $J \in \mathcal{Z}_{2k}(E)$, we denote by $\underline{\mathrm{J}}$ the complex structure in E defined by*

$$\begin{aligned} \ker(\underline{\mathrm{J}} - i\mathrm{Id}) &= \oplus_{j=1}^{k-1} \ker(J - \omega_{2k}^{-j}\mathrm{Id}) \\ \ker(\underline{\mathrm{J}} + i\mathrm{Id}) &= \oplus_{j=1}^{k-1} \ker(J - \omega_{2k}^{j}\mathrm{Id}) \end{aligned}$$

Remark 2.5 Let us remark that if $J \in \Sigma(E)$ is a complex structure then $\underline{\mathrm{J}} = -J$.

2.2 Homogeneous fibre bundles and Kaluza-Klein metrics

Convention. A Lie subgroup H of a Lie group G will always be a closed subgroup (otherwise we will say that H is a immersed subgroup).

2.2.1 Basic definitions

Let us consider a homogeneous fibre bundle. It means that are given $\pi_M : Q \to M$ a principal H-bundle (with H a Lie group) and K a Lie subgroup of H. We set $N = Q/K$, then the map $\pi_N : Q \to N$ is a principal K-bundle and we have $\pi_M = \pi \circ \pi_N$ where $\pi : N \to M$ is a fibre bundle with fibre H/K, which is naturally isomorphic to the associated bundle $Q \times_H H/K$. Moreover, following [36], we assume the following hypothesis:

(i) H/K is reductive: $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, and $\mathrm{Ad}K(\mathfrak{p}) \subset \mathfrak{p}$, where $\mathfrak{h}$ and $\mathfrak{k}$ are respectively the Lie algebras of H and K.

(ii) M is endowed with a Riemannian metric g

(iii) H/K is Riemannian: there exists a H-invariant Riemannian metric on H/K (equivalently an $\mathrm{Ad}K$-invariant (positive definite) inner product on $\mathfrak{p}$). Equivalently $\mathrm{Ad}_{\mathfrak{p}}K$ is compact.

(iv) The principal H-bundle $\pi_M : Q \to M$ is endowed with a connection. We denote by ω the corresponding $\mathfrak{h}$-valued connection form on Q.

Then the splitting $TQ = \mathcal{V}_0 \oplus \mathcal{H}_0$ defined by ω ($\mathcal{V}_0 = \ker d\pi_M$, $\mathcal{H}_0 = \ker \omega$) gives rise by $d\pi_N$, to the following decomposition $TN = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V} = \ker d\pi = d\pi_N(\mathcal{V}_0)$ and $\mathcal{H} = d\pi_N(\mathcal{H}_0)$. Let $\mathfrak{p}_Q := Q \times_K \mathfrak{p} \to N$ be the vector bundle associated to $\pi_N : Q \to N$ with fibre $\mathfrak{p}$. Let us denote by $[q,a] \in \mathfrak{p}_Q$ the element defined by $(q,a) \in Q \times \mathfrak{p}$. Then we have the following vector bundle isomorphism

$$I\colon \begin{array}{ccc} \mathcal{V} & \longrightarrow & \mathfrak{p}_Q \\ d\pi_N(q.a) & \longmapsto & [q,a] \end{array}$$

where $q \in Q$, $a \in \mathfrak{p}$ and as usual $q.a = \frac{d}{dt}_{|t=0} q.\exp(ta) \in T_qQ$. Decomposing $\omega = \omega_{\mathfrak{k}} + \omega_{\mathfrak{p}}$ following $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, then since H/K is reductive, $\omega_{\mathfrak{p}}$ is a K-equivariant ($\omega_{\mathfrak{p}}(X.h) = \mathrm{Ad}h\, \omega_{\mathfrak{p}}(X)$) and π_N-horizontal ($\omega_{\mathfrak{p}|\ker d\pi_N} = 0$) $\mathfrak{p}$-valued 1-form on Q and hence projects to a $\mathfrak{p}$-valued 1-form ϕ on N:

$$\phi(d\pi_N(X)) = [q, \omega_{\mathfrak{p}}(X)].$$

Then we have

$$\phi_{|\mathcal{V}} = I \quad \text{and} \quad \ker \phi = \mathcal{H}.$$

We can now construct a Riemannian metric $\bar{h}$ on N:

$$\bar{h} = \pi^* g + \langle \phi, \phi \rangle \tag{2.10}$$

where $\langle \ , \ \rangle$ is the fibre metric induced on $\mathfrak{p}_Q$ by the inner product on $\mathfrak{p}$.

In the same way, let Φ be the $\mathfrak{p}_Q$-valued 2-form on N defined by the component $\Omega_{\mathfrak{p}}$ of the curvature form Ω of ω. Since $\Omega_{\mathfrak{p}}$ is π_M-horizontal ($\Omega(X,Y) = 0$ if $X \in \mathcal{V}_0$ or $Y \in \mathcal{V}_0$), then Φ is π-horizontal: $\Phi(X,Y) = 0$, if $X \in \mathcal{V}$ or $Y \in \mathcal{V}$. Let us remark that under the identification $I\colon \mathfrak{p}_Q \to \mathcal{V}$, Φ is nothing but the curvature of the horizontal subbundle $\mathcal{H}$ (see [20, Rmk 4.2.2]), that is to say

$$\Phi = \phi \circ \mathrm{R}_{\mathcal{H}}.$$

The 1-form $\omega_{\mathfrak{k}}$ (which is a connection form in π_N because H/K is reductive) defines a connection in π_N. This connection induces a covariant derivative ∇^c in the associated bundle $\mathfrak{p}_Q$, with respect to which the fibre metric is parallel. It is important to remark that under the identification $I\colon \mathfrak{p}_Q \to \mathcal{V}$, ∇^c corresponds to some covariant derivative ∇^v on the bundle $\mathcal{V}$. This ∇^v will inherit the name of canonical vertical connection.

Definition 2.7 *We call the following data (Q,H,K,ω) a homogeneous fibre bundle structure on $\pi\colon N \to M$ (or on N, when the fibration π is considered as implicitly given). Moreover we say that the fibration $\pi\colon N \to M$ is a homogeneous fibre bundle. The tensor Φ is called the horizontal curvature. Finally, the metric $\bar{h}$ defined by* (2.10) *is called a Kaluza-Klein metric.*

Remark 2.6 A homogeneous fibre bundle $\pi\colon N \to M$ could be endowed with several homogeneous fibre bundle structures giving rise to this fibration an in particular to the same fibre H/K (which is a Riemannian homogeneous space). Example: suppose that Q admits a principal subbundle Q' with structure group H', a subgroup of H, such that H' acts transitively on H/K then the datas (Q', H', K') (where $K' = K \cap H'$) still gives rise to the fibration $\pi\colon N \to M$. Moreover if ω is reducible to Q' then the homogeneous fibre bundle structures (Q,H,K,ω) and (Q',H',K',ω') (where $\omega' = \omega_{|TQ'}$) give rise to the same splitting $TN = \mathcal{H} \oplus \mathcal{V}$.

Now, let us suppose that H/K is naturally reductive, $(\mathfrak{h}, \mathfrak{k})$ is effective and $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} = \mathfrak{k}$. Then according to theorem 7.2 (Appendix), there exists one and only one $\mathrm{Ad}H$-invariant symmetric bilinear form B^* on $\mathfrak{h}$ extending the inner product $\langle\ ,\ \rangle_{\mathfrak{p}}$ and such that $B^*(\mathfrak{k}, \mathfrak{p}) = 0$. Moreover, B^* is non singular on $\mathfrak{h}$ and hence on $\mathfrak{k}$. This pseudo-Riemannian metric B^* induces a fibre metric on $\mathfrak{h}_Q := Q \times_H \mathfrak{h} \to M$ that we denote by $\langle\ ,\ \rangle_{\mathfrak{h}_Q}$. This fibre metric lifts naturally to a fibre metric on the fibre bundle $\pi^*\mathfrak{h}_Q \to N$, which when restricted to $\mathfrak{p}_Q \cong \mathcal{V}$ gives rise to the fibre metric $\langle \cdot, \cdot \rangle_{\mathcal{V}} := \bar{h}_{|\mathcal{V}\times\mathcal{V}}$. Moreover the fibre metric $\langle\ ,\ \rangle_{\mathfrak{h}_Q}$ allows to define a Riemannian metric on $\mathfrak{h}_Q$ given by $\pi^*g + \langle\ ,\ \rangle_{\mathfrak{h}_Q}$.

Definition 2.8 *If H/K is naturally reductive, $(\mathfrak{h}, \mathfrak{k})$ is effective and $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} = \mathfrak{k}$, then the Riemannian metric $\pi^*g + \langle\ ,\ \rangle_{\mathfrak{h}_Q}$, where$\langle\ ,\ \rangle_{\mathfrak{h}_Q}$ is the fibre metric defined above, will be called the Kaluza-Klein metric on $\mathfrak{h}_Q$ induced by the Kaluza-Klein metric $\bar{h}$.*

2.2.2 Admissible subbundles.

Let (M, g) be an oriented even dimensional Riemannian manifold endowed with a metric connection ∇. We consider $\mathcal{SO}(M)$ the $SO(2n)$-bundle of positively oriented orthonormal frames of TM, and denote by $\pi\colon \mathcal{SO}(M) \to M$ the projection map. Let us consider the map

$$\begin{array}{rccl} \mathcal{E}\colon & \mathcal{SO}(M) \times \mathfrak{gl}(2n) & \longrightarrow & \mathrm{End}(TM) \\ & (e, A) & \longmapsto & a \,|\, \mathcal{M}at_e(a) = A. \end{array} \tag{2.11}$$

This map factors through to an isomorphism $\mathcal{SO}(M) \times_{SO(2n)} \mathfrak{gl}(2n) \cong \mathrm{End}(TM)$.

Let Q be a principal subbundle[3] of $\mathcal{SO}(M)$ with structure group H a Lie subgroup of $SO(2n)$. In other words, Q is a H-structure on (M, g). Then the restriction of $\mathcal{E}$ to $Q \times \mathfrak{gl}(2n)$ gives rise to an isomorphism $Q \times_H \mathfrak{gl}(2n) \cong \mathrm{End}(TM)$.

• Now, let $S \subset \mathfrak{gl}(2n) = \mathrm{End}(\mathbb{R}^{2n})$ be some H-submanifold, i.e. the H-orbit of some element $A_0 \in \mathfrak{gl}(2n)$: $S = \mathrm{Ad}H(A_0)$. Denoting by K the stabilizer of A_0 in H

$$K = \mathrm{Stab}_H(A_0) = \{h \in H |\, hA_0h^{-1} = A_0\},$$

we have $S = H/K$. Moreover the image of $Q \times S$ by $\mathcal{E}$ is a homogeneous fibre bundle $S(M) := \mathcal{E}(Q \times S) \cong Q \times_H S = Q/K$. Let us make precise that the former isomorphism, denoted $\mathcal{J}$, from Q/K onto $S(M)$ is obtained by factoring through, the restriction of $\mathcal{E}$ to $Q \times \{A_0\}$:

$$\mathcal{J}\colon e.K \longmapsto a \,|\, \mathcal{M}at_e(a) = A_0. \tag{2.12}$$

For example, let us take $S = \Sigma^+(\mathbb{R}^{2n})$, $A_0 = J_0 \in \Sigma^+(\mathbb{R}^{2n})$, $Q = \mathcal{SO}(M)$ and $H = SO(2n)$, then we have $S(M) = \Sigma^+(M) \cong \mathcal{SO}(M)/U(J_0)$.
A second example is obtained by taking an arbitrary H-structure Q on (M, g), $A_0 = J_0 \in \Sigma^+(\mathbb{R}^{2n})$ and $S = S_H(J_0) = \mathrm{Ad}H(J_0) \subset \Sigma^+(\mathbb{R}^{2n})$. Then the associated subbundle $S_H^{J_0}(M) = \mathcal{J}(Q/K)$, where $K = H \cap U(J_0)$, is a homogeneous fibre subbundle of $\Sigma^+(M)$.

• More generally, if $S \subset \mathfrak{gl}(2n)$ is any $\mathrm{Ad}H$-invariant submanifold, we can define

$$S(M) := \mathcal{E}(Q \times S) \cong Q \times_H S.$$

When S is a vector subspace, we will often prefer the notation $S(TM)$ instead of $S(M)$ (but we will sometimes use the notation $S(M)$ for shortness).
A first example is to take $S = H$, to obtain $H(M) = \mathcal{E}(Q \times H) \cong Q \times_H H = Q$. Then $H(M)$ is

[3] I.e. a reduced bundle with the terminology of [21, § I.5].

a subbundle of $\mathrm{End}(TM)$ with fibre H, and isomorphic to Q.
A second example is $H^0(M) = \mathcal{E}(Q \times H^0) \cong Q \times_H H^0$ is a subbundle of $H(M) \cong Q$ with fibre H^0.
Finally, a third example is given by $\mathfrak{h}(TM) \subset \mathfrak{so}(TM)$ where $\mathfrak{h} = \mathrm{Lie}(H)$.

• Now, let ∇ be a metric connection on (M, g) and let ω be the corresponding connection 1-form on $\mathcal{SO}(M)$. Suppose that ω is reducible to a connection into the subbundle Q, i.e. $\omega_{|TQ}$ takes values in $\mathfrak{h}$. This is equivalent to say that $Q \supset \mathrm{H}^\nabla(e_0)$ the holonomy bundle through $e_0 \in Q$, for any frame $e_0 \in Q$. In particular, in this case, the subbundle of endomorphisms $\mathfrak{h}(TM) \subset \mathfrak{so}(TM)$ is ∇-parallel.
Let us fix $e_0 \in Q$ and $x_0 = \pi(e_0) \in M$. Then the structure group of the holonomy bundle $\mathrm{H}^\nabla(e_0)$ is the holonomy group of ω at e_0, $\mathrm{Hol}^\nabla(e_0) \subset SO(2n)$, which we will identify to $\mathrm{Hol}^\nabla(x_0) \subset SO(T_{x_0}M)$, the holonomy group of ∇ at x_0, by $h \in SO(T_{x_0}M) \mapsto \mathcal{M}at_{e_0}(h) \in SO(2n)$. More precisely, we identify $T_{x_0}M = \mathbb{R}^{2n}$ via the frame e_0. We then set

$$\hat{H} := \mathrm{Hol}^\nabla(x_0) = \mathrm{Hol}^\nabla(e_0),$$

and $\hat{\mathfrak{h}} = \mathrm{Lie}(\hat{H})$. We can then define the homogeneous fibre subbundles $\hat{H}(M)$, $\hat{H}^0(M)$ and $\hat{\mathfrak{h}}(M)$ (which are bundles associated to $Q = \mathrm{H}^\nabla(e_0)$ with fibres $\hat{H}$, $\hat{H}^0$ and $\hat{\mathfrak{h}}$ respectively). Moreover, for any $J_0 \in \Sigma^+(\mathbb{R}^{2n})$, we can define the homogeneous fibre subbundle of $\Sigma^+(M)$, $S^{J_0}_{\hat{H}}(M) \cong \mathrm{H}^\nabla(e_0)/\hat{K}$ where $\hat{K} = U(J_0) \cap \hat{H}$.

About the metrics. Now, let us apply the considerations of §2.2.1 to the homogeneous fibre bundle $S^{J_0}_H(M) \to M$ defined over a Riemannian manifold (M, g) endowed with a metric H-structure Q, a connection on Q, and finally an element $J_0 \in \Sigma^+(M)$. We still denote by ∇ the metric linear connection on M defined by the connection on Q. Then $(S^{J_0}_H(M), \bar{h}) \to (M, g)$ is a homogeneous fibre bundle in the sense of definition 2.7, where $\bar{h}$ is a Kaluza-Klein metric defined, according to (2.10), by some H-invariant metric on $S_H(J_0)$.
Moreover, since H is compact (because a Lie subgroup of $SO(2n)$), then H/K is naturally reductive and if $(\mathfrak{h}, \mathfrak{k})$ is effective then we have $\mathfrak{h} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ (see prop. 7.9 in the Appendix). Moreover, any Kaluza-Klein metric $\bar{h}$ on $S^{J_0}_H(M)$ (i.e. in the form (2.10)) induces a unique Kaluza-Klein metric on $\mathfrak{h}(M)$ (Def. 2.8). We will denote by $b_{\mathfrak{h}(M)} \in \mathcal{C}(S^2(\mathfrak{h}(M)))$ the corresponding symmetric endomorphism, that is to say the endomorphism of $\mathfrak{h}(M)$ induced by the symmetric endomorphism $b \in S^2(\mathfrak{h})$ defined by B^* the AdH-invariant pseudo-Riemannian metric on $\mathfrak{h}$ extending $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$:

$$B^*(\cdot, \cdot) = B_0(b\,\cdot, \cdot) = B_0(\cdot, b\,\cdot)$$

where B_0 is the opposite of the Killing form of $\mathfrak{so}(2n)$, i.e.

$$B_0(a, b) = \mathrm{Tr}(a^t b) = -\mathrm{Tr}(ab).$$

Definition 2.9 *Let (M, g) be a Riemannian manifold of dimension $2n$, endowed with a metric H-structure Q and a connection on Q, i.e. a (metric) H-connection ∇. Let $J_0 \in \Sigma^+(\mathbb{R}^{2n})$. Then the homogeneous fibre bundle $(S^{J_0}_H(M), \bar{h}) \to (M, g)$ defined above will be called the admissible homogeneous fibre bundle over (M, g) defined by the data (Q, H, ∇, J_0).*

Remark 2.7 What about the choice of the metric $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$? After extension, such a metric corresponds to an invariant symmetric bilinear form on $\mathfrak{h}$, and hence to an element of $\mathfrak{h} \odot \mathfrak{h} = S^2(\mathfrak{h})$ fixed by the action of H. The space of fixed points is one of the component of the AdH-irreducible decomposition of $S^2(\mathfrak{h})$, and it contains in particular the Killing form of H.

2.2.3 The twistor bundle of almost complex structures $\Sigma^+(M)$.

Let (M,g) be an even dimensional oriented Riemannian manifold endowed with a metric connection ∇. Then this linear connection corresponds to a connection 1-form on the bundle of $SO(2n)$-bundle of positively oriented orthonormal frames of TM. Therefore according to the previous definitions, $\Sigma^+(M)$ is a homogeneous fibre bundle (see [20, §4.3.2] for more precisions). Moreover we can consider the isomorphism $\mathcal{J}\colon \mathcal{SO}(M)/U(J_0) \to \Sigma^+(M)$ defined by (2.12) where we take $A_0 = J_0 \in \Sigma^+(\mathbb{R}^{2n})$. Then we can express the vertical projection ϕ, the horizontal curvature Φ and the vertical connection ∇^c in terms of $\mathcal{J}$. Indeed, we have the following result (due to C.M. Wood [36, Prop. 4.1]; see also Rawnsley [26, Prop. 4.2]):

Theorem 2.1 [20, Th. 4.3.4] *If $A,B \in TN$, $V \in \mathcal{C}(\mathfrak{p}_Q)$ then:*

(i) $\phi(A) = \frac{1}{2}\mathcal{J}.\nabla_A\mathcal{J}$

(ii) $\Phi(A,B) = \frac{1}{2}\mathcal{J}[\pi^*R(A,B),\mathcal{J}]$*, where R is the curvature operator of* ∇.

(iii) $\nabla^c_A V = \frac{1}{2}\mathcal{J}[\nabla_A V,\mathcal{J}]$

An important remark about the identifications $\mathcal{SO}(M)/U(J_0) = \Sigma^+(M)$ and $\mathcal{V} = \mathfrak{p}_Q$. Let us consider the two fibre bundles $\pi\colon N = \mathcal{SO}(M)/U(J_0) \to M$ and $\pi_\Sigma\colon \Sigma^+(M) \to M$. We have seen that $\mathcal{J}\colon N = \mathcal{SO}(M)/U(J_0) \to \Sigma^+(M)$ is an isomorphism of bundle over M, and even an isometry (if $\Sigma^+(M)$ is endowed with the image metric $(\mathcal{J}^{-1})^*\bar{h}$, where $\bar{h}$ is given by (2.10)), so that we can identify these two bundles.
Moreover, $\mathcal{J}$ defines tautologically a canonical complex structure on $\pi^*TM \to N$, $\mathcal{J}\colon N \to \Sigma^+(\pi^*TM)$. Under this identification, the vertical subbundle $\mathcal{V}^\Sigma = \ker d\pi_\Sigma$ is given by $\mathcal{V}^\Sigma = \mathfrak{so}_-(\pi^*TM,\mathcal{J})$. Furthermore, we have $\mathfrak{h} = \mathfrak{so}(2n)$ and $\mathfrak{p} = \mathfrak{so}_-(J_0)$ so that $\mathfrak{h}_Q = \mathfrak{so}(TM)$, the bundle of skew-symmetric endomorphisms of TM, and $\mathfrak{p}_Q = \mathfrak{so}_-(\pi^*TM,\mathcal{J}) = \mathcal{V}^\Sigma$.
Now, we have two different isomorphism between $\mathcal{V} = \ker d\pi$ and $\mathfrak{so}_-(\pi^*TM,\mathcal{J})$: indeed, the restriction to $\mathcal{V}$ of $d\mathcal{J}\colon TN \to T\Sigma^+(M)$ which gives a isomorphism $d\mathcal{J}_{|\mathcal{V}}\colon \mathcal{V} \to \mathcal{V}^\Sigma$ and the canonical isomorphism $I = \phi_{|\mathcal{V}}\colon \mathcal{V} \to \mathfrak{p}_Q$. These are different since according to theorem 2.1, we have $\phi = \frac{1}{2}\mathcal{J}(d\mathcal{J})^v$ (because $\nabla\mathcal{J}$ is the vertical component of $d\mathcal{J}$ with respect to the decomposition $T\Sigma^+(M) = \mathcal{H}^\Sigma \oplus \mathcal{V}^\Sigma$ defined by ∇). Therefore $I = \frac{1}{2}\mathcal{J}d\mathcal{J}_{|\mathcal{V}}$ and the two previous isomorphisms differ by the factor $\frac{1}{2}\mathcal{J}$.
Convention. In the following, we consider that $N = \mathcal{SO}(M)/U(J_0)$ and $\Sigma^+(M)$ are identified via $\mathcal{J}$ so that we will write $N = \Sigma^+(M)$, and the splitting $TN = \mathcal{H}\oplus\mathcal{V}$ is identified via $d\mathcal{J}$ to the splitting $T\Sigma^+(M) = \mathcal{H}^\Sigma \oplus \mathcal{V}^\Sigma$, and in particular $\mathcal{V}$ and $\mathcal{V}^\Sigma$ are identified via $d\mathcal{J}$. Therefore, under this identification, we have $\forall J \in N, \forall V \in \mathcal{V}_J = \mathfrak{so}_-(TM,J)$, $\phi(V) = \frac{1}{2}JV$.
Remark that the metric $\bar{h}$ (see (2.10)) written on $\Sigma^+(M)$ is then given by

$$\bar{h} = \pi^*g + \frac{1}{4}\langle\cdot,\cdot\rangle_{\mathfrak{so}(TM)|\mathcal{V}}, \tag{2.13}$$

where the fibre metric $\langle\cdot,\cdot\rangle_{\mathfrak{so}(TM)}$ is the restriction of the tensor product metric defined by g on $\mathrm{End}(TM)$: $\langle A,B\rangle = \mathrm{Tr}(A^tB)$, $\forall A,B \in \mathrm{End}(TM)$.

2.2.4 The twistor bundle $\mathcal{Z}^{\alpha}_{2k}(M)$.

Again (M,g) is an even dimensional Riemannian manifold endowed with a metric connection ∇. Then $\mathcal{Z}^{\alpha}_{2k}(M)$ is a homogeneous fibre bundle (see [20, §4.3.3]).
We consider the isomorphism $\mathcal{J}\colon \mathcal{SO}(M)/\mathbb{U}(J_0) \to \mathcal{Z}^{\alpha}_{2k}(M)$ defined by (2.12) where we take $A_0 = J_0 \in \mathcal{Z}^{\alpha}_{2k}(\mathbb{R}^{2n})$. Then we have the following result (see [20, §4.3.3]):

Theorem 2.2 [20, Th. 4.3.6] *If $A, B \in TN$, $V \in \mathcal{C}(\mathfrak{p}_Q)$ then*

(i) $\nabla\mathcal{J} = -\mathrm{ad}\mathcal{J} \circ \phi$ *thus* $\phi(A) = -(\mathrm{ad}\mathcal{J})^{-1}\nabla_A\mathcal{J}$

(ii) $\Phi(A,B) = (\mathrm{ad}\mathcal{J})^{-1}[\mathcal{J}, \pi^*R(A,B)]$

(iii) $\nabla^{\mathfrak{p}}_A V = (\mathrm{ad}\mathcal{J})^{-1}[\mathcal{J}, \nabla_A V]$

2.2.5 Uniqueness of the connection ω defining the structure of homogeneous fibre bundle.

We want to answer the following question: Given an homogeneous fibre bundle, does the splitting $TN = \mathcal{H}\oplus\mathcal{V}$ determine the connection 1-form ω? In other words, given the horizontal distribution $\mathcal{H}$ can we recover ω? One difficulty is that H could act not effectively on the fibre H/K.

Let us consider a homogeneous fibre bundle and $S_0 = H/K$ its fibre which is, let us recall it, a reductive homogeneous space. We suppose that H is connected. Let be $\mathfrak{h}_1 = \mathfrak{h}(\mathfrak{p}) := \mathfrak{p} + [\mathfrak{p},\mathfrak{p}] = \mathfrak{p} + [\mathfrak{p},\mathfrak{p}]_{\mathfrak{k}}$, this is a ideal of $\mathfrak{h}$, and we have $\mathfrak{h}_1 = \sum_{h\in H} \mathrm{Ad}h(\mathfrak{p})$ (Appendix, prop. 7.2). Moreover, the subgroup of H generated by $\mathfrak{h}_1$ is normal, acts transitively on $S_0 = H/K$ and is generated by $\exp(\mathfrak{p})$. Denoting by $\mathfrak{k}_1$ the Lie algebra of the isotropy group K_1 for the transitive action of H_1, we have $\mathfrak{k}_1 = \mathfrak{k}\cap\mathfrak{h}_1 = [\mathfrak{p},\mathfrak{p}]_{\mathfrak{k}}$, and $\mathfrak{h}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}$, $\mathrm{Ad}K_1(\mathfrak{p}) = \mathfrak{p}$ (Appendix, prop. 7.2).

Definition 2.10 • *We call $\mathfrak{h}_1$ the essential component of $\mathfrak{h}$. Suppose that $\mathfrak{h}$ admits a decomposition $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ where $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$. We then say that $\mathfrak{h}_1$ admits a complement ideal $\mathfrak{h}_0$ in $\mathfrak{h}$. If $X \in \mathfrak{h}$, we call $X_{\mathfrak{h}_1}$ its essential component.*

• *We say that $\mathfrak{k}$ is regular in $\mathfrak{h}$ if there exists an $\mathrm{Ad}H$-invariant symmetric bilinear form B on $\mathfrak{h}$, non degenerate, for which $\mathfrak{k}$ is non singular. In this case, we will say that the $\mathrm{Ad}H$-invariant summand $\mathfrak{p} = \mathfrak{k}^{\perp}$ is a regular summand (of $\mathfrak{k}$ in $\mathfrak{h}$).*

Remark 2.8 • According to the prop. 7.5 (Appendix), if $(\mathfrak{h},\mathfrak{k})$ is effective, then $\mathfrak{p}$ is a regular summand w.r.t. a B which is positive definite on $\mathfrak{p}$ if and only if $\mathfrak{p}$ is natural and $\mathfrak{h} = \mathfrak{h}_1$.

Proposition 2.5 *Suppose that $\mathfrak{h}_1$ admits a complement ideal $\mathfrak{h}_0$ in $\mathfrak{h}$, and that $\mathfrak{p}$ is an $\mathrm{Ad}K$-invariant regular summand of $\mathfrak{k}_1$ in $\mathfrak{h}_1$. The connection 1-form ω on a homogeneous fibre bundle has its essential component $\omega_{\mathfrak{h}_1}$ which is completely determined by its component $\omega_{\mathfrak{p}}$.*

Proof. Decomposing the equation $\omega(\xi.h) = \mathrm{Ad}h^{-1}(\omega(\xi))$, $\forall h \in H$, following $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ leads to $\omega_{\mathfrak{h}_i}(\xi.h) = \mathrm{Ad}h^{-1}(\omega_{\mathfrak{h}_i}(\xi))$, $\forall h \in H$, $i = 0,1$ (because $\mathfrak{h}_0$ and $\mathfrak{h}_1$ are ideals).
Now, consider $X \in \mathfrak{h}_1$ and $X = X_{\mathfrak{k}_1} + X_{\mathfrak{p}}$ its decomposition, then $\forall h \in H$, $\mathrm{Ad}h(X) = \mathrm{Ad}h(X_{\mathfrak{k}_1}) + \mathrm{Ad}h(X_{\mathfrak{p}})$ is the decomposition of $\mathrm{Ad}h(X)$ following $\mathfrak{h}_1 = \mathrm{Ad}h(\mathfrak{k}_1) \overset{\perp}{\oplus} \mathrm{Ad}h(\mathfrak{p})$, so that $[\mathrm{Ad}h(X)]_{\mathrm{Ad}h(\mathfrak{p})} = \mathrm{Ad}h(X_{\mathfrak{p}})$. Therefore $[X]_{\mathrm{Ad}h(\mathfrak{p})} = \mathrm{Ad}h\left([\mathrm{Ad}h^{-1}(X)]_{\mathfrak{p}}\right)$. Now, let $\xi \in TQ$

and $\xi = \xi_{\mathcal{H}} + q.X$, where $X \in \mathfrak{h}$, its decomposition following $TQ = \ker \omega \oplus \ker d\pi =: \mathcal{H}^Q \oplus \mathcal{V}^Q$. Then we have $\forall h \in H$,

$$[\omega_{\mathfrak{h}_1}(\xi)]_{\mathrm{Ad}h(\mathfrak{p})} = [X_{\mathfrak{h}_1}]_{\mathrm{Ad}h(\mathfrak{p})} = \mathrm{Ad}h\left([\mathrm{Ad}h^{-1}(X_{\mathfrak{h}_1})]_{\mathfrak{p}}\right) = \mathrm{Ad}h\left([\mathrm{Ad}h^{-1}(\omega_{\mathfrak{h}_1}(\xi))]_{\mathfrak{p}}\right)$$
$$= \mathrm{Ad}h\left([(\omega_{\mathfrak{h}_1}(\xi.h))]_{\mathfrak{p}}\right) = \mathrm{Ad}h\left(\omega_{\mathfrak{p}}(\xi.h)\right). \quad (2.14)$$

Finally, the equality $\mathfrak{h}_1 = \sum_{h\in H} \mathrm{Ad}h(\mathfrak{p})$ allows us to conclude. Indeed, if ω' is another 1-form such that $\omega'_{\mathfrak{p}} = \omega_{\mathfrak{p}}$ then $[\omega'_{\mathfrak{h}_1}]_{\mathrm{Ad}h(\mathfrak{p})} = [\omega_{\mathfrak{h}_1}]_{\mathrm{Ad}h(\mathfrak{p})}$, $\forall h \in H$, hence $\omega'_{\mathfrak{h}_1} - \omega_{\mathfrak{h}_1} \perp \sum_{h\in H} \mathrm{Ad}h(\mathfrak{p}) = \mathfrak{h}_1$ hence $\omega'_{\mathfrak{h}_1} - \omega_{\mathfrak{h}_1} = 0$ (since B is non degenerate on $\mathfrak{h}_1$). This completes the proof. ☐

Proposition 2.6 *The $\mathfrak{p}$-component $\omega_{\mathfrak{p}}$ of the connection 1-form is completely determined by the vertical projection $\phi\colon TN \to \mathfrak{p}_Q$.*

Proof. By definition, we have $\forall \xi \in TQ$, $\phi(d\pi_N(\xi)) = [q, \omega_{\mathfrak{p}}(\xi)]$, moreover for all $q \in Q$, the map $[q,\cdot]\colon \mathfrak{p} \to (Q \times_H \mathfrak{p})_q$ is a linear bijection. Therefore

$$\omega_{\mathfrak{p}}(\xi) = ([q,\cdot])^{-1}(\phi(d\pi_N(\xi))).$$

This completes the proof. ☐

Combining the two previous propositions, we obtain:

Theorem 2.3 *Suppose that $\mathfrak{h}_1$ admits a complement ideal $\mathfrak{h}_0$ in $\mathfrak{h}$, and that $\mathfrak{p}$ is an $\mathrm{Ad}K$-invariant regular summand of $\mathfrak{k}_1$ in $\mathfrak{h}_1$. The horizontal distribution on the homogeneous fibre bundle determines completely the essential component $\omega_{\mathfrak{h}_1}$ (defined on the principal bundle Q).*

Let U be the maximal normal subgroup of H included in K, and $H' = H/U$, $K' = K/U$, so that $S_0 = H'/K'$ with (H', K') effective. Moreover, since U is normal then the canonical projection $\pi_{H'}\colon H \to H' = H/U$ is a morphism of groups and then induces a morphism of Lie algebras, $\pi_{\mathfrak{h}'}\colon \mathfrak{h} \to \mathfrak{h}' = \mathfrak{h}/\mathfrak{u}$ such that $\pi_{\mathfrak{h}'|\mathfrak{p}}\colon \mathfrak{p} \to \mathfrak{p}' = \pi_{\mathfrak{h}'}(\mathfrak{p})$ is a linear bijection. Therefore $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{p}'$ and $\mathrm{Ad}K'(\mathfrak{p}') = \mathfrak{p}'$. Moreover the $\mathrm{Ad}K$-invariant inner product on $\mathfrak{p}$ induces an $\mathrm{Ad}K'$-invariant inner product[4] on $\mathfrak{p}'$, for which $\pi_{\mathfrak{h}'|\mathfrak{p}}$ is an isometry. We conclude that the fibre $S_0 = H/K$ is naturally reductive (w.r.t. $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$) if and only if so is $S_0 = H'/K'$ (w.r.t. $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{p}'$).
Let $\mathfrak{h}'_1 = \mathfrak{h}'(\mathfrak{p}') = \mathfrak{p}' + [\mathfrak{p}', \mathfrak{p}'] = \pi_{\mathfrak{h}'}(\mathfrak{h}_1)$. Then $\mathfrak{h}'_1 = \sum_{h\in H'} \mathrm{Ad}h(\mathfrak{p}') = \pi_{\mathfrak{h}'}\left(\sum_{h\in H} \mathrm{Ad}h(\mathfrak{p})\right)$. Under the hypothesis of natural reductivity, according to the prop. 7.4 (Appendix), we can decompose $\mathfrak{h}' = \mathfrak{h}'_1 \oplus \mathfrak{h}'_0$ where $\mathfrak{h}'_0$ is a ideal of $\mathfrak{h}'$, and the invariant inner product on $\mathfrak{p}'$ can be extended (in a unique way) to a non degenerate invariant scalar product on $\mathfrak{h}'_1$ such that $\mathfrak{k}'_1 \perp \mathfrak{p}'$ (theorem 7.2, Appendix). In particular, $\mathfrak{p}'$ is an $\mathrm{Ad}K'$-invariant regular summand of $\mathfrak{k}'_1$ in $\mathfrak{h}'_1$ and we can then apply prop. 2.5.

