Finite volume approximation for an immiscible two-phase flow in porous media with discontinuous capillary pressure

Konstantin Brenner, Clément Cancès, Danielle Hilhorst

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Abstract

We consider an immiscible incompressible two-phase flow in a porous medium composed of two different rocks so that the capillary pressure field is discontinuous at the interface between the rocks. This leads us to apply a concept of multi-valued phase pressures and a notion of weak solution for the flow which have been introduced in [Cancès & Pierre, *SIAM J. Math. Anal*, 44(2):966–992, 2012]. We discretize the problem by means of a numerical algorithm which reduces to a standard finite volume scheme in each rock and prove the convergence of the approximate solution to a weak solution of the two-phase flow problem. The numerical experiments show in particular that this scheme permits to reproduce the oil trapping phenomenon.

Keywords : Finite volume schemes, degenerate parabolic, two-phase flow in porous media, discontinuous capillarity

AMS Classification : 35K65, 35R05, 65M12, 76M12

1 Introduction

1.1 Multivalued phase pressures

Models of incompressible immiscible two-phase flows are widely used in oil engineering to predict the motion of oil in the underground. They have been widely studied from a mathematical point of view (see e.g. [1], [2], [6], [7], [18]) as well as from a numerical point of view (see e.g. [17], [20], [21], [19], [30], [35]). In these models, sometimes referred to as *dead-oil* approximations, it is assumed that there are only two phases, oil and water, and that each phase is composed of a single component.

The governing equations are derived by substituting the Darcy-Muskat law in the conservation equations for both phases, so that we obtain for each phase $\alpha \in \{o, w\}$ (o corresponds to the oil phase, while w corresponds to the water phase):

$$\phi\partial_t s_\alpha - \operatorname{div}\left(K\frac{k_\alpha(s_\alpha)}{\mu_\alpha}\left(\nabla p_\alpha - \rho_\alpha \mathbf{g}\right)\right) = 0, \tag{1}$$

where $\phi = \phi(\mathbf{x})$ is the porosity of the rock ($\phi \in (0,1)$ in the domain Ω), s_α is the saturation of the phase α, the permeability of the porous medium K is supposed to be a positive scalar function, the relative permeability k_α of the phase α is an increasing function of the saturation s_α, satisfying $k_\alpha(0) = 0$ and $k_\alpha(1) = 1$, μ_α, p_α

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and ρ_α denote respectively the viscosity, the pressure and the density of the phase α, and $\mathbf{g}$ is the gravity vector. Assuming that the two phases occupy the whole porous volume, one has

$$s_o + s_w = 1, \tag{2}$$

so that we can eliminate the water saturation. We note $s := s_o$, so that $s_w = 1 - s$.

We suppose that the phase pressures satisfy the relation

$$p_o - p_w = \pi(s_o), \tag{3}$$

where π is the capillary pressure function, which is strictly increasing on $(0, 1)$.

It follows from [16] and [1] that the quantity

$$\sum_{\alpha\in\{o,w\}} \int_0^T \int_\Omega K \frac{k_\alpha(s_\alpha)}{\mu_\alpha} (\nabla p_\alpha)^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \tag{4}$$

is bounded. However, when the phase α vanishes, i.e. when $s_\alpha = 0$, this does not provide any control on the pressure p_α. This leads to define p_α as a graph, allowing it to take any value lower than a threshold value, for which the phase α would appear. This point of view, which has been developed in [16], leads to

$$p_o \in [-\infty, p_w + \pi(0)] \quad \text{if } s_o = 0 \tag{5}$$

and

$$p_w \in [-\infty, p_o - \pi(1)] \quad \text{if } s_o = 1. \tag{6}$$

We will take advantage of this multivalued formalism in order to deal with the case where the porous medium is composed of several rock types, and where the functions describing the porous medium depend of space in a discontinuous way.

