Parametrization of matrix-valued lossless functions based on boundary interpolation

Ralf L.M. Peeters, Martine Olivi, Bernard Hanzon

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***Abstract*— This paper is concerned with parametrization issues for rational lossless matrix valued functions. In the same vein as previous works, interpolation theory with metric constraints is used to ensure the lossless property. We consider here boundary interpolation and provide a new parametrization of balanced canonical forms in which the parameters are angular derivatives. We finally investigate the possibility to parametrize orthogonal wavelets with vanishing moments using these results.**

I. Introduction

Lossless systems and their transfer functions play a central role in system theory mainly, but not only, due to the Douglas-Shapiro-Shields factorization. Lossless matrix-valued functions generalize, in some sense, the notion of denominator to the matrix case. As such they are involved in many representations of stable systems. They are also present in many applications, as the scattering matrix of a resonant filter or the polyphase matrix of a filter bank, for example. Their parametrization is an important issue for many purposes going from optimization, model reduction to physical parameters recovery. In the same vein as previous works [3], [4], we aim to stress the connections between two complementary descriptions: canonical forms on one hand and Schur type algorithms on the other hand. Thereby, we get parametrizations which combine the computational interest of canonical forms and the conceptual interest of Schur analysis.

In [4], by making appropriate choices in the Schur algorithm, construction of balanced canonical forms is achieved in a recursive way with unitary matrix multiplications. Each step of the recursion involves an interpolation condition at a point located within the analyticity domain. However, for several reasons, it may prove helpful to allow for interpolation points on the boundary of the analyticity domain (the circle in discrete-time and the imaginary axis in continuous-time). In [7], the tridiagonal canonical form of Ober in continuous-time was obtained from a recursion involving interpolation conditions *at infinity*. Vanishing moments, diagonal Markov parameters, can also be interpreted in term of boundary interpolation conditions.

In this paper, we extend the work of [4] on *discrete-time lossless systems* to allow for interpolation conditions *on the unit circle*.

II. Parametrization of lossless matrices from the tangential Schur algorithm.

In this section, we briefly recall the results of [4]. The tangential Schur algorithm recursively handles an interpolation condition of the form

$$F(1/\bar{w})u = v, \tag{1}$$

in which $w \in \mathbb{D}$, u and v are m-vectors such that $\|u\| = 1$ and $\|v\| < 1$.

The solutions of this interpolation problem (see e.g. [1]) are given by means of a linear fractional transformation (LFT) of the form

$$\mathcal{T}_\Theta(G) = (\Theta_{22}G + \Theta_{21})(\Theta_{12}G + \Theta_{11})^{-1}, \tag{2}$$

where

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}$$

is a rational J-inner function of size $2m \times 2m$. More precisely,

Theorem 2.1: All the rational lossless functions $F(z)$ satisfying the interpolation condition (1) can be represented as a linear fractional transformation (LFT)

$$F = \mathcal{T}_{\Theta_{w,u,v}}(G) \tag{3}$$

of a lossless function $G(z)$. The symbol $\Theta_{w,u,v}$ is the J-inner function given, in terms of the interpolation data, by

$$\Theta_{w,u,v}(z) = \left[I_{2m} + \left(\frac{b_w(z)}{b_w(\nu)} - 1 \right) \frac{\begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}^* J}{1 - \|v\|^2} \right] H, \tag{4}$$

where $b_w(z)$ is the Blaschke factor

$$b_w(z) = \frac{z - w}{1 - \bar{w}z}, \tag{5}$$

and J the matrix

$$J = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}. \tag{6}$$

The unit complex number ν and the constant J-unitary matrix H ($HJH^* = J$) are arbitrary and should be specified. Moreover, we have that

$$\deg F = \deg G + 1.$$

Proof: see [4] ∎

R. Peeters is with the Department of Knowledge Engineering, Universiteit Maastrich, PO Box 616, 6200 MD Maastricht, The Netherlands ralf.peeters@math.unimaas.nl

M. Olivi is with INRIA Sophia-Antipolis, BP 93, 06902 Sophia-Antipolis Cedex, France Martine.Olivi@sophia.inria.fr

B. Hanzon is with the School of Mathematical Sciences, University College Cork, Cork, Ireland b.hanzon@ucc.ie

The tangential Schur algorithm yields from a lossless function $F(z)$ of McMillan degree n a finite sequence of interpolation vectors associated with

- a sequence $\mathbf{w} = (w_n, w_{n-1}, \ldots, w_1)$ of points of the unit disk
- a sequence $\mathbf{u} = (u_n, u_{n-1}, \ldots, u_1)$ of interpolation direction vectors

It works as follows: let $F_n = F$. Then, for $k = n, n-1, \ldots$ put $v_k = F_k(1/\bar{w}_k)u_k$;

- if $\|v_k\| < 1$, let
$$F_{k-1} = T^{-1}_{\Theta_{w_k,u_k,v_k}} F_k, \tag{7}$$
in which $\Theta_{w_k,u_k,v_k}(z)$ is defined by (4).
- if v_k has norm 1, then stop.

A local coordinate map $\varphi(\mathbf{w},\mathbf{u})$ can be associated with the sequences $\mathbf{w}$ and $\mathbf{u}$ by

$$\varphi(\mathbf{w},\mathbf{u}) : F(z) \in \mathcal{V}(\mathbf{w},\mathbf{u}) \mapsto (v_1, v_2, \ldots, v_k, F_0),$$

where the domain $\mathcal{V}(\mathbf{w},\mathbf{u})$ of the map consists with functions for which $\|v_k\| < 1$ for $k = n, n-1\ldots, 1$.