Corollary 2.1 *If H acts effectively on the fibre H/K and if H/K is naturally reductive (w.r.t. $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$) then $\omega_{\mathfrak{h}_1}$ is completely determined by $\omega_{\mathfrak{p}}$.*

In terms of the homogeneous fibre bundles, the following holds.

Proposition 2.7 *The fibration $Q/U \to M$ (resp. $Q/U \to N$) defines a principal H'-bundle (resp. a principal K'-bundle) which give rise to the initial homogeneous fibre bundle $\pi\colon N \to M$, but now the structure group H' acts effectively on the homogeneous fibre $S_0 = H'/K'$. Moreover, the new connection 1-form ω' on the principal bundle $Q' = Q/U$ is obtained by factoring through the $\mathfrak{h}'$-projection of the initial connection 1-form: $(\pi_{Q'})^*\omega' = \pi_{\mathfrak{h}'}(\omega)$, where $\pi_{Q'}\colon Q \to Q'$ is the canonical projection.*

[4]Which induces the same invariant metric on S_0.

Furthermore, it is useful to remark the following.

Lemma 2.5 *Under the condition of proposition 2.5, the homogeneous fibre bundle $\pi\colon N \to M$ endowed with its horizontal distribution can be defined, at least locally, by a new homogeneous fibre structure $(Q_1, H_1, K_1, \omega_1)$ with H_1 (the normal subgroup generated by $\mathfrak{h}_1 = \mathfrak{h}(\mathfrak{p})$) as structure group.*

Proof. Under the condition of proposition 2.5, we have $\omega_{\mathfrak{h}_1}(\xi.h) = \mathrm{Ad}h^{-1}(\omega_{\mathfrak{h}_1}(\xi))$, $\forall h \in H$. Moreover, locally there exists a subbundle of Q with structure group H_1: in other words, we can always reduce locally Q to a (local) subbundle with structure group H_1. Then on this (local) subbundle, $\omega_{\mathfrak{h}_1}$ defines therefore a connection 1-form.

If $\mathfrak{u}$ admits a complement ideal. Now, let us suppose that the surjective morphism $\pi_{\mathfrak{h}'}$ admits a global section $\mathfrak{h}' \to \mathfrak{h}$, the image of which will then be a Lie sub-algebra $\mathfrak{h}^* \subset \mathfrak{h}$ such that $\mathfrak{h}^* \cong \mathfrak{h}'$, $\mathfrak{h}^* \cap \mathfrak{u} = \{0\}$ and $\mathfrak{h} = \mathfrak{h}^* \oplus \mathfrak{u}$. Let H^* be the subgroup generated by $\mathfrak{h}^*$ then it is clear that $H^* \cap U =: U^*$ is discrete, $H^*U = UH^* = H$ and $H^*/U^* = H'$. Moreover H^* acts transitively on $S = H/K$ and we have $S = H^*/K^*$ where $K^* = K \cap H^*$ (and we have $K' = K^*/U^*$). Moreover we have also $\mathfrak{k} = \mathfrak{u} \oplus \mathfrak{k}^*$.
Furthermore, setting $\mathfrak{p}^* = \left(\pi_{\mathfrak{h}'|\mathfrak{h}^*}\right)^{-1}(\mathfrak{p}')$, we have $\mathrm{Ad}K^*(\mathfrak{p}^*) = \mathfrak{p}^*$ and $\mathfrak{h}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$ (remark that $\pi_{\mathfrak{h}'|\mathfrak{h}^*}$ is an isomorphism of Lie algebras). Therefore, H'/K' is naturally reductive w.r.t. $\mathfrak{h}' = \mathfrak{k}' \oplus \mathfrak{p}'$ if and only if so is H^*/K^* w.r.t. $\mathfrak{h}^* = \mathfrak{k}^* \oplus \mathfrak{p}^*$. Moreover, we remark that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}^*$. Therefore if we suppose now that $\mathfrak{h}^*$ is moreover a ideal of $\mathfrak{h}$ so that $\mathrm{Ad}K^*(\mathfrak{p}^*) = \mathfrak{p}^*$, [5] we can choose $\mathfrak{p} = \mathfrak{p}^*$, whence $\mathfrak{h}_1^* := \mathfrak{h}(\mathfrak{p}^*) = \mathfrak{h}(\mathfrak{p}) =: \mathfrak{h}_1$. Hence under the hypothesis of natural reductivity, since $\mathfrak{u}^* = \{0\}$, we can decompose $\mathfrak{h}^* = \mathfrak{h}_0^* \oplus \mathfrak{h}_1^*$ and the invariant inner product on $\mathfrak{p}$ can be extended to a non degenerate invariant scalar product on $\mathfrak{h}_1$ such that $\mathfrak{k}_1 \perp \mathfrak{p}$. Hence $\mathfrak{h} = \mathfrak{h}^* \oplus \mathfrak{u} = \mathfrak{h}_1 \oplus \mathfrak{h}_0^* \oplus \mathfrak{u} = \mathfrak{h}_1 \oplus \mathfrak{h}_0$ where $\mathfrak{h}_0 = \mathfrak{h}_0^* \oplus \mathfrak{u}$, and $\mathfrak{p}$ is an $\mathrm{Ad}K$-invariant regular summand of $\mathfrak{k}_1$ in $\mathfrak{h}_1$.
The following could be useful.

Proposition 2.8 *Suppose that $\mathfrak{u}$ admits a complement ideal, and moreover that $\mathfrak{k}^*$ is non singular w.r.t. some non degenerate symmetric bilinear form B^* on $\mathfrak{h}^*$. Then, setting $\mathfrak{p} = \mathfrak{k}^{*\perp_{B^*}}$, we have $\mathfrak{h}^* = \mathfrak{h}_1 := \mathfrak{h}(\mathfrak{p})$, so that $\mathfrak{p}$ is an $\mathrm{Ad}K$-invariant regular summand of $\mathfrak{k}_1$ in $\mathfrak{h}_1$.*
In particular, if B is some non degenerate symmetric bilinear form on $\mathfrak{h}$ such that $\mathfrak{h}^* \perp \mathfrak{u}$, then $B^* = B_{|\mathfrak{h}^*}$ is non degenerate, moreover $\mathfrak{k}^*$ is non singular w.r.t. B^* if and only if $\mathfrak{k}$ is non singular w.r.t. B, and then $\mathfrak{p} := \mathfrak{k}^{*\perp_{B^*}} = \mathfrak{k}^{\perp_B}$. Hence $\mathfrak{p}$ is also an $\mathrm{Ad}K$-invariant regular summand of $\mathfrak{k}$ in $\mathfrak{h}$.

Proof. • Since $\mathfrak{h}(\mathfrak{p})$ is an ideal in $\mathfrak{h}^*$, then $\mathfrak{h}(\mathfrak{p})^{\perp_{B^*}}$ is an ideal in $\mathfrak{h}^*$ and we have $\mathfrak{h}(\mathfrak{p})^{\perp_{B^*}} \subset \mathfrak{p}^{\perp_{B^*}} = \mathfrak{k}^*$. Therefore, since $(\mathfrak{h}^*, \mathfrak{k}^*)$ is effective we must have $\mathfrak{h}(\mathfrak{p})^{\perp_{B^*}} = 0$, that is to say $\mathfrak{h}^* = \mathfrak{h}(\mathfrak{p})$. This proves the first point.

• Since $\mathfrak{h} = \mathfrak{h}^* \overset{\perp}{\oplus} \mathfrak{u}$, then $\mathfrak{h}^*$ and $\mathfrak{u}$ are on singular w.r.t. B so that in particular $B^* = B_{|\mathfrak{h}^*}$ is non degenerate. Then the equivalence follows from the equality $\mathfrak{k} = \mathfrak{k}^* \overset{\perp}{\oplus} \mathfrak{u}$. This completes the proof of the second point. □

In the situation of proposition 2.8, $\omega_{\mathfrak{h}^*}$ is completely determined by $\omega_{\mathfrak{p}}$ (according to proposition 2.5).

[5] And since $\pi_{\mathfrak{h}'}(\mathfrak{p}^*) = \mathfrak{p}'$

Semisimple case. Let us for example suppose that H is semi-simple, then we can decompose $\mathfrak{h}$ as a direct sum of simple ideals: $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r$. Then $\mathfrak{u}$ must be a sum of $\mathfrak{h}_i$'s and by a change of indexation, one can suppose that $\mathfrak{u} = \mathfrak{h}_{k+1} \oplus \cdots \mathfrak{h}_r$, so that $\mathfrak{h}^* := \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$ is an ideal of $\mathfrak{h}$. We are therefore in the situation described by the previous paragraph.
Let us mention the following particular case:

Proposition 2.9 *Suppose that $\mathfrak{h}$ is semisimple. The isotropy Lie algebra $\mathfrak{k}^*$ is non singular w.r.t. to Killing form B^* of $\mathfrak{h}^*$ if and only if so is $\mathfrak{k}$ w.r.t. the Killing form B of $\mathfrak{h}$. Furthermore in this case, setting $\mathfrak{p} = (\mathfrak{k}^*)^{\perp_{B^*}} = \mathfrak{k}^{\perp_B}$ and $\mathfrak{h}_1 = \mathfrak{h}(\mathfrak{p})$, then we have $\mathfrak{h}_1 = \mathfrak{h}^*$ (and hence $\mathfrak{k}^* = \mathfrak{k}_1$) and $\mathfrak{p}$ is an $\mathrm{Ad}K$-invariant regular summand of $\mathfrak{k}_1$ in $\mathfrak{h}_1$.*

Proof. This follows immediately from proposition 2.5, since an ideal of a semisimple Lie algebra is semisimple, its killing form is nothing but the restriction to this ideal of the Killing form of the Lie algebra, and since two ideals in direct sum are orthogonal w.r.t. the Killing form. This completes the proof. □

Compact Case. Finally, let us consider the particular case where H is compact. In this case, we have a direct sum $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z(\mathfrak{h})$, where $Z(\mathfrak{h})$ is the center of $\mathfrak{h}$, and the ideal $\mathfrak{u}$ splits also $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] \oplus Z(\mathfrak{u})$ with $Z(\mathfrak{u}) \subset Z(\mathfrak{h})$ (Appendix, prop. 7.8). Moreover the semisimple ideal $[\mathfrak{h}, \mathfrak{h}]$ is a direct sum $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}^*_{ss} \oplus [\mathfrak{u}, \mathfrak{u}]$, where $\mathfrak{h}^*_{ss}$ is some semisimple ideal of $[\mathfrak{h}, \mathfrak{h}]$ (Appendix, prop. 7.7). Therefore, after choosing some subspace $Z(\mathfrak{h}^*)$ such that $Z(\mathfrak{h}) = Z(\mathfrak{u}) \oplus Z(\mathfrak{h}^*)$, we then have

$$\mathfrak{h} = \mathfrak{h}^* \oplus \mathfrak{u}$$

where $\mathfrak{h}^* = \mathfrak{h}^*_{ss} \oplus Z(\mathfrak{h}^*)$. Moreover, we have $[\mathfrak{h}^*, \mathfrak{h}^*] = \mathfrak{h}^*_{ss}$, and $Z(\mathfrak{h}^*)$ is the center of $\mathfrak{h}^*$. Therefore, we are in the situation described by the paragraph entitled *If $\mathfrak{u}$ admits a complement ideal.*
Moreover, the opposite of the Killing form of $\mathfrak{h}$, $-K$, is positive definite on $[\mathfrak{h}, \mathfrak{h}]$. Furthermore for any Euclidean scalar product $\langle \cdot, \cdot \rangle_{Z(\mathfrak{h})}$ on the vector space $Z(\mathfrak{h})$, the positive definite scalar product $B = -K + \langle \cdot, \cdot \rangle_{Z(\mathfrak{h})}$is $\mathrm{Ad}H$-invariant on $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z(\mathfrak{h})$. Then, for any such B, its restriction $B^* = B_{|\mathfrak{h}^*}$ is $\mathrm{Ad}H$-invariant and positive definite, so that setting $\mathfrak{p} = \mathfrak{k}^{*\perp_{B^*}}$, we have $\mathfrak{h}^* = \mathfrak{h}(\mathfrak{p})$. Applying proposition 2.5, we obtain that $\omega_{\mathfrak{h}^*}$ is completely determined by $\omega_{\mathfrak{p}}$.
Moreover, if we choose the Euclidean scalar product $\langle \cdot, \cdot \rangle_{Z(\mathfrak{h})}$ such that $Z(\mathfrak{u}) \perp Z(\mathfrak{h}^*)$ so that $\mathfrak{u} \perp \mathfrak{h}^*$ w.r.t. B, then we have $\mathfrak{p} = \mathfrak{h}^{*\perp_{B^*}} = \mathfrak{h}^{\perp_B}$.

In conclusion, in the situation of proposition 2.8 or proposition 2.9, or if H is compact, then proposition 2.5 can always be applied and we obtain that $\omega_{\mathfrak{h}^*}$ is completely determined by $\omega_{\mathfrak{p}}$, and if moreover $(\mathfrak{h}, \mathfrak{k})$ is effective (i.e. $\mathfrak{h} = \mathfrak{h}^*$) then ω is determined by $\omega_{\mathfrak{p}}$.

Corollary 2.2 *Consider the twistor bundle $\Sigma^+(M) \to M$ or on one of its subbundle $\pi \colon N \to M$ defined from the data (Q, H, ∇, J_0) by $N = S_H^{J_0}(M)$. If H acts effectively on the fibre $S_H(J_0)$, then the metric connection ∇ is completely determined by the horizontal distribution on $\pi \colon N \to M$.*

Theorem 2.4 *thm-convention-hk1 An admissible homogeneous fibre bundle $\pi \colon S_H^{J_0}(M) \to M$ endowed with its horizontal distribution, can always be defined, at least locally, by data (Q, H, ∇, J_0) such that H acts effectively on the fibre $S_H(J_0)$ and such that also $\mathfrak{h} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ for some reductive summand $\mathfrak{p}$.*

Proof. This results immediately from the lemma 2.5 and the study of the compact case above. □

Important Convention. In all the next of the paper, we will suppose that when an admissible subbundle $S_H^{J_0}(M)$ is given, the structure group H acts effectively on the fibre $S_H(J_0) = H/K$, so that at the level of the Lie algebra setting, we have $\mathfrak{h} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] = \sum_{h\in H} \mathrm{Ad}h(\mathfrak{p})$.

2.3 Metric f-manifolds and metric f-connections.

We need to recall some notions and properties about metric f-manifolds (N, F, h) presented in [20]. The aim is to identify the different geometric constraints on (N, F, h) for the existence of metric f-connections with skew-symmetric torsion. It is useful and even necessary for a good understanding of the meaning of these geometric constraints to divide our study in several steps. First of all, let us begin by recalling some definitions. We refer the reader to [20, §6] for details.

2.3.1 Some definitions.

Let us consider (N, F) *an f-manifold*, i.e. a manifold endowed with an f-structure, i.e. an endomorphism $F \in \mathcal{C}(\mathrm{End}(TN))$ such that $F^3 + F = 0$. Let us set $\mathcal{H} = \mathrm{Im}\, F$ and $\mathcal{V} = \ker F$, then we have $TN = \mathcal{H} \oplus \mathcal{V}$. If we put $P = -F^2$, then P is the projector on $\mathcal{H}$ along $\mathcal{V}$. Moreover $PF = FP = F$ and $F^2P = -P$. In particular, $\bar{J} := F_{|\mathcal{H}}$ is a complex structure in the vector bundle $\mathcal{H}$.
Let us denote also by $q := \mathrm{Id} - P$ the projector on $\mathcal{V}$ along $\mathcal{H}$. We denote by $X = X^{\mathcal{V}} + X^{\mathcal{H}}$, or sometimes simply by $X = X^v + X^h$, the decomposition of any element $X \in TN$.

We denote by $\Phi := \mathrm{R}_{\mathcal{H}}$ the curvature of $\mathcal{H}$ and by $\mathrm{R}_{\mathcal{V}}$ the curvature of $\mathcal{V}$.

Definition 2.11 *The Nijenhuis tensor N_F of F is defined by*

$$N_F(X,Y) = [FX, FY] - F[FX, Y] - F[X, FY] - P[X, Y],$$

where $X, Y \in \mathcal{C}(TN)$.

Then we obtain immediately ([17])

Proposition 2.10 *We have the following identities.*

$$\begin{array}{rcccl} N_F(qX, qY) & = & -P[qX, qY] & = & PN_F(qX, qY) \\ qN_F(X, Y) & = & q[FX, FY] & = & qN_F(pX, pY) \\ N_F(qX, PY) & = & -F[qX, FY] - P[qX, PY] & & \end{array}$$

so that

$$N_{|\mathcal{V}\times\mathcal{V}} = \mathrm{R}_{\mathcal{V}} \quad and \quad N^{\mathcal{V}} = -\mathrm{R}_{\mathcal{H}}(F\cdot, F\cdot),$$

where $\mathrm{R}_{\mathcal{V}}$ and $\mathrm{R}_{\mathcal{H}}$ are the curvature of $\mathcal{V}$ and $\mathcal{H}$ respectively. In particular, $N^{\mathcal{V}}(\mathcal{V}, \mathcal{V}) = N^{\mathcal{V}}(\mathcal{H}, \mathcal{V}) = \{0\}$ i.e

$$N(\mathcal{H}, \mathcal{V}) \subset \mathcal{H} \quad and \quad N(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}.$$

Moreover $N_{F|\mathcal{V}\times\mathcal{H}} = N^{\mathcal{H}}_{F\,|\mathcal{V}\times\mathcal{H}}$ satisfies the following property

$$N_F(X^v, \bar{J}Y^h) = -\bar{J}N_F(X^v, Y^h)$$

i.e. $N_F(X^v, \cdot)_{|\mathcal{H}}$ anti-commutes with $\bar{J}$.

Definition 2.12 *Let (N,F) be an f-manifold. Then for any $B \in \mathcal{T}$, we set*

$$B^{\varepsilon,\varepsilon'}(X,Y) = -\frac{1}{4}\left(\varepsilon\varepsilon' B(FX,FY) + \varepsilon FB(FX,Y) + \varepsilon' FB(X,FY) - B(X,Y)\right).$$

and $B^{2,0} := B^{++}$, $B^{1,1} := B^{+,-} + B^{-+}$ and $B^{0,2} := B^{--}$.

We will also set $\bar{B} = B^{\mathcal{H}}_{|\mathcal{H}^2}$ for any $B \in \mathcal{T}$.

2.3.2 Characterization of metric connections preserving the splitting.

Theorem 2.5 [20, Th.6.2.1] *Let (N,h) be a Riemannian manifold with an orthogonal decomposition $TN = \mathcal{H} \oplus \mathcal{V}$. Then a metric connection ∇ leaves invariant this decomposition (i.e. $\mathcal{H}$ and $\mathcal{V}$ are ∇-parallel) if and only if its torsion T satisfies*

$$T_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = \Phi, \qquad T_{|\mathcal{V}\times\mathcal{V}\times\mathcal{H}} = \mathrm{R}_{\mathcal{V}}$$

and

$$\begin{aligned}
\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(T_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}\right)(X^v, Y^h, Z^h) &= \mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(D\Omega_{q|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right)(Y^h, Z^h, X^v)\\
\mathrm{Sym}_{\mathcal{V}\times\mathcal{V}}\left(T_{|\mathcal{H}\times\mathcal{V}\times\mathcal{V}}\right)(X^h, Y^v, Z^v) &= \mathrm{Sym}_{\mathcal{V}\times\mathcal{V}}\left(D\Omega_{q|\mathcal{V}\times\mathcal{V}\times\mathcal{H}}\right)(Y^v, Z^v, X^h)
\end{aligned}$$

In particular, the components $\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(T_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}\right)$ and $\mathrm{Sym}_{\mathcal{V}\times\mathcal{V}}\left(T_{|\mathcal{H}\times\mathcal{V}\times\mathcal{V}}\right)$ are independent of ∇.

We see that the conditions on the torsion concerns only the components $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ and $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{V}\times\mathcal{V})}$. Let us call metric q-manifolds, the metric manifolds (N,h) endowed with a symmetric projector q (defining then a splitting $TN = \mathcal{H}\oplus\mathcal{V}$ where $\mathcal{V} = \mathrm{Im}\, q$ and $\mathcal{H} = \ker q = \mathcal{V}^\perp$). Now, let us see under which conditions on the metric q-manifold (N,h,q), there exists a connection preserving this structure and with skew-symmetric torsion.

Corollary 2.3 [20, Cor. 6.2.2] *Let (N,h) be a Riemannian manifold with an orthogonal decomposition $TN = \mathcal{H} \oplus \mathcal{V}$. Then, the following statements are equivalent*

(i) *There exists a metric connection ∇ leaving invariant the decomposition $TN = \mathcal{H} \oplus \mathcal{V}$, such that the component $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ of the torsion is skew-symmetric, i.e. $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})} = \mathrm{Skew}(\Phi)$.*

(ii) *For any metric connection ∇ leaving invariant the decomposition $TN = \mathcal{H} \oplus \mathcal{V}$, the component of the torsion $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ is skew-symmetric.*

(iii) *$D\Omega_{q|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}$ is skew-symmetric w.r.t. the two first variables, i.e. $Dq_{|\mathcal{H}\times\mathcal{H}} \in \mathcal{C}((\Lambda^2\mathcal{H}^*)\otimes\mathcal{V})$, or equivalently $D\Omega_q(P\cdot, P\cdot, q\cdot) = \frac{1}{2}\Phi$.*

(iv) *$D\Omega_{q|\mathcal{H}\times\mathcal{V}\times\mathcal{H}}$ is skew-symmetric w.r.t. the first and third variables, i.e. $D\Omega_q(PX, qY, PZ) = -\frac{1}{2}\Phi(Z,X,Y)$.*

(v) $\mathrm{Skew}\left(D\Omega_{q|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}\right) = 0$.

We then say that (N,q,h) is of type $\mathcal{H}^2\mathcal{V}$.

In the same way, we have

Corollary 2.4 [20, Cor. 6.2.3] *The corollary still holds if we replace everywhere $\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})$ by $\mathcal{S}(\mathcal{V}\times\mathcal{V}\times\mathcal{H})$ and Φ by $\mathrm{R}_\mathcal{V}$. We then say that (N,q,h) is of type $\mathcal{V}^2\mathcal{H}$ (if one of the corresponding equivalent statements holds).*

Definition 2.13 *We will say that the (orthogonal) decomposition on the Riemannian manifold (N,h) is **reductive** or that (N,q,h) is **reductive** if (N,q,h) is of type $\mathcal{H}^2\mathcal{V}$ and of type $\mathcal{V}^2\mathcal{H}$. This is equivalent to say that there exists a metric connection ∇ leaving invariant the decomposition $TN=\mathcal{H}\oplus\mathcal{V}$, and with skew-symmetric torsion.*

2.3.3 Characterization of metric f-connections.

Now, let us come back to the case of an f-manifolds. Then the condition $\nabla F=0$ is equivalent to the fact that ∇ leaves invariant the decomposition $TN=\mathcal{H}\oplus\mathcal{V}$ and moreover $\nabla^\mathcal{H}\bar{J}=0$, where $\nabla^\mathcal{H}$ is the connection induced by ∇ on $\mathcal{H}$. Heuristically, we have to add to the conditions of theorem 2.5:
• those which characterizes metric Hermitian connections in terms of their torsion (see [13, Prop. 2], or [20, Th. 5.3.3]),
• the condition $\nabla^\mathcal{H}_\mathcal{V}\bar{J}=0$.

Definition 2.14 *We will say that an f-structure F and a metric h on a manifold N are compatible if $\mathcal{H}\perp\mathcal{V}$ and if $\bar{J}$ is an orthogonal complex structure on $\mathcal{H}$ endowed with the metric induced by h. This is equivalent to say that F is skew-symmetric w.r.t. the metric h: $F\in\mathfrak{so}(TN)$, or equivalently that $I=\bar{J}\oplus\mathrm{Id}_\mathcal{V}$ is orthogonal: $I^*h=h$. We will then say that (N,F,h) is a metric f-manifold.*

Theorem 2.6 [20, Th. 6.2.2] *Let (N,F,h) be a metric f-manifold. Then a metric connection ∇ preserves the f-structure F if and only if all the following 3 statements hold:*

$$\begin{array}{lll}
\nabla\Omega_{F|\mathcal{H}^3}=0 & \iff & N_{\bar{J}}=4\left(T_{|\mathcal{H}^3}\right)^{0,2} \text{ and } \mathrm{Skew}\Big(\left(T_{|\mathcal{H}^3}\right)^{2,0}-\left(T_{|\mathcal{H}^3}\right)^{1,1}\Big)=\left(d^c\Omega_{F|\mathcal{H}^3}\right)^{**}\\
\nabla\Omega_{F|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}=0 & \iff & \left\{\begin{array}{c}\boxed{\begin{array}{c}\nabla\Omega_{F|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}=0\\ \nabla\Omega_{F|\mathcal{H}\times\mathcal{V}\times\mathcal{H}}=0\end{array}}\\ \nabla\Omega_{F|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}=0\end{array}\right. \iff \left\{\begin{array}{c}\boxed{\nabla\Omega_{q|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}=0}\\ T(X^v,FY^h,Z^h)+T(X^v,Y^h,FZ^h)=\\ N(X^v,Y^h,FZ^h)\end{array}\right.\\
\nabla\Omega_{F|\mathcal{S}(\mathcal{V}\times\mathcal{V}\times\mathcal{H})}=0 & \iff & \nabla\Omega_{q|\mathcal{S}(\mathcal{V}\times\mathcal{V}\times\mathcal{H})}=0
\end{array}$$

2.3.4 Skew-symmetric torsion: Existence of a characteristic connection.

The component $\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})$ of $\otimes^3 TN$.

Definition 2.15 *Let (N,F,h) be a metric f-manifold. We define the **extended Nijenhuis tensor** $\tilde{N}_F$ as the TN-valued 2-form on N defined by*

$$\tilde{N}_F:=N_F+\Phi+\mathrm{R}_\mathcal{V}(Z^v,X^v,Y^h)+\mathrm{R}_\mathcal{V}(Y^v,Z^v,X^h).$$

We remark that $\tilde{N}_{F|\mathcal{S}(\mathcal{V}\times\mathcal{V}\times\mathcal{H})}=\mathrm{Skew}(\mathrm{R}_\mathcal{V})$ is always skew-symmetric.

Proposition 2.11 [20, Prop. 6.2.5] *Let (N,F,h) be a metric f-manifold. Then the following statements are equivalent.*

(i) *There exists a metric f-connection ∇ (satisfying then $\nabla F=0$) with a torsion T such that ${T^{0,2}}_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ is skew-symmetric.*

(ii) *There exists a metric connection* ∇, *satisfying* $\nabla F_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})} = 0$, *with a torsion* T *such that* ${T^{0,2}}_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is skew-symmetric.*

(iii) $N_F(FY^h, Z^h, X^v) + N_F(Y^h, FZ^h, X^v) = N_F(X^v, Y^h, FZ^h)$.

(iv) *The extended Nijenhuis tensor* $\tilde{N}_F$ *satisfies:* $\tilde{N}_{F|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is skew-symmetric.*

Proposition 2.12 [20, Prop. 6.2.6] *Let* (N, F, h) *be a metric* f*-manifold. Then the following statements are equivalent.*

(i) *There exists a metric* f*-connection* ∇ *(satisfying then* $\nabla F = 0$*) with a torsion* T *such that* $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is skew-symmetric.*

(ii) *There exists a metric connection* ∇, *satisfying* $\nabla F_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})} = 0$, *with a torsion* T *such that* $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is skew-symmetric.*

(iii) (N, q, h) *is of type* $\mathcal{H}^2\mathcal{V}$, *and* $\tilde{N}_{F|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is skew-symmetric.*

Furthermore, under these statements, for any such connection satisfying (i) or (ii), then $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is unique (i.e. uniquely determined by the metric* f*-manifold* (N, F, h)*) and equal to* Skew(Φ). *Conversely, any extension* $T \in \mathcal{C}(\mathcal{T})$ *of this unique skew-symmetric trilinear form* Skew(Φ) *defines a metric connection* ∇, *satisfying* $\nabla F_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})} = 0$.

We are led to the following definition.

Definition 2.16 *We will say that a metric* f*-manifold* (N, F, h) *is reductive if* (N, q, h) *is reductive, where* q *is defined by* F.
We will say that a metric f*-manifold* (N, F, h) *is* reductively of type $\mathcal{G}_1$ *if* $\tilde{N}_{F|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ *is skew-symmetric.*

The component $\mathcal{H}^3$ of TN^3.

Definition 2.17 *We will say that a metric* f*-manifold is* horizontally of type $\mathcal{G}_1$ *or that it is* of horizontal type $\mathcal{G}_1$ *if one of the following equivalent statements holds.*

(i) *The* horizontal Nijenhuis tensor $N_{\bar{J}}$ *is skew-symmetric.*

(ii) *There exists a metric* f*-connection* ∇, *such that* $(T_{|\mathcal{H}^3})^{0,2}$ *is skew-symmetric.*

(iii) *There exists a metric connection* ∇, *satisfying* $\nabla F_{|\mathcal{H}^3} = 0$, *such that* $(T_{|\mathcal{H}^3})^{0,2}$ *is skew-symmetric.*

Proposition 2.13 [20, Prop. 6.2.7] *Let* (N, F, h) *be a metric* f*-manifold. Then the following statements are equivalent.*

(i) (N, F, h) *is horizontally of type* $\mathcal{G}_1$.

(ii) *There exists a metric* f*-connection* ∇, *such that* $T_{|\mathcal{H}^3}$ *is skew-symmetric.*

(iii) *There exists a metric connection* ∇, *satisfying* $\nabla F_{|\mathcal{H}^3} = 0$, *such that* $T_{|\mathcal{H}^3}$ *is skew-symmetric.*

In this case, for any such connection satisfying (ii) or (iii), then $T_{|\mathcal{H}^3}$ *is unique (i.e. uniquely determined by the metric* f*-manifold* (N, F, h)*). Conversely any extension* $T \in \mathcal{C}(\mathcal{T})$ *of this unique skew-symmetric trilinear form* $T_{|\mathcal{H}^3}$ *defines a metric connection* ∇, *satisfying* $\nabla F_{|\mathcal{H}^3} = 0$.

Definition 2.18 *Let (N, F, h) be a metric f-manifold of horizontal type $\mathcal{G}_1$. Then the unique horizontal 3-form $\bar{T} \in \mathcal{C}(\Lambda^3\mathcal{H}^*)$, such that the torsion T of any metric f-connection ∇ in N with a skew-symmetric component $T_{\mathcal{H}^3}$, satisfies $T_{\mathcal{H}^3} = \bar{T}$ (see proposition 2.13) will be called the **Horizontal torsion 3-form** of (N, F, h).*

Remark 2.9 According to proposition 2.13-(iii), the horizontal torsion 3-form is also the horizontal component $T_{\mathcal{H}^3}$ of the torsion of any metric connection ∇, satisfying $\nabla F_{|\mathcal{H}^3} = 0$ and $T_{\mathcal{H}^3}$ is skew-symmetric.

Conclusion about characteristic connections. Grouping together the hypothesis of skew-symmetry for each component of the tensor $\tilde{N}_F$, we define:

Definition 2.19 *A metric f-manifold (N, F, h) with skew-symmetric extended Nijenhuis tensor $\tilde{N}_F$ will be sayed **of global type** $\mathcal{G}_1$ or globally of type $\mathcal{G}_1$.*

Now, grouping together the previous results allows us to conclude with theorem 1.3, given in the introduction, which characterizes the existence of characteristic connections on metric f-manifolds.

2.3.5 Precharacteristic and paracharacteristic connections.

Sometimes the condition of global type $\mathcal{G}_1$ could be too much strong and it could happen that one needs the existence (and uniqueness up to the $\mathcal{V}^3$-component of the torsion) of some characteristic connection by supposing weaker conditions on the metric f-manifold (N, F, h).

Definition 2.20 *Let (N, F, h) be a metric f-manifold of horizontal type $\mathcal{G}_1$. Then any metric f-connection ∇ with a skew-symmetric component $T_{|\mathcal{H}^3}$ of its torsion, will be called a horizontal-characteristic connection.*
Moreover, if we suppose that (N, F, h) is also of type $\mathcal{V}^2\mathcal{H}$, then a metric f-connection ∇ with skew-symmetric components $T_{|\mathcal{H}^3}$, $T_{|\mathcal{S}(\mathcal{V}\times\mathcal{V}\times\mathcal{H})}$ and $T_{|\mathcal{V}^3}$ of its torsion, will be called a precharacteristic connection.

Remark 2.10 Let us remark that in a metric f-manifold of horizontal type $\mathcal{G}_1$, horizontal-charactersitic connections always exist, and the component $T_{|\mathcal{H}^3}$ of the torsion is unique. Moreover, if we suppose that (N, F, h) is also of type $\mathcal{V}^2\mathcal{H}$, then precharacteristic connections always exist and the components $T_{|\mathcal{H}^3}$ and $T_{|\mathcal{S}(\mathcal{V}\times\mathcal{V}\times\mathcal{H})}$ are unique.

The following properties will hold for the horizontal curvature in most the examples of interest for us.

