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***Abstract*—An almost ubiquitous assumption made in the stochastic-analytic approach to study of the quality of user-service in cellular networks is Poisson distribution of base stations, often completed by some specific assumption regarding the distribution of the fading (e.g. Rayleigh). The former (Poisson) assumption is usually (vaguely) justified in the context of cellular networks, by various irregularities in the real placement of base stations, which ideally should form a lattice (e.g. hexagonal) pattern. In the first part of this paper we provide a different and rigorous argument justifying the Poisson assumption under sufficiently strong log-normal shadowing observed in the network, in the evaluation of a natural class of the typical-user service-characteristics (including path-loss, interference, signal-to-interference ratio, spectral efficiency). Namely, we present a Poisson-convergence result for a broad range of stationary (including lattice) networks subject to log-normal shadowing of increasing variance. We show also for the Poisson model that the distribution of all these typical-user service characteristics does not depend on the particular form of the additional fading distribution. Our approach involves a mapping of 2D network model to 1D image of it "perceived" by the typical user. For this image we prove our Poisson convergence result and the invariance of the Poisson limit with respect to the distribution of the additional shadowing or fading. Moreover, in the second part of the paper we present some new results for Poisson model allowing one to calculate the distribution function of the SINR in its whole domain. We use them to study and optimize the mean energy efficiency in cellular networks.**

***Index Terms*—Wireless cellular networks, Poisson, Hexagonal, convergence, shadowing, fading, spectral/energy efficiency, optimization**

I. Introduction

Cellular networks are being extensively deployed and upgraded in order to cope with the steady rise of user-traffic. This has created the need for new and robust analytic techniques to study the quality of user-service. The ability to tractably model and calculate the quality of user-service related to the signal-to-interference-and-noise ratio (SINR, or SIR when the noise is neglected) will serve as the motivating force behind the work presented here.

In order to derive analytic techniques, various mathematical models have been proposed. A common and simplifying model assumption is that the base stations are located according to a Poisson process in the plane. In the first section of this paper we recall such a model with shadowing and/or fading, and present a simple yet very useful result, upon which the bulk of the remaining work here hinges. This result involves mapping the point process on $\mathbb{R}^2$ (that models the locations of base stations) through the distance-loss function and the shadowing and/or fading variables (that model the propagation losses between these stations and a typical user), to a *point process of propagation losses* on $\mathbb{R}^+$, which are experienced by this user. It allows one to study all typical-user characteristics, which can be expressed in terms of its propagation losses (or received powers, e.g. SINR, etc). We observe that *this new process also forms a Poisson process*, regardless of the shadowing and/or fading distribution, and is characterized only by the moment of this distribution of order $2/\beta$, where β is the distance-loss exponent.

In the context of an actual deployment of cellular networks, lattice (e.g. hexagonal) models for the base station placement are usually thought of as more pertinent. However, perfect lattice models do not seem to allow analytic techniques for the study of the SINR-based characteristics. Hence, the Poisson model is used and justified by positioning "irregularities" of the network. It is also considered as a "worst-case" scenario due to its complete randomness property. Although the validity of these arguments alone may be questioned, *we seek to support the Poisson assumption with a powerfully new convergence result when there is sufficiently strong log-normal shadowing in the network*. Namely, provided a network is represented by a sufficiently large homogeneous point pattern (whose definition will later be made precise, and includes general lattices and various "perturbed lattice" models), we show that as the variance of log-normal shadowing increases, the resulting propagation losses between the stations and the typical user form a stochastic process that converges to the aforementioned non-homogeneous Poisson process on $\mathbb{R}$, due to the Poisson model. In other words, the actual (large but not necessarily Poisson) network is perceived by a typical user as an equivalent (infinite) Poisson network, provided shadowing is strong enough, of logarithmic standard deviation greater than approximately 10dB, as shown by numerical evidence. This is a realistic assumption for outdoor and indoor wireless communications in many urban scenarios.

This result rigorously justifies using a Poisson point process to model typical-user characteristics in a wide range of network models, which includes the hexagonal model and a large class of perturbed lattice models, thus adding theoretical weight to the work being done under the Poisson assumption.

After stating this convergence result, we derive important complementary analytic tools for the study of the distribution of the SINR in the Poisson model, which remain valid for any distribution of the shadowing and/or fading random variables.

We use them to investigate the spectral and energy efficiency in cellular networks. In particular, we evaluate the mean energy efficiency as a function of the base station transmit power, which allows one to optimally tune this latter power. Poisson and hexagonal networks with and without shadowing are compared in this context.

Related work: It is beyond the scope of this short introduction to review all works with Poisson model of wireless networks. Some results and further references may be found in [1, 2]. More specifically, our results presented in this paper allow to extend to a general fading distribution the explicit expressions for the distribution function of the SINR derived in [3], where Rayleigh fading is assumed. Our representation of the network via the process of propagation losses is similar tho this considered in [4] for other purposes, mostly to study the effect of shadowing/fading on connectivity. Moreover, using the Laplace/Fourier analysis, we manage to characterize (no longer explicitly) the SINR over its entire range, whereas the aforementioned explicit expressions are valid only for SINR ≥ 1. The result of Lemma 1 appeared in [5]. The infinite Poisson model was statistically fitted in [6] to some real data provided by an operator regarding propagation losses in order to estimate the parameters of the propagation loss using a simple linear regression model. The spectral and energy efficiency in hexagonal networks without shadowing was studied in [7]. Finally, the convergence result presented here is in the spirit of classical limit theorems of point processes, which are detailed in [8, Chapter 11]. In particular, these theorems show that under specific conditions, the repeated superposition, thinning or translation of points of a point process will result in the point process converging in the limit to a Poisson process.

The remaining part of this paper is organized as follows. In section II we present the basic result on the network mapping to the process of propagation losses. The main convergence result is presented in Section III and its proof is given Appendix A. We revisit the Poisson model in order to study the SINR distribution in Section IV.

II. Infinite Poisson model

For motivation purposes, we first recall the usual cellular network model based on the Poisson process. In particular, we model the geographic locations of the base stations with an homogeneous Poisson point process $\Phi = \{X_i\}_{i\in\mathbb{N}}$ of intensity λ on $\mathbb{R}^2$, which we refer to as the *infinite Poisson* model.

We assume that the (typical) user is located at the origin without loss of generality due to the stationarity of $\{X_i\}_{i\in\mathbb{N}}$. Let $l(X_i)$ be the *distance loss* between a base station at X_i and the user, where $l(\cdot)$ is given by

$$l(x) = (K|x|)^{\beta} \tag{1}$$

for two given positive constants K and β. When we incorporate *shadowing* (and/or *fading*), the *propagation loss* is

$$L_{X_i} = \frac{l(X_i)}{S_{X_i}}$$

where $\{S_x\}_{x\in\mathbb{R}^2}$ is a collection of independent and identically distributed (iid) positive random variables. We will sometimes write also $S_{X_i} = S_i$ to simplify the notation. Let S be a random variable having the same distribution as any S_x.