Following the approach of [10] and [15], the capillary pressure function $s \mapsto \pi(s, \mathbf{x})$ has to be extended into a maximal monotone graph $\tilde\pi(\cdot, \mathbf{x})$ from $[0, 1]$ to $\overline{\mathbb{R}}$ defined by

$$\tilde\pi(s, \mathbf{x}) = \begin{cases} [-\infty, \pi(0, \mathbf{x})] & \text{if } s = 0, \\ \pi(s, \mathbf{x}) & \text{if } s \in (0, 1), \\ [\pi(1, \mathbf{x}), +\infty] & \text{if } s = 1, \end{cases}$$

so that the relations (5) and (6) imply that

$$p_o(\mathbf{x}, t) - p_w(\mathbf{x}, t) \in \tilde\pi(s(\mathbf{x}, t), \mathbf{x}) \quad \text{for } (\mathbf{x}, t) \in \Omega \times (0, T). \tag{7}$$

Note that relation (7) does not enforce a unique value for the phase pressures. Nevertheless, if $s_\alpha(\mathbf{x}, t) > 0$, the corresponding phase pressure $p_\alpha(\mathbf{x}, t)$ is uniquely defined since it is controlled by the quantity (4).

Now, focusing on the case where $\mathbf{x} \mapsto \pi(s, \mathbf{x})$ is discontinuous across a surface Γ separating two rocks Ω_1 and Ω_2, the problem turns to finding phase pressures on the interface such that the relation (7) is satisfied on both sides of Γ. Denoting by $\tilde\pi_i$ the capillary pressure graph in Ω_i and by s_i the one-sided trace of the saturation on Γ from Ω_i, then the phase pressures have to satisfy

$$p_o(\mathbf{x}, t) - p_w(\mathbf{x}, t) \in \tilde\pi_1(s_1(\mathbf{x}, t)) \cap \tilde\pi_2(s_2(\mathbf{x}, t)) \quad \text{for } (\mathbf{x}, t) \in \Gamma \times (0, T). \tag{8}$$

We stress that the one-sided traces $p_{\alpha,i}$ of the phase pressure p_α (if it exists) can be discontinuous across Γ, i.e. $p_{\alpha,1} \neq p_{\alpha,2}$, if $s_{\alpha,j} = 0$ on one side of the interface. However, there exist interface phase pressures $p_\alpha(\mathbf{x}, t)$ for $\mathbf{x} \in \Gamma$ such that (8) holds. It is important to notice that, for $\mathbf{x} \in \Gamma$, if $s_{\alpha,1}(\mathbf{x}, t)$ and $s_{\alpha,2}(\mathbf{x}, t)$ both belong to $(0, 1]$, the phase pressure $p_\alpha(\mathbf{x}, t)$ corresponds to the trace of the phase pressure p_α on both sides of the interface.

Finally, we prescribe the balance of the flux across the interface, i.e.,

$$\sum_{i\in\{1,2\}} K_i \frac{k_{\alpha,i}(s)}{\mu_\alpha} \left(\nabla p_{\alpha|_{\Omega_i}} - \rho_\alpha \mathbf{g}\right) \cdot \mathbf{n}_i = 0 \quad \text{on } \Gamma, \tag{9}$$

where $p_{\alpha|_{\Omega_i}}$ denotes the restriction of p_α to the domain Ω_i, $\mathbf{n}_i$ is the normal to Γ outward w.r.t. Ω_i, and K_i denotes the permeability of Ω_i.

1.2 A brief review of the state of the art

Since discontinuous capillarity play a crucial role in the qualitative behavior of the saturation field in heterogeneous rock, numerous contributions have already been published for proposing numerical methods and mathematical analysis tools on this subject.

In particular, as pointed out by C.J. van Duijn *et al.* [38], such capillarity discontinuities may be responsible of *oil-trapping*. The first rigorous existence and uniqueness results in the one dimensional case has been proposed by M. Bertsch *el al.* [8] for a particular choice of functions characterizing the porous medium. This existence and uniqueness frame was extended to general physical data in [15] and [11], but still in the one-dimensional frame, relying on the graph extension of the capillary pressure. Note that this graph extension was simultaneously and independently proposed in [10]. The concept of multivalued phase pressures, based on the graph extension of the capillary pressure, allowed to prove the global existence of a solution to the problem [16].