The freedom in the choice of ν and H has been used in the literature to serve various purposes. In [4] it is used to get a simple computation of balanced canonical forms. We introduce the Halmos extension $H(E)$ of a contractive matrix E

$$\begin{aligned} H(E) &= \begin{bmatrix} (I-EE^*)^{-1/2} & E(I-E^*E)^{-1/2} \\ E^*(I-EE^*)^{-1/2} & (I-E^*E)^{-1/2} \end{bmatrix} \\ &= \begin{bmatrix} (I-EE^*)^{-1/2} & (I-EE^*)^{-1/2}E \\ (I-E^*E)^{-1/2}E^* & (I-E^*E)^{-1/2} \end{bmatrix}. \end{aligned} \tag{8}$$

Proposition 1: Let

$$\hat{\Theta}_{w,u,v} = H(uv^*) \begin{bmatrix} I_m + (b_w(z)-1)uu^* & 0 \\ 0 & I_m \end{bmatrix} H(\bar{w}uv^*). \tag{9}$$

A balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of $\tilde{G} = \mathcal{T}_{\hat{\Theta}_{w,u,v}}(G)$ can be computed from a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ of G by the formula

$$\begin{bmatrix} \widetilde{D} & \widetilde{C} \\ \widetilde{B} & \widetilde{A} \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & I_{n-1} \end{bmatrix} \begin{bmatrix} D & 0 & C \\ 0 & 1 & 0 \\ B & 0 & A \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I_{n-1} \end{bmatrix}, \tag{10}$$

in which U and V are the unitary matrices

$$U = \begin{bmatrix} I_m - (1 + w\,\kappa(w,v))uu^* & \alpha(w,v)\,u \\ \alpha(w,v)\,u^* & \overline{w}\kappa(w,v) \end{bmatrix} \tag{11}$$

$$V = \begin{bmatrix} I_m - (1-\kappa(w,v))\frac{vv^*}{\|v\|^2} & \alpha(w,v)\,v \\ -\alpha_{w,v}\,v^* & \kappa(w,v) \end{bmatrix} \tag{12}$$

with

$$\alpha(w,v) = \frac{\sqrt{1-|w|^2}}{\sqrt{1-|w|^2\|v\|^2}}, \tag{13}$$

$$\kappa(w,v) = \frac{\sqrt{1-\|v\|^2}}{\sqrt{1-|w|^2\|v\|^2}}. \tag{14}$$

Proof: see [4] ∎

Note that formula (9) is obtained from (4) by choosing

$$\nu = \frac{1+w}{1+\bar{w}}, \quad H = H(uv^*)^{-1}H(\bar{w}uv^*).$$

III. Parametrization of lossless matrices from boundary data

Boundary interpolation of rational Schur functions is studied in [1]. We give here a specific treatment for lossless matrix functions. We are thus interested in rational lossless functions F that satisfy an interpolation condition at a given point ξ on the unit circle $\mathbb{T}$. As it turns out, also the first derivative of F at ξ plays a role.

A. The angular derivative

Note that at a point $\xi \in \mathbb{T}$ it holds that $F(\xi)$ is unitary, so that $F(\xi)^{-1} = F(\xi)^*$. This gives the identity

$$\xi F(\xi)^* F'(\xi) = \bar{\xi} F'(\xi)^* F(\xi),$$

which proves that the expression $\xi F(\xi)^* F'(\xi)$ is actually Hermitian.

This also shows that a lossless function F and its derivative F' have a particular relationship at a given point ξ on the unit circle. More precisely, we have the following result.

Lemma 3.1: Let $F(z)$ be a $m \times m$ rational lossless function, $\xi \in \mathbb{T}$ a point on the unit circle, and $u \in \mathbb{C}^m$ a given vector. Then the scalar expression

$$\rho = -\xi u^* F(\xi)^* F'(\xi) u, \tag{15}$$

is nonnegative real: $\rho \geq 0$.

Proof: To show this, we may use the Potapov decomposition [8] of F

$$F = U_0 B_{w_1,u_1} B_{w_2,u_2} \ldots B_{w_n,u_n} \tag{16}$$

where $w_1, w_2, \ldots, w_n$ belong to the open unit disk, $u_1, u_2, \ldots, u_n$ are unit vectors,

$$B_{w,u} = I + \left(b_w^{-1} - 1\right) uu^*, \quad b_w(z)^{-1} = \frac{1-\bar{w}z}{z-w}, \tag{17}$$

are elementary (degree one) Potapov factors, and U_0 is a constant unitary matrix. First note that F' is computed as:

$$F' = U_0 \sum_{k=1}^{n} \left(\prod_{l=1}^{k-1} B_{w_l,u_l} \right) (b_{w_k}^{-1})' u_k u_k^* \left(\prod_{l=k+1}^{n} B_{w_l,u_l} \right),$$

in which we have that

$$(b_{w_k}^{-1})'(z) = -\frac{1-|w_k|^2}{(z-w_k)^2}.$$

At the point $\xi \in \mathbb{T}$ we shall introduce the notation $U_k = B_{w_k,u_k}(\xi)$. Note that $b_{w_k}(\xi)$ has modulus 1, and that U_k is unitary. We now have:

$$F(\xi) = U_0 U_1 \ldots U_n,$$

and

$$F'(\xi) = -\sum_{k=1}^{n} \frac{1-|w_k|^2}{(\xi-w_k)^2} \left(\prod_{l=0}^{k-1} U_l \right) u_k u_k^* \left(\prod_{l=k+1}^{n} U_l \right),$$

Therefore, using the relation

$$U_k^* u_k u_k^* = \overline{b_{w_k}^{-1}(\xi)} u_k u_k^* = \frac{1-\bar{\xi} w_k}{\bar{\xi}-\bar{w}_k} u_k u_k^*,$$

we get:

$$-\xi F(\xi)^* F'(\xi) = \sum_{k=1}^{n} \frac{1-|w_k|^2}{|\xi - w_k|^2} \hat{U}_k^* u_k u_k^* \hat{U}_k,$$

with

$$\hat{U}_k = \prod_{l=k+1}^{n} U_l.$$

Obviously this expression is positive semi-definite, as it is a positively weighted sum of positive semi-definite matrix products. Therefore the scalar quantity $\rho = -\xi u^* F(\xi)^* F'(\xi) u$ is nonnegative real. ■