Note that the *power received* at the origin from the station X_i, transmitting with power P_{X_i}, is equal to

$$p_{X_i} = \frac{P_{X_i}}{L_{X_i}} = \frac{P_{X_i}S_{X_i}}{l(X_i)}\,.$$

In this paper we do not assume any power control, i.e., $P_{X_i} = P$ for some given positive constant $P > 0$. In this case (as well in a more general case of iid emitted powers), including P_{X_i} in the associated shadowing random variable, we retrieve an equivalent model in which the shadowing is $\tilde{S}_{X_i} := P_{X_i}S_{X_i}$ and the transmitted powers $\tilde{P}_{X_i} = 1$. Henceforth we assume, without loss of generality, that the transmitted powers are all equal to one, while keeping in mind that the shadowing random variables now include the effective transmitted powers. This transformation will slightly simplify our notation. However, when studying the energy efficiency in Section IV-B4 we will reintroduce emitted powers to our model.

A. Mapping of the propagation losses of the typical user from $\mathbb{R}^2$ to $\mathbb{R}^+$

Denote by $\mathcal{N} = \{L_{X_i}\}_{i\in\mathbb{N}}$ the *process of propagation losses* experienced by the typical user with respect to the stations in Φ. We consider $\mathcal{N}$ a *point process on* $\mathbb{R}^+$. Note that *all* characteristics of the typical user, which can be expressed in terms of its propagation losses (or received powers, under the aforementioned assumption on emitted powers, e.g. SIR, SINR, spectral and energy efficiency, etc) are determined by the distribution of $\mathcal{N}$. This motivates the following simple result that appeared, to the best of our knowledge for the first time, in [5]. In order to make this presentation more self-contained we present it with a proof.

Lemma 1: *Assume infinite Poisson model with distance-loss (1) and generic shadowing (and/or fading) variable satisfying*

$$\mathbf{E}[S^{\frac{2}{\beta}}] < \infty\,. \tag{2}$$

Then the process of propagation losses $\mathcal{N}$ experienced by the typical user is a non-homogeneous Poisson point process on $\mathbb{R}^+$ with intensity measure

$$\Lambda([0,t)) := \mathbf{E}[\mathcal{N}([0,t))] = at^{\frac{2}{\beta}} \tag{3}$$

where

$$a := \frac{\lambda\pi\mathbf{E}[S^{\frac{2}{\beta}}]}{K^2} \tag{4}$$

Proof: The point process $\mathcal{N}$ may be viewed as a transformation of the point process Φ by the probability kernel

$$p(x,A) = \mathbf{P}\left(\frac{l(x)}{S} \in A\right), \quad x \in \mathbb{R}^2, A \in \mathcal{B}(\mathbb{R}^+)$$

By the displacement theorem [1, Theorem 1.10], the point process $\mathcal{N}$ is Poisson on $\mathbb{R}^+$ with intensity measure

$$\begin{aligned}\Lambda([0,t)) &= \lambda \int_{\mathbb{R}^2} \mathbf{P}\left(\frac{l(x)}{S} \in [0,t)\right) dx \\ &= \lambda \int_{\mathbb{R}^2\times\mathbb{R}^+} \mathbf{1}\left\{\frac{l(x)}{s} < t\right\} dx \mathbf{P}_S(ds) \\ &= \lambda \int_{\mathbb{R}^+} \frac{\pi (st)^{\frac{2}{\beta}}}{K^2} \mathbf{P}_S(ds) = \frac{\lambda\pi \mathbf{E}\left[S^{\frac{2}{\beta}}\right]}{K^2} t^{\frac{2}{\beta}}\end{aligned}$$

which completes the proof. ∎

Remark 2: Note that the distribution of $\mathcal{N}$ is *invariant with respect to the distribution of the shadowing/fading S having same given value of the moment $\mathbf{E}[S^{2/\beta}]$*. This means that the infinite Poisson network with an arbitrary shadowing S is perceived at a given location statistically in the same manner as an "equivalent" infinite Poisson with "constant shadowing" equal to $s_{const} = (\mathbf{E}[S^{2/\beta}])^{\beta/2}$ (to have the same moment of order $2/\beta$). The model with such a "constant shadowing" boils down to the model without shadowing ($S \equiv 1$) and the constant K replaced by $\tilde{K} = K/\sqrt{\mathbf{E}[S^{2/\beta}]}$.

The above result requires condition (2), which is satisfied by the usual models, as e.g. the log-normal shadowing or Rayleigh fading, and we tacitly assume it throughout the paper.

III. Convergence results under log-normal shadowing

In this section we derive a powerful convergence result rigorously showing that the infinite Poisson model can be used to analyse the characteristics of the typical user in the context of any *fixed (deterministic!) placement of base stations*, meeting some *empirical homogeneity* condition, provided there is *sufficiently strong log-normal shadowing*.

A. Model description

Let $\phi = \{X_i\}_{i\in\mathbb{N}}$ be a locally finite deterministic point pattern on $\mathbb{R}^2$ and $B_0(r)$ the ball of radius r, centered at the origin. For $0 < \lambda < \infty$, as $r \to \infty$ we require the *empirical homogeneity condition*

$$\frac{\phi(B_0(r))}{\pi r^2} \to \lambda. \tag{5}$$

Note that the above condition is satisfied by any lattice (e.g. hexagonal) pattern ϕ, as well as by almost any realization of an arbitrary ergodic point process.

Let the shadowing $S_i^{(\sigma)}$ between the station X_i and the origin be iid (across i) log-normal random variables

$$S_i^{(\sigma)} = \exp(-\sigma^2/2 + \sigma Z_i), \tag{6}$$

where Z_i are standard normal random variables. Note that for such $S_i^{(\sigma)} = S^{(\sigma)}$, we have $\mathbf{E}[S] = 1$ and $\mathbf{E}\left[(S^{(\sigma)})^{2/\beta}\right] = \exp[\sigma^2(2-\beta)/\beta^2]$. [1]

[1] This also means that the path-loss from a given station X expressed in dB, i.e., $\mathrm{dB}(L_X(y))$, where $\mathrm{dB}(x) := 10\times\log_{10}(x)$ dB, is a Gaussian random variable with standard deviation $\sigma_{\mathrm{dB}} = \sigma 10/\log 10$, called *logarithmic standard deviation of the shadowing*.