Concerning the numerical approximation of the solution to the problem, let us mention first the contribution of B.G. Ersland *et al* [25] where a method based on the characteristic method combined with Finite Elements was proposed. In [23], G. Enchéry *et al.* proved the convergence of a Finite Volume scheme for a simplified model reducing to a single equation, but the convergence proof was performed in the multidimensional case. It was then shown in [11] that, in the one-dimensional case, and accounting the convection, a closely related scheme converges towards the unique one-dimensional solution to the problem. In [29], R. Eymard *et al.* studied general Finite Volume method based on a pressure–pressure formulation. The convergence of the method was proved under a non-degeneracy assumption. A numerical method based on Mixed Finite Element was developed by H. Hoteit and A. Firoozabadi [34], while a Discontinuous Galerkin method has been proposed by A. Ern *et al.* [24], and its effective implementation was discussed in the contribution of I. Mozolevski and L. Schuh [36]. As far as we know, our contribution is the first one where the convergence of the numerical approximation is proved without particular assumption, like non-degeneracy or reduction of the model to a single equation.

In their recent contribution [3], B. Amaziane *et al.* studied the case of a compressible two-phase flow. Another model enrichment, that consists in taking the dynamic capillary effects into account, has been studied in [32], [33], where numerical strategies are proposed for solving the degenerate pseudo-parabolic corresponding problem. Finally, let us mention the contribution of A. Papafotiou *et al.* [37] where a node centered Finite Volume method was built in order to take the hysteresis into account.

Finally, since the effects of the capillary diffusion are often negligible within the homogeneous rock, several contributions have been proposed for computing the vanishing capillarity solution. Let us mention in particular the contributions [12], [13], [14], where it has been established that the interaction between buoyancy and capillary pressure discontinuities can produce singular effects yielding oil trapping. In the recent contribution [5], it has been pointed out that, even if the capillarity seems to be neglected in the so-called *vanishing capillarity* regime, the capillary pressure curves have a strong influence on the behavior of the solution. A "cheap" Finite Volume scheme was proposed in [4] for simulating the vanishing capillarity solution in the multidimensional context.

1.3 The model problem and assumptions on the data

We assume that the porous medium Ω is a connected open bounded polygonal subset of $\mathbb{R}^d$, and is made of two disjoint homogeneous rocks Ω_i, $i \in \{1, 2\}$, which are both open polygonal subsets of $\mathbb{R}^d$. We denote by Γ the interface between Ω_1 and Ω_2, i.e.

$$\overline{\Gamma} = \partial\Omega_1 \cap \partial\Omega_2.$$

For all functions a depending on the physical characteristics of the rock, we use the notation $a_i = a(\cdot, \mathbf{x})$ if $\mathbf{x} \in \Omega_i$.

We assume that the initial phase distribution is known

$$s_{|_{t=0}} = s_0 \in L^\infty(\Omega; [0,1]). \tag{10}$$

We also assume the natural boundary conditions

$$K_i \frac{k_{\alpha,i}(s)}{\mu_\alpha} (\nabla p_\alpha - \rho_\alpha \mathbf{g}) \cdot \mathbf{n}_i = 0, \quad \text{on } (\partial\Omega \cap \partial\Omega_i) \times (0,T), \tag{11}$$

where the positive constant T is fixed but arbitrary. Nevertheless, it should be possible to deal with other types of boundary conditions, such as Dirichlet conditions on a part of the boundary and Neumann conditions on the remaining part.

We make the following assumptions on the capillary pressure functions.