Definition 2.21 *Let (N, F) be an f-manifold. Let $A \in \mathcal{C}(\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{V})$, $B \in \mathcal{C}(\mathcal{H}^* \otimes \mathcal{V}^* \otimes \mathcal{H})$, and $C \in \mathcal{C}(\mathcal{V}^* \otimes \mathcal{H}^* \otimes \mathcal{H})$. Then we will say respectively that A, B or C is pure if respectively*

(i) $A(\bar{J}X, Y) = A(X, \bar{J}Y)$, $\forall X, Y \in \mathcal{H}$.

(ii) $B(\bar{J}X, Y) = -\bar{J}B(X, Y)$, $\forall X \in \mathcal{H}, Y \in \mathcal{V}$.

(iii) $C(X, \bar{J}Y) = -\bar{J}B(X, Y)$, $\forall X \in \mathcal{V}, Y \in \mathcal{H}$.

If (N,F) is endowed with a compatible metric h, then this means that A, B or C considered as element of $\mathcal{C}(\mathcal{H}^*\otimes\mathcal{H}^*\otimes\mathcal{V})$, satisfies respectively $A^{1,1}=0$, $B^{1,1}=0$, $C^{1,1}=0$. Moreover, we will say that A, B or C resp. (considered as trilinear forms) is skew-symmetric in $\mathcal{H}\times\mathcal{H}$ if resp. A is skew-symmetric w.r.t. the 2 first variables, B w.r.t. the first and third variables, and C w.r.t. the 2 last variables.

Let us remark that $N_{F|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}$ is pure by definition of N_F (see proposition 2.10), and skew-symmetric in $\mathcal{H}\times\mathcal{H}$ if (N,F,h) is of type $\mathcal{H}^2\mathcal{V}$ (see [20, Rem. 6.2.5]).

Definition 2.22 • (N,F,h) *will be called almost of type $\mathcal{H}^2\mathcal{V}$ if one of the following equivalent statements holds*

(i) $\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(D\Omega_{q|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right)$ *is pure.*

(ii) $\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(N_{F|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}\right)=2\,\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(D\Omega_{q|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right)(Y^h,Z^h,X^v)$.

• *If moreover (N,F,h) is of type $\mathcal{V}^2\mathcal{H}$, then we will say that it is almost reductive.*

Remark 2.11 We remark that if (N,F,h) is of type $\mathcal{H}^2\mathcal{V}$ then it is almost of type $\mathcal{H}^2\mathcal{V}$, and therefore if it is reductive then it is, in particular, almost reductive.

Theorem 2.7 [20, Th. 6.2.4] *A metric f-manifold (N,F,h) admits a precharacteristic connection ∇ such that the component $T_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}$ of its torsion is pure if and only if it is almost reductive and horizontally of type $\mathcal{G}_1$. Moreover, in this case, for any $\alpha\in\mathcal{C}(\Lambda^3\mathcal{V}^*)$, there exists a unique precharacteristic connection ∇ such that the component $T_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}$ of its torsion is pure, and such that $T_{|\Lambda^3\mathcal{V}}=\alpha$. This unique connection is given by*

$$T=(-d^c\Omega_F+N_{F|\mathcal{H}^3})+T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}+\mathrm{Skew}(\mathrm{R}_\mathcal{V})+\alpha.$$

where $T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}=\Phi+\frac{1}{2}(N_F(X^v,Y^h,Z^h)-N_F(Y^v,X^h,Z^h))$.
Moreover if we impose also the component $T_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}$ to be skew-symmetric in $\mathcal{H}\times\mathcal{H}$, then this is possible if and only if (N,F,h) is reductive and of horizontal type $\mathcal{G}_1$.

Definition 2.23 *On a metric f-manifold (N,F,h), a precharacteristic connection ∇ such that the component $T_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}$ of the torsion is pure, will be called a* ***paracharacteristic*** *connection.*

Remark 2.12 Since a paracharacteristic connection is unique up to a element $\alpha\in\mathcal{C}(\Lambda^3\mathcal{V}^*)$, we will often speak about *'the'* paracharacteristic connection.

Proposition 2.14 [20, Prop. 6.2.10] *Let (N,F,h) be a reductive metric f-manifold of global type $\mathcal{G}_1$. Then the paracharacteristic connection (defined by some $\alpha\in\mathcal{C}(\Lambda^3\mathcal{V}^*)$) coincide with the characteristic connection (defined by the same $\alpha\in\mathcal{C}(\Lambda^3\mathcal{V}^*)$) if and only if the horizontal curvature Φ is pure.*

2.3.6 Riemannian submersion and metric f-manifolds of global type $\mathcal{G}_1$.

From now, until the end of this section, we consider the case where the vertical subbundle is the tangent space of the fibres of a fibration (or more generally a submersion) $\pi\colon N\to M$, i.e. $\mathcal{V}=\ker d\pi$. Let us first remark that in this case $\mathrm{R}_\mathcal{V}=0$, which leads to immediate simplifications in the preceding results.
Convention In all the next of the paper, all the submersions $\pi\colon N\to M$ are supposed to be surjective (i.e. the open set $\pi(N)$ coincides with M).

Proposition 2.15 [20, Prop. 6.3.1] *Let $\pi\colon (N,h) \mapsto (M,g)$ be a Riemannian submersion, over which we consider the natural orthogonal decomposition: $TN = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V} = \ker d\pi$ and $\mathcal{H} = \mathcal{V}^\perp$. Denote by D and D^g respectively the Levi-Civita connections of (N,h) and (M,g), respectively. Let $\widetilde{D^g}$ be the connection on $\mathcal{H}$ defined by the lift of D^g: $\widetilde{D^g_A}B = (d\pi_{|\mathcal{H}})^{-1}(D^g_A \pi_*(B)) \in \mathcal{H}$ for all $A, B \in \mathcal{C}(TN)$.*
Let us suppose that (N,q,h) is of type $\mathcal{V}^2\mathcal{H}$. Then the horizontal component of the Levi-Civita connection in N is related to $\widetilde{D^g}$ by the following formula:

$$\langle D_A B, H\rangle = \langle \widetilde{D^g_A} B, H\rangle + \frac{1}{2}\left(\Phi(A,H,B^v) + \Phi(B,H,A^v)\right)$$

forall $A, B \in \mathcal{C}(TN)$ and $H \in \mathcal{C}(\mathcal{H})$.

Proposition 2.16 *In the same situation as in proposition 2.15, we have*

$$\langle \widetilde{D^g_{X^v}} Y^h, Z^h\rangle = \frac{1}{2}\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(\nabla\Omega_{q|\mathcal{H}\times\mathcal{V}\times\mathcal{H}}\right)(Z^h, X^v, Y^h) + \langle [X^v, Y^h], Z^h\rangle \tag{2.15}$$

Moreover, this equation still holds if we replace D^g by any metric connection ∇^M on M.

Proof. By a direct computation using the characterization of Levi-Civita ([21, Prop. 2.3]), we can prove

$$\langle D_{X^v} Y^h, Z^h\rangle = \frac{1}{2}\Phi(Y^h, Z^h, X^v) + \frac{1}{2}\mathrm{Sym}_{\mathcal{H}\times\mathcal{H}}\left(\nabla\Omega_{q|\mathcal{H}\times\mathcal{V}\times\mathcal{H}}\right)(Z^h, X^v, Y^h) + \langle [X^v, Y^h], Z^h\rangle$$

Then according to the proposition 2.15, we obtain (2.15), and the second assertion follows from the fact that $(\widetilde{D^g - \nabla^M})$ is a horizontal trilinear form. This completes the proof. □

Definition 2.24 *Let $\pi\colon (N,h) \mapsto (M,g)$ be a Riemannian submersion, with the same notations and definitions as in the previous proposition. Let us suppose that some metric connection ∇^c on $\mathcal{V}$ is given, and denote by T^c its vertical connection:*

$$T^c(X,Y) = \nabla^c_X Y^v - \nabla^c_Y X^v - [X,Y]^v = d^{\nabla^c} q(X,Y),$$

where $q\colon TN \to \mathcal{V}$ is the vertical projection.
We call $T^c_{|\mathcal{V}\times\mathcal{H}}$ the reductivity term. *We say that (N,q) is ∇^c-reductive if the reductivity term $T^c_{|\mathcal{V}\times\mathcal{H}}$ vanishes.*

Proposition 2.17 [20, Cor. 6.3.2] *Let $\pi\colon (N,h) \mapsto (M,g)$ be a Riemannian submersion, endowed with its canonical orthogonal splitting $TN = \mathcal{V} \oplus \mathcal{H}$. Then if (N,q,h) is of type $\mathcal{V}^2\mathcal{H}$ then it is also of type $\mathcal{H}^2\mathcal{V}$ and thus it is reductive. In particular, if $\mathcal{V}$ can be endowed with a metric connection ∇^c with a vanishing reductivity term $T^c_{|\mathcal{V}\times\mathcal{H}}$, then (N,q,h) is reductive. In particular, a homogeneous fibre bundle is reductive.*

Proposition 2.18 [20, Cor. 6.3.3] *Let $\pi\colon (N,h) \mapsto (M,g)$ be a Riemannian submersion, endowed with its canonical orthogonal splitting $TN = \mathcal{V} \oplus \mathcal{H}$. Let us suppose that $\mathcal{H}$ is endowed with an orthogonal complex structure, that is to say N is endowed with a metric f-structure compatible*[6] *with the previous splitting.*
Let us suppose that there exists some metric connection ∇^c on $\mathcal{V}$ for which (N,q) is ∇^c-reductive, and that $T^c_{|\mathcal{V}^3}$ is skew-symmetric. Then the following statements are equivalent

[6] i.e. $\ker F = \mathcal{V}$ and $\mathrm{Im}\, F = \mathcal{H}$.

(i) *There exists a characteristic connection on* (N, F, h).

(ii) (N, F, h) *is of global type* $\mathcal{G}_1$.

(iii) *The canonical connection* ∇^c *can be extended to a characteristic connection.*

(iv) *There exists a Hermitian connection* $\nabla^{\mathcal{H}}$ *on* $\mathcal{H}$ *such that* $\nabla := \nabla^c \oplus \nabla^{\mathcal{H}}$ *has a skew-symmetric torsion.*

In particular, these equivalences hold when $\pi\colon (N, h) \mapsto (M, g)$ *is a homogeneous fibre bundle with a naturally reductive fibre* H/K.

Remark 2.13 In other words, if (N, q, h) is of type $\mathcal{V}^2\mathcal{H}$, then the existence of a characteristic connection is equivalent to the global type $\mathcal{G}_1$, and in this case, the set of metric connections ∇^c on the vertical subbundle $\mathcal{V}$ which can be extended to a characteristic connection, is the affine space

$$D^v + \mathcal{C}(\Lambda^3\mathcal{V}^*).$$

We can rewrite the proposition 2.18 for paracharacteristic connections.

Proposition 2.19 [20, Cor. 6.3.4] *In the same situation as in the previous proposition, the following statements are equivalent*

(i) *There exists a paracharacteristic connection on* (N, F, h).

(ii) (N, F, h) *is of horizontal type* $\mathcal{G}_1$.

(iii) *The canonical connection* ∇^c *can be extended to a paracharacteristic connection.*

In particular, these equivalences hold when $\pi\colon (N, h) \mapsto (M, g)$ *is a homogeneous fibre bundle with a naturally reductive fibre* H/K.

2.3.7 Metric f-submersions and Horizontal projectibility.

Definition 2.25 • *A map* $\pi\colon (N, F) \to M$ *from an* f*-manifold* (N, F) *to a manifold* M *is called a* ***f-submersion*** *if it is a submersion and* $\ker F = \ker d\pi$.
• *A map* $\pi\colon (N, F, h) \to (M, g)$ *from a metric* f*-manifold to a Riemannian manifold is called a* ***metric f-submersion*** *if* $\pi\colon (N, h) \to (M, g)$ *is a Riemannian submersion and if* $\pi\colon (N, F) \to M$ *is an* f*-submersion.*

Definition 2.26 • *We will say that* $\pi\colon (N, F) \to M$ *is an* f*-fibration if it is an* f*-submersion and a fibration.*
• *We will say that* $\pi\colon (N, F, h) \to (M, g)$ *is a homogeneous fibre* f*-bundle if* $\pi\colon (N, h) \to (M, g)$ *is a homogeneous fibre bundle and* $\pi\colon (N, F, h) \to (M, g)$ *is a metric* f*-submersion.*

Proposition 2.20 *Let* $\pi\colon (N, F, h) \to (M, g)$ *be a metric* f*-submersion. We suppose that* (N, F, h) *is reductive.* • *Then we have*

$$N_F(X^v, Y^h, Z^h) = \langle (D^g_{X^v}\bar{J})\bar{J}Y^h, Z^h\rangle$$

where $X, Y, Z \in TN$. *Moreover, this equation still holds if we replace* D^g *by any metric connection* ∇^M *on* M.

Proof. It suffices to apply proposition 2.16. □

Definition 2.27 *Let $\pi\colon (N,F,h) \to (M,g)$ be a metric f-submersion. Let us suppose that (N,F,h) is reductive and horizontally $\mathcal{G}_1$. A metric connection $\overline{\nabla}$ will be sayed to be the* ***canonical connection*** *on M (w.r.t. the f-submersion) if its torsion $\overline{T}$ satisfies the equation $\tilde{\overline{T}} = T_{\mathcal{H}^3}$ where $T_{\mathcal{H}^3}$ is the horizontal 3-form (see Def. 2.18) and $\tilde{\overline{T}}$ is the lift in $\mathcal{H}$ of $\overline{T}$. If such a connection exists, we will say that the metric f-submersion is* ***horizontally projectible****.*

Remark 2.14 The canonical connection is unique when it exists (since it is metric and its torsion is given). Moreover it has a skew-symmetric torsion. The metric f-submersion is **horizontally projectible** if and only if the horizontal 3-form $T_{\mathcal{H}^3}$ is projectible to a 3-form on M.

Proposition 2.21 *Let $\pi\colon (N,F,h) \to (M,g)$ be a metric f-submersion. Let us suppose that there exists a metric connection $\overline{\nabla}$ on M such that*

$$\overline{\nabla}_H \bar{J} = 0, \quad \forall H \in \mathcal{H}.$$

Then (N,F,h) is horizontally of type $\mathcal{G}_1$ if and only if the torsion $\overline{T}$ satisfies: $\overline{T}_{\bar{J}}^{0,2}$ is a 3-form on $\pi^*TM \cong \mathcal{H}$ i.e. a section of $\pi^*(\Lambda^3 T^*M) = \Lambda^3\mathcal{H}^*$.
In particular, if $\overline{\nabla}$ is moreover geodesically equivalent to the Levi-Civita connection, i.e. has a skew-symmetric torsion, then (N,F,h) is horizontally of type $\mathcal{G}_1$, and horizontally projectible, $\overline{\nabla}$ being then the canonical connection of M.

Proof. Any metric connection ∇ on N which satisfies

$$\langle \nabla_{H_1} H_2, H_3\rangle = \langle \tilde{\overline{\nabla}}_{H_1} H_2, H_3\rangle,$$

satisfies $\nabla F_{|\mathcal{H}^3} = 0$ and $T_{|\mathcal{H}^3} = \tilde{\overline{T}}$. Moreover, according to theorem 2.6, for any metric connection ∇, on N, such that $\nabla F_{|\mathcal{H}^3} = 0$, we have $4N_{\bar{J}} = \left(T_{|\mathcal{H}^3}\right)^{0,2}$. Hence $\left(\tilde{\overline{T}}\right)^{0,2}_{\bar{J}} = 4N_{\bar{J}}$. Finally $N_{\bar{J}}$ is a 3-form if and only if $\overline{T}_{\bar{J}}^{0,2}$ is a 3-form. This completes the proof. □

2.3.8 The canonical metric f-structure in the twistor bundles.

Let (M,g) be an (even dimensional) Riemannian manifold endowed with a metric connection ∇. Then we consider the homogeneous fibre bundle $\pi\colon (\mathcal{Z}_{2k}^{\alpha}(M), h) \to (M,g)$ associated to (M,g,∇), defined in §2.2. Let us consider also its canonical $2k$-structure $\mathcal{J} \in \mathcal{C}(\mathcal{Z}_{2k}^{\alpha}(\pi^*TM))$ to which corresponds the orthogonal complex structure $\underline{\mathcal{J}} \in \mathcal{C}\left(\Sigma(\pi^*TM)\right) \cong \mathcal{C}\left(\Sigma(\mathcal{H})\right)$ (according to Def. 2.6). This complex structure defines then a metric f-structure $\mathcal{F}$ on $\mathcal{Z}_{2k}^{\alpha}(M)$. Then the twistor bundle $(\mathcal{Z}_{2k}^{\alpha}(M), \mathcal{F}, h)$ is a reductive metric f-manifold (prop. 2.17, last statement), more precisely a homogeneous fibre f-bundle (see Def. 2.26).

Proposition 2.22 [20, Prop. 6.3.11] *Let (M,g) be a Riemannian manifold endowed with a metric connection $\overline{\nabla}$. Then we have*

$$\overline{\nabla}_H \underline{\mathcal{J}} = 0, \quad \forall H \in \mathcal{H}.$$

Remark 2.15 • According to remark 2.5, in the particular case of $\Sigma^+(M)$, one have $\underline{\mathcal{J}} = -\mathcal{J}$ (i.e. $\underline{\mathcal{J}}(J) = -J, \forall J \in \Sigma^+(M)$). In this case, we will use directly the canonical complex structure $\mathcal{J}$ whereas the complex structure $\underline{\mathcal{J}}$ will be used only for $k > 2$. In fact the minus sign between

$\underline{\mathcal{J}}$ and $\mathcal{J}$, in the case of $\Sigma^+(M)$, has strictly no importance for what we what do. All what follows holds identically for each of the two choices.
• Remark that by definition of $\mathcal{F}$, we have $\bar{J} = \underline{\mathcal{J}}$, and in particular in $\Sigma^+(M)$, $\bar{J} = \mathcal{J}$.

2.3.9 Variational formulation of stringy harmonic maps into metric f-manifolds.

In this subsection, we will recall the results obtained in [20, § 6.4.2]. These results will be improved later on the present paper in the following way: in [20], we suppose that the characteristic connection has a parallel torsion and here we will prove that for the kind of f-bundles we are interesting in, this hypothesis can be removed.

Here, (N, F, h) is a reductive metric f-manifold of global type $\mathcal{G}_1$. We suppose that one of its characteristic connections, ∇, has a parallel torsion $\nabla T = 0$.
The following lemma tells us how to compute the exterior derivative of ∇-parallel 3-forms.

Lemma 2.6 *Let α be a ∇-parallel 3-form. Then*

$$d\alpha(X, Y, V, Z) = \underset{X,Y,Z}{\mathfrak{S}} \alpha(T(V, Z), X, Y) + \alpha(T(X, Y), V, Z)$$

In particular, if α is horizontal then

$$\begin{aligned}
d\alpha_{|\Lambda^4\mathcal{V}} &= 0\\
d\alpha_{|(\Lambda^3\mathcal{V})\wedge\mathcal{H}} &= 0\\
d\alpha_{|(\Lambda^2\mathcal{V})\wedge\mathcal{H}^2}(V_0, V_1, H_2, H_3) &= \alpha(\mathrm{R}_{\mathcal{V}}(V_0, V_1), H_2, H_3)\\
d\alpha_{|\mathcal{V}\wedge(\Lambda^3\mathcal{H})}(V_0, H_1, H_2, H_3) &= \underset{i,j,k}{\mathfrak{S}} \alpha(T^{\mathcal{H}}(V_0, H_i), H_j, H_k)
\end{aligned}$$

Notation We set $\mathcal{S}(\mathcal{H}, \mathcal{V}) = \mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V}) \oplus \mathcal{S}(\mathcal{H}\times\mathcal{V}\times\mathcal{V})$. Then for any $B \in \mathcal{T}$, we set $\mathring{B} = B_{|\mathcal{S}(\mathcal{H},\mathcal{V})}$. Let us remark that we have

$$-F \circlearrowleft \mathring{T} = d\Omega_{F|\mathcal{S}(\mathcal{H},\mathcal{V})} \quad \text{and} \quad -\bar{J} \circlearrowleft \bar{T} = d\Omega_{F|\mathcal{H}^3}.$$

Now using the previous lemma and after some computations, we obtain the following results (see [20, § 6.4.2]).

Proposition 2.23 *The following identity holds:*

$$d(F\cdot N_F)_{|\mathcal{H}^4} = d\left(F \circlearrowleft (\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))\right)_{|\mathcal{H}^4}$$

Therefore the following statements are equivalent

(i) $dH_{\|\mathcal{H}^4} = 0$	**(iv)** $\mathfrak{S}_{X,Y,Z}\, \bar{\mathrm{A}}(X, Y)Z = 0$
(ii) $dH^\star_{\|\mathcal{H}^4} = 0$	**(v)** $\mathfrak{S}_{X,Y,Z}\, \bar{\mathrm{A}}^{(-)}(X, Y)Z = 0.$
(iii) $d(F\cdot N_F)_{\|\mathcal{H}^4} = 0$	

We will say that $\bar{J}$ is a cyclic derivation of the horizontal curvature when (iv) holds. ∎

Notation Given $B \in \mathcal{C}(\Lambda^2T^*N \otimes \mathcal{H})$, we denote simply $\mathrm{Im}\, B = \{B(X, Y) \in \mathcal{H}, X, Y \in TN\} \subset \mathcal{H}$. In particular, we have $\mathrm{Im}\, N_{\bar{J}} = N_{\bar{J}}(\mathcal{H}, \mathcal{H}) \subset \mathcal{H}$ and $\mathrm{Im}\, \mathrm{R}_{\mathcal{V}} = \mathrm{R}_{\mathcal{V}}(\mathcal{V}, \mathcal{V}) \subset \mathcal{H}$.
Moreover, we will also use the notations $\ker C = \{X \in \mathcal{H} |\, C(X, \cdot) = 0\}$ and $\mathrm{Supp}(C) = (\ker C)^\perp$, for any $C \in \mathcal{C}(\Lambda^2\mathcal{H}^* \otimes TN)$.

Proposition 2.24 *Let us supppose that* $\mathrm{R}_{\mathcal{V}} = 0$. *Then the following identities hold:*

(i) $dH_{|\mathcal{H}^2\times\mathcal{V}^2} = 0$ **(iii)** $d(F\cdot N_F)_{|\mathcal{H}^2\times\mathcal{V}^2} = 0$
(ii) $dH^\star_{|\mathcal{H}^2\times\mathcal{V}^2} = 0$ **(iv)** $d\left(F \circlearrowleft(\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))\right)_{|\mathcal{H}^2\times\mathcal{V}^2} = 0.$

Proposition 2.25 *Let us supppose that* $\mathrm{R}_{\mathcal{V}} = 0$ *and that the horizontal curvature* Φ *is pure. Then the following statements are equivalent*

(i) $dH_{|\mathcal{H}^3\times\mathcal{V}} = 0$ **(iii)** $d(F\cdot N_F)_{|\mathcal{H}^3\times\mathcal{V}} = 0$
(ii) $dH^\star_{|\mathcal{H}^3\times\mathcal{V}} = 0$ **(iv)** $\mathfrak{S}_{X,Y,Z}\,\bar{J}N_{\bar{J}}(\rho(V)X,Y,Z) = 0.$

Moreover these later are also equivalent to the following equivalent statements

(iv) $N_{\bar{J}}(X,Y,\rho(V)Z) = 0$.

(v) $N_{\bar{J}}(\mathcal{H},\mathcal{H}) \perp \rho(\mathcal{V})(\mathcal{H})$, *or equivalently* $\ker N_{\bar{J}} \perp \ker\Phi$, *i.e.* $\mathrm{Supp}(N_{\bar{J}}) \perp \mathrm{Supp}(\Phi)$.

We will say that the 2-forms $N_{\bar{J}}$ *and* Φ *have orthogonal supports, when (v) is satisfied.*

Finally,

Lemma 2.7 *The components in* $\Lambda^4\mathcal{V}^*$ *and* $(\Lambda^3\mathcal{V}^*)\wedge\mathcal{H}^*$ *of the following exterior derivatives:* dH, $dH^\star$, $d(F\cdot N_F)$ *and* $d\left(F\circlearrowleft(\mathrm{Skew}(\Phi) + \mathrm{Skew}(\mathrm{R}_{\mathcal{V}}))\right)$, *vanishes.*

Regrouping and summarizing what precedes, we obtain the theorem 1.5 given in the introduction.

3 Metric connection with parallel curvature.

We will wee in the next section that the hypothesis on the twistor bundle (or one of its admissible subbundles) to be reductively of type $\mathcal{G}_1$ (a fortiori to be of global type $\mathcal{G}_1$) implies that the curvature of the linear metric connection ∇ is parallel: $\nabla R = 0$. Therefore, we are led to study metric connections with parallel curvature.

3.1 Manifold with a 'scalar' curvature operator.

Lemma 3.1 *Let* (M,g) *be a Riemannian manifold endowed with a metric connection* ∇. *Suppose* $\dim M \geq 4$ *and that the curvature operator* R *of* ∇ *is given by*

$$R(X,Y)Z = k\left(\langle X,Z\rangle Y - \langle Y,Z\rangle X\right),$$

for some $k\in\mathcal{C}^\infty(M)$. *Then we have*

$$\begin{cases} T(X,Y) = \dfrac{1}{2k}\left((Y\cdot k)X - (X\cdot k)Y\right) & \text{on } \Omega = \{k\neq 0\},\\ dk = 0 \quad \text{on } \Omega^c. \end{cases}$$

In particular, k *is constant if and only if* ∇ *coincides with the Levi-Civita connection, and therefore* (M,g) *has a constant sectional curvature. Furthermore,* ∇ *is geodesically equivalent to Levi-Civita if and only if it coincides with Levi-Civita.*

Proof. Let us recall the Bianchi identities [21]:

$$\begin{cases} \mathfrak{S}\left\{R(X,Y)Z\right\} = \mathfrak{S}\left\{T(T(X,Y),Z) + (\nabla_X T)(Y,Z)\right\} \\ \mathfrak{S}\left\{(\nabla_X R)(Y,Z) + R(T(X,Y),Z)\right\} = 0 \end{cases}$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to X, Y and Z. Now, we suppose according to the hypothesis of the lemma that $R(X,Y)Z = k\left(\langle Z,X\rangle Y - \langle Z,Y\rangle X\right) = kR_0(X,Y)Z$ for some $k \in \mathcal{C}^\infty(M)$. Therefore the second Bianchi identity yields

$$(X\cdot k)R_0(Y,Z)+(Z\cdot k)R_0(X,Y)+(Y\cdot k)R_0(Z,X)+kR_0(T(X,Y),Z)+kR_0(T(Z,X),Y)+kR_0(T(Y,Z),X) = 0.$$

We take (X,Y,Z) orthonormal and we evaluate at $U = Z$:

$$-(X\cdot k)Y + (Y\cdot k)X - k\left(T(X,Y) + \langle T(X,Y),Z\rangle Z\right) + k\langle T(Z,X),Z\rangle Y + k\langle T(Y,Z),Z\rangle X = 0,$$

hence

$$kT(X,Y) = (Y\cdot k)X - (X\cdot k)Y + k\left(\langle T(X,Y),Z\rangle Z + k\langle T(Z,X),Z\rangle Y + \langle T(Y,Z),Z\rangle X\right) \quad (3.1)$$

Since the two hand sides of this equation are skew with respect to (X,Y), it follows that it holds for any $X,Y \in TM$ and $Z \in S(\{X,Y\}^\perp)$ (where $S(\{X,Y\}^\perp)$ denotes the sphere in $\{X,Y\}^\perp$). Now, let be $x \in M$ arbitrary. Then we have $kT(X,Y) = k\left[T(X,Y)\right]_{\mathrm{Span}(X,Y)} + k\left[T(X,Y)\right]_{\{X,Y\}^\perp}$, $\forall X,Y \in T_xM$. Therefore we have according to (3.1), either $k(x) = 0$ and then $dk(x) = 0$, or we have for all $Z \in S(\{X,Y\}^\perp)$,

$$\left[T(X,Y)\right]_{\{X,Y\}^\perp} \in \mathbb{R}Z, \forall Z \in S(\{X,Y\}^\perp)$$

i.e. $\left[T(X,Y)\right]_{\{X,Y\}^\perp} = 0$ (since $\dim M \geq 4$). Hence, in the open set $\Omega := \{k \neq 0\}$, we can write

$$T(X,Y) = \alpha(X,Y)X + \beta(X,Y)Y. \quad (3.2)$$

The coefficients $\alpha(X,Y)$ and $\beta(X,Y)$ are uniquely determined when (X,Y) is free. Let us prove that (3.2) implies the existence of some 1-form ω on Ω such that $T(X,Y) = \omega(Y)X - \omega(X)Y$. Since (3.2) is a pointwise algebraic identity, we can fix a point $x \in \Omega$ and therefore suppose that we are working on an Euclidean space $(E, \langle\cdot,\cdot\rangle)$ and dealing with a skew-symmetric E-valued 2-form $T \in \Lambda^2E^* \otimes E$ which satisfies (3.2). Then the skew-symmetry of T implies that

$$\beta(X,Y) = -\alpha(Y,X).$$

The bilinearity of T implies that

$$\begin{matrix} \alpha(X+X',Y) = \alpha(X,Y) = \alpha(X',Y) \quad \text{(i)} \\ \alpha(Y,X+X') = \alpha(Y,X) + \alpha(Y,X') \quad \text{(ii)} \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha(\lambda X,Y) = \alpha(X,Y) \quad \text{(iii)} \\ \alpha(X,\lambda Y) = \lambda\alpha(X,Y) \quad \text{(iv)} \end{matrix}$$

The equations (iii)-(iv) hold if (X,Y) is free whereas (i)-(iii) hold a priori when (X,X',Y) is free. Let us explain why these latters can be extended to any (X,X',Y). Indeed, according to (i), we have $\alpha(Z,Y) = \alpha(X,Y)$, $\forall Z \notin \mathrm{Span}(X,Y)$. It follows that $\alpha(\cdot,Y)$ is constant on $E \smallsetminus \mathbb{R}Y$, for any $Y \in E \smallsetminus \{0\}$, which allows us to set $\alpha(\mathbb{R}Y,Y) := \alpha(E \smallsetminus \mathbb{R}Y, Y)$. Remark that this is coherent with (iii). Therefore, we can set $\alpha(X,Y) =: \omega(Y)$, for all $X,Y \in E$. Now, the equations (ii) and (iv) tells us that ω is linear i.e $\omega \in E^*$. We have proven our assertion that

$$T(X,Y) = \omega(Y)X - \omega(X)Y, \quad \forall X,Y \in T\Omega,$$

for some 1-form ω on Ω. Now, let us come back to (3.1) which yields

$$kT(X,Y) = (Y\cdot k)X - (X\cdot k)Y + k\left(\omega(X)Y - \omega(Y)X\right),$$

hence

$$kT(X,Y) = \frac{1}{2}\left((Y\cdot k)X - (X\cdot k)Y\right).$$

This completes the proof. □

One can check that the first Bianchi identity is also satisfied by the couple (T,R) given by the lemma. Indeed, by a computation we check that

$$\mathfrak{S}\, T(T(X,Y),Z) = \frac{1}{4k^2}\,\mathfrak{S}\,(Z\cdot k)\left((Y\cdot k)X - (X\cdot k)Y\right) - \left([(Y\cdot k)X - (X\cdot k)Y]\cdot k\right)Z = 0,$$

and we have of course $\mathfrak{S}\, R(X,Y)Z = 0$. Moreover, we have

$$\mathfrak{S}\,\left(\nabla_X T\right)(Y,Z) = \mathfrak{S}\,\left(\nabla\omega(X,Z)Y - \nabla\omega(X,Y)Z\right) = \mathfrak{S}\,\left(\nabla\omega(Y,X)Z - \nabla\omega(X,Y)Z\right),$$

and

$$\begin{aligned}\nabla\omega(X,Y) - \nabla\omega(Y,X) &= X\cdot\omega(Y) - \omega(\nabla_X Y) - Y\cdot\omega(X) + \omega(\nabla_Y X)\\ &= d\omega(X,Y) + \omega([X,Y]) - \omega(\nabla_X Y - \nabla_Y X)\\ &= d\omega(X,Y) - \omega(T(X,Y)) = d\omega(X,Y). \quad (3.3)\end{aligned}$$

But $d\omega = d\left(\frac{dk}{2k}\right) = d\left(\frac{1}{2k}\right)\wedge dk + 0 = 0$. Therefore

$$\mathfrak{S}\,\left(\nabla_X T\right)(Y,Z) = 0.$$

3.2 Interpretation and corollaries of the second Bianchi identity for parallel curvatures.

Lemma 3.2 *Let ∇ be a linear connection on a manifold M. If $\nabla R = 0$, then the image $\operatorname{Im} R = R(\Lambda^2 TM)$ of the curvature operator $R\colon \Lambda^2 TM \to \operatorname{End}(TM)$ coincides with the holonomy Lie algebra $\operatorname{Im} R = \mathfrak{hol}(\nabla)$.* [7]
Let us now suppose that ∇ is a metric connection on a Riemannian manifold (M,g). Then the following statement are equivalent

(i) $R\colon \Lambda^2 TM \to \operatorname{End}(TM)$ *is injective,*

(ii) $R\colon \Lambda^2 TM \to \mathfrak{so}(TM) = \Lambda^2 TM$ *is bijective,*

(iii) $\operatorname{Hol}^0(\nabla) = SO(n)$,

(iii) $R \in \mathcal{C}(\mathbb{R}^* R_0)$, *where* $R_0(X\wedge Y) = \langle X,\cdot\rangle - \langle Y,\cdot\rangle X$.