Consider the distance-loss model (1) with the constant K replaced by the function of σ

$$K(\sigma) = K \exp\left(-\frac{\sigma^2(\beta-2)}{2\beta^2}\right), \tag{7}$$

where $K > 0$ and $\beta > 2$.

As in Section II, we consider the point process on $\mathbb{R}^+$ of propagation losses experienced by the typical user with respect to the stations in ϕ

$$\mathcal{N}^{(\sigma)} := \left\{\frac{K(\sigma)^\beta |X_i|^\beta}{S_i^{(\sigma)}} : X_i \in \phi\right\}. \tag{8}$$

We consider also the analogous process of propagation losses

$$\bar{\mathcal{N}}^{(\sigma)} := \left\{\frac{K(\sigma)^\beta |X_i|^\beta}{S_i^{(\sigma)}} : a_\sigma < |X_i| < b_\sigma, X_i \in \phi\right\}, \tag{9}$$

where the stations in ϕ that are closer than a_σ and farther than b_σ are ignored, for all sequences $0 \le a_\sigma < b_\sigma \le \infty$ satisfying

$$\frac{\log(a_\sigma)}{\sigma^2} \to 0, \tag{10}$$

$$\frac{\log(b_\sigma)}{\sigma^2} \to \infty. \tag{11}$$

B. Main result

We present now our main convergence result.

Theorem 3: *Given homogeneity condition (5), then $\mathcal{N}^{(\sigma)}$ converges weakly as $\sigma \to \infty$ to the Poisson point process on $\mathbb{R}^+$ with the intensity measure Λ given by (3) with $a = \lambda\pi/K^2$. Moreover, $\bar{\mathcal{N}}^{(\sigma)}$ also converges weakly ($\sigma \to \infty$) to the Poisson point process with the same intensity measure, provided conditions (10) and (11) are satisfied.*

The proof of Theorem 3 is deferred to Appendix A.

Remark 4: The above result, in conjunction with Lemma 1, says that *infinite Poisson model can be used to approximate the characteristics of the typical user for a very general class of homogeneous pattens of base stations*, including the standard hexagonal one. The second statement of this result says that this approximation remains valid for *sufficiently large but finite patterns*.

Remark 5: The distance-loss model (1) suffers from having a singularity at the origin. This issue is often circumvented by some appropriate modification of the distance-loss function within a certain distance from the origin. The second statement of Theorem 3 with $a_\sigma = const > 0$ shows that such a modification is not significant in the Poisson approximation.

To illustrate Theorem 3 and obtain some insight into the speed of convergence we used Kolmogorov-Smirnov (K-S) test, cf [9], to compare the cumulative distribution functions (CDF) of the SIR of the typical user (see Section IV-A2) in the infinite Poisson model versus hexagonal one consisting of $30 \times 30 = 900$ stations on a torus. We found that for 9/10 realizations of the network shadowing the K-S test does not allow to distinguish the empirical (obtained from simulations) CDF of the SIR from the CDF of SIR evaluated in the infinite Poisson model with the critical p-value fixed to $\alpha = 10\%$ provided $\sigma_{\mathrm{dB}} \gtrsim 10$dB. On Figure 1 we present a few examples of these CDF's. Similar numerical study was done for the CDF of the path-loss with respect to the serving station in [6].

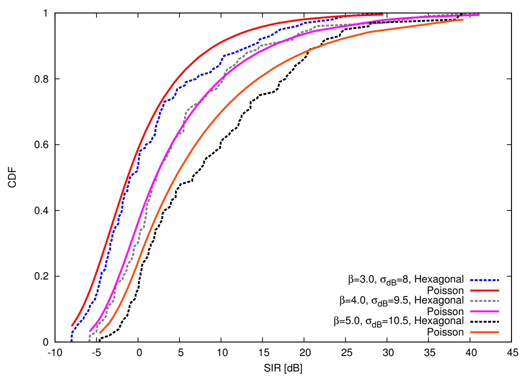

Fig. 1: Empirical CDF of SIR simulated in hexagonal network with shadowing and their Poisson approximations.

IV. Poisson model revisited

We now return to the model outlined in Section II and investigate it further. We show that Lemma 1 is very useful in studying random quantities of the network, independently of the shadowing/fading distribution. We do not pretend to develop a complete theory, which is well beyond the scope of this paper. As a general remark let us emphasize only that many, already known, results (cf *Related work* in the Introduction), were originally derived under specific assumptions on the distribution of shadowing and/or fading. By Lemma 1 they necessarily remain valid for a general distribution, with the appropriate specification of the value of the moment $\mathbf{E}[S^{2/\beta}]$.

Our specific goal in this section is to present a new representation of distribution function of the SINR, which in consequence will allow us to study the spectral and energy efficiency — two important engineering characteristics of the network. To the best of our knowledge, they have not yet been studied in the Poisson model. Moreover, as another illustration of our convergence result, we will compare these characteristics in Poisson and hexagonal network, with this latter studied by simulations.

Unless otherwise specified all results of this section regard Poisson model of Section II.

A. Path-loss and SIR

The novelty of our approach consists in representing the SINR in terms of the path-loss to the serving station and the respective SIR, rather than, as usual, the interference.

1) Path loss: The weakest propagation loss denoted by

$$L=\inf_{i\in\mathbb{N}}L_{X_i} \tag{12}$$

is often called *path-loss factor*. It may be interpreted as the propagation loss with the *serving* base station (that is, the one with strongest received power). Note by (3) that the number of points of the process $\mathcal{N}=\{L_{X_i}\}_{i\in\mathbb{N}}$ is almost surely finite in any finite interval. Hence, the above infimum is almost surely achieved for some base station; that is $L=\min_{i\in\mathbb{N}}L_{X_i}$. Moreover, Lemma 1 allows us to conclude the following characterisation of the distribution of L, which follows immediately from the well known expression for void probabilities of Poisson process $\mathbf{P}\{L\geq t\}=\mathbf{P}\{\mathcal{N}([0,t))=0\}=\exp[-\Lambda([0,t))]$.

Corollary 6: *The CDF of L is equal to $\mathbf{P}\{L\leq t\}=1-\exp[-at^{2/\beta}]$ where a is given by (4). Consequently, the probability density of L is given by*

$$\mathbf{P}_L\left(ds\right)=\frac{2a}{\beta}t^{\frac{2}{\beta}-1}e^{-at^{\frac{2}{\beta}}}\,dt\,. \tag{13}$$

This corresponds to a Fréchet distribution *with shape parameter $\frac{2}{\beta}$ and scale parameter $a^{\frac{\beta}{2}}$.*

2) Interference factor and SIR: The *interference factor* is defined by

$$f=L\sum_{i\in\mathbb{N}}\frac{1}{L_{X_i}}-1$$

where L is the path-loss factor. It may be interpreted as the interference to signal ratio; that is the *inverse of the SIR*. Introducing $I=\sum_{i\in\mathbb{N}}\frac{1}{L_{X_i}}$ (can be interpreted as the total received power) we can write $f=LI-1$. Also, $f=LI'$ where

$$I':=I-\frac{1}{L}=\sum_{i\in\mathbb{N}}\frac{1}{L_{X_i}}-\frac{1}{L}$$

may be viewed as the interference.