Assumption 1 *The functions π_i are increasing, locally Lipschitz continuous on $(0,1)$, and belong to $L^1(0,1)$.*

Their graph extensions, denoted by $\tilde{\pi}_i$, are defined by

$$\tilde{\pi}_i(s) = \begin{cases} [-\infty, \pi_i(0)] & \text{if } s = 0, \\ \pi_i(s) & \text{if } s \in (0,1), \\ [\pi_i(1), +\infty] & \text{if } s = 1. \end{cases}$$

Since $\tilde{\pi}_i$ are maximal monotone graphs from $[0,1]$ to $\overline{\mathbb{R}}$, they admit maximal monotone inverse graphs θ_i from $\overline{\mathbb{R}}$ to $[0,1]$, defined by

$$\theta_i(p) := \begin{cases} 0 & \text{if } p \le \pi_i(0), \\ \pi_i^{-1}(p) & \text{if } p \in (\pi_i(0), \pi_i(1)), \\ 1 & \text{if } p \ge \pi_i(1). \end{cases}$$

Due to the fact that π_i are supposed to be strictly increasing, the graphs θ_i are in fact nondecreasing continuous functions defined from $\overline{\mathbb{R}}$ to $[0,1]$. The following property holds:

$$\theta_i(p) = s \quad \text{iff} \quad p \in \tilde{\pi}_i(s). \tag{12}$$

Therefore, at the interface Γ, one has

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \quad \text{iff} \quad s_1 = \theta_1(\pi) \text{ and } s_2 = \theta_2(\pi). \tag{13}$$

The relations (13) are illustrated on Fig. 1.

We now state another crucial property of the functions θ_i, whose proof is given in [16].

Lemma 1.1 *It follows from Assumption 1 that*

$$\theta_i \in L^1(\mathbb{R}_-) \qquad \textit{and} \qquad (1-\theta_i) \in L^1(\mathbb{R}_+), \qquad i \in \{1,2\}.$$

We do also the following assumptions on the relative permeabilities.

Assumption 2 *For $\alpha \in \{o,w\}$, the relative permeabilities $k_{\alpha,i}$ of the phase α are the strictly increasing Lipschitz continuous functions of the saturation s_α, satisfying $k_{\alpha,i}(0) = 0$ and $k_{\alpha,i}(1) = 1$.*

The last assumption on the data we need concerns the Kirchhoff transform function, the will be introduced in Section 1.4.

Assumption 3 *For $i \in \{1,2\}$, the function $s \mapsto k_{o,i}(s)k_{w,i}(s)\pi_i'(s)$ belongs to $L^\infty(0,1)$.*

All along the paper, we denote by Q_T and $Q_{i,T}$ the space-time cylinders

$$Q_T := \Omega \times (0,T), \qquad Q_{i,T} := \Omega_i \times (0,T).$$

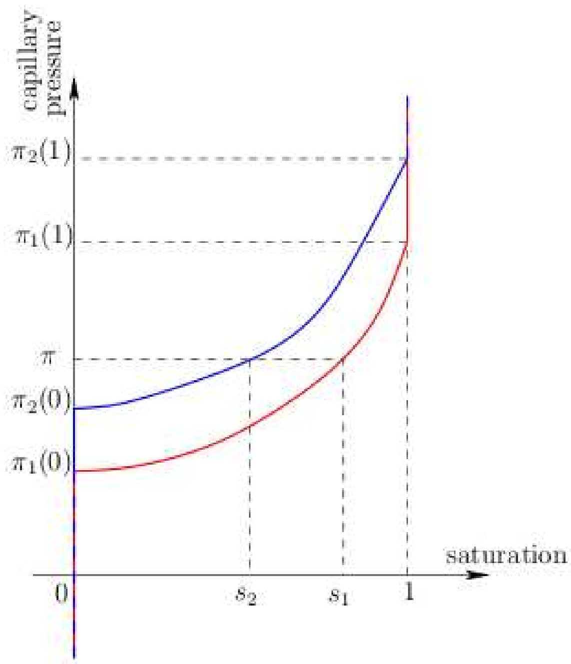

Figure 1: The capillary pressure graphs $\tilde{\pi}_i$ are obtained by extending the capillary pressure functions π_i by adding them the semi-axes $[-\infty, \pi_i(0)]$ and $[\pi_i(1), +\infty]$. At the interface Γ, to each capillary pressure level π correspond two values $s_i = \theta_i(\pi)$ that are the one-sided traces of the saturation on both sides of the interface.