Note that from the available expression for $\xi F'(\xi) F(\xi)^*$ we can quickly deduce the conditions under which ρ becomes equal to zero. This only happens when each of the terms in the weighted sum becomes equal to zero. Note that the weights are all strictly positive. Hence: $\rho = 0$ occurs if and only if

$$\begin{array}{rcl} u_n^* u & = & 0, \\ u_{n-1}^* U_n u & = & 0, \\ u_{n-2}^* U_{n-1} U_n u & = & 0, \\ \vdots & \vdots & \vdots \\ u_1^* U_2 \dots U_{n-1} U_n u & = & 0. \end{array}$$

But because of the structure of the matrix U_n , the condition $u_n^* u = 0$ implies that $U_n u = u$, whence the condition $u_{n-1}^* U_n u = 0$ becomes equivalent to the condition $u_{n-1}^* u = 0$. Continuing this argument it follows that $\rho = 0$ if and only if

$$u_n^* u = u_{n-1}^* u = \cdots = u_1^* u = 0.$$

This allows us to draw the following conclusions.

(i) In the case $n = 0$ no conditions apply and $\rho = 0$. Indeed, the inner function $F(z) = U_0$ is constant and its derivative $F'(z)$ is the zero matrix.

(ii) In the scalar case (with $n > 0$) the fact that $\|u_n\| = 1$ implies that $\rho > 0$ if the vector u is nonzero. (See also [1][Lemma 21.1.1.])

(iii) In the multivariable case (with $n > 0$) each of the conditions $u_k * u = 0$ generically does not hold, so that generically $\rho > 0$. If u is to be chosen from a basis for the space $\mathbb{C}^p$, then at least one such choice exists which achieves $u_k * u \neq 0$.

(iv) In the multivariable case (with $n > 0$) if the vectors $u_1, u_2, \dots, u_n$ happen to span the space $\mathbb{C}^p$ (which requires $n \geq p$), then $\rho > 0$ for every nonzero vector u . Otherwise, (and certainly for $0 < n < p$) a nontrivial subspace of vectors u does exist for which $\rho = 0$.

(v) The conditions under which $\rho = 0$ does or does not hold, are completely independent of the value of the interpolation point ξ. Therefore, a suitable choice of u which achieves strict positivity of ρ can be made independently of the choice of an interpolation point ξ .

B. Boundary interpolation of lossless functions.

A well-posed boundary interpolation problem includes the value of the angular derivative and can be stated as follows. Find all the lossless functions $F(z)$ such that

$$\left\{ \begin{array}{rcl} F(\xi) u & = & v \\ v^* F'(\xi) u & = & -\bar{\xi}\rho \end{array} \right. \tag{18}$$

where ξ belongs to the unit circle, u and v are unit vectors and $\rho > 0$ is the angular derivative. Any solution of the (general angular derivative) interpolation problem (18) can be represented as a linear fractional transformation of a lossless function G such that $G(\xi)u \neq v$ and the degree of G is equal to the degree of F minus one.

The J-inner symbol associated with the LFT is, up to a J-unitary constant factor

$$\Phi_{\xi,u,v,\rho} = I_{2m} - \frac{1}{2\rho} c_\xi \left[\begin{array}{c} u \\ v \end{array} \right] \left[\begin{array}{c} u \\ v \end{array} \right]^* J, \tag{19}$$

in which c_ξ is the Caratheodory function

$$c_\xi(z) = \frac{\xi + z}{\xi - z}. \tag{20}$$

Note that an elementary J-inner factor which is analytic in the open unit disk must be either of the form (4) or of the form (19) if its pole is on the disk [2].

Theorem 3.1: **Part I - increasing the degree**
Let $\xi \in \mathbb{T}$, $u \in \mathbb{C}^m, \|u\| = 1, v \in \mathbb{C}^m$ with $\|v\| = 1$, and $\rho \in \mathbb{R}^+$. Let $\Phi_{\xi,u,v,\rho}$ be the elementary J-inner factor given by (19).
Let G be a $m \times m$ rational lossless function of McMillan degree n such that $G(\xi)u \neq v$.
Let F be defined by $F = \mathcal{T}_{\Phi_{\xi,u,v,\rho}}(G)$.
Then F is a $m \times m$ rational lossless function of McMillan degree $n+1$ such that

$$\begin{array}{rcl} F(\xi) u & = & v, \\ v^* F'(\xi) u & = & -\bar{\xi}\rho. \end{array}$$

Part II - decreasing the degree
Let F be a $m \times m$ rational lossless function of McMillan degree $n + 1 \geq 1$.
Let $\xi \in \mathbb{T}$ and let $u \in \mathbb{C}^m$ with $\|u\| = 1$ be such that $\rho := -\xi u^* F(\xi)^* F'(\xi) u$ is strictly positive.
Let $v := F(\xi)u$ (which satisfies $\|v\| = 1$), and let $\Phi_{\xi,u,v,\rho}$ be the elementary J-inner factor (19).
Let G be defined by $G = \mathcal{T}^{-1}_{\Phi_{\xi,u,v,\rho}}(F)$.
Then G is a $m \times m$ rational lossless function of McMillan degree n such that $G(\xi)u \neq v$.