Define

$$\varphi_\beta\left(z\right):=e^{-z}+z^{\frac{2}{\beta}}\gamma\left(1-\frac{2}{\beta},z\right) \tag{14}$$

where $\gamma(\alpha,z)=\int_0^z t^{\alpha-1}e^{-t}dt$ is the lower incomplete gamma function. Another representation of the function $\varphi_\beta(z)$, used in evaluation of the Laplace transform of the f (cf proof of Corollary 8) is given in Appendix B.

Here is a key technical result for our approach.

Proposition 7: *The Laplace transform of the interference factor conditional to the path-loss factor is equal to*

$$\mathbf{E}\left[e^{-zf}|L=s\right]=e^{-a[\varphi_\beta(z)-1]s^{\frac{2}{\beta}}}\,. \tag{15}$$

Proof: We have

$$\mathbf{E}\left[e^{-zf}|L=s\right]=\mathbf{E}[e^{-zsI'}|L=s]\,. \tag{16}$$

Observe that I' is a shot-noise associated to the point process obtained from the Poisson point process $\mathcal{N}$ of the propagation losses by suppressing its smallest (nearest to the origin) point, which is located precisely at L. By the well-known property of the Poisson process, given $L=s$, this new process is also Poisson on (s,∞) with intensity measure

$$M\left((s,t)\right)=\Lambda\left([0,t)\right)-\Lambda\left([0,s]\right),\quad t\in\left(s,+\infty\right)\,.$$

Thus

$$M\left(dt\right)=\Lambda\left(dt\right)=\frac{2a}{\beta}t^{\frac{2}{\beta}-1}dt,\quad\text{on }\left(s,+\infty\right) \tag{17}$$

and by [1, Proposition 1.5]

$$\begin{aligned}\mathbf{E}\left[e^{-zsI'}|L=s\right]&=\exp\left[\int_s^\infty(e^{-\frac{zs}{t}}-1)M\left(dt\right)\right]\\&=\exp\left[-\frac{2a}{\beta}\int_s^\infty(1-e^{-\frac{zs}{t}})t^{\frac{2}{\beta}-1}dt\right]\,.\end{aligned} \tag{18}$$

Now we calculate the integral on the right-hand side of the above equation by making the change of variable $u := \frac{zs}{t}$, that is

$$\begin{aligned}
&\int_s^\infty (1-e^{-\frac{zs}{t}})t^{\frac{2}{\beta}-1}dt \\
&= \int_z^0 (1-e^{-u})\left(\frac{zs}{u}\right)^{\frac{2}{\beta}-1}\frac{-zs}{u^2}du \\
&= (zs)^{\frac{2}{\beta}}\int_0^z (1-e^{-u})u^{-\frac{2}{\beta}-1}du \\
&= s^{\frac{2}{\beta}}\frac{\beta}{2}\left[-1+\varphi_\beta(z)\right]
\end{aligned}$$

where the third equality is obtained by integration by parts. Combining the above equation with Equations (16), (18) gives the final result. ■

The following results can be derived from Proposition 7, although we will not use them in the remaining part of the paper.

Corollary 8: *The joint distribution of L and f is characterized by*

$$\mathbf{E}\left[\mathbf{1}\{L\geq u\}e^{-zf}\right]=\frac{1}{\varphi_\beta(z)}e^{-au^{\frac{2}{\beta}}\varphi_\beta(z)},\quad z\in\mathbb{R}^+.$$

The unconditional Laplace transform of the interference factor is

$$\mathbf{E}\left[e^{-zf}\right]=\frac{1}{\varphi_\beta(z)},\quad z\in\mathbb{R}^+. \tag{19}$$

Proof: For the first statement we have

$$\begin{aligned}
\mathbf{E}\left[\mathbf{1}\{L\geq u\}e^{-zf}\right] &= \int_u^\infty \mathbf{E}\left[e^{-zf}|L=s\right]\mathbf{P}_L(ds) \\
&= \int_u^\infty e^{-a[\varphi_\beta(z)-1]s^{\frac{2}{\beta}}}\frac{2a}{\beta}s^{\frac{2}{\beta}-1}e^{-as^{\frac{2}{\beta}}}ds \\
&= \frac{1}{\varphi_\beta(z)}e^{-au^{\frac{2}{\beta}}\varphi_\beta(z)}
\end{aligned}$$

where for the second equality we use (13) and (15). This completes the proof of the first statement. For the second statement, conditioning on the value of the path-loss factor yields

$$\mathbf{E}\left[e^{-zf}\right]=\int_{\mathbb{R}^+}\mathbf{E}\left[e^{-zf}|L=s\right]\mathbf{P}_L(ds)$$

where $\mathbf{P}_L(\cdot)$ is the distribution of L given by (13). Using the above equation and (15), we ascertain

$$\begin{aligned}
\mathbf{E}[e^{-zf}] &= \int_0^\infty e^{-a[\varphi_\beta(z)-1]s^{\frac{2}{\beta}}}\frac{2a}{\beta}s^{\frac{2}{\beta}-1}e^{-as^{\frac{2}{\beta}}}ds \\
&= \frac{2a}{\beta}\int_0^\infty s^{\frac{2}{\beta}-1}e^{-a\varphi_\beta(z)s^{\frac{2}{\beta}}}ds \\
&= \frac{1}{\varphi_\beta(z)}\int_0^\infty a\varphi_\beta(z)\frac{2}{\beta}s^{\frac{2}{\beta}-1}e^{-a\varphi_\beta(z)s^{\frac{2}{\beta}}}ds \\
&= \frac{1}{\varphi_\beta(z)}
\end{aligned}$$

where the third equality stems from $\varphi_\beta(z)\neq 0$ which follows from (40) in Appendix B and the assumption $\beta>2$. ■

Remark 9: For $t\geq 1$ the (complementary) CDF of the SIR admits the following explicit expression

$$\mathbf{P}\{\,\mathrm{SIR}\geq t\,\}=\mathbf{P}\{f\leq 1/t\}=\frac{t^{-2/\beta}}{C'(\beta)} \tag{20}$$

where

$$C'(\beta)=\frac{2\pi}{\beta\sin(2\pi/\beta)}=\frac{2\Gamma(2/\beta)\Gamma(1-2/\beta)}{\beta}$$

and $\Gamma(z)=\int_0^\infty e^{-z}z^{t-1}dt$ it the complete Gamma function. It was proved in [3] assuming exponential distribution of S. But in the infinite Poisson model, SIR is invariant with respect to λ, and K and consequently by Lemma 1 it is also invariant with respect to the value of $\mathbf{E}[S^{2/\beta}]$. Hence the result remains valid for *arbitrary distribution* of S, provided $\mathbf{E}[S^{2/\beta}]<\infty$. Other results of [3], including these involving the superposition of independent Poisson models (called K-tier cellular network model) can also be appropriately generalized using Lemma 1.