1.4 Global pressure formulation of the problem

The lack of control on the phase pressures, described in Section 1.1 and in [16], leads to important mathematical difficulties. A classical mathematical tool to circumvent some of them consists in introducing the so-called *global pressure* P as a new unknown function.

Define the total mobility M_i by $M_i(s) = K_i \left(\frac{k_{o,i}(s)}{\mu_o} + \frac{k_{w,i}(s)}{\mu_w} \right)$. Since the relative permeabilities $k_{\alpha,i}$ are supposed to be strictly monotone, one has $k_{\alpha,i}(s) > 0$ if $s \in (0, 1)$. As a consequence,

$$\text{there exists } \alpha_M > 0 \text{ such that, for } i \in \{1, 2\}, \text{ and for all } s \in [0, 1], \text{ one has } M_i(s) \geq \alpha_M. \tag{14}$$

Then, for $(\mathbf{x}, t) \in Q_{T,i}$ and $\pi \in \tilde{\pi}_i(s(\mathbf{x}, t))$, we set

$$P(\mathbf{x}, t) = p_w(\mathbf{x}, t) + \int_0^\pi \frac{k_{o,i}(\theta_i(a))}{k_{o,i}(\theta_i(a)) + \frac{\mu_o}{\mu_w} k_{w,i}(\theta_i(a))} \mathrm{d}a, \tag{15}$$

$$= p_o(\mathbf{x}, t) - \int_0^\pi \frac{k_{w,i}(\theta_i(a))}{k_{w,i}(\theta_i(a)) + \frac{\mu_w}{\mu_o} k_o(\theta_i(a))} \mathrm{d}a. \tag{16}$$

The global pressure P is built so that it satisfies

$$M_i(s) \nabla P = K_i \left(\frac{k_{o,i}(s)}{\mu_o} \nabla p_o + \frac{k_{w,i}(s)}{\mu_w} \nabla p_w \right).$$

While the phase pressures p_α shall be defined as multivalued, it has been pointed out in [16] that the global pressure P is always single valued (despite it seems to be defined up to a choice of $\pi \in \tilde{\pi}_i(s)$), and is therefore much easier to work with. Remark that P may however be discontinuous at the interface Γ. It is well known that in the case where the domain Ω is homogeneous ([17]), or if $\mathbf{x} \mapsto \pi(s, \mathbf{x})$ is a smooth fonction ([7], [18]), then the global pressure belongs to the space $L^\infty(0,T;H^1(\Omega))$. This regularity result does not remain true, as it will be shown in the sequel, in the case of a discontinuous capillary pressure.

Let us define the fractional flow function $f_i(s) = \dfrac{k_{o,i}(s)}{k_{o,i}(s) + \frac{\mu_o}{\mu_w} k_{w,i}(s)}$ and introduce the Kirchhoff transform

$$\varphi_i(s) = \int_0^s K_i \frac{k_{o,i}(a) k_{w,i}(a)}{\mu_w k_{o,i}(a) + \mu_o k_{w,i}(a)} \pi_i'(a) \mathrm{d}a, \qquad \forall s \in (0,1), \tag{17}$$

that we extend in a continuous way by constants outside of $(0,1)$. It follows from Assumption 2 that the functions f_i are Lipschitz continuous and increasing on $[0,1]$, with $f_i(0) = 0$ and $f_i(1) = 1$. Moreover, Assumptions 1, 2 and 3 imply that the functions φ_i are 1 and 2 that the functions φ_i are Lipschitz continuous and increasing on $[0,1]$.