Proof: **Part I.** The linear fractional transformation $F =$

$\mathcal{T}_{\Phi_{\xi,u,v,\rho}}(G)$ can be worked out as follows:

$$\begin{aligned} F &= \left(G + \frac{1}{2\rho} c_\xi v(v^* G - u^*)\right) \\ &\quad \left(I_m + \frac{1}{2\rho} c_\xi u(v^* G - u^*)\right)^{-1} \\ &= \left(G + \frac{1}{2\rho} c_\xi v(v^* G - u^*)\right) \\ &\quad \left(I_m - \frac{c_\xi}{2\rho + c_\xi(v^* G u - 1)} u(v^* G - u^*)\right) \end{aligned}$$

yielding

$$F(z) = G(z) - \frac{(z+\xi)(G(z)u - v)(v^* G(z) - u^*)}{2\rho(\xi - z) + (z+\xi)(v^* G(z)u - 1)}$$

Therefore we have that

$$F(z)u = v + \frac{2\rho(\xi - z)(G(z)u - v)}{2\rho(\xi - z) + (z+\xi)(v^* G(z)u - 1)}.$$

At the point $z = \xi$ it follows that $F(\xi)u = v$, which shows the first interpolation condition to be satisfied. For the second interpolation condition we have to perform differentiation. At the point $z = \xi$ we have:

$$F'(\xi)u = -\frac{\rho(G(\xi)u - v)}{\xi(v^* G(\xi)u - 1)}. \tag{21}$$

(Note that division by zero does not occur, because of the requirement $G(\xi)u \neq v$). Upon premultiplication by v^* it is obtained that:

$$v^* F'(\xi)u = -\bar{\xi}\rho,$$

which shows the second interpolation condition to be satisfied.

Part II.

The linear fractional transformation $F = \mathcal{T}_{\Phi^{-1}_{\xi,u,v,\rho}}(Q)$ can be worked out as under Part I, upon noting that algebraically the inverse of the J-inner matrix is obtained as

$$\Phi^{-1}_{\xi,u,v,\rho} = \Phi_{\xi,u,v,-\rho}$$

i.e., by reversing the sign of ρ. Therefore:

$$G(z) = F(z) + \frac{(z+\xi)(F(z)u - v)(v^* F(z) - u^*)}{2\rho(\xi - z) - (z+\xi)(v^* F(z)u - 1)}$$

From the definition $v = F(\xi)u$ it is clear that pole-zero cancellation takes place at $z = \xi$. In fact, a double pole-zero cancellation takes place at that point. To see this, we proceed as follows. As under Part I it holds that

$$G(z)u = v + \frac{2\rho(\xi - z)(F(z)u - v)}{2\rho(\xi - z) - (z+\xi)(v^* F(z)u - 1)}.$$

which can be rewritten as:

$$G(z)u = v + \frac{2\rho \frac{F(z)u - v}{z - \xi}}{\frac{2\rho + (z+\xi)\frac{v^* F(z)u - 1}{z - \xi}}{z - \xi}}.$$

From the definitions $F(\xi)u = v$ and $\rho = -\xi v^* F'(\xi)u$, we now have that:

$$\begin{aligned} G(z)u &= v + \frac{2\rho F'(\xi)u}{\frac{-2\xi v^* F'(\xi)u + (z+\xi)\frac{v^*(F(z) - F(\xi))u}{z-\xi}}{z-\xi}} \\ &= v + \frac{2\rho F'(\xi)u}{v^*\left(F'(\xi) + (z+\xi)\frac{F(z) - F(\xi) - F'(\xi)(z-\xi)}{(z-\xi)^2}\right)u}. \end{aligned}$$

At the point $z = \xi$ it follows that :

$$G(\xi)u = v + \frac{2\rho F'(\xi)u}{-\bar{\xi}\rho + \xi v^* F''(\xi)u}.$$

Upon multiplication by v, this yields

$$v^* G(\xi)u = 1 + \frac{2\bar{\xi}\rho^2}{\bar{\xi}\rho - \xi v^* F''(\xi)u}$$

where the second term is nonzero (as it has a nonzero numerator), so that it follows that $G(\xi)u \neq v$. ∎

C. Computing balanced realizations

In [6], a unified framework is presented in which linear fractional transformations on transfer functions are represented by corresponding linear fractional transformations on state-space realization matrices.

Theorem 3.2: Let G be a $m \times m$ rational matrix function which is proper and let (A, B, C, D) be a minimal state-space realization of G. Let

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

be a $2m \times 2m$ rational matrix function of McMillan degree d (block-partitioned into blocks of size m) and such that $\Theta(1/z)$ is proper. Let

$$(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) = \left(\mathcal{A}, \begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_2 \end{bmatrix}, \begin{bmatrix} \mathcal{C}_1 \\ \mathcal{C}_2 \end{bmatrix}, \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix}\right)$$

be a block partitioned minimal state-space realization of $\Theta(1/z)$. Then a state-space realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ for $F = \mathcal{T}_\Theta(G)$ is provided by

$$\begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix} = \mathcal{T}_\Delta(R_e),$$

where

$$R_e = \begin{bmatrix} D & 0 & C \\ 0 & I_m & 0 \\ B & 0 & A \end{bmatrix}$$

is an extended realization matrix for G and

$$\Delta = \begin{bmatrix} \mathcal{D}_{11} & \mathcal{C}_1 & 0 & \mathcal{D}_{12} & 0 & 0 \\ \mathcal{B}_1 & \mathcal{A} & 0 & \mathcal{B}_2 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 \\ \mathcal{D}_{21} & \mathcal{C}_2 & 0 & \mathcal{D}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{bmatrix}$$

Proof: see [6, th. 8] ∎

This theorem can be applied directly to the situation at hand. A (J-balanced) state-space realization $(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})$ of $\Phi_{\xi,u,v,\rho}$ is given by

$$\begin{aligned} \mathcal{A} &= \bar{\xi} & \mathcal{B} &= \tfrac{1}{\sqrt{\rho}}\begin{bmatrix}u\\v\end{bmatrix}^* J \\ \mathcal{C} &= -\tfrac{\bar{\xi}}{\sqrt{\rho}}\begin{bmatrix}u\\v\end{bmatrix} & \mathcal{D} &= I_{2m} - \tfrac{1}{\sqrt{2\rho}}\begin{bmatrix}u\\v\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}^* J. \end{aligned}$$