B. Distribution of SINR, spectral and energetic efficiency

1) SINR: In this section we are primarily interested in the *signal to interference and noise ratio*

$$\mathrm{SINR}=\frac{\frac{1}{L}}{N+\left(\sum_{i\in\mathbb{N}}\frac{1}{L_{X_i}}-\frac{1}{L}\right)}=\frac{1}{NL+f}, \tag{21}$$

where N is the noise power. We will show how CDF of the SINR can be evaluated, which opens a way for the study of functionals of SINR.

2) Spectral efficiency: An important characteristic of a wireless cellular network is its *spectral efficiency*, defined in the simplest case of additive white Gaussian noise (AWGN) and the optimal theoretical link performance as

$$\mathcal{S}:=\log(1+\mathrm{SINR})\,.$$

It tells us how many bits per second and per Hertz can be sent to the typical user of the network.

3) Energy efficiency: Up to now we have considered a unit transmitted power. Assume now that base stations transmit some power $P\geq 0$. In fact, in order for a base station to be able to transmit this power to the mobile, it needs to be powered (i.e., consumes energy per second) at the level $P'>P$. Following [10], assume that these two quantities are related through a simple linear relation $P'=cP+d$ for some positive constants c and d. The *energy efficiency* is defined by

$$\mathcal{E}:=\mathcal{E}(P)=\frac{W\,\log\left(1+\frac{1}{NL/P+f}\right)}{cP+d}\,.$$

where W is the bandwidth expressed in Hz. Thus $\mathcal{E}$ is equal to $W\log(1+\mathrm{SINR}(P))/P'$, where $\mathrm{SINR}(P)$ takes into account the transmitted power, and P' is the consumed power. It tells us how many bits per second per Watt of consumed power can be sent by a base station to the typical user. Note that $\mathcal{E}(0)=\mathcal{E}(\infty)=0$ and thus $\mathcal{E}(P)$ admits a non-trivial optimization in P.

4) Evaluation of the CDF of the SINR: Proposition 7 in conjunction with Corollary 6 completely characterizes the joint distribution of L and f. It does not, however, allow for an explicit expression for the CDF of the SINR in the whole domain (cf Remark 9). In this section we describe a practical way for numerical computation of this CDF. We will use it to study the spectral and energy efficiency. In fact, we will compute the CDF of the random variable $Y := NL + f$, which is sufficient, in view of (21).

Proposition 10: *The cumulative distribution function of Y is given by*

$$\mathbf{P}(Y < x) = \int_0^\infty F_s(x - Ns)\mathbf{P}_L(ds) \tag{22}$$

where $\mathbf{P}_L(ds)$ is given by (13), and $F_s(y) := \mathbf{P}(f < y | L = s)$ may be expressed by

$$F_s(y) = 1 - \frac{2e^{\gamma t}}{\pi} \int_0^\infty \mathcal{R}\left(\overline{\mathcal{L}}_{F_s}(\gamma + iu)\right) \cos ut \, du\,, \tag{23}$$

where $\gamma > 0$ is an arbitrary constant, $\mathcal{R}$ is the real part if it complex argument and

$$\overline{\mathcal{L}}_{F_s}(z) = \frac{1}{z} - \frac{1}{z} e^{-a[\varphi_\beta(z) - 1]s^{\frac{2}{\beta}}} \tag{24}$$

with $\varphi_\beta(\cdot)$ is given by (14).

Proof: The expression (22) is trivial. The expression (23) of the (conditional) CDF F_s is based on the Bromwich contour inversion integral of the Laplace transform $\overline{\mathcal{L}}_{F_s}$ of $1 - F_s$. The expression (24) for this latter follows from (15). ■

Remark 11: As shown in [11], the integral in (23) can be numerically evaluated using the trapezoidal rule, with the parameter γ allowing control of the approximation error. The function F_s may be also retrieved from (15) using other inversion techniques.

Remark 12: For $t \geq 1$ the (complementary) CDF of the SINR admits the following expression

$$\mathbf{P}\{\mathrm{SINR} \geq t\} = \frac{2t^{-2/\beta}}{\Gamma(1 + \frac{2}{\beta})} \int_0^\infty r e^{-r^2\Gamma(1-2/\beta) - Na^{-\beta/2}r^\beta} dr \tag{25}$$

with a given by (4). It follows from [3, Theorem 1] (where the exponential distribution is assumed for S) and our Lemma 1. Note on Figure 2 that the expression is not valid for $t < 1$.

5) Numerical examples: In what follows we assume the following parameter values. If not otherwise specified we take log-normal shadowing with logarithmic standard deviation $\sigma_{\mathrm{dB}} = 12$dB (cf the footnote on page 3). The parameters of the distance-loss model are $K = 4250\mathrm{Km}^{-1}$, $\beta = 3.52$ (which corresponds to the COST-Hata model for urban environment with the BS height 30m, mobile height 1.5m, carrier frequency 805MHz and penetration loss 19dB). For the hexagonal network model simulations, we consider 900 base stations (30x30) on a torus, with the cell radius $R = 0.26$Km (i.e., the surface of the hexagonal cell is equal to this of the disk of radius R). System bandwidth is $W = 10$MHz, noise power $N = -93$dBm.

Figure 2 shows the CDF of the SINR assuming the base station power $P = 58.5$dBm. We obtain results by simulations

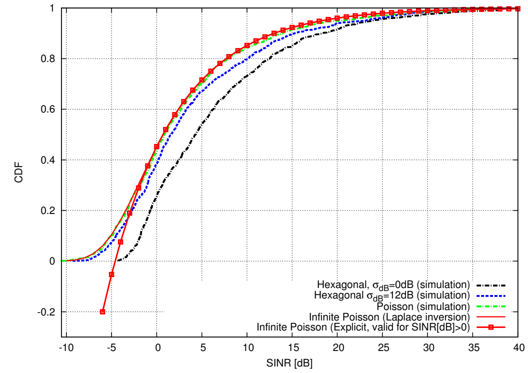

Fig. 2: CDF of the SINR for (finite) hexagonal network with and without ($\sigma_{\mathrm{dB}} = 0$dB) shadowing, finite Poisson as well as for the infinite Poisson model. The last in the legend curve corresponds to (20).