It is well known (see [17]) that the system (1)–(3) can be formally rewritten in $Q_{i,T}$ under the form

$$\begin{cases} \phi_i \partial_t s + \operatorname{div}\left(f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s)\right) = 0, \\ \operatorname{div}\mathbf{q}_i = 0, \\ \mathbf{q}_i = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g}, \end{cases} \tag{18}$$

where

$$\gamma_i(s) = K_i(\rho_o - \rho_w)\frac{k_{o,i}(s) k_{w,i}(s)}{\mu_w k_{o,i}(s) + \mu_o k_{w,i}(s)} \tag{19}$$

and

$$\zeta_i(s) = K_i\left(\frac{k_{o,i}(s)}{\mu_o}\rho_o + \frac{k_{w,i}(s)}{\mu_w}\rho_w\right).$$

The boundary conditions on the phase fluxes (11) are given by

$$\mathbf{q}_i \cdot \mathbf{n}_i = 0, \qquad \left(f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s)\right) \cdot \mathbf{n}_i = 0, \quad \text{on } (\partial\Omega \cap \partial\Omega_i) \times (0,T). \tag{20}$$

Concerning the transmission conditions on the interface Γ, we look for two phase pressures so that the relation (7) holds. This leads us to require the existence of a capillary pressure π such that

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2), \tag{21}$$

$$P_1 - W_1(\pi) = P_2 - W_2(\pi), \tag{22}$$

where

$$W_i(p) = \int_0^p f_i \circ \theta_i(u) du.$$

In view of (15), the function W_i is such that $P - W_i(\pi) = p_{w,i}$ for any $\pi \in \tilde{\pi}_i(s)$. Therefore, Eq. (22) is nothing but the requirement of the continuity of the water pressure in an extended sense. Indeed, if s_1 and s_2 both belong to $[0,1)$, water is present on both sides of the interface, and (22) requires the continuity of the water pressure. But if s_1 or s_2 is equal to 1, then (21)–(22) only enforce the existence of an interface water pressure such that (8) holds. By adding π (given by (21)) on both sides in (22), we deduce from (16) that the continuity in the same extended sense of the oil pressure is also required by the system (21)–(22).

The conservation of the total mass and of the oil mass give

$$\sum_{i\in\{1,2\}} \mathbf{q}_i \cdot \mathbf{n}_i = 0 \quad \text{on } \Gamma, \tag{23}$$

$$\sum_{i\in\{1,2\}} \left(f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} - \nabla\varphi_i(s)\right) \cdot \mathbf{n}_i = 0 \quad \text{on } \Gamma, \tag{24}$$

where $\mathbf{n}_i$ denotes the outward normal to Γ with respect to Ω_i.

Since the global pressure P is defined up to a constant, we have to impose a condition to select a solution. Let $m_{\Omega_i}(P)(t)$ denote a mean value of a global pressure in the subdomain i

$$m_{\Omega_i}(P)(t) := \frac{1}{m(\Omega_i)} \int_{\Omega_i} P(\mathbf{x}, t)\mathrm{d}\mathbf{x} \text{ for } i \in \{1, 2\}$$

We impose that

$$m_{\Omega_1}(P)(t) = 0, \quad \text{for a.e. } t \in (0, T). \tag{25}$$

The global pressure jump at the interface is fixed by the relation (22), so that the mean value $m_{\Omega_2}(P)$ of P on Ω_2 is locked by (25).

We now define a weak solution of Problem (18)-(25).

Definition 1.1 *We say that a function pair (s, P) is a weak solution of Problem (18)-(25) if:*

1. *$s \in L^\infty(Q_T; [0, 1])$ and $\varphi_i(s) \in L^2(0, T; H^1(\Omega_i))$;*
2. *$P \in L^2(0, T; H^1(\Omega_i))$, with $m_{\Omega_1}(P)(t) = 0$ for almost every $t \in (0, T)$;*
3. *there exists a measurable function π on $\Gamma \times (0, T)$ such that, for a.e. $(\mathbf{x}, t) \in \Gamma \times (0, T)$*