Proof. Let $x\in M$ be a fixed point, we consider the holonomy group at this point, and we will denote the curvature operator R_x at this point simply by R.
The equation $\nabla R = 0$ implies that R is invariant by the group $H = \operatorname{Hol}(\nabla)$ i.e. $hR(X,Y)h^{-1} = R(hX,hY)$, $\forall h\in H$, $\forall X,Y\in T_xM$. Therefore we have $h\operatorname{Im} R\, h^{-1} = \operatorname{Im} R$ so that $[\mathfrak{h},\operatorname{Im} R]\subset \operatorname{Im} R$, where $\mathfrak{h} = \mathfrak{hol}(\nabla)$. Furthermore, $\mathfrak{h}$ is generated by $\{h^{-1}R(hX,hY)h, h\in H, X,Y\in T_xM\}$

[7] I.e. $\operatorname{Im} R_x = \mathfrak{hol}_x(\nabla)$, $\forall x\in M$.

which contains $\operatorname{Im} R$ and is included in $\operatorname{Ad}H(\operatorname{Im} R) = \operatorname{Im} R$, and therefore $\mathfrak{h}$ is generated by $\operatorname{Im} R$. Moreover we have $[\operatorname{Im} R, \operatorname{Im} R] \subset [\mathfrak{h}, \operatorname{Im} R] \subset \operatorname{Im} R$ so that $\operatorname{Im} R = \mathfrak{h}$.
The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious, and (iii) $\Rightarrow$ (i) follows from $\operatorname{Im} R = \mathfrak{hol}(\nabla)$. It is clear that (iv) implies (i) and (ii). Conversely, if (iii) holds then R_x is $SO(T_xM)$-invariant therefore $R_x = k\, R_{0x}$ for some $k \in \mathbb{R}^*$ because the set of fixed point of $\Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{so}(n)$, under the action of $SO(n)$, is $\mathbb{R}\, R_0$. This completes the proof. □

Lemma 3.3 *Let E be an vector space of finite dimension ≥ 4. Let us set $\mathcal{T}(E) = \Lambda^2 E^* \otimes E$. Then for each $T \in \mathcal{T}(E)$, we consider the map $\mathcal{S}(T) \colon \Lambda^3 E \to \Lambda^2 E$ defined by*

$$\mathcal{S}(T)(X \wedge Y \wedge Z) = T(X \wedge Y) \wedge Z + T(Y \wedge Z) \wedge X + T(Z \wedge X) \wedge Y.$$

Then the map $\mathcal{S} \colon T \in \mathcal{T}(E) \mapsto \mathcal{S}(T) \in \Lambda^3 E^* \otimes \Lambda^2 E$ is injective.

Proof. Let $\langle \cdot, \cdot \rangle$ be an Euclidean inner product in E. Then using the identification $R_0 \colon \Lambda^2 E \to \mathfrak{so}(E)$, the equation $\mathfrak{S}\, T(X \wedge Y) \wedge Z = 0$, $\forall X, Y, Z \in E$, is equivalent to $\mathfrak{S}\, R_0(T(X,Y), Z) = 0$, $\forall X, Y, Z \in E$. Proceeding exactly as in the proof of lemma 3.1, we obtain that $T = 0$ (or more simply just take $k = 1$ in the proof of lemma 3.1). This completes the proof. □

Lemma 3.4 *Let ∇ be a linear connection on a manifold M. If $\nabla R = 0$, then the second Bianchi identity means that*

$$R \circ \mathcal{S}(T) = 0, \quad \textit{i.e.}\ \operatorname{Im} \mathcal{S}(T) \subset \operatorname{Ker} R.$$

Theorem 3.1 *Let ∇ be a linear connection on M with parallel curvature $\nabla R = 0$. If the curvature operator $R \colon \Lambda^2 TM \to \operatorname{End}(TM)$ is injective then the torsion vanishes $T = 0$.*
In particular if ∇ is a metric connection with respect to some Riemannian metric g on M, then still under the hypothesis $\nabla R = 0$, the following statements are equivalent

(i) *R is injective,*

(ii) $\operatorname{Hol}^0(\nabla) = SO(n)$

(iii) *this metric connection is the Levi-Civita connection, $\nabla = \nabla^g$, and $\operatorname{Hol}^0(\nabla^g) = SO(n)$,*

(iv) *$\nabla = \nabla^g$ and M has a constant sectionnal curvature.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is given by the equivalence (i) $\Leftrightarrow$ (iii) in lemma 3.2. According to lemma 3.4, if R is parallel and injective, the second Bianchi identity yields $R \circ \mathcal{S}(T) = 0 \Rightarrow \mathcal{S}(T) = 0 \Rightarrow T = 0$, according to lemma 3.3. This proves that (i) $\Rightarrow$ (iii). Moreover (iii) $\Rightarrow$ (ii) is obvious. Finally the equivalence (iii) $\Leftrightarrow$ (iv) follows from the equivalence (iii) $\Leftrightarrow$ (iv) in lemma 3.2. This completes the proof. □

Proposition 3.1 *Let ∇ be a linear connection on M with parallel curvature $\nabla R = 0$. If $\mathcal{S}(T) \colon \Lambda^3 TM \to \Lambda^2 TM$ is surjective then $R = 0$. In other words, if $\nabla R = 0$ and $R_x \neq 0$, for some $x \in M$, then $\mathcal{S}(T)_x$ is not surjective.*

Let us compare the dimensions of $\Lambda^3 E$ and $\Lambda^2 E$. Since $\dim \Lambda^3 E = \dfrac{n(n-1)(n-2)}{6}$ and $\dim \Lambda^2 E = \dfrac{n(n-1)}{2}$ we then have that $\dim \Lambda^3 E \geq \dim \Lambda^2 E \Leftrightarrow n \geq 5$. This yields:

Proposition 3.2 *If $\dim M \leq 4$, then $\mathcal{S}(T)$ is not surjective. Therefore, the second Bianchi identity does not imposes algebraic obstruction to the existence of a metric connection with non vanishing torsion and non vanishing parallel curvature.*

3.3 Study of the $GL(E)$-invariant injective linear maps $\mathcal{S}\colon \Lambda^2E^*\otimes E \mapsto \Lambda^3E^*\otimes\Lambda^2E$.

We want to study the map $\mathcal{S}\colon T\mapsto\mathcal{S}(T)$ defined by lemma 3.3.

Let us compute the dimensions of the vector spaces in concerns: $\dim(\Lambda^2E^*\otimes E)=\dfrac{n(n-1)}{2}n=\dfrac{n^2(n-1)}{2}$ and $\dim(\Lambda^3E^*\otimes\Lambda^2E)=\dfrac{n(n-1)(n-2)}{6}\times\dfrac{n(n-1)}{2}=\dfrac{n^2(n-1)^2(n-2)}{12}$. We then verify that $\dim(\Lambda^2E^*\otimes E)\le\dim(\Lambda^3E^*\otimes\Lambda^2E)$ if and only if $n\ge 4$.

Now let $(e_i)_{1\le i\le n}$ be a basis of the vector space E. We endow canonically E^* with the dual basis and any tensor product of E and E^* (Λ^2E, Λ^3E^*, $\mathcal{T}(E)$ ...) with the corresponding canonical basis. Then we compute that

$$\mathcal{S}(e_i^*\wedge e_j^*\otimes e_k)(e_l\wedge e_p\wedge e_q)=\delta_{\{i,j\},\{l,p\}}e_k\wedge e_q+\delta_{\{i,j\},\{p,q\}}e_k\wedge e_l+\delta_{\{i,j\},\{q,l\}}e_k\wedge e_p,$$

i.e.

$$\begin{array}{rcll}\mathcal{S}(e_i^*\wedge e_j^*\otimes e_k)(e_i\wedge e_j\wedge e_q) &=& e_k\wedge e_q & \\ \mathcal{S}(e_i^*\wedge e_j^*\otimes e_k)(e_l\wedge e_p\wedge e_q) &=& 0 & \text{if } \{i,j\}\not\subseteq\{l,p,q\},\end{array}$$

that is to say

$$\mathcal{S}(e_i^*\wedge e_j^*\otimes e_k)=\sum_{q\neq i,j,k} e_i^*\wedge e_j^*\wedge e_q^*\otimes e_k\wedge e_q. \tag{3.4}$$

We will prove the following.

Theorem 3.2 *Let $n\ge 5$ be an integer and E an Euclidean space of dimension n.*
Let $\mathrm{L}^*(\Lambda^3E,\Lambda^2E)$ be the open set, in $\mathrm{L}(\Lambda^3E,\Lambda^2E)=\Lambda^3E^*\otimes\Lambda^2E$, of surjective maps $\Lambda^3E\to\Lambda^2E$. Then $\mathcal{U}(E):=\mathcal{S}^{-1}\left(\mathrm{L}^*(\Lambda^3E,\Lambda^2E)\right)$ is non empty and is therefore an open dense set in $\Lambda^2E^*\otimes E$.

Corollary 3.1 *Let M be a manifold of dimension $n\ge 5$. Let ∇ be a linear connection on M with parallel curvature $\nabla R=0$, and we still denote by T its torsion. The set $\{x\in M|T_x\in\mathcal{U}(T_xM)\}$ is an open set in M. In particular if there exists $x_0\in M$ such that $T_{x_0}\in\mathcal{U}(T_{x_0}M)$ then we have $R=0$ in all a neighbourhood of $x_0\in M$.*

Proof of the theorem. • First of all, we note that if $\mathcal{U}(E)$ is non empty then it is of course open and dense. Indeed, $\mathcal{S}\colon\mathcal{T}(E)\to\mathrm{L}(\Lambda^3E,\Lambda^2E)$ is injective therefore $\mathcal{S}^{-1}\colon\mathcal{S}(\mathcal{T})\to\mathcal{T}$ is a homeomorphism so that $\mathcal{S}^{-1}(\overline{\mathcal{B}})=\overline{\mathcal{S}^{-1}(\mathcal{B})}$ for any subset $\mathcal{B}\subset\mathcal{S}(\mathcal{T})$. In particular, we have [8]

$$\overline{\mathcal{U}}=\overline{\mathcal{S}^{-1}(\mathcal{S}(\mathcal{T})\cap\mathrm{L}^*(\Lambda^3,\Lambda^2))}=\mathcal{S}^{-1}(\overline{\mathcal{S}(\mathcal{T})\cap\mathrm{L}^*(\Lambda^3,\Lambda^2)})=\mathcal{S}^{-1}(\mathcal{S}(\mathcal{T}))=\mathcal{T}$$

because $\mathcal{S}(\mathcal{T})\cap\mathrm{L}^*(\Lambda^3,\Lambda^2))$ is dense in $\mathcal{S}(\mathcal{T})$ according to lemma 2.1.

• Now, let us prove by recurrence on $n=\dim E$, that there exists $T\in\mathrm{L}(\Lambda^2E,E)$ such that $\mathrm{rank}(\mathcal{S}(T))=\dim\Lambda^2E$. Let $(e_i)_{1\le i\le n}$ be a basis of E and let us set $T_{ijk}=e_i^*\wedge e_j^*\otimes e_k$, for $1\le i,j,k\le n$, $i<j$. Remark that any $T\in\Lambda^2E^*\otimes E$ is then written in the form

$$T=\sum_{(i<j,k)} a_{ijk}\,T_{ijk},$$

where of course $\sum_{(i<j,k)}$ means a sum on all the 3-tuples (ij,k) such that $i<j$.

[8]We have forgotten the symbol 'E' in $\mathcal{U}(E)$, Λ^kE ... to do not weigh down the equation.

a) If $\dim E = 5$, then $\dim \Lambda^3 E = \dim \Lambda^2 E = 10$. Consider the element

$$T = -T_{125} + T_{154} + T_{142} + T_{231} + T_{451}.$$

Then $\mathcal{S}(T) \in \mathrm{L}(\Lambda^3 E, \Lambda^2 E)$ is invertible. Indeed, we compute easily that its matrix w.r.t. the canonical basis is

$$\mathcal{M}at_{\wedge^3 e, \wedge^2 e}(\mathcal{S}(T)) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where $\wedge^3 e$ and $\wedge^2 e$ are respectively the canonical basis of $\Lambda^3 E$ and $\Lambda^2 E$. This proves the assertion for $n = 5$.

b) Let E be of dimension $n+1$. Then according to the recurrence hypothesis, there exists an element $T^n = \sum_{1 \le i,j,k \le n} a_{ijk} T_{ijk}$ such that the family $\mathcal{S}(T^n)(e_i \wedge e_j \wedge e_q)$, $1 \le i < j < q \le n$, generates the vector space $\Lambda^2 E_n$ where $E_n = \mathrm{span}(e_i, 1 \le i \le n)$. Moreover, let us remark that we have $\Lambda^2 E = \Lambda^2 E_n \oplus (E_n \wedge e_{n+1})$ and $\Lambda^3 E = \Lambda^3 E_n \oplus (\Lambda^2 E_n \wedge e_{n+1})$, so that

$$\Lambda^3 E^* \otimes \Lambda^2 E = \Lambda^3 E_n^* \otimes \Lambda^2 E_n \oplus \Lambda^3 E_n^* \otimes (E_n \wedge e_{n+1}) \oplus (\Lambda^2 E_n \wedge e_{n+1})^* \otimes \Lambda^2 E_n \oplus (\Lambda^2 E_n \wedge e_{n+1})^* \otimes (E_n \wedge e_{n+1}).$$

Let us consider the projection of $\mathcal{S}(T^n)$ on $(\Lambda^2 E_n \wedge e_{n+1})^* \otimes (E_n \wedge e_{n+1})$, according to the previous decomposition:

$$[\mathcal{S}(T^n)]_4 = \sum_{1 \le i,j,k \le n} a_{ijk}\, e_i^* \wedge e_j^* \wedge e_{n+1}^* \otimes e_k \wedge e_{n+1}, \tag{3.5}$$

where we have used (3.4).

Now, suppose that $T^n \colon \Lambda^2 E_n \to E_n$ is surjective, i.e. $\forall k$, $\exists\, 1 \le i < j \le n$ such that $a_{ijk} \neq 0$. Then we see, according to (3.5), that $\mathcal{S}(T^n)(\Lambda^2 E_n \wedge e_{n+1}) = E_n \wedge e_{n+1}$, so that $\mathcal{S}(T^n)(\Lambda^3 E) = \Lambda^2 E$, since we already have $\mathcal{S}(T^n)(\Lambda^3 E_n) = \Lambda^2 E_n$. Therefore it suffices to find $T^n \in \mathrm{L}(\Lambda^2 E_n, E_n)$ such that $T^n \colon \Lambda^2 E_n \to E_n$ and $\mathcal{S}(T^n) \colon \Lambda^3 E_n \to \Lambda^2 E_n$ are surjective. But since $\mathcal{U}(E_n)$ is dense in $\mathrm{L}(\Lambda^2 E_n, E_n)$ (by the recurrence hypothesis) and $\mathrm{L}^*(\Lambda^2 E_n, E_n)$ is an open set, then $\mathcal{U}(E_n) \cap \mathrm{L}^*(\Lambda^2 E_n, E_n) \neq \emptyset$. This completes the proof. □

3.4 Vectorial torsion and parallel curvature.

Let ∇ be a metric connection on (M, g) with a vectorial non vanishing torsion $T \in \mathcal{C}(\mathcal{T}_1(TM))$, i.e.

$$T(X,Y) = \alpha(X)Y - \alpha(Y)X$$

where $\alpha = \langle \xi, \cdot \rangle$, $\xi \in \mathcal{C}(TM)$. Let us set $F = \ker \alpha = \xi^\perp$, $L = \mathbb{R}\xi$ and $\xi = |\xi| e_1$. Then $T(\Lambda^2 F) = \{0\}$, so that $T(\Lambda^2 TM) = T(L \wedge F) = F$. Moreover we compute that $\mathrm{Im}\, \mathcal{S}(T) = \Lambda^2 F$. Therefore, if $\nabla R = 0$, the second Bianchi identity yields $\ker R \supset \Lambda^2 F$ (lemma 3.4). Hence

$(\ker R)^{\perp} \subset L \wedge F$. But since according to prop. 2.4, R is symmetric, we have $(\ker R)^{\perp} = \operatorname{Im} R$, therefore $\operatorname{Im} R$ is a subalgebra of $\mathfrak{so}(TM)$ (lemma 3.2) such that $\operatorname{Im} R \subset L \wedge F$. This implies that $\operatorname{Im} R$ must be of dimension 1 (if $R \neq 0$).
From now, we suppose that R does not vanish. Set $\operatorname{Im} R = \mathbb{R} L \wedge L_2$ where $L_2 \subset F$ is a $\mathbb{R}$-line. Let us set $L_2 = \mathbb{R} e_2$, where $|e_2| = 1$ (remark that e_2 is not unique since there is two choices: $\pm e_2$, so that we can only define locally a vector field e_2, whereas L_2 is well defined globally). Therefore we have $\operatorname{Im} R = \mathbb{R} e_1 \wedge e_2$. Moreover we obtain

$$R = \lambda R_P$$

where $P = L \oplus L_2$, $\lambda \in C^{\infty}(M)$ (does not vanish by hypothesis), and R_P is the element of $\Lambda^2 P \otimes \mathfrak{so}(P)$ corresponding to $\operatorname{Id}_{\Lambda^2 P}$ (via $\Lambda^2 P \otimes \mathfrak{so}(P) \cong \operatorname{End}(\Lambda^2 P)$). Moreover we see that $\lambda = \dfrac{|R|}{2}$. More concretely, we have

$$R(A) = \frac{\lambda}{4} \langle A, e_1 \wedge e_2 \rangle e_1 \wedge e_2, \quad \forall A \in \Lambda^2 TM.$$

Let us write the equation $\nabla R = 0$

$$\left(d\lambda \langle \cdot, e_1 \wedge e_2 \rangle + \lambda \langle \cdot, \nabla (e_1 \wedge e_2) \rangle \right) e_1 \wedge e_2 + \lambda \langle \cdot, e_1 \wedge e_2 \rangle \nabla (e_1 \wedge e_2) = 0$$

but $\nabla(e_1 \wedge e_2) \perp e_1 \wedge e_2$ so that the previous equation is equivalent to $d\lambda = 0$ and $\lambda \nabla(e_1 \wedge e_2) = 0$ i.e. $d\lambda = 0$ and $\nabla(e_1 \wedge e_2) = 0$ (since $\lambda \neq 0$). Therefore λ is constant (we suppose that M is connected).
Thus $\nabla e_1 \in \mathbb{R} e_2$ and $\nabla e_2 \in \mathbb{R} e_1$. In particular, $P = \operatorname{span}(e_1, e_2)$ is ∇-parallel and idem for $P^{\perp}$. Let $(e_3, \cdots, e_n)$ be a (local) orthonormal basis of $P^{\perp}$ (so that $(e_1, \cdots, e_n)$ is an orthonormal basis of TM). Moreover, we have $T(e_i, e_j) = \nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j] = \alpha(e_j) e_i - \alpha(e_i) e_j$ so that

$$\begin{array}{l}
[e_1, e_2] = \nabla_{e_1} e_2 - \nabla_{e_2} e_1 + |\xi| e_2 \in P, \quad \text{hence } [P, P] \subset P \\
[e_1, e_j] \in P^{\perp} \quad \text{if } j > 2, \text{ i.e. } [e_1, P^{\perp}] \subset P^{\perp} \\
[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i \in P^{\perp} \quad \text{if } i, j > 2, \text{ i.e. } [P^{\perp}, P^{\perp}] \subset P^{\perp} \\
[e_2, e_j] = \nabla_{e_2} e_j - \nabla_{e_j} e_2 \in \mathbb{R} e_1 \oplus P^{\perp} \quad \text{if } j > 2.
\end{array}$$

Theorem 3.3 *Let ∇ be a metric connection on (M, g) with a vectorial non vanishing torsion $T \in \mathcal{C}(\mathcal{T}_1(TM))$ and a parallel non vanishing curvature. Then there exists an orthogonal splitting $TM = P \oplus P^{\perp}$ such that the distributions P and $P^{\perp}$ are ∇-parallel and integrable. Moreover, $R = \lambda R_P$ where λ is a constant and R_P is the element of $\Lambda^2 P \otimes \mathfrak{so}(P)$ corresponding to* $\operatorname{Id}_{\Lambda^2 P}$ *(via $\Lambda^2 P \otimes \mathfrak{so}(P) \cong \operatorname{End}(\Lambda^2 P)$).*

Remark 3.1 • Metric connection ∇ with a curvature $R = 0$ are all obtained (locally) as follows: take a orthonormal basis $(e_1, \ldots, e_n)$ of (M, g) and set $\nabla e_i = 0$.
• One can prove that metric connection ∇ with a vectorial non vanishing torsion and a vanishing curvature $R = 0$ are all obtained (locally) as follows: take $(x_1, \ldots, x_n)$ a local system of coordinates of M and $k \in C^{\infty}(U, \mathbb{R}^*)$, then set $e_i = \dfrac{1}{k} \dfrac{\partial}{\partial x_i}$ and define the couple (g, ∇) as follows

$$g(e_i, e_j) = \delta_{ij}, \quad \text{and} \quad \nabla e_i = 0.$$

The proof consists on using the first Bianchi identity (and $R = 0$) to obtain that the 1-form $\alpha = \langle \xi, \cdot \rangle$ is closed then we set (locally) $\alpha = (dk)/k = d(\ln |k|)$. Then we compute that $[k e_i, k e_j] = 0$.

4 The Twistor space $\Sigma(M)$ of orthogonal almost complex structures.

The present section 4 is devoted to the proof of the theorem 4.1 below.

4.1 Statements of the results.

Recall that the twistor bundle $\Sigma^+(M)$ is by convention endowed with its standard fibre metric, i.e. the one induced by the canonical inner product B_0 in $\mathfrak{so}(\mathbb{R}^{2n})$ (equation (2.2)). Since any invariant metric on $\Sigma^+(\mathbb{R}^{2n})$ can be extended in an unique way to an invariant symmetric bilinear form on $\mathfrak{so}(2n)$ and since we know that these latter are exactly the elements of the line $\mathbb{R}B_0$, it follows that all the possible invariant metric on $\Sigma^+(\mathbb{R}^{2n})$ are the $k\langle\cdot,\cdot\rangle$, where $k \in \mathbb{R}^*$ and $\langle\cdot,\cdot\rangle$ is the standard invariant metric (induced by B_0). We denote by $\bar{h}_k$ the corresponding Kaluza-Klein metric $\bar{h}_k = \pi^* g + k\langle\phi,\phi\rangle_{\mathcal{V}}$. Throughout the present subsection 4.1, we allow all this possible invariant metrics. In the following, we will sometimes denote simply by h the Kaluza-Klein metric meaning that $h = \bar{h}_k$ for some $k \in \mathbb{R}^*$. Moreover, in the proofs, we will take $k = 1$, without loss of generality.

Theorem 4.1 *Let (M, g) be a Riemannian manifold of dimension $2n \geq 6$, endowed with a metric connection ∇. Let us consider the homogeneous fibre f-bundle $(\Sigma^+(M), \mathcal{F}, h)$ defined in §2.3.8. Then $(\Sigma^+(M), \mathcal{F}, h)$ is globally of type $\mathcal{G}_1$ if and only if $\nabla = \nabla^g$ and (M, g) has a constant non vanishing sectional curvature. More precisely, $(\Sigma^+(M), \mathcal{F}, \bar{h}_k)$ is globally of type $\mathcal{G}_1$ if and only if $\nabla = \nabla^g$ and (M, g) has a constant sectional curvature equal to $-\frac{k^{-1}}{2}$. In this case, $\Sigma^+(M)$ is a locally 4-symmetric space and the corresponding 4-symmetric fibration is $\Sigma^+(M) \to M$.*

This theorem will follow from the two following results.

Proposition 4.1 *Let (M, g) be a Riemannian manifold of even dimension, endowed with a metric connection ∇. Then $(\Sigma^+(M), \mathcal{F}, h)$ satisfies the condition that $\tilde{N}_{F|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ is skew-symmetric if and only if the curvature R of ∇ satisfies the following condition: for all $J \in \Sigma^+(M)$ and $V \in \mathcal{V}_J = \mathfrak{so}_-(TM, J)$,*

$$\langle R_J^{--}(X,Y), V\rangle = -\langle VX, Y\rangle \quad \forall X, Y \in T_xM.$$

Theorem 4.2 *If $n \geq 3$, let $R \in \mathfrak{so}(2n) \otimes \mathfrak{so}(2n)$ such that $\forall J \in \Sigma^+(\mathbb{R}^{2n})$, $R_J^{--} = 0$, then we have $R = 0$. In other words,*

$$\sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathfrak{so}_-(J) \otimes \mathfrak{so}_-(J) = \mathfrak{so}(2n) \otimes \mathfrak{so}(2n).$$

Proof of theorem 4.1. Let us suppose proven theorem 4.2 and proposition 4.1. Then under the hypothesis of theorem 4.1, the proposition 4.1 tells us, according to equations (2.9) and (2.1), that the curvature is such that $R + \frac{1}{2}R_0$ satisfies the hypothesis of theorem 4.2 and therefore $R = -\frac{1}{2}R_0$ which allows to conclude according to lemma 3.1.
Conversely, if (M, g) has a constant non vanishing sectional curvature then prop. 4.1 tells us that $(\Sigma^+(M), \mathcal{F}, h)$ is reductively of type $\mathcal{G}_1$. Moreover we have $T = T^g = 0$ then in particular T is a 3-form therefore $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally of type $\mathcal{G}_1$ (see [20, Th. 6.3.2] or theorem 5.1 below).

Finally the fact that $(\Sigma^+(M), \mathcal{F}, h)$ is locally 4-symmetric if (M, g) has a constant sectional curvature, is the content of proposition 7.19 in the Appendix. $\square$

Remark 4.1 If (M, g) has a constant sectional curvature, $\mathcal{F}$ coincides with the canonical f-structure of the locally 4-symmetric space $\Sigma^+(M)$. Moreover, if (M, g) has a sectional curvature equal to zero then $\Sigma^+(M)$ is not reductively of type $\mathcal{G}_1$.

Proof of proposition 4.1. According to [20, prop. 6.2.5 (iii)], $\tilde{N}_{F|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}$ is skew-symmetric if and only if

$$N_F(FX^h, Y^h, Z^v) + N_F(X^h, FY^h, Z^v) = N_F(Z^v, X^h, FY^h), \quad \forall X, Y, Z \in T\Sigma^+(M), \qquad (4.1)$$

which is equivalent to

$$\Phi(FH_1, H_2, V) + \Phi(H_1, FH_2, V) = N_F(V, H_1, FH_2) \quad \forall H_1, H_2 \in \mathcal{H}, V \in \mathcal{V},$$

since, according to proposition 2.10, we have $N_{F|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = \Phi$.
Furthermore, according to theorem 2.1, $\Phi(H_1, H_2) = \frac{1}{2} J\,[\pi^* R(H_1, H_2), J]$, $\forall J \in \Sigma^+(M), \forall H_1, H_2 \in \mathcal{H}_J$. Therefore (4.1) is equivalent to

$$\frac{1}{2}\langle J\,[R(X, JY) + R(JX, Y), J]\,, \phi(V)\rangle = N_F(V, \tilde{X}, F\tilde{Y}), \qquad \forall J \in \Sigma^+(M), \forall V \in \mathcal{V}_J,\ \forall X, Y \in T_xM,$$

where $\tilde{X}, \tilde{Y} \in (\pi^*TM)_J = \mathcal{H}_J$ are defined by $X = \pi_*\tilde{X}$, $Y = \pi_*\tilde{Y}$, and $x = \pi(J)$. But we have $N_F(V, \tilde{X}, \tilde{Y}) = \langle(\widetilde{D}^g_V \bar{J})\bar{J}\tilde{X}, \tilde{Y}\rangle = \langle(\widetilde{\nabla}_V \bar{J})\bar{J}\tilde{X}, \tilde{Y}\rangle$ according to proposition 2.20 (reductivity of $\Sigma^+(M)$). Hence the previous equation means

$$\frac{1}{2}\langle J\,[R(X, JY) + R(JX, Y), J]\,, \phi(V)\rangle = \langle(\nabla_V \bar{J})\bar{J}\tilde{X}, \bar{J}\tilde{Y}\rangle = -\langle(\nabla_V \bar{J})\tilde{X}, \tilde{Y}\rangle$$

i.e.

$$\langle R_J^{--}(X, Y), \phi(V)\rangle = -\frac{1}{2}\langle \bar{J}(\nabla_V \bar{J})\tilde{X}, \tilde{Y}\rangle.$$

Finally, according to theorem 2.2(i), we have $\frac{1}{2}\bar{J}(\nabla_V \bar{J}) = \phi(V)$, so that

$$\langle R_J^{--}(X, Y), V\rangle = -\langle VX, Y\rangle, \quad \forall J \in \Sigma^+(M), \forall V \in \mathcal{V}_J,\ \forall X, Y \in T_xM.$$

This completes the proof. $\square$

Remark 4.2 In fact, the proposition 4.1 and the theorem 4.2 imply more than what the theorem 4.1 says. Indeed, these imply that if $(\Sigma^+(M), \mathcal{F}, h)$ is reductively of type $\mathcal{G}_1$ (Def. 2.16) then $\nabla = \nabla^g$ and (M, g) has a constant sectional curvature $\neq 0$.

The 3 next subsections are devoted to the proof of theorem 4.2.

4.2 About the subspaces $\mathfrak{so}_-(J)$.

We prove easily the following properties.

Proposition 4.2 *If $n \geq 3$, the following identity holds*

$$\sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathfrak{so}_-(J) = \mathfrak{so}(\mathbb{R}^{2n}).$$

Proposition 4.3 *Let be an endomorphism $A \in \mathfrak{so}(\mathbb{R}^{2n})$. For all $\lambda \in \mathrm{Spec}(A) \setminus \{0\} \subset i\mathbb{R}$, set $\lambda_0 = -i\lambda \in \mathbb{R}$ and let $\mathfrak{m}_A(\lambda)$ be the real subspace such that $\mathfrak{m}_A(\lambda)^{\mathbb{C}} = \ker(A-\lambda\mathrm{Id}) \oplus \ker(A+\lambda\mathrm{Id})$. We denote by $J_A \in \Sigma(\mathbb{R}^{2n})$ a complex structure such that $A_{|\mathfrak{m}_A(\lambda)} = \lambda^0 J_{A|\mathfrak{m}_A(\lambda)}$, $\forall \lambda \in \mathrm{Spec}(A) \setminus \{0\}$. Then $A \in \mathfrak{so}(\mathbb{R}^{2n})$ belongs to some subspace $\mathfrak{so}_-(J)$ for some $J \in \Sigma^+(\mathbb{R}^{2n})$ if and only if the following conditions are satisfied*

- *for all $\lambda \in \mathrm{Spec}(A) \setminus \{0\}$, we have* $\dim \mathfrak{m}_A(\lambda) \in 4\mathbb{N}$.
- *if $0 \notin \mathrm{Spec}(A)$ and if for all $\lambda \in \mathrm{Spec}(A)$, $\dim \mathfrak{m}_A(\lambda) = 4$, then the complex structure J_A, which is then unique, is positive: $J_A \in \Sigma^+(\mathbb{R}^{2n})$.*

Proposition 4.4 *Let (e_i) be an orthonormal basis in $\mathbb{R}^{2n}$, $n \geq 3$.*
- *Then for any 4-tuple $\{i,j,k,l\}$, there exist $J, J' \in \Sigma^+(\mathbb{R}^{2n})$ such that the endomorphisms $(e_i \wedge e_j) + (e_k \wedge e_l)$ and $(e_i \wedge e_j) - (e_k \wedge e_l)$ belong respectively to the subspaces $\mathfrak{so}_-(J)$ and $\mathfrak{so}_-(J')$.*
- *Moreover, for any 6-tuple $\{i,j,k,l,p,q\}$, there exists $J, J' \in \Sigma^+(\mathbb{R}^{2n})$ such that $(e_i \wedge e_j) + (e_k \wedge e_l) \in \mathfrak{so}_-(J)$ and $e_p \wedge e_q \in \mathfrak{so}_+(J)$ whereas $(e_i \wedge e_j) - (e_k \wedge e_l) \in \mathfrak{so}_-(J')$ and $e_p \wedge e_q \in \mathfrak{so}_+(J')$.*

Proposition 4.5 *Let $J \in \Sigma^+(\mathbb{R}^{2n})$. Let $e_1, e_2 \in S^{2n-1}$ such that $e_2 \perp (e_1 \oplus Je_1)$ and set $E_i = e_i \oplus Je_i$ for $i = 1,2$. Then there exists $A \in \mathcal{S}_-(E_1) \oplus \mathcal{S}_-(E_2) \subset \mathcal{S}_-(E_1 \oplus E_2)$ and $B \in \mathfrak{so}_+(E_1 \oplus E_2)$ such that $\{A, B\} \neq 0$.*

4.3 $O(E)$-invariant irreducible decomposition of $\mathfrak{so}(E) \otimes \mathfrak{so}(E)$.

Let E be a vector space of finite dimension. When E is supposed to be Euclidean, we denote by g its metric. We still denote by g the corresponding element of $S^2(E)$ under the identification $S^2(E) \cong S^2(E^*)$ defined by the metric g. Our reference for the present subsection are [4, Chap. I] and [27, Chap. 4].

4.3.1 Equivariant maps.

Definition 4.1 *Let the* bianchi map *b be the following idempotent map of $\otimes^4 E$*

$$b(R)(X,Y,Z,T) = \frac{1}{3}\left(R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T)\right),$$

for any R in $\otimes^4 E$ and X, Y, Z, T in E^*.

The projector b is $GL(E)$-equivariant and maps $S^2(\Lambda^2 E)$ into itself so that we have the $GL(E)$-invariant decomposition

$$S^2(\Lambda^2 E) = \mathrm{Ker}\, b \oplus \mathrm{Im}\, b.$$

Moreover, we have $b(\otimes^2 \Lambda^2 E) \subset \Lambda^3 E \otimes E$.

Definition 4.2 • *We define the Ricci contraction as the $O(g)$-equivariant map $c\colon \otimes^4 E \to \otimes^2 E$ defined by*

$$c(R)(X,Y) = \mathrm{Tr} R(X,\cdot,Y,\cdot)$$

- *The Kulkarni-Nomizu product of the two 2-tensors $A, B \in \otimes^2 E$ is the 4-tensor $A \owedge B \in \otimes^4 E$ given by*

$$(A \owedge B)(X,Y,Z,T) = A(X,Z)B(Y,T) + A(Y,T)B(X,Z) - A(X,T)B(Y,Z) - A(Y,Z)B(X,T),$$

for any X, Y, Z, T in E^*. Remark that $A \owedge B \in \otimes^2 \Lambda^2 E$.

The Ricci contraction needs the choice of a metric contrary to the Kulkarni-Nomizu product. From now until the end of § 4.3.1, E is supposed to be Euclidean.