Theorem 3.3: A balanced state-space realization $(\hat{A},\hat{B},\hat{C},\hat{D})$ for $F = \mathcal{T}_\Theta(G)$ of McMillan degree $n+1$ is computed from a given balanced realization (A,B,C,D) of G by

$$\left[\begin{array}{c|c} \hat{D} & \hat{C} \\ \hline \hat{B} & \hat{A} \end{array}\right] = \left[\begin{array}{c|cc} D & 0 & C \\ \hline 0 & -\xi & 0 \\ B & 0 & A \end{array}\right] + \frac{\left[\begin{array}{c} Du-v \\ \hline \dfrac{}{2\sqrt{\rho}\xi} \\ Bu \end{array}\right]\left[\begin{array}{c} D^*v-u \\ \hline 2\sqrt{\rho} \\ C^*v \end{array}\right]^*}{1-v^*Du+2\rho} \tag{22}$$

D. Parametrization issue.

Theorem 3.1 is the basis for a Schur type recursive algorithm and for a parametrization of lossless functions of degree n in terms of a sequence of boundary interpolation data. Let $\sigma = (\xi_k, u_k)_{k=1,\dots,n}$, $|\xi_k| = 1$, $\|u_k\| = 1$, be a sequence of boundary interpolation data. Given a lossless function $F(z)$ of McMillan degree n, the algorithm handles the sequence of boundary interpolation conditions as follows: $F_n = F$ and for $k = n, n-1, \dots 2$ let

$$\begin{aligned} v_k &= F_k(\xi_k)u_k, \\ \rho_k &= -\xi_k v_k^* F'(\xi_k)u_k. \end{aligned}$$

If $\rho_k > 0$, then put

$$F_{k-1} = T^{-1}_{\Phi_{\xi_k,u_k,v_k,\rho_k}}(F_k), \tag{23}$$

otherwise stop. Recall that generically, $\rho_k > 0$, as explained in section III-A.

The sequences $(v_1, v_2, \dots, v_n)$ and $(\rho_1, \rho_2, \dots \rho_n)$ can be used as parameters, the function $F(z)$ is then computed by reversing the algorithm. However, all the sequences of v_i's are not allowed since the conditions

$$v_k \neq F_{k-1}(\xi_k)u_k, \quad k = 1, \dots, n,$$

must be satisfied.

IV. Boundary interpolation as a limit of classical interpolation

We now prove that, for an appropriate choice of the J-unitary constant factor, the linear fractional transformation for the boundary problem arises through a suitable limiting process from a sequence of LFT's for interior problems. With this choice, (22) recovers the form described in [4], that is a product of unitary matrices. In this section, we normalize the Blaschke product as

$$\zeta_w(z) = -\frac{|w|}{w} b_w(z). \tag{24}$$

and the J-inner function (4) as

$$\Theta_{w,u,v}(z) = I_{2p} + \frac{(\zeta_w(z)-1)}{1-\|v\|^2}\begin{bmatrix}u\\v\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}^* J. \tag{25}$$

It is obtained by choosing $\nu = -\frac{w}{|w|}$ and $H = I_{2m}$ in (4).

Lemma 4.1: The J-inner matrix $\Theta_{w,u,v}$ given by (25) is related to the J-inner matrix $\hat{\Theta}_{w,u,v}$ through the relation

$$\begin{aligned} \hat{\Theta}_{w,u,v}(z) &= \Theta_{w,u,v}(z) H\left(\frac{1-|w|}{1-|w|\|v\|^2}uv^*\right) \\ & \quad \begin{bmatrix} I_p + (-\frac{w}{|w|}-1)uu^* & 0 \\ 0 & I_p \end{bmatrix}, \end{aligned}$$

where $H(E)$ denotes the Halmos extension (8) of a contractive matrix E.

Proof: The two J-inner functions may only differ by a right J-unitary constant matrix H

$$\hat{\Theta}_{w,u,v}(z) = \Theta_{w,u,v}(z)H,$$

which satisfies $H = \hat{\Theta}_{w,u,v}\left(-\frac{w}{|w|}\right)$. Note that $b_w\left(-\frac{w}{|w|}\right) = -\frac{w}{|w|}$ and that for every contractive matrix E and unitary matrix P, we have

$$\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} H(E) = H(PE)\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}$$

We thus have

$$\begin{aligned} H &= H(uv^*)\begin{bmatrix} I_p + (-\frac{w}{|w|}-1)uu^* & 0 \\ 0 & I_p \end{bmatrix} H(\bar{w}uv^*) \\ &= H(uv^*)H(-|w|uv^*)\begin{bmatrix} I_p + (-\frac{w}{|w|}-1)uu^* & 0 \\ 0 & I_p \end{bmatrix} \end{aligned}$$

It is easily verified that

$$H(uv^*)H(-|w|uv^*) = H\left(\frac{1-|w|}{1-|w|\|v\|^2}uv^*\right)$$

■

Proposition 2: Let $\Theta_{w,u,v}$ be the J-inner function defined by (25) and F be a rational lossless function; let $\xi \in \mathbb{T}$, and

$$\begin{aligned} v &= F(\xi)u, \\ \rho &= -\xi v^* F'(\xi)u. \end{aligned} \tag{26}$$

Assume $\rho > 0$ and let $\Phi_{\xi,u,v,\rho}$ be the J-inner function defined by (19). We then have

$$\lim_{w\to\xi} \frac{1-\|F(1/\bar{w})u\|^2}{1-|w|^2} = \rho \tag{27}$$

and for $z \neq \xi$,

$$\lim_{w\to\xi} \Theta_{w,u,F(1/\bar{w})u}(z) = \Phi_{\xi,u,v,\rho}(z) \tag{28}$$

$$\lim_{w\to\xi} \hat{\Theta}_{w,u,F(1/\bar{w})u}(z) = \Phi_{\xi,u,v,\rho}(z)\hat{H}, \tag{29}$$

where

$$\hat{H} = H\left(\frac{uv^*}{1+2\rho}\right)\begin{bmatrix} I_m - (1+\xi)uu^* & 0 \\ 0 & I_m \end{bmatrix}$$