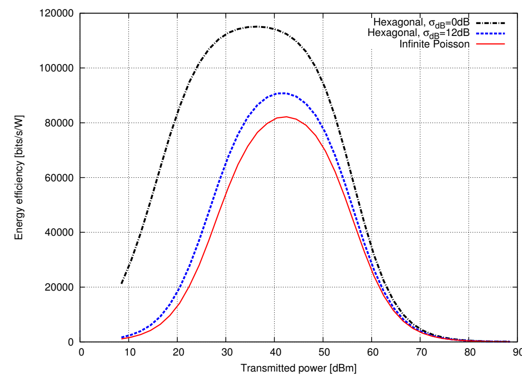

Fig. 3: Mean energy efficiency of finite hexagonal model with and without ($\sigma_{\mathrm{dB}} = 0$dB) shadowing as well as the infinite Poisson in function of the transmitted power.

for finite hexagonal network on the torus with and without shadowing as well as the finite Poisson network on the same torus. For the infinite Poisson model we use our method based on the inversion of the Laplace transform. For comparison, we plot also the curve corresponding the explicit expression (20), which is valid only for SINR > 1.

Figure 3 shows the expected energy efficiency $\mathbf{E}(\mathcal{E})$ of the finite hexagonal model with and without shadowing as well as the infinite Poisson, assuming the affine relation between consumed and emitted power with constants $c = 21,45$ and $d = 354.44$W.

On both figures we see that the infinite Poisson model gives a reasonable approximation of the (finite) hexagonal network provided the shadowing is high enough. In particular, the value of the transmitted power at which the hexagonal network with the shadowing attains the maximal expected energy efficiency is very well predicted by the Poisson model.

CONCLUSIONS

We present a powerful mathematical convergence result rigorously justifying the fact that *any* actual (including regular hexagonal) network is perceived by a typical user as an equivalent (infinite) Poisson network, provided log-normal shadowing variance is sufficiently high. Good approximations are obtained for logarithmic standard deviation of the shadowing greater than approximately 10dB, which is a realistic assumption in many urban scenarios. Moreover, the equivalent infinite Poisson representation is invariant with respect to an additional fading distribution. Using this representation we study the distribution of the SINR of the typical user. In particular, we evaluate and optimize the mean energy efficiency as a function of the base station transmit power.

APPENDIX

A. Proof of Theorem 3

In order to simplify the notation we set $n := \sigma^2$. Moreover, without loss of generality we assume that n takes positive integer values. Also, it is more convenient to study the point process of propagation losses on the logarithmic scale, which we do in what follows. In this regard denote by $\Lambda_{\log}$ the image of the measure Λ given by (3) with $a = \lambda\pi/K^2$ through the logarithmic mapping

$$\Lambda_{\log}((-\infty, s]) := \int_{R^+} 1(\log(t) \leq s)\,\Lambda(dt) = \frac{\pi}{K^2} e^{\frac{2s}{\beta}}$$

for $s \in \mathbb{R}$.

For a given i, we first observe by (6) that

$$\begin{aligned}\nu_n(s, |X_i|) &:= \mathbf{P}\left[\log\left(\frac{K(n)^\beta |X_i|^\beta}{S_i^{(n)}}\right) \leq s\right] \\ &= \mathbf{P}\left[Z_i \leq \frac{s - \beta\log(K|X_i|) - n/\beta}{\sqrt{n}}\right] \\ &= G\left[\frac{s - \beta\log(K|X_i|) - n/\beta}{\sqrt{n}}\right], \qquad (26)\end{aligned}$$

where G is the CDF of the standard Gaussian random variable.

Let $B_0(r) = \{x \in \mathbb{R}^2 : |x| < r\}$. We now need to derive two results.

Lemma 13:

$$\begin{aligned}\lim_{n\to\infty} \int_{B_0(b_n)\setminus B_0(a_n)} \nu_n(s, |x|)dx &= \lim_{n\to\infty} \int_{\mathbb{R}^2} \nu_n(s, |x|)dx \\ &= \Lambda_{\log}((-\infty, s])\end{aligned}$$

provided that a_n and b_n satisfy (10) and (11).

Proof: We start with the integral over $\mathbb{R}^2$ in polar form

$$\int_{\mathbb{R}^2} \nu_n(s, |x|)dx = 2\pi \int_0^\infty rG\left[\frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}\right] dr$$

Introduce a change of variables

$$t = \frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}, \qquad dt = -\frac{\beta}{\sqrt{n}}\frac{dr}{r},$$

$$r = \frac{1}{K}\exp\left[\frac{1}{\beta}\left(s - t\sqrt{n} - n/\beta\right)\right], \qquad dr = -\frac{\sqrt{n}r}{\beta}dt,$$

hence

$$\begin{aligned}&2\pi \int_0^\infty rG\left[\frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}\right] dr \\ &= 2\pi \int_{-\infty}^\infty \frac{1}{K^2}\exp\left[\frac{2}{\beta}\left(s - t\sqrt{n} - n/\beta\right)\right] G(t)\frac{\sqrt{n}}{\beta}dt \\ &= 2\pi\frac{\sqrt{n}}{\beta}\frac{\exp\left[\frac{2}{\beta}(s - n/\beta)\right]}{K^2} \int_{-\infty}^\infty \exp\left[\frac{-2t\sqrt{n}}{\beta}\right] G(t)dt\end{aligned}$$

Noting that $G(-t) = 1 - G(t) := \bar{G}(t)$, apply integration by parts on the above integral

$$\begin{aligned}\int_{-\infty}^\infty \exp\left[\frac{-2t\sqrt{n}}{\beta}\right] G(t)dt &= \int_{-\infty}^\infty \exp\left[\frac{2t\sqrt{n}}{\beta}\right] \bar{G}(t)dt \\ &= \frac{\beta}{2\sqrt{n}}\exp\left[\frac{2t\sqrt{n}}{\beta}\right] \bar{G}(t)\Big|_{-\infty}^\infty \\ &+ \frac{\beta}{2\sqrt{n}}\int_{-\infty}^\infty \exp\left[\frac{2t\sqrt{n}}{\beta}\right] g(t)dt,\end{aligned}$$

where $g(t)$ is the normal probability density, and, via the inequality

$$\int_y^\infty e^{-x^2}dx < \frac{e^{-y^2}}{(1+y)}, \quad y \geq 0, \qquad (27)$$