Lemma 4.1 *Under the identification* $\mathrm{End}(E) \cong \otimes^2 E$, *the Kulkarni-Nomizu product of two 2-tensors* $A, B \in \mathrm{End}(E)$ *is the 4-tensor* $A \owedge B \in \mathrm{End}(\Lambda^2 E)$ *given by*

$$A \owedge B(X \wedge Y) = AX \wedge BY + BX \wedge AY$$

Remark 4.3 It is important to keep in mind that under the identification $\mathrm{End}(E) \cong \otimes^2 E$, Id_E corresponds to g.

Lemma 4.2 *The following identities holds*

• $c(A \wedge B) = B^t A - A^t B$, $\forall A, B \in \mathrm{End}(E)$. *In particular,* $c(A \wedge B) = [A, B]$ *if* $A, B \in \mathfrak{so}(E)$ *and* $c(A \wedge B) = -[A, B]$ *if* $A, B \in \mathcal{S}(E)$.

• $c(A \odot B) = B^t A + A^t B$, $\forall A, B \in \mathrm{End}(E)$. *In particular,* $c(S^2(\mathrm{End}(E))) = \mathcal{S}(E)$, *and the restriction* $c\colon S^2(\Lambda^2 E) \to \mathcal{S}(E)$ *is given by* $c(A \odot B) = -(AB + BA)$.

• $c(A \owedge B) = \mathrm{Tr}(B)A + \mathrm{Tr}(A)B - (BA + AB)$, $\forall A, B \in \mathrm{End}(E)$. *In particular,* $c(\mathrm{Id} \owedge A) = (n-2)A + \mathrm{Tr}(A)\mathrm{Id}$, $\forall A \in \mathrm{End}(E)$.

Proposition 4.6 *Let* $A, B \in \mathrm{End}(E)$. *Then the following holds:*

a) $(A \owedge B)^t = A^t \owedge B^t$,

b) $A \owedge B = B \owedge A$,

c) $g \owedge g$ *corresponds to* $\mathrm{Id} \owedge \mathrm{Id} = 2\,\mathrm{Id}_{\Lambda^2 E} = 2\, R_0$ *(through the identification* $\mathrm{End}(\Lambda^2 E) = \otimes^2 \Lambda^2 E$*).*

d) $\mathrm{Id} \owedge \cdot : \mathrm{End}(E) \to \mathrm{End}(\Lambda^2 E)$ *is injective.*

Proposition 4.7 • *The map*

$$\begin{array}{ccc} A & \longmapsto & \dfrac{1}{n-2}\,\mathrm{Id} \owedge A \\ \mathfrak{so}(E) & \longrightarrow & \Lambda^2 \mathfrak{so}(E) \end{array}$$

is a section of $c_{|\Lambda^2 \mathfrak{so}(E)} = [\cdot, \cdot]\colon \Lambda^2 \mathfrak{so}(E) \to \mathfrak{so}(E)$ *whence the following* $O(E)$*-invariant decomposition*

$$\Lambda^2 \mathfrak{so}(E) = \ker[\cdot, \cdot] \oplus (\mathrm{Id} \owedge \mathfrak{so}(E)) .$$

• *The subspace* $\mathrm{Id} \owedge \mathcal{S}(E)$ *is a* $O(E)$*-invariant complement, in* $S^2(\Lambda^2 E)$, *of the kernel of the Ricci-contraction* $c\colon S^2(\Lambda^2 E) \to \mathcal{S}(E)$. *More precisely, the following* $O(E)$*-invariant decomposition holds*

$$S^2(\Lambda^2 E) = \mathbb{R}\,(\mathrm{Id} \owedge \mathrm{Id}) \oplus (\mathrm{Id} \owedge \mathcal{S}_0(E)) \oplus \ker c_{|S^2(\Lambda^2 E)} .$$

4.3.2 Irreducible decomposition of $S^2(\Lambda^2 E)$.

In the present subsection, we still denote by b and c resp. the restrictions of b and c to $S^2(\Lambda^2 E)$. Let us set $\mathcal{R}_b(E) = \mathrm{Ker}\, b$ (in $S^2(\Lambda^2 E)$).

Proposition 4.8 [4, Chap. I]

a) $A \owedge B \in \mathcal{R}_b(E)$, $\forall A, B \in \otimes^2 E$

b) $g \owedge \cdot : S^2(E) \to \mathcal{R}_b(E)$ *is injective.*

c) $b(\operatorname{Ker} c) \subset \operatorname{Ker} c$*.*

d) $\operatorname{Im} b = \Lambda^4 E$*.*

Proposition 4.9 [4, Chap. I] *If* $\dim E \geq 4$*, the* $O(g)$*-module* $\mathcal{R}_b(E)$ *has the following irreducible decomposition*

$$\mathcal{R}_b(E) = (\mathbb{R}g \owedge g) \oplus \big(g \owedge S^2_0(E)\big) \oplus (\operatorname{Ker} c \cap \ker b)\,.$$

Proposition 4.10 [12, Th. 19.2] *The standard representation of* $\mathfrak{so}(2n)$ *on* $\Lambda^k\mathbb{R}^{2n}$ *is irreducible for* $k = 1, 2, \dots, n-1$*. For* $k = n$*, it has two irreducible factors.*
The standard representation of $\mathfrak{so}(2n+1)$ *on* $\Lambda^k\mathbb{R}^{2n+1}$ *is irreducible for* $k = 1, 2, \dots, n$*.*

Combining what precedes we have

Proposition 4.11 *If* $\dim E \geq 9$*, the irreducible decomposition of the* $O(g)$*-module* $S^2(\Lambda^2 E)$ *is given by*

$$S^2(\Lambda^2 E) = (\mathbb{R}g \owedge g) \oplus \big(g \owedge S^2_0(E)\big) \oplus (\operatorname{Ker} c \cap \ker b) \oplus \Lambda^4 E.$$

4.3.3 Irreducible decomposition of $\Lambda^2(\Lambda^2 E)$.

Proposition 4.12 [27, Chap. 4] *The* $O(E)$*-invariant decomposition* $\Lambda^2\mathfrak{so}(E) = \ker[\cdot,\cdot] \oplus (\operatorname{Id} \owedge \mathfrak{so}(E))$ *is irreducible.*

Remark 4.4 We have $\ker b \cap \Lambda^2(\Lambda^2 E) = \{0\}$ so that $b\colon \Lambda^2(\Lambda^2 E) \to \Lambda^3 E \otimes E$ is injective. Therefore, since $b(\otimes^2\Lambda^2 E) = \Lambda^3 E \otimes E$, we have $\Lambda^3 E \otimes E \cong \ker c_{|\Lambda^2(\Lambda^2 E)} \oplus \Lambda^2 E \oplus \Lambda^4 E$.

4.3.4 Conclusion.

Combining what precedes we have

Proposition 4.13 *If* $\dim E \geq 9$*, the irreducible decomposition of the* $O(g)$*-module* $\mathfrak{so}(E) \otimes \mathfrak{so}(E)$ *is given by the last equality in the following successive splitting*

$$\begin{aligned}\mathfrak{so}(E) \otimes \mathfrak{so}(E) &= \Lambda^2\mathfrak{so}(E) \oplus S^2(\mathfrak{so}(E)) = \Lambda^2\mathfrak{so}(E) \oplus \operatorname{Ker} b_s \oplus \operatorname{Im} b_s \\ &= \ker[\cdot,\cdot] \oplus (\operatorname{Id} \owedge \mathfrak{so}(E)) \oplus (\mathbb{R}\operatorname{Id} \owedge \operatorname{Id}) \oplus \big(\operatorname{Id} \owedge \mathcal{S}^2_0(E)\big) \oplus (\operatorname{Ker} c_s \cap \ker b_s) \oplus \Lambda^4 E \end{aligned} \quad (4.2)$$

where b_s *and* c_s *are respectively the restrictions of* b *and* c *to* $S^2(\Lambda^2 E)$*.*

4.4 Proof of theorem 4.2.

If the present subsection, $E = \mathbb{R}^{2n}$ and (e_i) is an orthonormal basis of E.

The vector subspace $\mathfrak{so}^{\otimes}(\Sigma) := \sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathfrak{so}_-(J) \otimes \mathfrak{so}_-(J)$ is $SO(2n)$-invariant. Therefore it is a sum of irreducible subspaces and hence it suffices to prove that it has a non-trivial intersection with each irreducible factor of the decomposition (4.2) (or more simply that $\mathfrak{so}^{\otimes}(\Sigma)$ is not orthogonal to any irreducible factor of this decomposition).

• Let us remark that we have $\mathfrak{so}^{\otimes}(\Sigma) = \mathfrak{so}^{\odot}(\Sigma) \oplus \mathfrak{so}^{\wedge}(\Sigma)$ where we have set

$$\begin{aligned} \mathfrak{so}^{\odot}(\Sigma) &= \sum_{J\in\Sigma^+(\mathbb{R}^{2n})} \mathfrak{so}_-(J) \odot \mathfrak{so}_-(J) = S^2(\mathfrak{so}(E)) \cap \mathfrak{so}^{\otimes}(\Sigma) \\ \mathfrak{so}^{\wedge}(\Sigma) &= \sum_{J\in\Sigma^+(\mathbb{R}^{2n})} \mathfrak{so}_-(J) \wedge \mathfrak{so}_-(J) = \Lambda^2(\mathfrak{so}(E)) \cap \mathfrak{so}^{\otimes}(\Sigma). \end{aligned}$$

Moreover, each factor (the symmetric and the antisymmetric one) is non vanishing. Indeed, take any $J \in \Sigma^+(E)$ and choose $A, B \in \mathfrak{so}_-(J) \setminus \{0\}$ such that (A, B) is free. Then we have $A \odot A \in \mathfrak{so}^{\odot}(\Sigma) \setminus \{0\}$ and $A \wedge B \in \mathfrak{so}^{\wedge}(\Sigma) \setminus \{0\}$. Hence, $\mathfrak{so}^{\odot}(\Sigma)$ and $\mathfrak{so}^{\wedge}(\Sigma)$ are non vanishing and $SO(2n)$-invariant.

4.4.1 Proof of $\mathfrak{so}^{\odot}(\Sigma) = \odot^2\mathfrak{so}(E)$.

• $\mathrm{Id} \owedge \mathrm{Id} = 2\mathrm{Id}_{\mathfrak{so}(E)} = 2\sum_i v_i \otimes v_i$ where (v_i) is any orthonormal basis of $\mathfrak{so}(E)$. According to proposition 4.2, one can choose a basis such that each v_i is in some $\mathfrak{so}_-(J)$, for some $J \in \Sigma^+(E)$. Therefore $\mathrm{Id} \owedge \mathrm{Id} \in \mathfrak{so}^{\otimes}(\Sigma) \setminus \{0\}$.

• For any $A \in S^2(E)$, we have

$$(\mathrm{Id} \owedge A)\,(X \wedge Y) = X \wedge AY + AX \wedge Y$$

in other words $\mathrm{Id} \owedge A$ is nothing but the Lie algebra action of A on $\Lambda^2 E$. Therefore, setting $A_i = e_i^* \otimes e_i$, we have

$$\mathrm{Id} \owedge A_i = \sum_{k\neq i} (e_i \wedge e_k) \otimes (e_i \wedge e_k) = \sum_{k\neq i} A_{ik} \otimes A_{ik}$$

where we have set $A_{ij} = e_i \wedge e_j$. Fix i and let $k_0 \neq i$ and $j < l$ such that $\{j, l\} \cap \{i, k_0\} = \varnothing$. Now, set $Q_0 = (A_{ik_0} + A_{jl}) \otimes (A_{ik_0} + A_{jl})$, then according to proposition 4.4, there exist $J \in \Sigma^+(E)$ such that $A_{ik_0} + A_{jl} \in \mathfrak{so}_-(J)$, therefore $Q_0 \in \mathfrak{so}_-(J) \otimes \mathfrak{so}_-(J)$. Moreover, we have

$$\langle \mathrm{Id} \owedge A_i, Q_0 \rangle = 4$$

(remember that $\langle A_{ij}, A_{ij} \rangle = 2$), and since $\mathrm{Id} \owedge \mathrm{Id} = \sum_{i<j} A_{ij} \otimes A_{ij}$ then

$$\langle \mathrm{Id} \owedge \mathrm{Id}, Q_0 \rangle = 2 \times 4,$$

so that

$$\langle \mathrm{Id} \owedge A_i - \frac{1}{2n} \mathrm{Id} \owedge \mathrm{Id}, Q_0 \rangle = 4\left(1 - \frac{2}{2n}\right) = 4\left(1 - \frac{1}{n}\right) \neq 0 \quad \text{if } n \geq 2.$$

Therefore since $\mathrm{Id} \owedge A_i - \dfrac{1}{2n}\mathrm{Id} \owedge \mathrm{Id} \in \mathrm{Id} \owedge \mathcal{S}_0^2(E)$, we have proven that $\mathfrak{so}^{\otimes}(\Sigma)$ is not orthogonal to $\mathrm{Id} \owedge \mathcal{S}_0^2(E)$.

• Let $\{i, j, k, l\}$ be a subset of 4 elements in $[\![1, 2n]\!]$ and consider $Q = A_{ij} \odot A_{kl} \in S^2(\Lambda^2 E)$. Then we have $c(Q) = 0$. Moreover, remarking that

$$(A + B) \odot (A + B) - (A - B) \odot (A - B) = 4A \odot B, \quad \forall A, B \in \Lambda^2 E,$$

we conclude that $Q \in \mathfrak{so}^{\otimes}(\Sigma)$ (according to proposition 4.4) and more precisely we have $Q \in \mathfrak{so}^{\odot}(\Sigma)$. Now, let us consider $b(Q)$. Since it is a linear combinaison of elements of the form

$A_{i'j'} \odot A_{k'l'}$ for some sets of 4 integers i', j', k', l' we have $b(Q) \in \mathfrak{so}^{\odot}(\Sigma)$. Moreover, obviously $b(Q) \notin \{0, Q\}$ so that $b(Q) \in \mathfrak{so}^{\odot}(\Sigma) \cap \operatorname{Im} b \smallsetminus \{0\}$ and $Q - b(Q) \in \mathfrak{so}^{\odot}(\Sigma) \cap \operatorname{Ker} b \smallsetminus \{0\}$. Using proposition 4.8-c), we conlude that $Q - b(Q) \in \mathfrak{so}^{\odot}(\Sigma) \cap \operatorname{Ker} b \cap \operatorname{Ker} c \smallsetminus \{0\}$. We have proven that $\mathfrak{so}^{\odot}(\Sigma)$ intersects $\operatorname{Im} b \smallsetminus \{0\}$ and $\operatorname{Ker} b \cap \operatorname{Ker} c \smallsetminus \{0\}$. This completes the proof of the equality $\mathfrak{so}^{\odot}(\Sigma) = \odot^2 \mathfrak{so}(E)$, if $n \geq 5$ (according to prop. 4.11).

Now, if $n = 3$ or 4, we have to prove that $\mathfrak{so}^{\odot}(\Sigma) \supset \operatorname{Im} b = \Lambda^4 E$. In fact, if $n \geq 3$, we have $\operatorname{Im} b = \operatorname{span}\{b(A_{ij} \odot A_{kl}), |\{i, j, k, l\}| = 4\}$ since $b(A_{ij} \odot A_{kl}) = 0$ if $|\{i, j, k, l\}| \leq 3$. But we have seen that $b(A_{ij} \odot A_{kl}) \in \mathfrak{so}^{\odot}(\Sigma)$ if $|\{i, j, k, l\}| = 4$. Therefore, $\operatorname{Im}(b) \subset \mathfrak{so}^{\odot}(\Sigma)$ if $n \geq 3$. This completes the proof of the equality $\mathfrak{so}^{\odot}(\Sigma) = \odot^2 \mathfrak{so}(E)$.

4.4.2 Proof of $\mathfrak{so}^{\wedge}(\Sigma) = \Lambda^2 \mathfrak{so}(E)$.

Let $J \in \Sigma^+(E)$ and $A, B \in \mathfrak{so}_-(J)$ such that $[A, B] \neq 0$. That exists because $[\mathfrak{so}_-(J), \mathfrak{so}_-(J)] = \mathfrak{u}(J) \neq 0$ if $n \geq 3$ (and $[\mathfrak{so}_-(J), \mathfrak{so}_-(J)] = \mathfrak{su}(J) \neq 0$ if $n = 2$). Then $A \wedge B \in \mathfrak{so}^{\wedge}(\Sigma) \smallsetminus \{0\}$ is not in $\ker[\cdot, \cdot]$ and therefore $\mathfrak{so}^{\wedge}(\Sigma) \supset \operatorname{Id} \oslash \mathfrak{so}(E)$.
Now, the rank of the symmetric space $\Sigma^+(\mathbb{R}^{2n}) = SO(2n)/U(n)$ is $\left[\frac{n}{2}\right] \geq 2$ if $n \geq 4$, so that there exists $A, B \in \mathfrak{so}_-(J)$, non colinear such that $[A, B] = 0$. This proves that $\mathfrak{so}^{\wedge}(\Sigma) \supset \ker[\cdot, \cdot]$ if $n \geq 4$.
Then, if $n = 3$ we just have to prove that $\mathfrak{so}^{\wedge}(\Sigma) \neq \operatorname{Id} \oslash \mathfrak{so}(E)$ i.e. that there exists $J \in \Sigma^+(\mathbb{R}^6)$ such that $\Lambda^2 \mathfrak{so}_-(J) \not\subseteq \operatorname{Id} \oslash \mathfrak{so}(E)$, or in other words $\Lambda^2 \mathfrak{so}_-(J) \not\subseteq \operatorname{Id} \oslash [\mathfrak{so}_-(J), \mathfrak{so}_-(J)]$. Indeed, according to lemma 4.2 (see also prop. 4.7) we have

$$A \wedge B = \operatorname{Id} \oslash C \Rightarrow [A, B] = (n-2)C,$$

$\forall A, B, C \in \mathfrak{so}(E)$, and therefore $\Lambda^2 \mathfrak{so}_-(J) \not\subseteq \operatorname{Id} \oslash \mathfrak{so}(E) \iff \Lambda^2 \mathfrak{so}_-(J) \not\subseteq \operatorname{Id} \oslash [\mathfrak{so}_-(J), \mathfrak{so}_-(J)]$. But $\dim[\mathfrak{so}_-(J), \mathfrak{so}_-(J)] = \dim \mathfrak{u}(J) = 9$ and $\dim \Lambda^2 \mathfrak{so}_-(J) = 15$. Therefore $\Lambda^2 \mathfrak{so}_-(J) \not\subseteq \operatorname{Id} \oslash [\mathfrak{so}_-(J), \mathfrak{so}_-(J)]$. Finally we have proven that $\mathfrak{so}^{\wedge}(\Sigma) = \Lambda^2 \mathfrak{so}(E)$.

This completes the proof of theorem 4.2.

5 Complementary results about the horizontal type $\mathcal{G}_1$ and the parallelness of the torsion on $\Sigma(M)$, and towards the study of its subbundles.

In this subsection we want to characterize the manifolds (M, g, ∇) such that the associated twistor bundle has some particular properties like: respectively the horizontal type $\mathcal{G}_1$, the parallelness of the torsion of the paracharacteristic connection, and finally the pureness of the horizontal curvature. To do that we will need some algebraic results. To do not weight the presentation and the proofs of the main results of the present subsection we put all these algebraic tools and results with their own proofs, in the last part, § 5.4, of the current subsection.

The aim of the present study is to begin to understand what happens for admissible subbundles of $\Sigma^+(M)$. Indeed, in the case of $\Sigma^+(M)$, only one particular hypothesis suffices to impose strong constraints on (M, g) (like the fact to be reductively of type $\mathcal{G}_1$ implies that (M, g) has a constant sectional curvature, see § 4.1). Therefore, we want to understand what each particular property (among those listed above) implies on $\Sigma^+(M)$, with the aim to generalise the obtained results for admissible subbundles of $\Sigma^+(M)$.

A remark about the notations. In section 2.3, when we had a Riemannian submersion $\pi \colon (N, h) \to (M, g)$, a metric connection on N was denoted by ∇ whereas a metric connection on M was denoted by $\overline{\nabla}$ (because the first one was more often used that the second one). From now, and until the end of the paper, we will only consider the fibration of the twistor bundle $(\Sigma^+(M), \mathcal{F}, h)$ (or more generally $(\mathcal{Z}_{2k}^\alpha(M), \mathcal{F}, h)$) over the Riemannian manifold (M, g): in this situation, we will denote by ∇ a metric connection on (M, g) (the more often written one) whereas a metric f-connection on $(\Sigma^+(M), \mathcal{F}, h)$ (which will often be the paracharacteristic connection) will be denoted by $\hat{\nabla}$. These notations will anyway always be clearly precised.

5.1 Characterization of the horizontal type $\mathcal{G}_1$.

Let (M, g) be a Riemannian manifold endowed with a metric connection ∇. We already know that if the metric connection ∇ has a skew-symmetric torsion T, then $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally of type $\mathcal{G}_1$ (see [20, Th. 6.3.2] or the next theorem below). We want to prove the converse statement.
Let $\mathcal{T}(\mathbb{R}^{2n}) = \mathcal{T}' \oplus (\ker b)_0 \oplus \Lambda^3 E^*$ be the $SO(2n)$-irreducible decomposition of $\mathcal{T}(\mathbb{R}^{2n})$, recalled at lemma 2.2. Again b is the Bianchi projector: $b(T) = \frac{1}{3}\mathrm{Skew}(T)$.

Theorem 5.1 *Let (M, g) be a Riemannian manifold endowed with a metric connection ∇. Let us consider the homogeneous fibre f-bundle $(\Sigma^+(M), \mathcal{F}, h)$ defined in §2.3.8. Then $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally of type $\mathcal{G}_1$ if and only if the torsion T of ∇ satisfies $T \in \mathcal{C}\left(\Lambda^3 T^*M \oplus \mathcal{T}'\right)$.*
Moreover if the torsion T of ∇ is a 3-form, then $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally projectible and ∇ is then the canonical connection of M.

Proof. of theorem 5.1. According to propositions 2.21 and 2.22, we have the following

Lemma 5.1 *Let (M, g) be a Riemannian manifold endowed with a metric connection ∇. Then the homogeneous fibre f-bundle $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally of type $\mathcal{G}_1$ if and only if the torsion T of ∇ satisfies: for all $J \in \Sigma^+(M)$, $T_J^{0,2}$ is a 3-form.*
In particular, if T is a 3-form, then $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally of type $\mathcal{G}_1$, and horizontally projectible, ∇ being then the canonical connection of M.

Now, we are left with an algebraic problem. Using the irreducible decomposition of $\mathcal{T}(\mathbb{R}^{2n}) = \mathfrak{so}(2n) \otimes \mathbb{R}^{2n}$ recalled at lemma 2.2, we have:

Proposition 5.1 *Let $\mathcal{T}(\mathbb{R}^{2n}) = \mathcal{T}' \oplus (\ker b)_0 \oplus \Lambda^3 E^*$ be the $SO(2n)$-irreducible decomposition of $\mathcal{T}(\mathbb{R}^{2n})$, where b is the Bianchi projector: $b(T) = \frac{1}{3}\mathrm{Skew}(T)$.*
- *Let $T \in \mathcal{T}(\mathbb{R}^{2n}) = \mathfrak{so}(2n) \otimes \mathbb{R}^{2n}$ such that for all $J \in \Sigma^+(\mathbb{R}^{2n})$, $T_J^{0,2}$ is a 3-form, then $[T]_{(\ker b)_0} = 0$, i.e. $T \in \Lambda^3 E^* \oplus \mathcal{T}'$.*
- *Moreover, if $T \in \mathcal{T}(\mathbb{R}^{2n})$ satisfies $T_J^{0,2} = 0$, $\forall J \in \Sigma^+(\mathbb{R}^{2n})$, then $T \in \mathcal{T}'$.*

Proof. We want to prove that if $T \in \mathcal{T}$ satisfies $\forall J \in \Sigma^+(\mathbb{R}^{2n})$, $T_J^{0,2} \in \mathrm{Im}\, b = \Lambda^3 E^*$, then $T \in \mathrm{Im}\, b \oplus \mathcal{T}'$. We have $[T]_{\ker b} = T - b(T)$ and if $T_J^{0,2} \in \mathrm{Im}\, b$ then $T_J^{0,2} = b\left(T_J^{0,2}\right)$. Moreover, we have $b(T_J^{0,2}) = b(T)_J^{0,2}$, for any $T \in \mathcal{T}$ and $J \in \Sigma^+(\mathbb{R}^{2n})$, according to equation (2.6). Therefore if $T_J^{0,2} \in \mathrm{Im}\, b$, then $(T - b(T))_J^{0,2} = T_J^{0,2} - b(T)_J^{0,2} = b(T^{0,2}) - b(T)^{0,2} = 0$. In other words, if $\forall J \in \Sigma^+(\mathbb{R}^{2n})$, $T_J^{0,2} \in \mathrm{Im}\, b$, then $[T]_{\ker b} \in \left(\sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathcal{T}^{0,2}(J)\right)^\perp$. Furthermore the subspace $\sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathcal{T}^{0,2}(J)$ and its orthogonal are $SO(2n)$-invariant. Moreover, we see easily that for any $J \in \Sigma^+(\mathbb{R}^{2n})$, the subspace $\mathcal{T}_J^{0,2}$ intersects $\ker b \setminus \{0\}$ and $\mathrm{Im}\, b \setminus \{0\}$ non trivially. According to lemma 2.4, we have $\mathcal{T}_J^{0,2} \subset (\ker b)_0 \oplus \mathrm{Im}\, b$. Therefore $\sum_{J \in \Sigma^+(\mathbb{R}^{2n})} \mathcal{T}^{0,2}(J) = (\ker b)_0 \oplus \mathrm{Im}\, b$.

This completes the proof of prop. 5.1. □

Let us come back to the proof of theorem 5.1. Let us suppose that $(N, F, h) := (\Sigma^+(M), \mathcal{F}, h)$ is horizontally of type $\mathcal{G}_1$. According to lemma 5.1 and proposition 5.1, the torsion of ∇ is written $T = T_a + T_\xi$ where $T_a \in \mathcal{C}(\Lambda^3 T^* M)$ is a 3-form and $T_\xi \in \mathcal{C}(\mathcal{T}')$ i.e. $T_\xi(X, Y) = R_0(X \wedge Y)\xi$.
Now, let $\hat{\nabla}$ be a paracharacteristic connection on (N, F, h), or more generally any metric f-connection such that $\hat{T}_{|\mathcal{H}^3}$ is a 3-form (proposition 2.13). Let us consider the two following equations: (i) $\hat{\nabla} F_{|\mathcal{H}^3} = 0$, i.e. $\hat{\nabla}^{\mathcal{H}}_H \bar{J} = 0$, $\forall H \in \mathcal{H}$; (ii) $\nabla_H \bar{J} = 0$, $\forall H \in \mathcal{H}$ (prop. 2.22).
Since $\hat{\nabla}_{|\mathcal{H}^3} = D^h_{|\mathcal{H}^3} + \frac{1}{2}\hat{T}_{|\mathcal{H}^3}$, the first equation can be written

$$\widetilde{D^g_{|H}} \bar{J} + \frac{1}{2}\left(\hat{T}_{\mathcal{H}^3}(\cdot, \bar{J}\cdot, \cdot) + \hat{T}_{\mathcal{H}^3}(\cdot, \cdot, \bar{J}\cdot)\right) = 0 \tag{5.1}$$

where we have used the fact that $D^h_{|\mathcal{H}^3} = \widetilde{D^g}_{|\mathcal{H}^3}$ according to proposition 2.15.
Now using the fact that $\nabla = D^g + \mathrm{A}(T) = D^g + \mathrm{A}(T_a + T_\xi) = D^g + \frac{1}{2}T_a + A_\xi$, the second equation becomes

$$\widetilde{D^g_{|H}} \bar{J} + \frac{1}{2}\left(T_a(\cdot, \bar{J}\cdot, \cdot) + T_a(\cdot, \cdot, \bar{J}\cdot)\right) + A_\xi(\cdot, \bar{J}\cdot, \cdot) + A_\xi(\cdot, \cdot, \bar{J}\cdot)) = 0. \tag{5.2}$$

Therefore

$$-\widetilde{D^g_{|H}} \bar{J} = \hat{T}^{*,(2,0)+(0,2)}_{\mathcal{H}^3}(\cdot, \bar{J}\cdot) = \tilde{T}^{*,(2,0)+(0,2)}_a(\cdot, \bar{J}\cdot) + 2A^{*,(2,0)+(0,2)}_\xi(\cdot, \bar{J}\cdot).$$

Furthermore, we have $(\mathcal{T}')^{0,2} = 0$ according to lemma 2.4, therefore

$$\begin{cases} \hat{T}^{(0,2)}_{\mathcal{H}^3} = \tilde{T}^{0,2}_a, \\ \hat{T}^{*,(2,0)}_{\mathcal{H}^3} = \tilde{T}^{*,(2,0)}_a + 2A^{*,(2,0)}_\xi, \end{cases} \quad \forall J \in \Sigma^+(M) \tag{5.3}$$

Applying the operation Skew to the second equation yields

$$\hat{T}^{(2,0)+(1,1)}_{\mathcal{H}^3} = \tilde{T}^{(2,0)+(1,1)}_a + 2\,\mathrm{Skew}(A^{*,(2,0)}_\xi).$$

Finally, we obtain

$$\hat{T}_{\mathcal{H}^3} = \tilde{T}_a + 2\,\mathrm{Skew}(A^{*,(2,0)}_\xi) \tag{5.4}$$

Conversely, let us suppose that the torsion of ∇ is written $T = T_a + T_\xi$ where $T_a \in \mathcal{C}(\Lambda^3 T^* M)$ is a 3-form and $T_\xi \in \mathcal{C}(\mathcal{T}')$. We want to prove that (N, F, h) is horizontally of type $\mathcal{G}_1$. Let $\hat{\nabla}$ be any metric connection on (N, h) leaving invariant the splitting $TN = \mathcal{H} \oplus \mathcal{V}$ and such that $\hat{\nabla}_{|\mathcal{H}^3} = D^h_{|\mathcal{H}^3} + \frac{1}{2}\hat{T}_{|\mathcal{H}^3}$, where $\hat{T}_{|\mathcal{H}^3}$ is the horizontal 3-form defined by (5.4). Then $\hat{T}_{|\mathcal{H}^3}$ is nothing but the horizontal component of the torsion $\hat{T}$ of $\hat{\nabla}$. Moreover, by definition of $\hat{T}_{|\mathcal{H}^3}$ and since (5.2) always holds (because of prop. 2.22), then $\hat{T}_{|\mathcal{H}^3}$ satisfies (5.1) i.e. $\hat{\nabla} F_{|\mathcal{H}^3} = 0$. Therefore (N, F, h) is horizontally of type $\mathcal{G}_1$ according to proposition 2.13. This completes the proof of theorem 5.1. □

Remark 5.1 According to what precedes, when $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally $\mathcal{G}_1$ then its horizontal torsion 3-form $\hat{T}_{\mathcal{H}^3}$ is then given (in terms of the torsion T of ∇) by the equation (5.4).

About the new section of endomorphisms: Skew($\mathbf{A}_\xi^{*,(\mathbf{2},\mathbf{0})}$). Let us compute Skew($A_\xi^{*,(2,0)}$). One has $A_\xi^{*,(0,2)} = 0$ so that $A_\xi^{*,(2,0)} = A_\xi - A_\xi^{*,(1,1)} = A_\xi - \frac{1}{2}(A_\xi + A_\xi(\cdot, J\cdot, J\cdot)) = \frac{1}{2}(A_\xi - A_\xi(\cdot, J\cdot, J\cdot))$. Furthermore, we compute

$$\begin{aligned}
2\mathrm{Skew}\left(A_\xi^{*,(2,0)}\right)(X,Y,Z) &= \mathrm{Skew}(A_\xi - A_\xi(\cdot, J\cdot, J\cdot))(X,Y,Z) = -\mathrm{Skew}(A_\xi(\cdot, J\cdot, J\cdot))(X,Y,Z) \\
&= -A_\xi(X,JY,JZ) - A_\xi(Z,JX,JY) - A_\xi(Y,JZ,JX) = \\
-\langle X,JY\rangle\langle\xi,JZ\rangle + \langle\xi,JY\rangle\langle X,JZ\rangle &- \langle Z,JX\rangle\langle\xi,JY\rangle + \langle\xi,JX\rangle\langle Z,JY\rangle - \langle Y,JZ\rangle\langle V,JX\rangle + \langle\xi,JZ\rangle\langle Y,JX\rangle \\
&= 2\left(-\langle X,JY\rangle\langle\xi,JZ\rangle + \langle\xi,JY\rangle\langle X,JZ\rangle + \langle\xi,JX\rangle\langle Z,JY\rangle\right) \\
&= -2\,\mathfrak{S}\left(\langle X,JY\rangle\langle\xi,JZ\rangle\right) = 2\,\mathfrak{S}\left(\langle X,JZ\rangle\langle\xi,JY\rangle\right) \qquad (5.5)
\end{aligned}$$

5.2 Characterization of the parallelness of the torsion.

Theorem 5.2 *Suppose that the metric connection ∇ has a skew-symmetric torsion, so that in particular $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally $\mathcal{G}_1$. Let $\hat\nabla$ be the paracharacteristic connection on $(\Sigma^+(M), \mathcal{F}, h)$ then:*

- $\hat\nabla_\mathcal{H}\hat T = 0$ *if and only if* $\nabla T = 0$ *and* $\nabla R = 0$.
- $\hat\nabla_\mathcal{V}\hat T = 0$ *if and only if* ∇ *coincides with the Levi-Civita connection and* (M,g) *has a constant sectional curvature.*

Proof. We set $(N,F,h) := (\Sigma^+(M), \mathcal{F}, h)$. Since the torsion T of ∇ is skew-symmetric, then according to theorem 5.1, $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally $\mathcal{G}_1$ and $\hat T_{|\mathcal{H}^3} = \widetilde T$. Moreover, according to theorem 2.7, the torsion of the paracharacteristic connection is given by

$$\hat T = \hat T_{|\mathcal{H}^3} + \Phi - \frac{1}{2}\left(N_F(X^v,Y^h,Z^h) - N_F(Y^v,X^h,Z^h)\right) + \alpha$$

Furthermore, we can decompose $\hat\nabla = \nabla^\mathcal{H} \oplus \nabla^v$, where the vertical component ∇^v is then the canonical vertical connection of the homogeneous fibre bundle $\Sigma^+(M)$ and moreover $\nabla^\mathcal{H} \in \widetilde\nabla + \mathcal{C}(\mathcal{V}^* \otimes \mathfrak{so}(\mathcal{H}))$. Indeed we have

$$\left(\hat\nabla - \widetilde\nabla\right)_{|\mathcal{H}^3} = \left(\widetilde{D^g_{|H}} + \frac{1}{2}T_{\mathcal{H}^3}\right) - \left(\widetilde{D^g_{|H}} + \frac{1}{2}\tilde T\right) = 0. \qquad (5.6)$$

1) Now, since $\mathcal{H}$ and $\mathcal{V}$ are $\hat\nabla$-parallel, we have the following equivalence:

$$\hat\nabla_\mathcal{H}\hat T = 0 \Longleftrightarrow \left[\ \hat\nabla_\mathcal{H}\left(\hat T_{|\mathcal{H}^3}\right) = 0, \quad \hat\nabla_\mathcal{H}\left(\hat T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})}\right) = 0 \quad \text{and} \quad \hat\nabla_\mathcal{H}\left(\hat T_{|\mathcal{V}^3}\right) = 0\ \right].$$

1. a) Moreover, we have $\hat\nabla_H\left(\hat T_{|\mathcal{H}^3}\right) = \widetilde\nabla_H\widetilde T = \widetilde{\nabla_{\bar H}T}$, $\forall H \in \mathcal{H}$, and we have set $\bar H = \pi_* H$. Hence $\left[\hat\nabla_\mathcal{H}\hat T\right]_{|\mathcal{H}^3} = 0 \Leftrightarrow \nabla T = 0$.