Proof: Write the Taylor series of $F(z)$ about $\xi \in \mathbb{T}$

$$F(z) = F(\xi) + (z-\xi)F'(\xi) + o(|z-\xi|).$$

Using (26) we get

$$\begin{aligned} 1-\|F(z)u\|^2 &= 1-u^*F(z)^*F(z)u \\ &= -2\mathrm{Re}\left[(z-\xi)v^*F'(\xi)u\right]+o(|z-\xi|^2), \end{aligned}$$

and since

$$\lim_{w\to\xi}\frac{1-|w|}{\mathrm{Re}(\xi/w-1)}=1$$

we get (27). Note that

$$\zeta_w(z)-1=(|w|-1)\frac{|w|z+w}{w-|w|^2z},$$

so that

$$\lim_{w\to\xi}\frac{\zeta_w(z)-1}{1-\|F(1/\bar{w})u\|^2}=-\frac{1}{2\rho}c_\xi(z)$$

which achieves the proof of (28). Writing

$$1-|w|\|v\|^2=\frac{1}{2}[(1-|w|)(1+\|v\|^2)+(1+|w|)(1-\|v\|^2),$$

we get

$$\lim_{w\to\xi}\frac{1-|w|\|v\|^2}{1-|w|}=1+2\rho,$$

which gives (29). ■

Theorem 4.1: Let (A,B,C,D) be a balanced realization of a lossless function G. A balanced realization $(\hat{A},\hat{B},\hat{C},\hat{D})$ of $\hat{G}=\mathcal{T}_{\hat{\Phi}_{\xi,u,v,\rho}}(G)$ is given by

$$\left[\begin{array}{cc}\hat{D} & \hat{C}\\ \hat{B} & \hat{A}\end{array}\right]=\left[\begin{array}{cc}\hat{V} & 0\\ 0 & I_n\end{array}\right]\left[\begin{array}{ccc}D & 0 & C\\ 0 & 1 & 0\\ B & 0 & A\end{array}\right]\left[\begin{array}{cc}\hat{U}^* & 0\\ 0 & I_n\end{array}\right]. \tag{30}$$

where $\hat{U}$ and $\hat{V}$ are *unitary* $(p+1)\times(p+1)$ complex matrices depending on ξ,u,v,ρ as follows

$$\hat{U}=\left[\begin{array}{cc}\sqrt{\frac{1}{1+\rho}}\,u & I_p-(1+\xi\sqrt{\frac{\rho}{1+\rho}})uu^*\\ \bar{\xi}\sqrt{\frac{\rho}{1+\rho}} & \sqrt{\frac{1}{1+\rho}}u^*\end{array}\right], \tag{31}$$

$$\hat{V}=\left[\begin{array}{cc}\sqrt{\frac{1}{1+\rho}}\,v & I_m-(1-\sqrt{\frac{\rho}{1+\rho}})uu^*\\ \sqrt{\frac{\rho}{1+\rho}} & -\sqrt{\frac{1}{1+\rho}}v^*\end{array}\right]. \tag{32}$$

Proof: As previously, the limits of (13) and (14) are easily computed

$$\lim_{w\to\xi}\alpha(w,v)=\sqrt{\frac{1}{1+\rho}},\ \lim_{w\to\xi}\kappa(w,v)=\sqrt{\frac{\rho}{1+\rho}},$$

and the matrices U and V in Theorem 1 converge to $\hat{U}$ and $\hat{V}$. Formula (30) can also be checked directly using Theorem 3.2. ■

V. Parametrization of orthogonal wavelets with vanishing moments

We start from the theory of orthogonal wavelets as discussed in [9]. However, we do not require the wavelets to have a compact support. We consider a multi-resolution structure for $L^2(\mathbb{R})$, consisting of a nested sequence of linear subspaces $V_k, k\in\mathbb{Z}$ satisfying the properties

(i) nesting: $\forall\, k\in\mathbb{Z}, V_k\subset V_{k+1}$
(ii) empty intersection: $\cap_{k\in\mathbb{Z}}V_k=0$
(iii) completness: $\overline{\cup_{k\in\mathbb{Z}}V_k}=L^2(\mathbb{R})$
(iv) shift invariance:

$$\forall\, f\in L^2(\mathbb{R}),\forall\, k,\ m\in\mathbb{Z},\ \ f(t)\in V_k\Rightarrow f(t-m)\in V_k$$

(v) scale invariance:

$$\forall\, f\in L^2(\mathbb{R}),\forall\, k\in\mathbb{Z},\ \ f(t)\in V_k\Rightarrow f(2t)\in V_{k+1}$$

The spaces V_k are called *approximation spaces*. The *detail spaces* W_k are the orthogonal complement of V_k in the enveloping spaces V_{k+1}.

We shall hypothesize that there exists a function $\phi(z)$ (resp. $\psi(z)$), called the *scaling function* (resp. *wavelet function*), which generates a shift invariant orthonormal basis $\{\phi(t-m), m\in\mathbb{Z}\}$ (resp. $\{\psi(t-m), m\in\mathbb{Z}\}$) for the approximation space V_0 (resp. W_0). It follows from the properties of the multi-resolution structure that there exists sequences of coefficients C_k and D_k, $k\in\mathbb{Z}$, which allows to represent the scaling and the wavelet functions as a linear combination of the orthogonal basis functions of V_1:

$$\left\{\begin{array}{lcl}\phi(t) &=& \sqrt{2}\sum_{k\in\mathbb{Z}}C_k\phi(2t+k),\\ \psi(t) &=& \sqrt{2}\sum_{k\in\mathbb{Z}}D_k\phi(2t+k),\end{array}\right. \tag{33}$$

with

$$\left\{\begin{array}{lcl}C_k &=& \sqrt{2}\int\phi(t)\phi(2t+k)\ dt,\\ D_k &=& \sqrt{2}\int\psi(t)\phi(2t+k)\ dt.\end{array}\right.$$

Orthonormality of the basis for the spaces V_k and W_k allows us to derive important properties of the coefficient sequences (C_k) and (D_k). The reverse approach, which consists in starting from such coefficients and looking for the scaling and wavelet functions, is a highly non-trivial task that has received quite some attention in the literature.