(cf [12, Section 7.8]) the final integrated term on the left vanishes. Hence

$$\begin{aligned}&\int_{-\infty}^\infty \exp\left[\frac{-2t\sqrt{n}}{\beta}\right] G(t)dt \\ &= \frac{\beta}{2\sqrt{n}}\int_{-\infty}^\infty \exp\left[-\frac{t^2}{2} + \frac{2t\sqrt{n}}{\beta}\right]\frac{dt}{\sqrt{2\pi}} \\ &= \frac{\beta}{2\sqrt{n}}\exp\left[\frac{2n}{\beta^2}\right]\int_{-\infty}^\infty \exp\left[-\frac{1}{2}\left(t - \frac{2\sqrt{n}}{\beta}\right)^2\right]\frac{dt}{\sqrt{2\pi}} \qquad (28) \\ &= \frac{\beta}{2\sqrt{n}}\exp\left[\frac{2n}{\beta^2}\right],\end{aligned}$$

which gives

$$2\pi\int_0^\infty rG\left[\frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}\right] dr = \frac{\pi}{K^2}e^{\frac{2s}{\beta}}.$$

We proceed similarly with the other integral in Lemma 13

$$\begin{aligned}&\int_{B_0(b_n)\setminus B_0(a_n)} \nu_n(s, |x|)dx \\ &= 2\pi\int_{a_n}^{b_n} rG\left[\frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}\right] dr.\end{aligned}$$

The change of variables $t = \frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}$ gives

$$\begin{aligned}&2\pi\int_{a_n}^{b_n} rG\left[\frac{s - \beta\log(Kr) - n/\beta}{\sqrt{n}}\right] dr \\ &= 2\pi\int_{v_n}^{u_n} \frac{1}{K^2}\exp\left[\frac{2}{\beta}\left(s - t\sqrt{n} - n/\beta\right)\right] G(t)\frac{\sqrt{n}}{\beta}dt \\ &= 2\pi\frac{\sqrt{n}}{\beta}\frac{\exp\left[\frac{2}{\beta}(s - n/\beta)\right]}{K^2}\int_{v_n}^{u_n} \exp\left[\frac{-2t\sqrt{n}}{\beta}\right] G(t)dt\end{aligned}$$

where

$$u_n = \frac{s - \beta\log(Ka_n) - n/\beta}{\sqrt{n}},$$
$$v_n = \frac{s - \beta\log(Kb_n) - n/\beta}{\sqrt{n}}.$$

Moreover

$$\int_{v_n}^{u_n} \exp\left[\frac{-2t\sqrt{n}}{\beta}\right] G(t)dt$$
$$= \int_{-u_n}^{-v_n} \exp\left[\frac{2t\sqrt{n}}{\beta}\right] \bar{G}(t)dt$$
$$= \frac{\beta}{2\sqrt{n}} \exp\left[\frac{2t\sqrt{n}}{\beta}\right] \bar{G}(t)\Big|_{-u_n}^{-v_n} + \frac{\beta}{2\sqrt{n}} \int_{-u_n}^{-v_n} \exp\left[\frac{2t\sqrt{n}}{\beta}\right] g(t)dt$$
$$= \frac{\beta}{2\sqrt{n}} \exp\left[\frac{2t\sqrt{n}}{\beta}\right] \bar{G}(t)\Big|_{-u_n}^{-v_n}$$
$$+ \frac{\beta}{2\sqrt{n}} \exp\left[\frac{2n}{\beta^2}\right] \int_{-u_n}^{-v_n} \exp\left[-\frac{1}{2}\left(t - \frac{2\sqrt{n}}{\beta}\right)^2\right] \frac{dt}{\sqrt{2\pi}}.$$

We now derive the conditions of a_n and b_n which ensure that the above integral converges and the integrated term disappears. The latter can be achieved, given inequality (27), in the limit as $-u_n$ and $-v_n$ both approach infinity, or equivalently $\frac{\beta \log(Kb_n)}{\sqrt{n}} + \frac{\sqrt{n}}{\beta} \to \infty$ and $\frac{\beta \log(Ka_n)}{\sqrt{n}} + \frac{\sqrt{n}}{\beta} \to \infty$ as $n \to \infty$, which agrees with conditions (10) and (11). Further conditions are revealed after the change of variable $w = t - 2\sqrt{n}/\beta$, yielding the integral

$$\int_{-u_n}^{-v_n} e^{-\frac{1}{2}\left(t-\frac{2\sqrt{n}}{\beta}\right)^2} \frac{dt}{\sqrt{2\pi}} = \int_{-u_n-\frac{2\sqrt{n}}{\beta}}^{-v_n-\frac{2\sqrt{n}}{\beta}} e^{-\frac{w^2}{2}} \frac{dw}{\sqrt{2\pi}}$$

whose limits of integration imply, in light of the earlier integral (28), that as $n \to \infty$ the following $\frac{\beta \log(Kb_n)}{n} > 1/\beta$ and $\frac{\beta \log(Ka_n)}{n} < 1/\beta$ are required, of which both conditions (10) and (11) satisfy.

■

Lemma 14: *Assume (5), (10) and (11), then*

$$\lim_{n\to\infty} \sum_{X_i \in \phi \cap (B_0(b_n) \setminus B_0(a_n))} \nu_n(s, |X_i|) \tag{29}$$
$$= \lim_{n\to\infty} \sum_{X_i \in \phi} \nu_n(s, |X_i|) = \Lambda_{\log}((-\infty, s]). \tag{30}$$

Proof: For $k \geq 0$ and a fixed $\epsilon > 0$, let $r_k = e^{\epsilon k}$ and $A_k = B_0(r_{k+1}) \setminus B_0(r_k)$, and write the summation in (30) as

$$\sum_{X_i \in \phi} \nu_n(s, |X_i|)$$
$$= \sum_{X_i \in \phi \cap B_0(r_{k_0})} \nu_n(s, |X_i|) + \sum_{k=k_0}^{\infty} \sum_{X_i \in \phi \cap A_k} \nu_n(s, |X_i|), \tag{31}$$

for some $k_0 \geq 0$, whose value will be fixed later on. In the limit of $n \to \infty$, the first summation in (31) disappears

$$\sum_{X_i \in \phi \cap B_0(r_{k_0})} \nu_n(s, |X_i|)$$
$$= \sum_{X_i \in \phi \cap B_0(r_{k_0})} \mathbf{P}\left[Z \leq \frac{s - \beta \log(K|X_i|) - n/\beta}{\sqrt{n}}\right]$$
$$= \sum_{X_i \in \phi \cap B_0(r_{k_0})} G\left[\frac{s - \beta \log(K|X_i|) - n/\beta}{\sqrt{n}}\right]$$
$$\leq \phi(B_0(r_{k_0}))G\left[\frac{s - \beta \log(K|X_*|) - n/\beta}{\sqrt{n}}\right]$$
$$\to 0 \qquad (n \to \infty).$$

where X_* gives the maximum of $G\left[\frac{s-\beta \log(K|X|)-n/\beta}{\sqrt{n}}\right]$ over $X \in \phi \cap B_0(r_{k_0})$ which exists since ϕ is (by our assumption) a locally finite point measure. For the second summation in (31) we write $\nu_n(s, |X_i|) = \nu_n(s, |x|\frac{|X_i|}{|x|})$, hence

$$\nu_n(s, |X_i|)$$
$$= \frac{1}{|A_k|} \int_{A_k} \nu_n(s, |x|\frac{|X_i|}{|x|})dx.$$