1. b) Furthermore, using the abuse of notation consisting to denote by $A(X^v,Y^h,Z^h)$ the trilinear form $A_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}$ for any $A \in \mathcal{C}(\otimes^3 T^*N)$, we have

$$\hat T_{|\mathcal{S}(\mathcal{H}\times\mathcal{H}\times\mathcal{V})} = \Phi - \frac{1}{2}\left(N_F(X^v,Y^h,Z^h) - N_F(Y^v,X^h,Z^h)\right)$$

so that $\forall H \in \mathcal{H}$, $\hat{\nabla}_H \left(\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right) = \hat{\nabla}_H \Phi$. Then we compute

$$\begin{aligned}\left(\hat{\nabla}_H \Phi\right)(X^h, Y^h) &= \hat{\nabla}^v_H(\Phi(X^h,Y^h)) - \Phi(\hat{\nabla}^{\mathcal{H}}_H X^h, Y^h) - \Phi(X^h, \hat{\nabla}^{\mathcal{H}}_H Y^h) \\ &= \frac{1}{2}\bar{J}\left[\nabla_H(\Phi(X^h,Y^h)), \bar{J}\right] - \left(\Phi(\nabla_H X^h, Y^h) + \Phi(X^h, \nabla_H Y^h)\right),\end{aligned}$$

where we used the theorem 2.1-(iii) at the second line. Moreover

$$\frac{1}{2}\bar{J}\left[\nabla_H(\Phi(X^h,Y^h)), \bar{J}\right] = \nabla_H(\Phi(X^h,Y^h)) \tag{5.7}$$

because $\nabla_H \bar{J} = 0$, so that

$$\left(\hat{\nabla}_H \Phi\right)(X^h, Y^h) = (\nabla_H \Phi)(X^h, Y^h)$$

Furthermore, using again $\nabla_H \bar{J} = 0$,

$$\begin{aligned}\nabla_H \Phi &= \hat{\nabla}_H\left(\frac{1}{2}\bar{J}\left[\tilde{R}, \bar{J}\right]\right) = \frac{1}{2}\bar{J}\left[\nabla_H \tilde{R}, \bar{J}\right] \\ &= \frac{1}{2}\bar{J}\left[\widetilde{\nabla_{\bar{H}} R}, \bar{J}\right].\end{aligned} \tag{5.8}$$

so that

$$\hat{\nabla}_H \Phi = \left[\widetilde{\nabla_{\bar{H}} R}\right]_{\mathfrak{so}_-(\bar{J})}.$$

Therefore $\left[\hat{\nabla}_{\mathcal{H}} \hat{T}\right]_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = 0 \Leftrightarrow \nabla R = 0$, according to proposition 4.2.

1. c) Finally, still using the same abuse of notation, we have

$$\hat{\nabla}_{\mathcal{H}}\left(N_F(X^v, Y^h, Z^h)\right) = \hat{\nabla}_{\mathcal{H}}\left(\langle(\tilde{\nabla}_{X^v}\bar{J})\bar{J}Y^v, Z^h\rangle\right) = -2\hat{\nabla}_{\mathcal{H}}\langle\phi(X^v)Y^h, Z^h\rangle = 0,$$

where ϕ is the vertical projection defined in subsubsection 2.2.1 (and given in our case by theorem 2.1). We have used proposition 2.20 in the first equality . Moreover we have indeed $\hat{\nabla}_{\mathcal{H}}\phi = 0$ since $\mathcal{V}$ is $\hat{\nabla}$-parallel. This last point need in fact to be made more precise. We have to consider that ϕ is an element of $\mathcal{C}(T^*N \otimes \mathfrak{so}(\pi^*TM)) = \mathcal{C}(T^*N \otimes \mathfrak{so}(\mathcal{H}))$, and moreover since ϕ vanishes on $\mathcal{H}$, $\phi \in \mathcal{C}(\mathcal{V}^* \otimes \mathfrak{so}(\mathcal{H})) \subset \mathcal{C}(T^*N \otimes \mathfrak{so}(\mathcal{H}))$.[9] Therefore for all $A \in TN$, $\hat{\nabla}_A \phi \in \mathcal{C}(\mathcal{V}^* \otimes \mathfrak{so}(\mathcal{H}))$ because $\mathcal{V}$ and $\mathcal{H}$ are parallel.
Moreover we have:

$$\begin{aligned}\left(\hat{\nabla}_H \phi\right)(X) &= \hat{\nabla}^{\mathcal{H}}_H(\phi(X)) - \phi(\hat{\nabla}_H X) = \tilde{\nabla}_H(\phi(X^v)) - \nabla^c_H(\phi(X^v)) \\ &= \tilde{\nabla}_H(\phi(X^v)) - \frac{1}{2}\bar{J}\left[\tilde{\nabla}_H(\phi(X^v)), \bar{J}\right] = \left[\tilde{\nabla}_H(\phi(X^v))\right]_{\mathfrak{so}_+(\bar{J})} = 0,\end{aligned}$$

where $[\cdot]_{\mathfrak{so}_+(\bar{J})}$ is the projection on $\mathfrak{so}_+(\bar{J})$ along $\mathfrak{so}_-(\bar{J})$. At the last equality of the first line, we have used that by definition of ∇^v, we have $\phi(\hat{\nabla}^v X) = \nabla^c(\phi(X^v))$. Moreover, at the second line, the last term vanishes because since $\tilde{\nabla}\bar{J} = 0$, $\forall H \in \mathcal{H}$, then $\mathfrak{so}_\pm(\bar{J}) \subset \pi^*\mathfrak{so}(TM)$ are resp. ∇-parallel.

[9] In other words, we consider $\mathcal{V}$ as a subbundle of $(\mathfrak{so}(\mathcal{H}), \hat{\nabla}^{\mathcal{H}})$, more precisely $\mathcal{V} = \mathfrak{so}_-(\mathcal{H}, \bar{J})$.

We have proven the equivalence $\hat{\nabla}_{\mathcal{H}}\hat{T} = 0 \iff \begin{cases} \nabla T = 0 \\ \nabla R = 0 \end{cases}$. This completes the proof of the first point.

2) Let us now prove the second point. We have $\hat{\nabla} = D^h + \mathrm{A}(\hat{T})$. Then according to proposition 2.15, we have

$$\hat{\nabla}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} = \widetilde{D^g}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} + \frac{1}{2}\Phi(Y^h, Z^h, X^v) + \frac{1}{2}\left(\hat{T}(X^v, Y^h, Z^h) + \hat{T}(Z^h, X^v, Y^h) + \hat{T}(Z^h, Y^h, X^v)\right)$$

where we use the abuse of notation explained in the proof of the first point above. But we have $\Phi(Y^h, Z^h, X^v) = -\hat{T}(Z^h, Y^h, X^v)$ according to theorem 2.5. Therefore

$$\begin{aligned}\left(\hat{\nabla} - \widetilde{\nabla}\right)_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} &= \left(\hat{\nabla} - \widetilde{D^g}\right)_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} = \frac{1}{2}\left(\hat{T}(X^v, Y^h, Z^h) + \hat{T}(Z^h, X^v, Y^h)\right) \\ &= \frac{1}{4}\left(N_F(X^v, Y^h, Z^h) - N_F(X^v, Z^h, Y^h)\right) = \frac{1}{4}\left(\langle(\nabla_{X^v}\bar{J})\bar{J}Y^h, Z^h\rangle - \langle(\nabla_{X^v}\bar{J})\bar{J}Z^h, Y^h\rangle\right) \\ &= \frac{1}{2}\langle(\nabla_{X^v}\bar{J})\bar{J}Y^h, Z^h\rangle = -\langle\phi(X^v)\bar{Y}, \bar{Z}\rangle. \quad (5.9)\end{aligned}$$

We have used proposition 2.20 in the equality before the last, and theorem 2.2(i) in the last one. We have set $\bar{Y} = \pi_* Y$ and $\bar{Z} = \pi_* Z$.

2. a) Since we have $\forall V \in \mathcal{V}$, $\widetilde{\nabla}_V\tilde{T} = 0$, therefore the equation $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}^3}\right) = 0$ is equivalent to

$$T(V.X, Y, Z) + T(X, V.Y, Z) + T(X, Y, V.Z) = 0, \quad \forall V \in \mathcal{V}, \tag{5.10}$$

according to (5.9). In other words, for each $x \in TM$, we have: $T(V.X, Y, Z) + T(X, V.Y, Z) + T(X, Y, V.Z) = 0$, $\forall V \in \mathcal{V}_J = \mathfrak{so}_-(T_xM, J)$, $\forall J \in \Sigma^+(T_xM)$, $\forall X, Y, Z \in T_xM$. Therefore, according to proposition 4.2, we obtain

$$T(V.X, Y, Z) + T(X, V.Y, Z) + T(X, Y, V.Z) = 0, \quad \forall V \in \mathfrak{so}(TM).$$

This implies $T = 0$ according to proposition 5.3. Therefore $\nabla = \nabla^g$.

2. b) Furthermore, again since $\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = \Phi$, the equation $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right) = 0$ leads us to compute:

$$\begin{aligned}\hat{\nabla}_V\Phi(X, Y) &= \hat{\nabla}^v_V\left(\Phi(X, Y)\right) - \Phi(\hat{\nabla}^{\mathcal{H}}_V X, Y) - \Phi(X, \hat{\nabla}^{\mathcal{H}}_V Y) \\ &= \frac{1}{2}\bar{J}\left[\nabla_V\left(\Phi(X, Y)\right), \bar{J}\right] - \left(\Phi(\hat{\nabla}^{\mathcal{H}}_V X, Y) + \Phi(X, \hat{\nabla}^{\mathcal{H}}_V Y)\right),\end{aligned} \tag{5.11}$$

$\forall X, Y \in \mathcal{C}(\mathcal{H})$, $\forall V \in \mathcal{V}$. Moreover, according to (5.9), we have,

$$\hat{\nabla}^{\mathcal{H}}_V\left(\Phi(X, Y)\right) = \nabla_V\left(\Phi(X, Y)\right) - \left[\phi(V), \Phi(X, Y)\right],$$

therefore, because of $[\phi(V), \Phi(X, Y)] \in \mathfrak{so}_+(\bar{J})$, we obtain

$$\frac{1}{2}\bar{J}\left[\hat{\nabla}^{\mathcal{H}}_V\left(\Phi(X, Y)\right), \bar{J}\right] = \frac{1}{2}\bar{J}\left[\nabla_V\left(\Phi(X, Y)\right), \bar{J}\right],$$

and hence (5.11) becomes

$$\hat{\nabla}_V\Phi = \frac{1}{2}\bar{J}\left[\hat{\nabla}^{\mathcal{H}}_V\Phi, \bar{J}\right]. \tag{5.12}$$

Moreover, since $\hat{\nabla}^{\mathcal{H}}\bar{J}=0$, $\hat{\nabla}^{\mathcal{H}}_V\Phi$ takes values in $\mathcal{V}=\mathfrak{so}_-(\bar{J})\subset\mathfrak{so}(\mathcal{H})$, so that finally $\hat{\nabla}_V\Phi=\hat{\nabla}^{\mathcal{H}}_V\Phi$.
Now, we have to compute

$$\begin{aligned}\hat{\nabla}^{\mathcal{H}}_V\Phi &= \hat{\nabla}^{\mathcal{H}}_V\left(\frac{1}{2}\bar{J}\left[\tilde{R},\bar{J}\right]\right)=\frac{1}{2}\bar{J}\left[\hat{\nabla}^{\mathcal{H}}_V\tilde{R},\bar{J}\right]\\ &= \left[\hat{\nabla}^{\mathcal{H}}_V\tilde{R}\right]_{\mathfrak{so}_-(\bar{J})},\end{aligned}$$

where we used that $\hat{\nabla}^{\mathcal{H}}\bar{J}=0$ in the second equality in the first line. Besides, according to (5.9) we have $\hat{\nabla}^{\mathcal{H}}_V\tilde{R}=\tilde{\nabla}_V\tilde{R}-\phi(V)\cdot\tilde{R}=-\phi(V)\cdot\tilde{R}$. Finally, we obtain

$$\hat{\nabla}_V\Phi=-\left[\phi(V)\cdot\tilde{R}\right]_{\mathfrak{so}_-(\bar{J})}.$$

Therefore the equation $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right)=0$ is equivalent to

$$\left[V\cdot\tilde{R}\right]_{\mathfrak{so}_-(\bar{J})}=0,\quad\forall V\in\mathcal{V}=\mathfrak{so}_-(\bar{J}),$$

which implies $R=kR_0$ for some $k\in\mathcal{C}^\infty(M)$ according to proposition 5.5.

2. c) Now, let us consider the equation $\hat{\nabla}_{\mathcal{V}}\hat{T}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}=0$, i.e. $\hat{\nabla}_{\mathcal{V}}\phi=0$ (ϕ considered as an element of $\mathcal{C}(T^*N\otimes\mathfrak{so}(\mathcal{H}))$) by the same computations as in the part **1. c)** of the present proof. We will need the following lemma

Lemma 5.2 *The following equality holds*

$$\left[\nabla_A\left(\phi(X^v)\right)\right]_{\mathfrak{so}_+(\bar{J})}=\left[\phi(A^v),\phi(X^v)\right].$$

Proof. We will prove this equality by lifting it in the bundle of orthonormal frames $\mathcal{SO}(M)$. We use the notation of § 2.2. Then the covariant derivative $\nabla_A\left(\phi(X^v)\right)$ lifts into

$$D_A\left(\omega_{\mathfrak{p}}(X^v)\right)+[\omega(A),\omega_{\mathfrak{p}}(X^v)],$$

where $\mathfrak{p}=\mathfrak{so}_-(J_0)$ according to the definition of ϕ (§ 2.2.1). Therefore projecting on $\mathfrak{k}=\mathfrak{so}_+(J_0)$ along $\mathfrak{so}_-(J_0)=\mathfrak{p}$, we obtain

$$[\omega_{\mathfrak{p}}(A^v),\omega_{\mathfrak{p}}(X^v)].$$

This completes the proof of the lemma. □

Now, let us compute $\hat{\nabla}_{\mathcal{V}}\phi$. Since, we have $\hat{\nabla}^{\mathcal{H}}_V=\tilde{\nabla}_V-\phi(V)$, $\forall V\in\mathcal{V}$, we obtain

$$\begin{aligned}\left(\hat{\nabla}_V\phi\right)(X) &= \hat{\nabla}^{\mathcal{H}}_V\left[\phi(X)\right]-\phi(\hat{\nabla}_VX)=\tilde{\nabla}_V\left(\phi(X)\right)-\left[\phi(V),\phi(X)\right]-\nabla^c_V\left(\phi(X^v)\right)\\ &= \tilde{\nabla}_V\left(\phi(X)\right)-\frac{1}{2}\bar{J}\left[\tilde{\nabla}_V\left(\phi(X^v)\right),\bar{J}\right]-\left[\phi(V),\phi(X^v)\right]\\ &= \left[\tilde{\nabla}_V\left(\phi(X)\right)\right]_{\mathfrak{so}_+(\bar{J})}-\left[\phi(V),\phi(X^v)\right]=0,\end{aligned}$$

according to the lemma 5.2. Therefore, the condition $\hat{\nabla}_{\mathcal{V}}\hat{T}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}}=0$ is always satisfied. Remark that we did not used the fact that T is skew-symmetric in this **2.c)**. This completes the proof. □

Remark 5.2 We have in fact proven that if T is skew-symmetric, then the following holds:
(i) $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}^3}\right) = 0 \Leftrightarrow \nabla = \nabla^g$,
(ii) $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right) = 0 \Leftrightarrow R = kR_0$,
(iii) $\left[\hat{\nabla}_{\mathcal{H}}\hat{T}\right]_{|\mathcal{H}^3} = 0 \Leftrightarrow \nabla T = 0$,
(iv) $\left[\hat{\nabla}_{\mathcal{H}}\hat{T}\right]_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = 0 \Leftrightarrow \nabla R = 0$,
(v) $\hat{\nabla}\hat{T}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} = 0$ always holds.

Remark 5.3 If we suppose moreover that $(\Sigma^+(M), \mathcal{F}, h)$ is of global type $\mathcal{G}_1$ then the previous theorem holds also for the characteristic connection. Indeed if the twistor bundle is of global type $\mathcal{G}_1$, then the torsion of the characteristic connection is given by

$$\hat{T} = \hat{T}_{|\mathcal{H}^3} + \operatorname{Skew}(\Phi) + \alpha,$$

so that all the previous considerations still hold.

In fact we can improve the previous theorem by removing the hypothesis that ∇ has a skew-symmetric torsion.

Theorem 5.3 *Suppose that $(\Sigma^+(M), \mathcal{F}, h)$ is horizontally $\mathcal{G}_1$ (resp. of global type $\mathcal{G}_1$). Let $\hat{\nabla}$ be the canonical paracharacteristic (resp. characteristic) connection on $(\Sigma^+(M), \mathcal{F}, h)$ then:*

- $\hat{\nabla}_{\mathcal{H}}\hat{T} = 0$ *if and only if $\nabla T = 0$, $\nabla R = 0$ and T is skew-symmetric.*

- $\hat{\nabla}_{\mathcal{V}}\hat{T} = 0$ *if and only if ∇ coincides with the Levi-Civita connection and (M, g) has a constant sectional curvature.*

Proof. Now, $\hat{T}_{|\mathcal{H}^3}$ is not simply given by $\tilde{T}$ but, according to (5.4) by $\hat{T}_{|\mathcal{H}^3} = \tilde{T}_a + 2\operatorname{Skew}\left(A_\xi^{*,(2,0)}\right)$, where $T = T_a + T_\xi$, $T_a \in \mathcal{C}(\Lambda^3 T^*M)$ and $T_\xi \in \mathcal{C}(\mathcal{T}')$, according to theorem 5.1.

• *The second point.* First of all, we remark that in the proof of the second point in theorem 5.2, we used the fact that T is skew-symmetric only at the **2. a)**, i.e. for the study of the condition $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}^3}\right) = 0$. Therefore, to prove the second point of theorem 5.3, we only have to study this last equation. Instead of (5.10), this equation $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}^3}\right) = 0$ leads to

$$V \cdot T_a - 2\operatorname{Skew}\left(A_{V.\xi}^{*,(2,0)}\right) = 0, \forall V \in \mathcal{V}. \tag{5.13}$$

Indeed, to derive this equation, we only have to use that $2\operatorname{Skew}\left(A_\xi^{*,(2,0)}\right)(X,Y,Z) = 2\,\mathfrak{S}\left(\langle X, \bar{J}Z\rangle\langle \xi, \bar{J}Y\rangle\right)$ (equation (5.5)) and then to compute:

$$\begin{aligned}\hat{\nabla}_V \operatorname{Skew}\left(A_\xi^{*,(2,0)}\right) &= \mathfrak{S}\left(\langle X, \hat{\nabla}_V^{\mathcal{H}} \bar{J}Z\rangle\langle \xi, \bar{J}Y\rangle\right) + \mathfrak{S}\left(\langle X, \bar{J}Z\rangle\langle \hat{\nabla}_V^{\mathcal{H}} \xi, \bar{J}Y\rangle\right) + \mathfrak{S}\left(\langle X, \bar{J}Z\rangle\langle \xi, \hat{\nabla}_V^{\mathcal{H}} \bar{J}Y\rangle\right)\\ &= \mathfrak{S}\left(\langle X, \bar{J}Z\rangle\langle \hat{\nabla}_V^{\mathcal{H}} \xi, \bar{J}Y\rangle\right) = -\mathfrak{S}\left(\langle X, \bar{J}Z\rangle\langle \phi(V).\xi, \bar{J}Y\rangle\right) = -\operatorname{Skew}\left(A_{\phi(V).\xi}^{*,(2,0)}\right).\end{aligned}$$

In the second equality, we used $\hat{\nabla}^{\mathcal{H}}\bar{J} = 0$ because $\hat{\nabla}F = 0$, and in the third one, we used that $\hat{\nabla}_V^{\mathcal{H}} = \tilde{\nabla}_V - \phi(V)$, $\forall V \in \mathcal{V}$, according to equation (5.9). Moreover, this last equation also yields $\hat{\nabla}_V^{\mathcal{H}}\tilde{T}_a = \phi(V) \cdot \tilde{T}_a$, and finally we obtain (5.13). Now, this equation (5.13) leads us to prove the following:

Lemma 5.3 *Let E be a Euclidean space of dimension $2n \geq 6$. Consider the following map*

$$\begin{array}{rccl} \mathcal{A}\colon & \Sigma^+(E) \times E & \longrightarrow & \Lambda^3 E^* \\ & (J, \xi) & \longmapsto & \mathcal{A}^J_\xi := 2\,\mathrm{Skew}\left(A^{*,(2,0)}_\xi\right). \end{array}$$

Then we have: if $\dfrac{\partial A^J_\xi}{\partial J} = 0$ *(at one point (J,ξ)) then $\xi = 0$.*

Proof of the lemma. The equation $\dfrac{\partial A^J_\xi}{\partial J} = 0$ is equivalent to

$$\mathfrak{S}\langle X, WZ\rangle\langle \xi, JY\rangle + \langle X, JZ\rangle\langle \xi, WY\rangle = 0, \quad \forall W \in T_J\Sigma^+(E) = \mathfrak{so}_-(J), \forall X, Y, Z \in E,$$

i.e.

$$\begin{aligned}\langle X, WZ\rangle\langle \xi, JY\rangle + \langle Z, WY\rangle\langle \xi, JX\rangle + \langle Y, WX\rangle\langle \xi, JZ\rangle + \langle X, JZ\rangle\langle \xi, WY\rangle + \langle Z, JY\rangle\langle \xi, WX\rangle \\ + \langle Y, JX\rangle\langle \xi, WZ\rangle = 0, \qquad \forall W \in \mathfrak{so}_-(J), \forall X, Y, Z \in E.\end{aligned}$$

Equivalently,

$$-\langle \xi, JY\rangle WX + \langle \xi, JX\rangle WY + \langle X, WY\rangle J\xi - \langle \xi, WY\rangle JX + \langle \xi, WX\rangle JY + \langle X, JY\rangle W\xi = 0,$$

$\forall W \in \mathfrak{so}_-(J), \forall X, Y \in E$. Therefore,

$$\langle X, WY\rangle J\xi = 0, \qquad \forall W \in \mathfrak{so}_-(\{\xi, J\xi\}^\perp, J), \forall X, Y \in \{\xi, J\xi\}^\perp.$$

But since $\dim E = 2n \geq 6$, then $\dim\{\xi, J\xi\}^\perp \geq 4$ and there exists $W \in \mathfrak{so}_-(\{\xi, J\xi\}^\perp, J)$ such that $W \neq 0$, therefore there exists $X, Y \in \{\xi, J\xi\}^\perp$ such that $\langle X, WY\rangle \neq 0$, and finally we obtain $\xi = 0$. This completes the proof of the lemma. □

Let us come back to the equation (5.13) which can be written

$$V \cdot T_a = \mathcal{A}^J_{V.\xi}, \quad \forall V \in \mathfrak{so}_-(J), \forall J \in \Sigma^+(M). \tag{5.14}$$

Now, let us fix $x \in M$, set $E = T_xM$ and $f(J, V) = V \cdot T_a$, $\forall J \in \Sigma^+(E)$, $V \in \mathfrak{so}(E)$. Then we have $\dfrac{\partial A^J_{V.\xi}}{\partial J} = \dfrac{\partial f}{\partial J} = 0$ and the lemma implies that $V.\xi = 0$, and this holds $\forall V \in \mathfrak{so}_-(J), \forall J \in \Sigma^+(E)$. Therefore $\xi = 0$, since $\sum_{J \in \Sigma^+(E)} \mathfrak{so}_-(E, J) = \mathfrak{so}(E)$ (prop. 4.2). This leads to $V \cdot T_a = 0$, $\forall V \in \mathfrak{so}_-(J)$ (according to (5.14)), which implies that $T_a = 0$ since $\sum_{J \in \Sigma^+(E)} \mathfrak{so}_-(E, J) = \mathfrak{so}(E)$ and according to proposition 5.3.

Therefore, the equation $\hat{\nabla}_\mathcal{V}\left(\hat{T}_{|\mathcal{H}^3}\right) = 0$ is equivalent to $T = 0$. We obtain the same conclusion as in theorem 5.2 (part **2. a)** of the proof of this theorem). This completes the proof of the second point of the theorem 5.3.

• *The first point.* **1. a)** We have the following relation which generalizes (5.6)

$$\begin{aligned}\left(\hat{\nabla} - \tilde{\nabla}\right)_{|\mathcal{H}^3} &= \left(D_{|H^3} + \frac{1}{2}\hat{T}_{\mathcal{H}^3}\right) - \left(\widetilde{D^g_{|H^3}} + \frac{1}{2}\tilde{T}_a + \tilde{A}_\xi\right) \\ &= \frac{1}{2}\left(\tilde{T}_a + 2\,\mathrm{Skew}\left(A^{*,(2,0)}_\xi\right)\right) - \left(\frac{1}{2}\tilde{T}_a + \tilde{A}_\xi\right) \\ &= \mathrm{Skew}(A^{*,(2,0)}_\xi) - \tilde{A}_\xi =: A_{(\xi, \bar{J})}. \end{aligned} \tag{5.15}$$

Therefore, the equation $\hat{\nabla}_{\mathcal{H}}\hat{T}_{\mathcal{H}^3} = 0$ is equivalent to

$$\left(\tilde{\nabla}_X + A_{(\xi,\bar{J})}(X)\right)\left(\tilde{T}_a + 2\,\mathrm{Skew}\left(A_\xi^{*,(2,0)}\right)\right) = 0,$$

i.e. $\forall X \in T_xM$, $\forall J \in \Sigma^+(T_xM)$,

$$\nabla_X T_a + 2\,\mathrm{Skew}\left(A^{*,(2,0)}_{\nabla_X\xi}\right) + A_{(\xi,J)}(X)\cdot T_a + 2\,A_{(\xi,J)}(X)\cdot \mathrm{Skew}\left(A_\xi^{*,(2,0)}\right) = 0 \qquad (5.16)$$

1. b) We follow the same procedure as in the proof of theorem 5.2 (**1. b**). We have $\forall H \in \mathcal{H}$, $\hat{\nabla}_H\left(\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right) = \hat{\nabla}_H\Phi$. Then we compute: $\forall X, Y \in \mathcal{C}(\mathcal{H})$,

$$\begin{aligned}
\hat{\nabla}_H\Phi(X,Y) &= \hat{\nabla}^v_H(\Phi(X,Y)) - \Phi(\hat{\nabla}^{\mathcal{H}}_H X, Y) - \Phi(X, \hat{\nabla}^{\mathcal{H}}_H Y)\\
&= \frac{1}{2}\bar{J}\left[\nabla_H\left(\Phi(X,Y)\right), \bar{J}\right] - \left(\Phi(\nabla_H X, Y) + \Phi(X, \nabla_H Y)\right)\\
&\quad - \left(\Phi(A_{(\xi,\bar{J})}(\bar{H})X, Y) + \Phi(X, A_{(\xi,\bar{J})}(\bar{H})Y)\right)\\
&= \left(\nabla_H\Phi\right)(X,Y) - \left(\Phi(A_{(\xi,\bar{J})}(\bar{H})X, Y) + \Phi(X, A_{(\xi,\bar{J})}(\bar{H})Y)\right)
\end{aligned}$$

where we have used (5.7). We then obtain, according to (5.8),

$$\hat{\nabla}_H\Phi = \left[\widetilde{\nabla_{\bar{H}}R}\right]_{\mathfrak{so}_-(\bar{J})} - \left(\Phi(A_{(\xi,\bar{J})}(\bar{H})\cdot,\cdot) + \Phi(\cdot, A_{(\xi,\bar{J})}(\bar{H})\cdot)\right).$$

Therefore the equation $\left[\hat{\nabla}_{\mathcal{H}}\hat{T}\right]_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = 0$ is equivalent to

$$\left[\widetilde{\nabla_{\bar{H}}R}\right]_{\mathfrak{so}_-(\bar{J})} - \left[R(A_{(\xi,\bar{J})}(\bar{H})\cdot,\cdot) + R(\cdot, A_{(\xi,\bar{J})}(\bar{H})\cdot)\right]_{\mathfrak{so}_-(\bar{J})} = 0. \qquad (5.17)$$

1. c) Concerning the study of the equation: $\hat{\nabla}_{\mathcal{H}}\hat{T}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} = 0$, i.e. $\hat{\nabla}_{\mathcal{H}}\phi = 0$ (ϕ still considered as an element of $\mathcal{C}(T^*N \otimes \mathfrak{so}(\mathcal{H}))$), it will suffice to prove the following:

Lemma 5.4 *The following assertions are equivalent,*

(i) $\hat{\nabla}\phi = 0$,

(ii) $\hat{\nabla}_{\mathcal{H}}\phi = 0$,

(iii) $\forall x \in M$, $\exists J \in N_x = \Sigma^+(T_xM) \mid \hat{\nabla}_H\phi = 0$, $\forall H \in \mathcal{H}_J$,

(iv) *the torsion T of ∇ is skew-symmetric, i.e. $\xi = 0$.*

Proof. Remark that the equivalence between (i) and (ii) follows from the fact that $\hat{\nabla}_{\mathcal{V}}\phi = 0$ always holds according to the point **2. c** in the proof of theorem 5.2, but it will also be a direct consequence of the present proof.
According to equations (5.9) and (5.15), we have

$$\hat{\nabla}^{\mathcal{H}}_X = \tilde{\nabla}_X + A_{(\xi,\bar{J})}(X) - \phi(X).$$

Therefore proceeding as in the previous computations, we compute:

$$\begin{aligned}
\left(\hat{\nabla}_X\phi\right)(Y) &= \hat{\nabla}^{\mathcal{H}}_X\left[\phi(Y)\right] - \phi(\hat{\nabla}_X Y)\\
&= \nabla_X\left(\phi(Y)\right) + \left[A_{(\xi,\bar{J})}(X^h), \phi(Y^v)\right] - \left[\phi(X^v), \phi(Y^v)\right] - \frac{1}{2}\bar{J}\left[\nabla_X\phi(Y), \bar{J}\right]\\
&= \left[\nabla_X\phi(Y^v)\right]_{\mathfrak{so}_+(\bar{J})} - \left[\phi(X^v), \phi(Y^v)\right] + \left[A_{(\xi,\bar{J})}(X^h), \phi(Y^v)\right]\\
&= \left[A_{(\xi,\bar{J})}(X^h), \phi(Y^v)\right]
\end{aligned}$$

according to lemma 5.2. Therefore $\hat{\nabla}\phi = 0 \Leftrightarrow \hat{\nabla}_{\mathcal{H}}\phi = 0 \Leftrightarrow \left[A_{(\xi,J)}(X), V\right] = 0$, $\forall X \in TM$, $\forall J \in \Sigma^+(T_xM)$, $\forall V \in \mathfrak{so}_-(J)$ (where of course $x = \pi(X)$). Moreover, since $[\mathfrak{so}_-(J), \mathfrak{so}_-(J)] = \mathfrak{so}_+(J)$, we therefore obtain, according to the Jacobi identity,

$$\begin{aligned} \hat{\nabla}_{\mathcal{H}}\phi = 0 \quad &\Leftrightarrow \quad \left[A_{(\xi,J)}(X), \mathfrak{so}(T_xM))\right] = 0, \quad \forall X \in TM, \forall J \in \Sigma^+(T_xM) \\ &\Leftrightarrow \quad A_{(\xi,J)}(X) = 0, \quad \forall X \in TM, \forall J \in \Sigma^+(T_xM). \end{aligned}$$

And we conclude according to the following lemma

Lemma 5.5 *Let E be a Euclidean space of dimension $2n \geq 4$ and $J \in \Sigma^+(E)$. Then the linear space maps $\xi \in E \mapsto \mathrm{Skew}(A_\xi^{*,(2,0)})$ and $v \in E \mapsto A_{(\xi,J)} := \mathrm{Skew}(A_\xi^{*,(2,0)}) - A_\xi$ are injective.*

Proof. • According to $\mathrm{Skew}\left(A_\xi^{*,(2,0)}\right)(X,Y,Z) = \mathfrak{S}\left(\langle X, JZ\rangle\langle \xi, JY\rangle\right)$ (equation (5.5)), we have $A_{(\xi,J)} = 0$ if and only if

$$-\langle X, JY\rangle\langle \xi, JZ\rangle + \langle \xi, JY\rangle\langle X, JZ\rangle + \langle \xi, JX\rangle\langle Z, JY\rangle - \langle X, Y\rangle\langle \xi, Z\rangle + \langle \xi, Y\rangle\langle X, Z\rangle = 0,$$

for all $X, Y, Z \in E$, or equivalently

$$\langle X, JY\rangle J\xi - \langle \xi, JY\rangle JX + \langle \xi, JX\rangle JY - \langle X, Y\rangle\xi + \langle \xi, Y\rangle X = 0,$$

for all $X, Y \in E$. Therefore,

$$\forall X, Y \in \{\xi, J\xi\}^\perp, \qquad \langle X, JY\rangle J\xi - \langle X, Y\rangle\xi = 0$$

which implies that if $\dim E > 2$, then $\xi = 0$.
• In the same way, we have $\mathrm{Skew}(A_\xi^{*,(2,0)}) = 0$ if and only if

$$\langle X, JY\rangle J\xi - \langle \xi, JY\rangle JX + \langle \xi, JX\rangle JY,$$

for all $X, Y \in E$. Therefore, $\forall X, Y \perp J\xi$, $\langle X, JY\rangle J\xi = 0$, so that if $\dim E > 2$, then $\xi = 0$. This complete the proof of the lemma. □

Conclusion about the first point. According to **1. c** above , the equation $\hat{\nabla}_{\mathcal{H}}\hat{T} = 0$ implies that $\xi = 0$ i.e. the torsion T is skew-symmetric, so that **1. a** and **1. b** respectively tell us that the equation $\hat{\nabla}_{\mathcal{H}}\hat{T} = 0$ implies that $\nabla T = 0$ and $\nabla R = 0$ respectively. One could also directly apply theorem 5.2, since we know that T is skew-symmetric, according to **1. c**. This completes the proof of theorem 5.3. □

Remark 5.4 The points **1. a** and **1. b** have their own interest since these tell us to what are equivalent respectively the equations $\hat{\nabla}_{\mathcal{H}}\hat{T}_{|\mathcal{H}^3} = 0$ and $\hat{\nabla}_{\mathcal{H}}\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = 0$. Namely, these are equivalent to equations (5.16) and (5.17) respectively. Moreover, one can prove by arguments of (basic) algebraic geometry (like Zariski density) that equation (5.16) is in fact equivalent to $\nabla T = 0$ and $\xi = 0$, and that equation (5.17) is equivalent to $\nabla R = 0$ and $\xi = 0$.