When a signal $s(t)$ is to be decomposed into an approximation signal $a(t)\in V_k$ and a detail signal $d(t)\in W_k$ satisfying $s(t)=a(t)+d(t)$, this can be performed directly from the coefficient sequences of these signals with respect to the various orthonormal bases. Let

$$s(t) = \sqrt{2}\sum_{k\in\mathbb{Z}}s_k\phi(2t-k) \tag{34}$$

$$a(t) = \sum_{k\in\mathbb{Z}}a_k\phi(t-k) \tag{35}$$

$$d(t) = \sum_{k\in\mathbb{Z}}d_k\phi(t-k). \tag{36}$$

If we introduce the filters

$$C(z)=\sum_{k\in\mathbb{Z}}C_kz^{-k},\ D(z)=\sum_{k\in\mathbb{Z}}D_kz^{-k}, \tag{37}$$

it follows that

$$a_k=\sum_{m\in\mathbb{Z}}C_{2k-m}s_m, d_k=\sum_{m\in\mathbb{Z}}D_{2k-m}s_m.$$

The filters $C(z)$ and $D(z)$ are first used to filter the sequence (s_k), after which down-sampling is applied to the outcome. An equivalent procedure, by first splitting the sequence (s_k)

into its even and its odd phase, can be interpreted in terms of the associated polyphase filter $H(z)$. It is given by

$$H(z) = \begin{bmatrix} C_e(z) & C_o(z) \\ D_e(z) & D_o(z) \end{bmatrix} \tag{38}$$

where

$$\begin{array}{rclrcl} C_e(z) &=& \sum_{k\in\mathbb{Z}} C_{2k} z^{-k} & D_e(z) &=& \sum_{k\in\mathbb{Z}} D_{2k} z^{-k} \\ C_o(z) &=& \sum_{k\in\mathbb{Z}} C_{2k+1} z^{-k} & D_o(z) &=& \sum_{k\in\mathbb{Z}} D_{2k+1} z^{-k} \end{array} \tag{39}$$

Filtering the input sequence $u_k = \begin{bmatrix} s_{2k} \\ s_{2k+1} \end{bmatrix}$ by $H(z)$ produces the output sequence $y_k = \begin{bmatrix} a_k \\ d_k \end{bmatrix}$.

The orthogonality conditions on the filter coefficients are in fact equivalent to the condition that the polyphase matrix $H(z)$ is lossless. This provides us with a very useful and concise description of the class of filter banks leading to orthogonal wavelets, which allows us to parametrize this class directly with orthonormality built in.

A wavelet $\psi(t)$ is said to have a *vanishing moment* of order p, if it holds that

$$\int t^p \psi(t)\ dt = 0. \tag{40}$$

If a wavelet has m vanishing moments, for order $p = 0, 1, \ldots, m-1$, then the scaling and wavelets functions have continuous derivative up to order $m-1$. These are very desirable properties in view of practical applications. A vanishing moment of order 0 is a standard admissibility condition for all wavelets. One famous class of orthogonal wavelets exhibiting vanishing moments, is the class of Daubechies wavelets. This class is obtained by restricting to FIR filters $C(z)$ and $D(z)$ of finite order $2n-1$, which allows to built in n vanishing moments. We shall see that vanishing moment conditions can be reformulated in terms of boundary interpolation conditions for the lossless polyphase matrix function $H(z)$.

We introduce the notation

$$v_p = \int t^p \phi(t)\ dt, \quad w_p = \int t^p \psi(t)\ dt.$$

Note that orthonormality of the basis impose that $v_0 = 1$. Multiplying the equations (33) by t^p and integrating, we get the following relations

$$\begin{bmatrix} v_m \\ w_m \end{bmatrix} = \sum_{p=0}^{m} \begin{pmatrix} m \\ p \end{pmatrix} v_{m-p} \begin{bmatrix} C^{(p)}(1) \\ D^{(p)}(1) \end{bmatrix}, \tag{41}$$

where $\begin{pmatrix} m \\ p \end{pmatrix}$ is the binomial coefficient and $C(z)$ and $D(z)$ the filters (37) and $(.)^{(p)}$ denotes the pth derivative. To derive interpolation conditions on the polyphase matrix $H(z)$ we shall use the equality

$$\begin{bmatrix} C(z) \\ D(z) \end{bmatrix} = H(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

A. One vanishing moment

For $m = 0$, (41) yields

$$\begin{bmatrix} 1 \\ \frac{w_0}{v_0} \end{bmatrix} = H(1) \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Since $H(z)$ is lossless, it takes orthogonal values at every point on the unit circle, and in particular $H(1)$ is orthogonal. It is thus norm preserving and we must have $w_0 = 0$. Therefore, a vanishing moment of order 0 is automatically built. It also follows that

$$H(1) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

up to a change of sign in the second row.

If we impose the McMillan degree of $H(z)$ to be 1, then only a single solution exists: $H(z) = H(1)$. It is the polyphase matrix associated with the Haar wavelet.

To proceed with vanishing moments of higher order, we focus our attention on the lossless matrix

$$\tilde{H}(z) = H(z)H(1)^T,$$

which satisfies the property $\tilde{H}(1) = I_2$.