Then the bounds

$$e^{-\epsilon} = \frac{r_k}{r_{k+1}} \leq \frac{|X_i|}{|x|} \leq \frac{r_{k+1}}{r_k} = e^{\epsilon},$$

and form of ν_n, which implies $\nu_n(s, |x|e^{\epsilon}) = \nu_n(s - \beta\epsilon, |x|)$, lead to the lower bound

$$\sum_{k=k_0}^{\infty} \sum_{X_i \in \phi \cap A_k} \nu_n(s, |X_i|) \geq \sum_{k=k_0}^{\infty} \frac{\phi(A_k)}{|A_k|} \int_{A_k} \nu_n(s-\beta\epsilon, |x|)dx, \tag{32}$$

and the upper bound

$$\sum_{k=k_0}^{\infty} \sum_{X_i \in \phi \cap A_k} \nu_n(s, |X_i|) \leq \sum_{k=k_0}^{\infty} \frac{\phi(A_k)}{|A_k|} \int_{A_k} \nu_n(s+\beta\epsilon, |x|)dx. \tag{33}$$

Moreover, we write

$$\frac{\phi(A_k)}{|A_k|} = \frac{\phi(B_0(r_{k+1})) - \phi(B_0(r_k))}{|B_0(r_{k+1})| - |B_0(r_k)|}$$
$$= \frac{\frac{\phi(B_0(r_{k+1}))}{|B_0(r_{k+1})|} - \frac{\phi(B_0(r_k))}{|B_0(r_k)|}\frac{|B_0(r_k)|}{|B_0(r_{k+1})|}}{1 - \frac{|B_0(r_k)|}{|B_0(r_{k+1})|}}$$
$$= \frac{\frac{\phi(B_0(r_{k+1}))}{|B_0(r_{k+1})|} - \frac{\phi(B_0(r_k))}{|B_0(r_k)|}e^{-2\epsilon}}{1 - e^{-2\epsilon}},$$

and requirement (5) yields $\lim_{k\to\infty} \frac{\phi(A_k)}{|A_k|} = \lambda$. Hence, for any fixed $\delta > 0$, there exists a $k_0(\delta)$ such that for all $k \geq k_0$, the bounds

$$(1 - \delta)\lambda \leq \frac{\phi(A_k)}{|A_k|} \leq (1 + \delta)\lambda,$$

hold. Lower bound (32) becomes

$$\sum_{k=k_0}^{\infty} \sum_{X_i \in \phi \cap A_k} \nu_n(s, |X_i|) \geq \sum_{k=k_0}^{\infty} (1 - \delta)\lambda \int_{A_k} \nu_n(s - \beta\epsilon, |x|)dx,$$
$$= (1 - \delta)\lambda \int_{|x| \geq r_{k_0}} \nu_n(s - \beta\epsilon, |x|)dx$$

Finally, Lemma 13 allows us to set $a_n = r_{k_0}$ and $b_n = \infty$, hence

$$\lim_{n\to\infty} \sum_{k=k_0}^{\infty} \sum_{X_i\in\phi\cap A_k} \nu_n(s,|X_i|) \geq (1-\delta)\frac{\pi\lambda}{K^2} e^{\frac{2s-\beta\epsilon}{\beta}},$$

and similarly the upper bound (33) becomes

$$\lim_{n\to\infty} \sum_{k=k_0}^{\infty} \sum_{X_i\in\phi\cap A_k} \nu_n(s,|X_i|) \leq (1-\delta)\frac{\pi\lambda}{K^2} e^{\frac{2s+\beta\epsilon}{\beta}},$$

and indeed $\epsilon \to 0$ and $\delta \to 0$ completes the proof of (30). The other result, (29), can be proved by a straightforward modification of the above arguments. ∎

Proof of Theorem 3: We use a classical convergence result [8, Theorem 11.22V] which in our setting requires verification of the following two conditions (cf [8, (11.4.2) and (11.4.3)])

$$\sup_i \nu_n^i(A) \to 0 \quad (n\to\infty). \tag{34}$$

and

$$\sum_i \nu_n^i(A) \to \Lambda_{\log}(A) \quad (n\to\infty), \tag{35}$$

for all bounded Borel sets $A \subset \mathbb{R}$, where $\nu_n^i(\cdot)$ is the (probability) measure on $\mathbb{R}$ defined by setting $\nu_n^i((-\infty,s]) := \nu_n(s,|X_i|)$. The first condition, (34), clearly holds by (26) for any locally finite ϕ. The second condition, (35) follows from Lemma 13 and 14, which establish the required convergence for $A = (-\infty, s]$ and any $s \in \mathbb{R}$. This is enough to conclude the convergence for all bounded Borel sets. ∎

B. Representation of the function $\varphi_\beta(z)$ given by (14)

Lemma 15: *The function $\varphi_\beta(z)$ given by (14) can be written as*

$$\varphi_\beta(z) = -\frac{2}{\beta} z^{\frac{2}{\beta}} \gamma\left(-\frac{2}{\beta}, z\right) \tag{36}$$

$$= \Gamma\left(1-\frac{2}{\beta}\right)\gamma^*\left(-\frac{2}{\beta}, z\right) \tag{37}$$

where

$$\gamma^*(\alpha,z) = \frac{z^{-\alpha}}{\Gamma(\alpha)}\gamma(\alpha,z) \tag{38}$$

$$= e^{-z}\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha+k+1)} \tag{39}$$

is an analytic function on $\mathbb{C}\times\mathbb{C}$. Moreover

$$\varphi_\beta(z) > 0, \quad \forall\beta > 2, \forall z \in \mathbb{R}^+ \tag{40}$$

Proof: Using (14) and recurrence formula [13, 6.5.22] we obtain (36). Introducing the modified incomplete gamma function γ^* defined by (38) (which is holomorphic on $\mathbb{C}\times\mathbb{C}$; see [13, 6.5.4]) we obtain (37). The expansion (39) is given in [13, 6.5.29]. This expansion shows that if $\alpha > -1$ then for any $z\in\mathbb{R}^+$, $\gamma^*(\alpha,z) > 0$, and therefore by (37) $\varphi_\beta(z) > 0$ for any $\beta > 2, z\in\mathbb{R}^+$. ∎

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