Remark 5.5 We have in fact proven the following: (i) $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}^3}\right) = 0 \Leftrightarrow \nabla = \nabla^g$,
(ii) $\hat{\nabla}_{\mathcal{V}}\left(\hat{T}_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}}\right) = 0 \Leftrightarrow R = kR_0$
(iii) $\left[\hat{\nabla}_{\mathcal{H}}\hat{T}\right]_{|\mathcal{H}^3} = 0 \Leftrightarrow$ equation (5.16) $\Leftrightarrow$ (by arguments of algebraic geometry not developed here) $\nabla T = 0$ and $\xi = 0$,
(iv) $\left[\hat{\nabla}_{\mathcal{H}}\hat{T}\right]_{|\mathcal{H}\times\mathcal{H}\times\mathcal{V}} = 0 \Leftrightarrow$ equation (5.17) $\Leftrightarrow$ (by the same kind of arguments) $\nabla R = 0$ and $\xi = 0$,
(v) $\hat{\nabla}_{\mathcal{H}}\hat{T}_{|\mathcal{V}\times\mathcal{H}\times\mathcal{H}} = 0 \Leftrightarrow T$ is skew-symmetric.

5.3 Characterization of the pureness of the horizontal curvature Φ.

Proposition 5.2 *Let (M, g) be a Riemannian manifold endowed with a metric connection ∇. Let us consider the homogeneous fibre f-bundle $(\Sigma^+(M), \mathcal{F}, h)$. Its horizontal curvature Φ is pure if and only if we have*

$$R_J^{+-} = 0, \quad \forall J \in \Sigma^+(M).$$

In other words, Φ is pure if and only if for all $x \in M$, R_x is orthogonal to $\sum_{J\in\Sigma^+(T_xM)} \mathfrak{so}_+(J) \otimes \mathfrak{so}_-(J)$ in $\otimes^2\mathfrak{so}(T_xM)$.

Remark 5.6 Equivalently, Φ is pure if and only if for all $J \in \Sigma^+(M)$, $\mathfrak{so}_+(J)$ and $\mathfrak{so}_-(J)$ are "orthogonal" with respect to the bilinear form $R \in \mathcal{C}(\mathfrak{so}(TM)^* \otimes \mathfrak{so}(TM)^*)$.

Proof. According to definition 2.21, Φ is pure if and only if $\Phi(\bar{J}\cdot, \cdot) = \Phi(\cdot, \bar{J}\cdot)$ which means

$$\frac{1}{2}\bar{J}[R(X, \bar{J}Y), J] = \frac{1}{2}\bar{J}[R(\bar{J}X, Y), \bar{J}]$$

i.e. $R_J^{+-} = 0, \quad \forall J \in \Sigma^+(M)$. This completes the proof. □

Theorem 5.4 *Φ is pure if and only if $R = kR_0$ for some $k \in \mathcal{C}^\infty(M)$.*

Proof. This results immediately from the proposition 5.4. □

Remark 5.7 According to theorem 4.1, the fact that $(\Sigma^+(M), h, \mathcal{F})$ is reductively of type $\mathcal{G}_1$ implies that: it is of global type $\mathcal{G}_1$, Φ is pure; and if moreover (M, g) is complete, it implies that the characteristic connection has a parallel torsion, and that $(\Sigma^+(M), h, \mathcal{F})$ has a closed stringy structure (Def. 1.1). Indeed, according to [20, Th 6.4.4], any $(2k)$-symmetric space has a closed stringy structure.

5.4 Some algebraic tools.

Let E be an Euclidean vector space. We consider the linear actions of $\mathfrak{so}(E)$ on the different tensorial product of E and its dual E^*. In particular we consider the action of $\mathfrak{so}(E)$ respectively on $\mathcal{T}(E)$ and $\mathrm{End}(\mathfrak{so}(E))$: $\forall V \in \mathfrak{so}(E)$, $T \in \mathcal{T}(E)$, $R \in \mathrm{End}(\mathfrak{so}(E))$,

$$V \cdot T = V \cdot T(\cdot, \cdot) - T(V\cdot, \cdot) - T(V\cdot, \cdot) \qquad V \cdot R = [V, R(\cdot)] + R([\cdot, V]).$$

Remark that under our usual identification $\mathcal{T}(E) = \Lambda^2E^* \otimes E^* \subset \otimes^3E^*$, one has $V \cdot T = -\left(T(V\cdot, \cdot, \cdot) + T(\cdot, V\cdot, \cdot) + T(\cdot, \cdot, V\cdot)\right)$. In the same way, under the identification $\mathrm{End}(\mathfrak{so}(E)) = \Lambda^2E^* \otimes \mathfrak{so}(E) = \mathrm{L}(\Lambda E, \mathfrak{so}(E))$, one has $V \cdot R = [V, R(\cdot)] - (R(V\cdot, \cdot) + R(\cdot, V\cdot))$.
These actions are also the Lie algebra action corresponding of $\mathfrak{so}(E)$ corresponding to the following Lie group actions of $SO(E)$:

$$\begin{aligned} g \cdot T &= gT(g^{-1}\cdot, g^{-1}\cdot) = T(g^{-1}\cdot, g^{-1}\cdot, g^{-1}\cdot) \\ g \cdot R &= \mathrm{Ad}g \circ R \circ \mathrm{Ad}g^{-1} = \mathrm{Ad}g \circ R(g^{-1}\cdot, g^{-1}\cdot) \end{aligned}$$

Proposition 5.3 *If $T \in \mathcal{T}(E)$ satisfies $V \cdot T = 0$, $\forall V \in \mathfrak{so}(E)$ then $T = 0$.*

Proof. If $V \cdot T = 0$, $\forall V \in \mathfrak{so}(E)$, then $g \cdot T = T$, $\forall g \in SO(E)$, and according to the $SO(E)$-irreducible decomposition of $\mathcal{T}(E)$, $T = 0$ (the only fixed point being 0). □

Proposition 5.4 *Let E be a Euclidean space. Then $R \in \mathrm{End}(\mathfrak{so}(E))$ satisfies $R_J^{+-} = 0$, $\forall J \in \Sigma^+(E)$ if and only if $R \in \mathrm{Id} \owedge \mathrm{Id}$ i.e. $R = kR_0$ for some $k \in \mathbb{R}^*$.*

Proof. The vector space

$$\mathcal{E} = \sum_{J \in \Sigma^+(E)} \mathfrak{so}_+(J) \otimes \mathfrak{so}_-(J)$$

is $SO(2n)$-invariant. It is a matter of using the irreducible decomposition of $\otimes^2\mathfrak{so}(E)$ (equation (4.2)) and to proceed as in the proof of theorem 4.2. It is clear that $(R_0)_J^{+-} = 0, \forall J \in \Sigma^+(E)$, i.e. $\mathbb{R}R_0 = \mathbb{R}\mathrm{Id} \owedge \mathrm{Id} \perp \mathcal{E}$.
Moreover, since $\mathcal{E}$ is $SO(2n)$-invariant, it coincides with the direct sum of its respective projections on $\Lambda^2\mathfrak{so}(E)$ and $S^2\mathfrak{so}(E)$ (according to the splitting $\otimes^2\mathfrak{so}(E) = \Lambda^2\mathfrak{so}(E) \oplus S^2\mathfrak{so}(E)$). Hence

$$\begin{aligned} \mathcal{E} &= \sum_{J \in \Sigma^+(E)} \mathfrak{so}_+(J) \wedge \mathfrak{so}_-(J) \oplus \mathfrak{so}_+(J) \odot \mathfrak{so}_-(J) \\ &= \sum_{J \in \Sigma^+(E)} \mathfrak{so}_+(J) \otimes \mathfrak{so}_-(J) \oplus \mathfrak{so}_-(J) \otimes \mathfrak{so}_+(J). \end{aligned}$$

For any $A, B \in \mathrm{End}(E)$, $A \owedge B$ considered as a element of $\mathrm{End}(\mathfrak{so}(E))$ (i.e. when identified with $R_0 A \owedge B R_0^{-1}$) satisfies: $A \owedge B(V) = BVA^t + AVB^t$, $\forall V \in \mathfrak{so}(E)$. In particular, we have

$$\mathrm{Id} \owedge B(V) = BV + VB^t = \begin{cases} [B, V] & \text{if} \quad B \in \mathfrak{so}(E) \\ \{B, V\} & \text{if} \quad B \in \mathcal{S}(E) \end{cases}$$

Therefore:
• If $A \in \mathfrak{so}(E)$, we have

$$\begin{aligned} \langle \mathrm{Id} \owedge A, \mathfrak{so}_+(J) \wedge \mathfrak{so}_-(J) \rangle = \langle \mathrm{Id} \owedge A(\mathfrak{so}_+(J)), \mathfrak{so}_-(J) \rangle &= \langle [A, \mathfrak{so}_+(J)], \mathfrak{so}_-(J) \rangle \\ &= \langle [A_-, \mathfrak{so}_+(J)], \mathfrak{so}_-(J) \rangle \neq 0 \end{aligned}$$

if A satisfies $[A_-, \mathfrak{so}_+(J)] \neq 0$ which holds if $A_- \neq 0$ since then $[A_-, J] \neq 0$.
• If $A \in \mathcal{S}(E)$, we have

$$\begin{aligned} \langle \mathrm{Id} \owedge A, \mathfrak{so}_+(J) \odot \mathfrak{so}_-(J) \rangle = \langle \mathrm{Id} \owedge A(\mathfrak{so}_+(J)), \mathfrak{so}_-(J) \rangle &= \langle \{A, \mathfrak{so}_+(J)\}, \mathfrak{so}_-(J) \rangle \\ &= \langle \{A_-, \mathfrak{so}_+(J)\}, \mathfrak{so}_-(J) \rangle \neq 0 \end{aligned}$$

if A satisfies $\{A_-, \mathfrak{so}_+(J)\} \neq 0$ which holds for some $A \in \mathcal{S}(E)$ (if $n \geq 2$) according to the proposition 4.5. Moreover, we remark that $\forall A \in \mathcal{S}(E)$, $A_- = (A_0)_- = (A_-)_0 \in \mathcal{S}_0(E)$, where the index "0" denote the component in $\mathcal{S}_0(E)$ according to the decomposition $\mathcal{S}(E) = \mathcal{S}_0(E) \oplus \mathbb{R}\mathrm{Id}_E$.
Therefore we have proven that $\mathcal{E} \supset \mathrm{Id} \owedge \mathfrak{so}(E) \oplus \mathrm{Id} \owedge \mathcal{S}_0(E)$.
Furthermore, we also have $\ker[\cdot,\cdot]_{\Lambda^2\mathfrak{so}(E)} \subset \mathcal{E}$. Indeed, according to the proposition 4.4, there exists $U \in \mathfrak{so}_+(J) \smallsetminus \{0\}, V \in \mathfrak{so}_+(J) \smallsetminus \{0\}$ such that $[U, V] = 0$, i.e. $\ker[\cdot,\cdot]_{\Lambda^2\mathfrak{so}(E)} \cap \mathcal{E} \neq \{0\}$.
Finally let us prove that $\mathcal{E} \supset \ker b_s \cap \ker c_s \oplus \mathrm{Im}\, b_s$, with the notation of proposition 4.13. Let $\{i, j, k, l, p, q\}$ be a subset of 6 elements in $[\![1, 2n]\!]$. Then $A_{ij} \odot (A_{kl} + A_{pq}) \in \mathfrak{so}_+(J) \odot \mathfrak{so}_-(J)$ for some $J \in \Sigma^+(E)$ (proposition 4.4), $(\mathrm{Id} - b)((A_{ij} \odot A_{kl}) \in \ker b_s \cap \ker c_s$ and $b((A_{ij} \odot A_{kl}) \in \mathrm{Im}\, b_s = \Lambda^4 E$. Moreover we have

$$\langle A_{ij} \odot (A_{kl} + A_{pq}), (\mathrm{Id} - b)(A_{ij} \odot A_{kl}) \rangle = \frac{2}{3} |A_{ij} \odot A_{kl}|^2 \neq 0$$

and

$$\langle A_{ij} \odot (A_{kl} + A_{pq}), b(A_{ij} \odot A_{kl})\rangle = \frac{1}{3}|A_{ij} \odot A_{kl}|^2 \neq 0.$$

Therefore $\mathcal{E} \supset \ker b_s \cap \ker c_s$ and if $n \geq 5$, $\mathcal{E} \supset \operatorname{Im} b_s$ (prop. 4.11). It remains to prove that $\mathcal{E} \supset \operatorname{Im} b_s$ if $n = 3$ or 4. We know that if $n \geq 3$, $\operatorname{Im} b_s = \operatorname{span}\{b(A_{ij} \odot A_{kl}), |\{i,j,k,l\}| = 4\}$ (see the end of § 4.4.1) and moreover we have $b(A_{ij} \odot A_{kl}) \in \mathcal{E}$. Indeed, it suffices to remark that

$$2A_{ij} \odot A_{kl} = (A_{ij} + A_{pq}) \odot A_{kl} + (A_{ij} - A_{pq}) \odot A_{kl}$$

and then we conclude by using proposition 4.4.
We have proven that $\mathcal{E} = \operatorname{Id} \otimes \mathfrak{so}(E) \oplus \operatorname{Id} \otimes \mathcal{S}_0(E) \oplus \ker c_s \cap \ker b_s \oplus \operatorname{Im} b_s$, therefore $\mathcal{E}^\perp = \mathbb{R}\operatorname{Id} \otimes \operatorname{Id}$. This completes the proof.

Proposition 5.5 *Let E be an Euclidean space of even dimension $2n$, $n \geq 3$. If $R \in \operatorname{End}(\mathfrak{so}(E))$ satisfies*

$$[V \cdot R, J] = 0, \qquad \forall V \in \mathfrak{so}_-(J), \forall J \in \Sigma^+(E),$$

then $R \in \mathbb{R}R_0$.

Proof. It suffice to proceed as in the previous proposition or as in the proof of theorem 4.2 (§ 4.4) by using again the irreducible decomposition of $\otimes^2 \mathfrak{so}(E)$. □

6 Subbundles of the Twistor space $\Sigma(M)$.

6.1 Adapted structure Lie groups and associated subbundles.

6.1.1 Twistor Algebraic identities.

In the present subsection, E is an Euclidean vector space of dimension $2n$.

Definition 6.1 *Let $H \subset SO(E)$ be a compact Lie group, and let $\mathfrak{h}$ be its Lie algebra. Let us suppose that there exists some orthogonal complex structure $J_0 \in \Sigma^+(E)$ such that $J_0 H J_0^{-1} = H$: we say that J_0 is adapted to H. Then, we consider the compact symmetric submanifold of complex structures defined by the H-orbit of J_0:*

$$S_H(J_0) = \operatorname{Ad}(H)J_0 = \{hJ_0h^{-1}, h \in H\} = H/K, \tag{6.1}$$

where $K = \operatorname{Stab}_H(J_0) = \mathbb{U}(E, J_0) \cap H$. We will then say that H satisfies the (adapted) twistor 2-tensorial algebraic identity (w.r.t. J_0) if

$$\mathfrak{h} \otimes \mathfrak{h} = \sum_{J \in S_H(J_0)} \mathfrak{h}_-(J) \otimes \mathfrak{h}_-(J). \tag{6.2}$$

where $\mathfrak{h}_-(J) = \{A \in \mathfrak{h} | AJ + JA = 0\}$. In particular, we will call respectively the symmetric and the skew-symmetric twistor 2-tensorial algebraic identities, the following respectively

$$\mathfrak{h} \wedge \mathfrak{h} = \sum_{J \in S_H(J_0)} \mathfrak{h}_-(J) \wedge \mathfrak{h}_-(J) \tag{6.3}$$

$$\mathfrak{h} \odot \mathfrak{h} = \sum_{J \in S_H(J_0)} \mathfrak{h}_-(J) \odot \mathfrak{h}_-(J). \tag{6.4}$$

Definition 6.2 *Let us consider the same situation as in the previous definition. Then we will say that H satisfies the twistor torsional algebraic identity if*

$$\operatorname{Ker} b \cap \left(\sum_{J \in S(H)} \mathcal{T}^{0,2}(J) \right)^{\perp} = \mathcal{T}', \tag{6.5}$$

where $b\colon \mathcal{T}(E) = \Lambda^2 E \otimes E \to \Lambda^3 E$ is the Bianchi projector: $b(T) = \frac{1}{3}\operatorname{Skew}(T)$.

Definition 6.3 *Let us consider the same situation as in the previous definitions. But we suppose that J_0 is not adapted to H : $J_0 H J_0^{-1} \neq H$ and we set $\mathfrak{h}_-(J) = J[\mathfrak{h}, J]$ for any $J \in S_H(J_0)$. We then say that J_0 is unadapted to H and the identities (6.2,6.3-6.4,6.5) will inherit the term* unadapted*: e.g. unadapted twistor 2-tensorial algebraic identity as concerns* (6.2)*. Moreover, if we do not specify the terms adapted/undapted, it means either that we do not know a priori if J_0 is adapted or not to H, or that this is already clear from the context. Remark that in general the orbit $S_H(J_0)$ defined in* (6.1) *is not symmetric.*

Remark 6.1 Let us remark that according to theorem 4.2 and proposition 5.1, $SO(2n)$ satisfies the twistor 2-tensorial algebraic identity as well as the twistor torsional algebraic identity.

Proposition 6.1 *Let (M, g) be a Riemannian manifold of dimension $2n$, endowed with a metric H-structure Q and a connection on Q, i.e. a (metric) H-connection ∇. Let $J_0 \in \Sigma^+(\mathbb{R}^{2n})$. Let $(N, \bar{F}, \bar{h})$ be the admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ defined by these datas.*
• If the torsion T of ∇ is a 3-form, then the subbundle $(N, \bar{F}, \bar{h})$ is horizontally of type $\mathcal{G}_1$. Moreover the horizontal torsion 3-form $\hat{T}_{|\mathcal{H}^3}$ (def. 2.18) is given by $\hat{T}_{|\mathcal{H}^3} = \widetilde{T}$.
• If H satisfies the twistor torsional algebraic identity, then the subbundle $(N, \bar{F}, \bar{h})$ is horizontally of type $\mathcal{G}_1$ if and only if the torsion T of ∇ is a 3-form.

6.1.2 Adapted subbundles.

6.2 Statements of the results.

Proposition 6.2 *Let (M, g) be a Riemannian manifold of dimension $2n$, endowed with a metric H-structure Q and a connection on Q, i.e. a (metric) H-connection ∇. Let $J_0 \in \Sigma^+(\mathbb{R}^{2n})$. Let $(N, \bar{F}, \bar{h})$ be the admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ defined by these datas. If this subbundle $(N, \bar{F}, \bar{h})$ is reductively of type $\mathcal{G}_1$ then so is the holonomy twistor subbundle $(\hat{N}, \hat{F}, \hat{h})$, where the metric $\hat{h}$ is induced by $\bar{h}$.*
Furthermore, if there exists an admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ which is reductively of type $\mathcal{G}_1$ then it is unique and it coincides with the holonomy twistor subbundle. Moreover, the metric $\bar{h} = \hat{h}$ is then unique and determined by the curvature operator of the connection.

The proof will use the following:

Lemma 6.1 *In the situation described by proposition 6.2, the subbundle $(N, \bar{F}, \bar{h})$ is reductively of type $\mathcal{G}_1$ if and only if we have for all $J \in N \subset \Sigma^+(M)$ and $V \in \mathcal{V}^N_J$,*

$$\langle R^{--}_J(X, Y), V \rangle_{\bar{h}} = -\langle VX, Y \rangle_g \quad \forall X, Y \in T_x M, \tag{6.6}$$

where of course $\mathcal{V}^N$ is the vertical subbundle of N.

Proof of the lemma. It follows immediately from the proof of proposition 4.1 by using the fact that $i * \Phi^N = \Phi^\Sigma$ and $i^* N_{\mathcal{F}} = N_{\bar{F}}$ (see [20, Lemma 6.2.2]) where $i\colon N \to \Sigma^+(M)$ is the inclusion, and Φ^N and Φ^Σ are resp. the horizontal curvature of N and $\Sigma^+(M)$. $\square$

Proof of proposition 6.2. Let $x_0 \in M$ be a reference point and denote simply by R the curvature operator R_{x_0} at x_0. Since $\operatorname{Im} R \subset \hat{\mathfrak{h}} \subset \mathfrak{h}$, the lemma yields $\frac{1}{2}J[R(\mathfrak{so}_-(J)), J] = \mathfrak{h}_-(J)$ so that

$$\frac{1}{2}J[\hat{\mathfrak{h}}, J] = \mathfrak{h}_-(J), \quad \forall J \in N_{x_0} = S_H(J_0)$$

since $\operatorname{Im} R \subset \hat{\mathfrak{h}} \subset \mathfrak{h}$. Therefore the two homogeneous reductive space $\hat{H}/\hat{K}$ and H/K (where $K = U(J_0) \cap H$ and $\hat{K} = U(J_0) \cap \hat{H}$) have the same tangent space at the reference point $o = J_0$ (or in other words the same reductive summand $\mathfrak{p} = \hat{\mathfrak{h}}_-(J_0) = \mathfrak{h}_-(J_0)$). More generally, any $\hat{H}$-orbit of an element $J \in S_H(J_0)$, and in particular $\hat{H}/\hat{K}$, has the same tangent space at each of its elements, as $S_H(J_0) = H/K$. Therefore any such orbit, and in particular $\hat{H}/\hat{K}$, is open in $S_H(J_0) = H/K$, but then any such orbit is also closed and $S_H(J_0)$ is connected. Hence any such orbit coincides with $S_H(J_0)$, and in particular $\hat{H}/\hat{K} = H/K$.
The fact that the metric is determined by R follows immediately from the lemma.This completes the proof. $\square$

Theorem 6.1 *Let (M, g) be a Riemannian manifold of dimension $2n$, endowed with a metric H-structure Q and a connection on Q, i.e. a (metric) H-connection ∇. Let $J_0 \in \Sigma^+(\mathbb{R}^{2n})$. Let us suppose that H satisfies the twistor 2-tensorial algebraic identity (Def. 6.1). Let $(N, \bar{F}, \bar{h})$ be the admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ defined by the previous datas. Then $(N, \bar{F}, \bar{h})$ is reductively of type $\mathcal{G}_1$ if and only if the curvature R of ∇ is given by*

$$R = b_{\mathfrak{h}(M)}^{-1} \circ \operatorname{pr}_{\mathfrak{h}(M)} \tag{6.7}$$

where $\operatorname{pr}_{\mathfrak{h}(M)}\colon \mathfrak{so}(TM) \to \mathfrak{h}(TM)$ is the orthogonal projection and $b_{\mathfrak{h}(M)} \in \mathcal{C}(S^2(\mathfrak{h}(M)))$ is the symmetric endomorphism defined by the Kaluza-Klein metric on $\mathfrak{h}(M)$ induced by $\bar{h}$.
In particular, in this case we have $\nabla R = 0$ and H coincides with the holonomy group $H = \hat{H}$. Moreover the horizontal curvature of $(N, \bar{F}, \bar{h})$ is pure.

Proof. The assertion concerning the curvature operator follows from the lemma 6.1, the definition of the twistor 2-tensorial algebraic identity (Def. 6.1), and the consideration about the Kaluza-Klein presented at the end of §2.2.1 and in the paragraph 'About the metric' at §2.2.2. The other assertions are immediate consequences of the equation (6.7). This completes the proof. $\square$

According to lemma 3.2, the previous theorem means that an admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ with a structure group H satisfying the twistor 2-tensorial algebraic identity, is then reductively of type $\mathcal{G}_1$ if and only if it is defined by a connection ∇ with a parallel symmetric curvature operator: $R\colon \Lambda^2 TM \to \Lambda^2 TM$, $\nabla R = 0$ and $R \in \mathcal{C}\left(S^2(\Lambda^2 TM)\right)$.
Therefore, according to proposition 6.1, we have

Theorem 6.2 *An admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ with a structure group H satisfying the twistor 2-tensorial algebraic identity, is of global type $\mathcal{G}_1$ if and only if it is defined by a connection ∇ with a skew-symmetric torsion and a parallel symmetric curvature operator.*

One can also generalize theorem 5.2 to admissible subbundles

Theorem 6.3 *Let (M, g) be a Riemannian manifold of dimension $2n$, endowed with a metric H-structure Q and a connection on Q, i.e. a (metric) H-connection ∇. Let $J_0 \in \Sigma^+(\mathbb{R}^{2n})$. Let us suppose that H satisfies the twistor 2-tensorial algebraic identity (Def. 6.1) and the twistor torsional algebraic identity (Def. 6.2). Let $(N, \bar{F}, \bar{h})$ be the admissible subbundle of $(\Sigma^+(M), \mathcal{F}, h)$ defined by the previous datas. Then if $(N, \bar{F}, \bar{h})$ is of horizontal (resp. global) type $\mathcal{G}_1$, then its paracharacteristic (resp. characteristic) connection has a parallel torsion if and only if the connection ∇ (which already has a skew-symmetric torsion) satisfies*

$$\nabla T = 0 \quad \textit{and } \nabla R = 0.$$

Therefore, (M, g) is locally homogeneous. Moreover, if (M, g) is supposed to be complete then ∇ is its canonical connection. Furthermore N is a locally homogeneous space and the paracharacteristic (resp. characteristic) connection is complete, i.e. its universal cover is a reductive homogeneous $\tilde{N} = G/K$, and the paracharacteristic (resp. characteristic) connection coincides with the canonical connection of N.

Remark 6.2 • If we suppose that the metric connection ∇ on (M, g) has a skew-symmetric torsion we do not need to suppose that H satisfies the twistor torsional algebraic identity. Indeed, the use of this last hypothesis is to obtain the implication: '$(N, \bar{F}, \bar{h})$ is of horizontal type $\mathcal{G}_1$' $\Rightarrow$ '∇ has a skew-symmetric torsion'.
• In the case $(N, \bar{F}, \bar{h})$ is of global type $\mathcal{G}_1$, the equation $\nabla R = 0$ is already given by theorem 6.2.

Proof. A concerns the first assertion, it suffices to follow the proof of theorem 5.2. The proof of the equation $\nabla T = 0$ is exactly the same whereas for the equation $\nabla R = 0$ one just need to use that H satisfies the twistor torsional algebraic identity to conclude that $(\nabla R)_J^{--} = 0$, $\forall J \in N$ implies $\nabla R = 0$.
The next assertions results from the recalls and statements (more particulary proposition 7.11) presented in §7.2 in the Appendix. This completes the proof. □

7 Appendix

7.1 Reductive and naturally reductive homogeneous spaces

7.1.1 Effectiveness, reductivity and other generalities.

Let $M = H/K$ be a homogeneous space with H a real Lie group and K a closed subgroup of H. H acts transitively on M in a natural manner which defines a natural representation: $\phi\colon g \in H \mapsto (\phi_g \colon x \in M \mapsto g.x) \in \mathrm{Diff}(M)$. Then $\ker\phi$ is the maximal normal subgroup of H contained in K. Further, let us consider the linear isotropy representation:

$$\rho_{x_0}\colon h \in K \mapsto d\phi_h(x_0) \in GL(T_{x_0}M)$$

where $x_0 = 1.K$ is the reference point in M. Then we have $\ker\rho_{x_0} \supset \ker\phi$. Moreover the linear isotropy representation is faithful (i.e. ρ is injective) if and only if H acts freely on the bundle of linear frames $L(M)$.
We can always suppose without loss of generality that the action of H on M is effective (i.e. $\ker\phi = \{1\}$) but it does not imply in general that the linear isotropy representation is faithful. However if there exists on M a H-invariant linear connection, then the linear isotropy representation is faithful provided that H acts effectively on M. (Indeed, given a manifold M with a linear connection, and $x \in M$, an affine transformation f of M is determined by $(f(x), df(x))$, i.e. f is the identity if and only if it leaves one linear frame fixed).

We will say that H/K is reductive if there exists a decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ such that $\mathfrak{p}$ is $\mathrm{Ad}K$-invariant: $\forall k \in K, \mathrm{Ad}k(\mathfrak{p}) = \mathfrak{p}$. Such a decomposition is called a reductive decomposition. If H/K is reductive, the surjective map $\xi \in \mathfrak{h} \mapsto \xi.x_0 \in T_{x_0}M$ has $\mathfrak{k}$ as kernel and so its restriction to $\mathfrak{p}$ is an isomorphism $\mathfrak{p} \cong T_{x_0}M$. This provides an isomorphism of the associated bundle $H \times_K \mathfrak{p}$ with TM.

On a reductive homogeneous space $M = H/K$, the $\mathrm{Ad}(K)$-invariant summand $\mathfrak{p}$ provides by left translation in H, a H-invariant distribution $\mathcal{H}(\mathfrak{p})$, given by $\mathcal{H}(\mathfrak{p})_g = g.\mathfrak{p}$ which is horizontal for $\pi\colon H \to M$ and right K-invariant and thus defines a H-invariant connection in the principal bundle $\pi\colon H \to M$. In fact this procedure defines a bijective correspondence between reductive summands $\mathfrak{p}$ and H-invariant connections in $\pi\colon H \to M$ (see [21], chap. 2, Th 11.1). Then the $\mathfrak{k}$-valued connection 1-form ω on H, corresponding to this H-invariant connection, is the $\mathfrak{k}$-component of the left invariant Maurer-Cartan form of H. This connection 1-form ω induces a covariant derivative in the associated bundle $H \times_K \mathfrak{p} \cong TM$ and thus a H-invariant covariant derivative ∇^0 in the tangent bundle TM. In particular, we can conclude that if H/K is reductive then the linear isotropy representation is faithful, provided that K acts effectively, or equivalently that if H/K is reductive then $\ker \mathrm{Ad}_{\mathfrak{p}} = \ker \rho_{x_0} = \ker \phi$.

Convention. The Lie group H will always be supposed connected.

Proposition 7.1 [23, Chap. I] *Let H/K be a reductive homogeneous space, with a reductive decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$. Then the subspace $\mathfrak{h}_1 = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is an ideal of $\mathfrak{h}$ and the corresponding normal connected subgroup $H_1 \subset H$ acts transitively on H/K. Moreover, H_1 is generated by the set $\exp(\mathfrak{p}) \subset H$.*

Proposition 7.2 *Let $\mathfrak{h}$ be Lie algebra, $\mathfrak{k}$ a subalgebra, and $\mathfrak{p}$ an $\mathrm{ad}\mathfrak{k}$-invariant complement of $\mathfrak{k}$: $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ and $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. We define the following ideal of $\mathfrak{h}$*

$$\mathfrak{h}(\mathfrak{p}) := \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}}$$

that will sometimes be also denoted simply by $\mathfrak{h}_1$ (when there is no ambiguity on the choice of $\mathfrak{p}$).
•Then for any connected Lie group H with Lie algebra $\mathfrak{h}$, we have

$$\mathfrak{h}(\mathfrak{p}) = \sum_{h \in H} \mathrm{Ad}h(\mathfrak{p}).$$

•Moreover $\mathfrak{k}_1 := \mathfrak{h}(\mathfrak{p}) \cap \mathfrak{k}$ satisfies $\mathfrak{h}(\mathfrak{p}) = \mathfrak{k}_1 \oplus \mathfrak{p}$, $[\mathfrak{k}_1, \mathfrak{p}] \subset \mathfrak{p}$ and $\mathfrak{k}_1 = [\mathfrak{h}_1]_{\mathfrak{k}} = [\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}}$, where $[\cdot]_{\mathfrak{k}}\colon \mathfrak{h} \to \mathfrak{k}$ is the projection on $\mathfrak{k}$ along $\mathfrak{p}$.
•Furthermore, if $M = H/K$ is a reductive homogeneous space, w.r.t. the reductive decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, then $\mathfrak{k}_1$ is the Lie algebra of the isotropy group K_1 under the transitive action of H_1 on M and the decomposition $\mathfrak{h}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}$ is a reductive decomposition for the H_1-homogeneous space $H_1/K_1 = M$.