B. Two vanishing moments

For $m = 1$ and imposing $w_1 = 0$, (41) yields, after some computations, the following interpolation condition for the first derivative of $\tilde{H}(z)$

$$\tilde{H}^{(1)}(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2v_1 + 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\rho_1 \\ -\frac{1}{4} \end{bmatrix}. \tag{42}$$

From section III-B, we have a description of all the lossless functions satisfying boundary interpolation conditions of the form (18). The lossless matrix $\tilde{H}$ satisfies such a condition with $\xi = 1, u = v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\rho_1 = \frac{1}{4}\beta_1 = -\frac{1}{4}(2v_1 + 1)$ whihc is a positive number if and only if $v_1 < 1/2$. Then β_1 appears as a free parameter and we may use Theorem 3.1 to describe the set of solutions in terms of a linear fractional transformation of an arbitrary lossless function G. The condition for a vanishing moment of order 1 happens to be

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \tilde{H}^{(1)}(1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{4}. \tag{43}$$

Using the symmetry of $\tilde{H}^{(1)}(1)$, this condition translates into an equivalent condition on the structure of the matrix $G(1)$.

Proposition 3: Any 2×2 lossless function $\tilde{H}$ satisfying $\tilde{H}(1) = I_2$ together with (43) can be represented as $\tilde{H} = \mathcal{T}_{\Phi_1}(G)$, where Φ_1 is given by (19) with $\xi = 1, \rho = \rho_1$ and $u = v = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $G(z)$ is a 2×2 lossless function satisfying

$$G(1) = \begin{bmatrix} \frac{1-\beta_1^2}{1+\beta_1^2} & -\frac{2\beta_1}{1+\beta_1^2} \\ -\frac{2\beta_1}{1+\beta_1^2} & -\frac{1-\beta_1^2}{1+\beta_1^2} \end{bmatrix}, \quad \beta_1 = 4\rho_1 \tag{44}$$

Proof: The value of $G(1)$ is obtained form (43) using (21) and the fact that $G(1)$ is orthogonal. ∎

If we consider the case where the McMillan degree of $\tilde{H}$ equals 1, and there are two vanishing moments built in, we then arrive at an explicit parametrization

$$\tilde{H}(z) = I_2 - \frac{2(z-1)}{z(1+4\beta_1+\beta_1^2)-(1-4\beta_1+\beta_1^2)} \begin{bmatrix}\beta_1\\1\end{bmatrix}\begin{bmatrix}\beta_1\\1\end{bmatrix}^T.$$

This shows that the pole of $\tilde{H}$ is located at $z = \frac{1-4\beta_1+\beta_1^2}{1+4\beta_1+\beta_1^2}$, which for $\beta_1 > 0$ has a real value in the interval $]-1,1[$. We arrive at a lossless FIR system if and only if this pole is located at $z=0$. This implies that $\beta_1 = 2\pm\sqrt{3}$. The first solution corresponds to the polyphase matrix associated with the Daubechies db2 wavelet, the second solution provides an alternative with reversed filter coefficients.

C. Three vanishing moments

We define the lossless matrix $\tilde{G}$ by $\tilde{G}(z) = G(z)G(1)$ where $G(1)$ is given by (44). For $m=2$ and imposing $w_2 = 0$, (41) yields the following condition for a vanishing moment of order 2

$$\begin{bmatrix}1 & 0\end{bmatrix} \tilde{H}^{(2)}(1) \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{4}\left(\frac{\beta_1}{2}+1\right). \tag{45}$$

This condition translates into the equivalent condition on $\tilde{G}$

$$\begin{bmatrix}1 & 0\end{bmatrix} \tilde{G}^{(2)}(1) \begin{bmatrix}1\\-\beta_1\end{bmatrix} = \frac{\beta_1}{4}\frac{1-\beta_1^2}{1-\beta_1^2}. \tag{46}$$

Proposition 4: Any 2×2 lossless function $\tilde{H}(z)$ satisfying the three vanishing moment conditions $\tilde{H}(1) = I_2$, (43) and (45) can be represented as $\tilde{H} = \mathcal{T}_{\Phi_1}(\mathcal{T}_{\Phi_2}(F)K_1)$ where Φ_i is the J-inner matrix (19) with $\xi=1, \rho=\rho_i$ and $u = v = \begin{bmatrix}1 & 0\end{bmatrix}^T$, K_1 is given by (44) and F is a 2×2 lossless function satisfying

$$F(1) = \begin{bmatrix} \frac{1-\beta_2^2}{1+\beta_2^2} & -\frac{2\beta_2}{1+\beta_2^2} \\ -\frac{2\beta_2}{1+\beta_2^2} & -\frac{1-\beta_2^2}{1+\beta_2^2}\end{bmatrix},$$

with

$$\beta_2 = \frac{4\beta_1\rho_2(1+\beta_1^2)}{\beta_1(1-\beta_1^2)+4\rho_2(1+\beta_1^2)}.$$

Proof: The proof relies on technical computations with Mathematica. ■

If we consider the case where the McMillan degree of $\tilde{H}$ equals 2, and there are three vanishing moments built in, we then arrive at an explicit parametrization in terms of ρ_1 and ρ_2. Imposing the lossless function to be FIR, and working out the expressions with Mathematica, we find two pairs of values (ρ_1, ρ_2) one of which gives the Dauberchies db3 wavelet.

VI. Conclusion and future works

In this paper we have got parametrization of (balanced realizations) of lossless functions in which the parameters are boundary interpolation value and angular derivatives. This is interesting form an application viewpoint since it seems that a number of meaningful quantities (vanishing moments, Markov parameters, etc.) can be interpreted in terms boundary interpolation. In this connection, we have got explicit parametrizations of 2×2 polyphase matrices of arbitrary order n with 4 vanishing moments built in (the results have been presented up to 3), in terms of angular derivative (positive) parameters. However, the conditions are cleverly handled in an unusual recursive fashion that we still do not completely understand. In our opinion, restriction to the FIR class would be an interesting target for further research.

These results could also be used to derive parametrizations for Schur functions. Indeed, a Schur function can be viewed as a sub-block of a lossless function of higher dimension. A parametrization of Schur functions could thus be deduced from that of lossless functions. The use of boundary interpolation could be proved necessary in this approach to handle functions which are not strictly Schur.

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