Proof. •We remark that $\mathfrak{h}(\mathfrak{p})$ is by definition the smallest ideal containing $\mathfrak{p}$. Moreover, $\sum_{h \in H} \mathrm{Ad}h(\mathfrak{p})$ is $\mathrm{Ad}H$-invariant and therefore an ideal of $\mathfrak{h}$. Since it contains $\mathfrak{p}$, it contains also $\mathfrak{h}(\mathfrak{p})$. Conversely, since $\mathfrak{h}(\mathfrak{p})$ is an ideal, we have $\mathfrak{p} \subset \mathfrak{h}(\mathfrak{p}) \Rightarrow \mathrm{Ad}H(\mathfrak{p}) \subset \mathfrak{h}(\mathfrak{p})$ and therefore $\sum_{h \in H} \mathrm{Ad}h(\mathfrak{p}) \subset \mathfrak{h}(\mathfrak{p})$.
•We have $\mathfrak{h}_1 \oplus \mathfrak{p} \subset \mathfrak{h}(\mathfrak{p})$. Conversely, $\forall X \in \mathfrak{h}(\mathfrak{p})$, $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, then $X_{\mathfrak{k}} = X - X_{\mathfrak{p}} \in \mathfrak{h}(\mathfrak{p})$, therefore $\mathfrak{h}(\mathfrak{p}) \subset \mathfrak{k}_1 \oplus \mathfrak{p}$ and $X_{\mathfrak{k}_1} = X_{\mathfrak{k}}$.
•We have clearly $\mathfrak{k} \cap \mathfrak{h}_1 \subset \mathrm{Lie}\,(H_1)$. Moreover, $\dim \mathfrak{k}_1 = \dim \mathfrak{h}_1 - \dim \mathfrak{p} = \dim H_1 - \dim H_1/K_1 = \dim K_1$, therefore $\mathfrak{k}_1 = \mathrm{Lie}\,(H_1)$. Finally, since $\mathfrak{p}$ is $\mathrm{Ad}K$-invariant then it is $\mathrm{Ad}K_1$-invariant so that $\mathfrak{h}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}$ is a reductive decomposition for the H_1-homogeneous space $H_1/K_1 = M$. This completes the proof. □

7.1.2 Riemannian Homogeneous spaces.

It is well known that a Riemannian homogeneous space $M = H/K$ is reductive. But in general to prove this, in the literature, one suppose that H is effective on M, which amounts to factoring out H by some normal subgroup. But in our situation, we can in general change the group H, since this one is in our context the structure group of some principal bundle, and in particular could be the holonomy group of some linear connection. Consequently, we need to make sure that the Lie algebra $\mathfrak{h}$ itself admits a AdK-invariant decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ because the existence of such a decomposition for a quotient of $\mathfrak{h}$ will not be enough for us.

Let us recall the following

Theorem 7.1 *[4] Let M be a Riemannian manifold.*

(a) *The group* $\mathrm{Is}(M)$ *of all the isometries of M is a Lie group and acts differentiably on M.*

(b) *Let $x_0 \in M$, then an isometry f of M is determined by the image $f(x_0)$ of the point x_0 and the corresponding tangent map $T_{x_0}f$ (i.e. if $f(x_0) = g(x_0)$ and $T_{x_0}f = T_{x_0}g$ then $f = g$).*

(c) *The isotropy subgroup* $\mathrm{Is}_{x_0}(M) = \{f \in \mathrm{Is}(M); f(x_0) = x_0\}$ *is a closed subgroup of* $\mathrm{Is}(M)$ *and the linear isotropy representation $\rho_{x_0} \colon f \in \mathrm{Is}_{x_0}(M) \mapsto T_{x_0}f \in O(T_{x_0}M)$ is an isomorphism from* $\mathrm{Is}_{x_0}(M)$ *onto a closed subgroup of $O(T_{x_0}M)$. Hence* $\mathrm{Is}_{x_0}(M)$ *is a compact subgroup of* $\mathrm{Is}(M)$.

If $M = H/K$ is Riemannian, and if H acts effectively on M, we can consider that H is a (immersed) subgroup of $\mathrm{Is}(M)$. Therefore, we can consider its closure $\bar{H}$ in $\mathrm{Is}(M)$, and it will be to remark that $\mathrm{Ad}\bar{H}(\mathfrak{h}) = \mathfrak{h}$.
Then If $M = H/K$ is Riemannian, the image K_{x_0} of the isotropy group K by the linear isotropy representation ρ_{x_0}, is compact (because a closed subgroup of $\mathrm{Is}_{x_0}(M)$). If $M = H/K$ is Riemannian (i.e. there exists a H-invariant Riemannian metric on H/K), then the Levi-Civita connection (of any invariant metric) is an invariant linear connection, therefore we have $\ker \rho_{x_0} = \ker \phi$ (see § 7.1.1) so that

$$K' := K/\ker \phi \cong K_{x_0}.$$

Moreover, the image K_{x_0} of the isotropy group K by the linear isotropy representation ρ_{x_0}, is relatively compact. Therefore, considering K' as a subgroup of $0(T_{x_0})$, then its closure is compact and any

Then, since K' is compact, there exists an AdK'-invariant complement $\mathfrak{p}'$ of $\mathfrak{k}'$ in $\mathfrak{h}'$.
Let us set $U := \ker \phi$, $\mathfrak{u} := \mathrm{Lie}\,(U)$, and denote by $\pi \colon \mathfrak{h} \to \mathfrak{h}'$ the canonical projection. Then K' acts naturally on the set of complements $\mathfrak{p}$ of $\mathfrak{u}$ in $\pi^{-1}(\mathfrak{p}')$ as follows: since $\pi \colon \mathfrak{h} \to \mathfrak{h}'$ induces an isomorphism $\pi_{\mathfrak{p}} \colon \mathfrak{p} \to \mathfrak{p}'$, we can set

$$f_{\mathfrak{p}}(k') = (\pi_{\mathfrak{p}})^{-1} \circ \mathrm{Ad}k' \circ \pi_{\mathfrak{p}}, \qquad \forall k' \in K',$$

this defines a linear action $f \colon K' \to \mathrm{GL}(\mathfrak{p})$. Therefore by an averaging procedure, there exists an AdK'-invariant complement $\mathfrak{p}$ of $\mathfrak{u}$ in $\pi^{-1}(\mathfrak{p}')$, which is then an AdK'-invariant complement of $\mathfrak{k}$ in $\mathfrak{h}$, and hence an AdK-invariant complement of $\mathfrak{k}$ in $\mathfrak{h}$.

Proposition 7.3 *If H/K is Riemannian then there exists $\mathfrak{p} \subset \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ and* $\mathrm{Ad}K(\mathfrak{p}) = \mathfrak{p}$ *(even if H does not acts effectively on H/K). In other words a Riemannian homogeneous space is reductive.*

7.1.3 Naturally reductive metrics.

Definition 7.1 *Let H/K be a Riemannian homogeneous space, with a reductive decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$. We will say a H-invariant metric on H/K is naturally reductive if the corresponding inner product on $\mathfrak{p}$ satisfies:*

$$\langle [Z, X]_{\mathfrak{p}}, Y \rangle = -\langle X, [Z, Y]_{\mathfrak{p}} \rangle, \quad \forall X, Y, Z \in \mathfrak{p} \tag{7.1}$$

If such a metric exists on H/K, we will say that H/K is naturally reductive and consider that it is endowed by a naturally reductive Riemannian metric.
In this case, the torsion of the canonical connection of H/K is totally skew-symmetric w.r.t. this naturally reductive metric.

Definition 7.2 *Let H/K be a Riemannian homogeneous space, and $\mathfrak{p}$ such that $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$. We say that $\mathfrak{p}$ is natural if it is* $\mathrm{Ad}K$*-invariant and admits an* $\mathrm{Ad}K$*-invariant naturally reductive inner product. In other words $\mathfrak{p}$ is natural if it defines a reductive decomposition with respect to which H/K is naturally reductive.*
We will also say in this case that the decomposition $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ is natural.

Definition 7.3 [22] *Let us consider an* $\mathrm{ad}\mathfrak{k}$*-invariant decomposition: $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$. We will say that a inner product B on $\mathfrak{p}$ is strictly invariant if $\mathrm{ad}_{\mathfrak{p}}\mathfrak{k} \subset \mathfrak{so}(\mathfrak{p}, B)$ and it satisfies equation* (7.1)*; i.e. $[\mathrm{ad}_{\mathfrak{p}} X]_{\mathfrak{p}}$ is skew-symmetric for all $X \in \mathfrak{h}$ (where $[\cdot]_{\mathfrak{p}} \colon \mathfrak{h} \to \mathfrak{p}$ is the projection on $\mathfrak{p}$ along $\mathfrak{k}$).*

Theorem 7.2 [22] *Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $(\mathfrak{h}, \mathfrak{k})$ effective. Let B be a strictly invariant inner product on $\mathfrak{p}$. There exists one and only one invariant symmetric bilinear form B^* on $\mathfrak{h}(\mathfrak{p}) = \mathfrak{k}_1 \oplus \mathfrak{p}$ extending B and such that $B^*(\mathfrak{k}_1, \mathfrak{p}) = 0$. Moreover, B^* is non-singular on $\mathfrak{h}(\mathfrak{p})$ and hence on $\mathfrak{k}_1$.*

Definition 7.4 [22] *We will say, in the situation of the previous theorem that $\mathfrak{p}$ is* pervasive *in $\mathfrak{h}$ if $\mathfrak{h}(\mathfrak{p}) = \mathfrak{h}$. This is also equivalent to $[\mathfrak{p}, \mathfrak{p}]_{\mathfrak{k}} = \mathfrak{k}$.*
A Riemannian Homogeneous space H/K is semi-normal if it admits a reductive decomposition w.r.t. which $\mathfrak{p}$ is natural and pervasive. In other words, there exists a non-degenerate invariant symmetric bilinear B on $\mathfrak{h}$ such that $\mathfrak{k}$ is non-singular and if we set $\mathfrak{p} := \mathfrak{k}^{\perp_B}$, then $B_{|\mathfrak{p}}$ is positive definite (and thus defines a naturally reductive Riemannian metric on H/K).

According to this theorem, we see that if B is strictly invariant on $\mathfrak{p}$, and $\mathfrak{h}(\mathfrak{p}) = \mathfrak{h}$, then $\mathfrak{p}$ is automatically $\mathrm{Ad}K$-invariant (and not only $\mathrm{Ad}K^0$-invariant) for any subgroup $K \subset H$ with Lie algebra $\mathfrak{k}$: indeed, B is H-invariant (H is connected, according to our convention) and therefore, since $\mathfrak{k}$ is $\mathrm{Ad}K$-invariant its orthogonal $\mathfrak{p}$ is $\mathrm{Ad}K$-invariant too. Therefore the fact for $\mathfrak{p}$ to be natural w.r.t. H/K is a purely Lie algebra concept.

Proposition 7.4 [22, Corollary 4] *Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ be natural and $(\mathfrak{h}, \mathfrak{k})$ effective. There exists a complement ideal $\mathfrak{h}_0$ of $\mathfrak{h}_1$ in $\mathfrak{h}$: $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$.*

Proposition 7.5 [22, Corollary 5] *Let $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $(\mathfrak{h}, \mathfrak{k})$ effective. Then $\mathfrak{p}$ is natural and pervasive if and only if there exists a non-singular invariant bilinear form B^* on $\mathfrak{h}$ such that B^* is positive definite on $\mathfrak{p}$ and $B^*(\mathfrak{h}, \mathfrak{p}) = 0$.*

7.1.4 Semi-simple and compact Lie group.

Proposition 7.6 [15, p. 131] *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{a}$ a ideal of $\mathfrak{g}$. Then $\mathfrak{a}$ is semisimple, the Killing form of $\mathfrak{a}$ is the restriction to $\mathfrak{a}$ the Killing form B of $\mathfrak{g}$, and if $\mathfrak{a}^\perp$ is the orthogonal of $\mathfrak{a}$ w.r.t. B, we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ (and $\mathfrak{a}^\perp$ is a semisimple ideal).*

Proposition 7.7 *Every semi-simple Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g} = \oplus_1^k \mathfrak{g}_i$ of simple ideals $\mathfrak{g}_i$. Each ideal $\mathfrak{a}$ of $\mathfrak{g}$ is the direct sum of certain $\mathfrak{g}_i$. In particular, each ideal $\mathfrak{a}$ admits a complement ideal in $\mathfrak{g}$.*

Proposition 7.8 *Every compact Lie algebra $\mathfrak{g}$ is the direct sum $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ is the center of $\mathfrak{g}$ and the ideal $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple and compact.*
Moreover, any ideal $\mathfrak{a}$ of $\mathfrak{g}$ is also a direct sum $\mathfrak{a} = [\mathfrak{a}, \mathfrak{a}] \oplus Z(\mathfrak{a})$, and $Z(\mathfrak{a}) \subset Z(\mathfrak{g})$.

Proof. For the first assertion see [15]. For the second one, since $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple, we can write $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{a}, \mathfrak{a}] \oplus \mathfrak{h}$ where $\mathfrak{h}$ is a semi-simple ideal of $\mathfrak{g}$. Moreover, $Z(\mathfrak{a})$ commutes with $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$ and with $\mathfrak{h}$ therefore it commutes with $[\mathfrak{g}, \mathfrak{g}]$ and thus with $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus Z(\mathfrak{g})$. This completes the proof. □.

Proposition 7.9 *If H is compact, then H/K is naturally reductive and if $(\mathfrak{h}, \mathfrak{k})$ is effective then we have $\mathfrak{h} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$.*

Proof. We have a direct sum $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z(\mathfrak{h})$, where $Z(\mathfrak{h})$ is the center of $\mathfrak{h}$. Moreover, the opposite of the Killing form of $\mathfrak{h}$, $-K$, is positive definite on $[\mathfrak{h}, \mathfrak{h}]$. Furthermore for any Euclidean scalar product $\langle \cdot, \cdot \rangle_{Z(\mathfrak{h})}$ on the vector space $Z(\mathfrak{h})$, the positive definite scalar product $B = -K + \langle \cdot, \cdot \rangle_{Z(\mathfrak{h})}$ is AdH-invariant on $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus Z(\mathfrak{h})$. Moreover if $(\mathfrak{h}, \mathfrak{k})$ is effective, then $\mathfrak{h} = \mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ according to proposition 7.5.

7.2 Covering, linear connection and locally homogeneous space.

7.2.1 Affine context.

Definition 7.5 *An affine manifold (M, ∇) is a manifold endowed with a linear connection.*
An affine manifold (M, ∇) is called locally homogeneous if it satisfies $\nabla T = 0$ and $\nabla R = 0$.

Theorem 7.3 [21, Th 2.8, Chap. X] *Let (M, ∇) be a complete affine manifold such that $\nabla T = 0$ and $\nabla R = 0$. If M is simply connected then M is a reductive homogeneous space and the connection ∇ is the canonical connection. Moreover, there exists a group G of affine transformations of $(\tilde{M}, \tilde{\nabla})$ which acts simply transitively on the holonomy bundle of $(\tilde{M}, \tilde{\nabla})$.*

Definition 7.6 *Let $\pi\colon \tilde{M} \to M$ be a submersion. Let $\tilde{\nabla}$ and ∇ be linear connections on $\tilde{M}$ and M respectively. We say that ∇ lifts into $\tilde{\nabla}$ or equivalently that $\tilde{\nabla}$ projects on ∇ if $\pi\colon (\tilde{M}, \tilde{\nabla}) \to (M, \nabla)$ is an affine map*[10].

Lemma 7.1 *Let $\pi\colon \tilde{M} \to M$ be a covering. Then any linear connection, ∇, on M, can be lifted to a linear connection on $\tilde{M}$.*

Proof. It suffices to set
$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = (d\pi)^{-1} \left(\nabla_{\tilde{X}} d\pi(\tilde{Y}) \right),$$
for any $\tilde{X}, \tilde{Y} \in \mathcal{C}(T\tilde{M})$. This completes the proof. □

[10]See [21, Chap. VI, §1] for a definition of affine maps.

Proposition 7.10 *Let* $\pi\colon (\tilde{M},\tilde{\nabla}) \to (M,\nabla)$ *be an affine covering. The following holds*
• *The associated map between the respective bundles of linear frames gives rise to a surjective map* $\pi_*\colon \mathrm{H}^{\tilde{\nabla}}(\tilde{M},\tilde{e}_0) \to \mathrm{H}^{\nabla}(M,e_0)$ *between the holonomy bundles.*
• $(\tilde{M},\tilde{\nabla})$ *is complete if and only if* (M,∇) *is so.*
• (M,∇) *is locally homogeneous (i.e.* $\nabla T = 0$ *and* $\nabla R = 0$*) if and only if so is* $(\tilde{M},\tilde{\nabla})$.

Furthermore, if an affine manifold (M,∇) *is complete and locally homogeneous then its universal cover* $(\tilde{M},\tilde{\nabla})$ *is a reductive homogeneous space and the lifted connection* $\tilde{\nabla}$ *is the canonical connection.*

This allows to define:

Definition 7.7 *Let* M *be a connected manifold, then there exists at most one linear connection* ∇ *on* M *such that* (M,∇) *is complete and locally homogeneous. If this connection exists, we call it the canonical connection of* M.

Proof. • According to [21, Chap. VI, §1], the map $d\pi\colon T\tilde{M} \to TM$ maps a horizontal curves into a horizontal curve and therefore so is for its associated map $d\pi\colon L(\tilde{M}) \to L(M)$ between the bundles of linear frames. Therefore according to the definition of the holonomy bundle ([21, Chap. II, §7]), $\mathrm{H}^{\tilde{\nabla}}(\tilde{M},\tilde{e}_0)$ is sent into $\mathrm{H}^{\nabla}(M,e_0)$. Now, let us prove the surjectivity of $d\pi\colon \mathrm{H}^{\tilde{\nabla}}(\tilde{M},\tilde{e}_0) \to \mathrm{H}^{\nabla}(M,e_0)$. Let $x\colon I \to M$ be a curve, $x_0 = x(0)$ and $e_0 \in L(M)_{x_0}$. Let $\gamma\colon I \to L(M)$ be the parallel transport of e_0 along x. Let y be a lift of x, $y_0 = y(0)$ and $\tilde{e}_0 = d\pi(y_0)^{-1}.e_0$. Consider $\tilde{\gamma}\colon I \to L(N)$ the parallel transport of $\tilde{e}_0$ along y. Then we have $d\pi \circ \tilde{\gamma} = \gamma$ (because $d\pi \circ \tilde{\gamma}$ is horizontal, its projection on M is $\pi \circ y = x$ and its values at $t = 0$ is e_0). This proves the surjectivity and completes the proof of the first point.
• The proof of the second point is exactly the same as in [21, Th. 4.6, Chap. IV].
• The third point follows immediately from [21, Prop. 1.2, Chap. VI].
Finally the last assertion results from the two previous points and from Theorem 7.3. This completes the proof. □

Proposition 7.11 *Let* (M,∇) *be a locally homogeneous and complete affine manifold, and* $(\tilde{M},\tilde{\nabla})$ *its universal cover. Then* $(\tilde{M},\tilde{\nabla})$ *is a reductive homogeneous space and there exists a group* G *of affine transformations of* $(\tilde{M},\tilde{\nabla})$ *which acts simply transitively on the holonomy bundle of* $(\tilde{M},\tilde{\nabla})$.
Let $J_0 \in \Sigma^+(\mathbb{R}^{2n})$ *and consider the associated holonomy admissible subbundles* N *and* $\tilde{N}$ *respectively over* (M,∇) *and* $(\tilde{M},\tilde{\nabla})$ *respectively. Then* $\tilde{N}$ *is a reductive* G*-homogeneous space and a covering of* N *which is therefore complete and locally homogeneous.*

Proof. According to the previous proposition, $(\tilde{M},\tilde{\nabla})$ is complete and locally homogeneous, therefore since it is also simply connected, Theorem 7.3 allows to conclude concerning the first assertion.
The covering $d\pi\colon T\tilde{M} \to TM$ induces an isomorphism on the fibres (i.e. an isomorphism from T_yM onto $T_{\pi(y)}M$, for any $y \in \tilde{M}$), and so is for its associated map $d\pi\colon L(\tilde{M}) \to L(M)$, and therefore so is also for the induced map $\pi_*\colon \mathrm{H}^{\tilde{\nabla}}(\tilde{M},\tilde{e}_0) \to \mathrm{H}^{\nabla}(M,e_0)$. In particular the results of that is that the holonomy group of M is a discrete covering of the holonomy group of $\tilde{M}$, $H/\tilde{H} = \Gamma$ the group of desk transformations of π, where $H = \mathrm{Hol}^{\nabla}(M,e_0)$, $\tilde{H} = \mathrm{Hol}^{\tilde{\nabla}}(\tilde{M},\tilde{e}_0)$. Moreover, according to the first assertion, we have $\mathrm{H}^{\tilde{\nabla}}(\tilde{M},\tilde{e}_0) = G.\tilde{e}_0 \cong G$ so that we have the

following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_*} & \mathrm{Hol}^{\nabla}(e_0) \\ \downarrow & & \downarrow \\ G/H & \xrightarrow{\pi} & \mathrm{Hol}^{\nabla}(e_0)/H \end{array} .$$

Moreover, setting $\tilde{K} := \tilde{H} \cap U(J_0)$ and $H \cap U(J_0) =: K$ we have the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_*} & \mathrm{Hol}^{\nabla}(e_0) \\ \downarrow & & \downarrow \\ G/\tilde{K} & \longrightarrow & \mathrm{Hol}^{\nabla}(e_0)/K \end{array} ,$$

and therefore $\tilde{N} = G/\tilde{K}$ is a discrete covering of $N = \mathrm{Hol}^{\nabla}(e_0)/K$. This completes the proof of the second assertion and hence of the proposition. $\square$

7.2.2 Riemannian context.

Definition 7.8 *Let (M, g) be a Riemannian manifold endowed with a metric connection ∇ such that (M, ∇) is locally homogeneous. Then we say that (M, g, ∇) id a Riemannian homogeneous space, and that g is a invariant metric on (M, ∇).*

7.3 Locally k-symmetric spaces.

7.3.1 Recalls about symmetric spaces.

Let us recall that

Proposition 7.12 *An affine manifold (M, ∇) is locally symmetric if and only if the torsion and the curvature of ∇ satisfies $T = 0$ and $\nabla R = 0$.*
A Riemannian manifold (M, g) is a Riemannian locally symmetric space if and only if ∇^g, its Levi-Civita connection, has a parallel curvature: $\nabla^g R^g = 0$.

Proposition 7.13 *Let (M, g) be a Riemannian locally symmetric space. Then we have* $\mathrm{Hol}^0(\nabla^g) = SO(n)$ *if and only if (M, g) has a constant sectional curvature.*

7.3.2 Affine context.

Definition 7.9 *Let $\tau \colon G \to G$ be an automorphism of finite order k of a Lie group G. Let G_0 be a subgroup such that $(G^\tau)^0 \subset G_0 \subset G^\tau$, where G^τ is the set of points fixed by τ. Then the reductive homogeneous space G/G_0 is called a k-symmetric space.*
If $k = 2p$, let us set $\sigma = \tau^2$ and let H be a subgroup such that $(G^\sigma)^0 \subset H \subset G^\sigma$. Then G/H is called the p-symmetric space corresponding to the $2p$-symmetric space G/G_0 and the fibration $G/G_0 \to G/H$ (whose the fibre H/G_0 is symmetric) is called the $2p$-symmetric fibration (with symmetric fibre) associated to G/G_0.

The eigenspace decomposition of the Lie algebra automorphism $\tau \colon \mathfrak{g} \to \mathfrak{g}$ induces a reductive decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{n}$ where $\mathfrak{g}_0 = \ker(\tau - \mathrm{Id})$ and $\mathfrak{n}$ is the unique real subspace of $\mathfrak{g}$ such that $\mathfrak{n}^{\mathbb{C}} = \oplus_{J \in \mathbb{Z}_k \setminus \{0\}} \mathfrak{g}_j^{\mathbb{C}}$, with $\mathfrak{g}_j^{\mathbb{C}} = \ker(\tau - \omega_k^j \mathrm{Id})$, ω_k is a k-th primitive root of unity. In particular, $\mathfrak{g}_0^{\mathbb{C}} = (\mathfrak{g}_0)^{\mathbb{C}}$. *Therefore a k-symmetric space is a reductive homogeneous space (in a canonical way) and hence it admits a canonical connection.*

Definition 7.10 *Let (M, ∇, J) be an affine manifold endowed with some endomorphism $J \in \mathcal{C}(\mathrm{End}TM)$ satisfying $J^k = \mathrm{Id}$, $J^p \neq \mathrm{Id}$ if $p < k$ and $\ker(J - \mathrm{Id}) = \{0\}$. Then we will say that (M, ∇, J) is a locally k-symmetric space if*

$$\nabla T = 0, \quad \nabla R = 0, \quad \nabla J = 0,$$

J is invariant by the group of affine transformations of ∇, and J leaves invariant the torsion and the curvature tensors:

$$T(J\cdot, J\cdot) = JT(\cdot,\cdot) \quad R(J\cdot, J\cdot) = JR(\cdot,\cdot)J^{-1}$$

Remark 7.1 In particular if $k = 2$, we have $J = -\mathrm{Id}_{TM}$ and therefore $T = -T$ i.e. $T = 0$ so that we recover the following characterization of locally symmetric space (M, ∇): $T = 0$ and $\nabla R = 0$.

Proposition 7.14 *A k-symmetric space is locally k-symmetric. Moreover the corresponding canonical connections coincide.*

Proof. The element $J_0 \in \mathrm{End}(T_{y_0}N)$ corresponding to $\tau_{|\mathfrak{n}}$ under the identification $T_{y_0}N = \mathfrak{n}$, satisfies $\forall g \in G_0$, $gJ_0g^{-1} = J_0$. Therefore, it defines a G-invariant section $J \colon G/G_0 \to \mathrm{End}(TG/G_0)$, $g.y_0 \mapsto gJ_0g^{-1}$. Then J being G-invariant, it is parallel w.r.t. the canonical connection: $\nabla J = 0$. Moreover, since τ is an automorphism, we have easily that J leaves the torsion and the curvature. This completes the proof. □

Definition 7.11 *Let (M, ∇, J) and (M', ∇', J') be locally k-symmetric spaces.*
A diffeomorphism $\Phi \colon (M, \nabla, J) \to (M', \nabla', J')$ is called a similarity or an isomorphism if it satisfies $\Phi^*\nabla' = \nabla$ and $\Phi^*J = J'$.
A local diffeomorphism $\Phi \colon U \to U'$ (where $U \subset M$, $U' \subset M'$) is called a local isomorphism of (M, ∇, J) into (M', ∇', J') if it is an isomorphism from (U, ∇, J) into (U', ∇', J').
We will say (M, ∇, J) and (M', ∇', J') are locally isomorphic if for every points $x \in M$, $y \in M'$ there is a local isomorphism from a neighbourhood of x onto a neighbourhood of y.

Proposition 7.15 *Let (M, ∇, J) be a locally k-symmetric space and $\pi \colon M' \to M$ be a covering. Then there exists, up to a similarity, a unique locally k-symmetric space (M', ∇', J') such that π is a local isomorphism.*

Proposition 7.16 *Let (M, ∇, J) be a locally k-symmetric space and $\pi \colon M \to M'$ be a covering. If ∇ projects on some connection ∇' on M' and J projects on some J', then there exists, up to a similarity, a unique locally k-symmetric space (M', ∇', J') such that π is a local isomorphism.*

Corollary 7.1 *Let (M, ∇, J) be a locally k-symmetric space whose the canonical connection ∇ is complete then its universal cover $(\tilde{M}, \tilde{\nabla}, \tilde{J})$ is a k-symmetric space.*

Theorem 7.4 *Let $(N, \hat{\nabla}, \hat{J})$ be a locally k-symmetric space, suppose that $k = 2p$ and set $\mathcal{V} = \ker(\hat{J} + \mathrm{Id})$. Then we have $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (where $[\cdot,\cdot]$ is the bracket of vector fields) so that $\mathcal{V}$ is integrable. Moreover $\hat{J}^2$ is of order p and $\hat{J}^2_{|\mathcal{V}} = \mathrm{Id}_{\mathcal{V}}$. Suppose that the canonical connection $\hat{\nabla}$ is complete. Then let us denote by $\tilde{N} = G/G_0$ the universal cover of N and let us consider the associated $2p$-symmetric fibration $\tilde{\pi} \colon G/G_0 \to G/H$, with $H = (G^{\tau^2})^0$ so that G/H is simply connected. Denote by $\hat{p} \colon G/G_0 \to N$ the covering map which is also a local isomorphism. Then* *there exists a covering $p\colon G/H \to M$ of G/H such that there is a commutative diagram in the form:*

$$\begin{array}{ccc} G/G_0 & \xrightarrow{\hat{p}} & N \\ {\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ G/H & \xrightarrow{p} & M \end{array}$$

and a structure of locally p-symmetric space (M, ∇, J) such that $d\pi \circ \hat{J}^2 = J \circ d\pi$.
Then we will say that $\pi\colon (N, \hat{\nabla}, \hat{J}) \to (M, \nabla, J)$ is a $2p$-symmetric submersion associated to the locally $2p$-symmetric space $(N, \hat{\nabla}, \hat{J})$.

Remark 7.2 In this case, the fibre of π is a locally symmetric space.

7.3.3 Riemannian context.

Definition 7.12 [20, §2.1.2] *Let G/G_0 be a k-symmetric space. If G/G_0 is Riemannian as a G-homogeneous space then there exists an $\mathrm{Ad}G_0$-invariant inner product on $\mathfrak{n}$ for which $\tau_{|\mathfrak{n}}$ is an isometry, i.e. there exists a G-invariant metric on G/G_0 such that $J \in \mathcal{C}(O(TM, g))$.*

Definition 7.13 *Let (M, g, ∇) be a Riemannian locally homogeneous space endowed with $J \in \mathcal{C}(O(TM, g))$ such that (M, ∇, J) is locally k-symmetric. We say that (M, g, ∇, J) is a Riemannian locally k-symmetric space.*

Theorem 7.5 *Let (N, h, ∇, J) be a Riemannian locally $2p$-symmetric space, and F its canonical f-structure. The following statements are equivalent:*

(i) *(N, F, h) is reductive of global type $\mathcal{G}_1$. In this case, ∇ is its characteristic connection.*

(ii) *Any locally $2p$-symmetric space (N', ∇', J') which is locally isomorphic to (N, ∇, J) admits an invariant Riemannian metric h' for which J' is orthogonal, such that (N', h', ∇') is reductive of global type $\mathcal{G}_1$. In this case, ∇' is the characteristic connection of (N', h', F').*

(iii) *The unique (up to similarity) simply connected $2p$-symmetric space $(\tilde{N}, \tilde{\nabla}, \tilde{J})$ defined by the infinitesimal model of (N, ∇, F) admits (as a reductive homogeneous space) an invariant naturally reductive metric $\tilde{h}$ for which $\tilde{J}$ is orthogonal (so that $(\tilde{N}, \tilde{h}, \tilde{\nabla}, \tilde{J})$ is a Riemannian $2p$-symmetric space). In this case, $(\tilde{N}, \tilde{h}, \tilde{F})$ is reductive of global type $\mathcal{G}_1$ and $\tilde{\nabla}$ is its characteristic connection.*

7.3.4 The example of 4-symmetric spaces.

Definition 7.14 *Let (M, g, ∇) be Riemannian manifold endowed with a metric connection. Consider $\Sigma^+(M)$ and let us consider the endomorphism $\hat{J} \in \mathrm{End}(T\Sigma^+(M))$, defined by $\hat{J} = \pi^*\mathcal{J} \oplus -\mathrm{Id}_{\mathcal{V}}$ following the direct sum $T\Sigma^+(M) = \mathcal{H} \oplus \mathcal{V}$ defined by ∇. We call it the canonical 4-structure on $\Sigma^+(M)$.*

Proposition 7.17 *Let $\pi\colon (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian covering. Then there exists a morphism $\Sigma(\tilde{M}) \to \Sigma(M)$ over π, defined by $J \mapsto d\pi \circ J \circ d\pi^{-1}$. Moreover, this morphism π^{Σ} is a Riemannian covering from $(\Sigma^+(\tilde{M}), \tilde{h})$ to $(\Sigma^+(M), h)$, where $\tilde{h}$ and h are respectively the standard Kaluza-Klein metric (2.13), and it preserves the canonical 4-structure: $(\pi^{\Sigma})^*\mathcal{J} = \widetilde{\mathcal{J}}$.*

Hence we have the following commutative diagram:

$$\begin{array}{ccc} (\tilde{M},\tilde{g}) & \xrightarrow{\pi} & (M,g) \\ \uparrow & & \uparrow \\ (\Sigma^+(\tilde{M}),\tilde{h},\widetilde{\mathcal{J}}) & \xrightarrow{\pi^\Sigma} & (\Sigma^+(M),h,\mathcal{J}) \end{array}.$$

Proposition 7.18 [19] *Let (M,g) be a space form. Then the twistor bundle $(\sigma^+(M),h)$ is a Riemannian 4-symmetric space. Moreover the metric standard Kaluza-Klein metric h is naturally reductive if (M,g) is not the Euclidean space. More concretely, we have $\Sigma^+(S^{2n}) = SO(2n+1)/U(n)$, $\Sigma^+(H^{2n}) = SO^+(1,2n)/U(n)$ and $\Sigma^+(\mathbb{R}^{2n}) = \Sigma_0^+(\mathbb{R}^{2n}) \ltimes \mathbb{R}^{2n} = SO(2n) \ltimes \mathbb{R}^{2n}/U(n)$.*

Corollary 7.2 *Let (M,g) be a space form. Then the structure $(\Sigma^+(M),\hat{\nabla},\hat{J})$ of locally 4-symmetric space on $(\Sigma^+(M),h)$ is given as follows: $\hat{J}$ is the canonical 4-structure on $\Sigma^+(M)$ and the canonical connection $\hat{\nabla}$ is its paracharacteristic connection which is also its characteristic connection.*

Proposition 7.19 *Let (M,g) be a complete Riemannian manifold with a constant sectional curvature. Then the twistor bundle $(\Sigma^+(M),h)$ is a complete locally symmetric space and the corresponding fibration (over a locally symmetric space) is the fibration $\Sigma^+(M) \to M$.*
Moreover, the metric h on $\Sigma^+(M)$ is naturally reductive and $\mathcal{F}$ coincides with the canonical f-structure of the locally 4-symmetric space $\Sigma^+(M)$, which allows to conclude that the homogeneous fibre f-bundle $(\Sigma^+(M),\mathcal{F},h)$ is of global type $\mathcal{G}_1$.

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