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2), \tag{26}$$

$$P_1 - W_1(\pi) = P_2 - W_2(\pi). \tag{27}$$

4. *for all $\psi \in \mathcal{C}_c^\infty\left(\overline{\Omega} \times [0, T)\right)$, the following integral equalities hold:*

$$\int_0^T \sum_{i\in\{1,2\}} \int_{\Omega_i} \mathbf{q}_i \cdot \nabla\psi \mathrm{d}\mathbf{x}\mathrm{d}t = 0, \tag{28}$$

and

$$\int_0^T \int_\Omega \phi s \partial_t \psi \mathrm{d}\mathbf{x}\mathrm{d}t + \int_\Omega \phi s_0 \psi(\cdot, 0)\mathrm{d}\mathbf{x} = \int_0^T \sum_{i\in\{1,2\}} \int_{\Omega_i} \left(f_i(s)\mathbf{q}_i + \gamma_i(s)\mathbf{g} + \nabla\varphi_i(s)\right) \cdot \nabla\psi \mathrm{d}\mathbf{x}\mathrm{d}t, \tag{29}$$

where

$$\mathbf{q}_i = -M_i(s)\nabla P + \zeta_i(s)\mathbf{g}.$$

We will use several time the following lemma, which ensures that the global pressure jump $P_1 - P_2$ at the interface belongs to $L^\infty(\Gamma \times (0, T))$.

Lemma 1.2 *The function $p \mapsto W_1(p) - W_2(p)$ belongs to $\mathcal{C}^1(\mathbb{R}; \mathbb{R})$, is uniformly bounded on $\mathbb{R}$ and admits finite limits as $p \to \pm\infty$.*

Proof: Define

$$\widehat{W}_i(p) = \begin{cases} \displaystyle\int_0^p (f_i \circ \theta_i(p) - 1)\, \mathrm{d}p & \text{if } p \geq 0, \\ \displaystyle\int_0^p f_i \circ \theta_i(p)\ \mathrm{d}p & \text{if } p < 0, \end{cases} \tag{30}$$

therefore $W_1(p) - W_2(p) = \widehat{W}_1(p) - \widehat{W}_2(p)$. Hence, we deduce that if $\widehat{W}_1(p), \widehat{W}_2(p)$ have finite limits for $p \to \pm\infty$, then $W_1 - W_2$ also does, since $f_i(1) = 1$. Since $\widehat{W}_1, \widehat{W}_2$ are nonincreasing functions, it only remains to check that they are bounded. Let $p \geq 0$, then

$$\begin{aligned} 0 \geq \widehat{W}_i(p) &\geq -\int_0^p |f_i \circ \theta_i(p) - f_i(1)|\mathrm{d}p \\ &\geq -L_{f_i} \int_0^p |\theta_i(p) - 1|\mathrm{d}p \geq -L_{f_i}\|\theta_i - 1\|_{L^1(\mathbb{R}_+)}. \end{aligned}$$

Similarly, for $p < 0$, one has

$$0 \le \widehat{W}_i(p) \le L_{f_i} \|\theta_i\|_{L^1(\mathbb{R}_-)}.$$

We conclude the proof of Lemma 1.2 by applying Lemma 1.1. □

2 The Finite Volume approximation

2.1 Discretization of $\mathbf{Q_T}$

Definition 2.1 *An admissible mesh of Ω is given by a set $\mathcal{T}$ of open bounded convex subsets of Ω called control volumes, a family $\mathcal{E}$ of subsets of $\overline{\Omega}$ contained in hyperplanes of $\mathbb{R}^d$ with strictly positive measure, and a family of points $(\mathbf{x}_K)_{K\in\mathcal{T}}$ (the "centers" of control volumes) satisfying the following properties:*

1. *there exists $i \in \{1,2\}$ such that $K \subset \Omega_i$. We note $\mathcal{T}_i = \{K \in \mathcal{T}, K \subset \Omega_i\}$;*
2. *$\overline{\bigcup_{K\in\mathcal{T}_i} K} = \overline{\Omega_i}$. Thus, $\overline{\bigcup_{K\in\mathcal{T}} K} = \overline{\Omega}$;*
3. *for any $K \in \mathcal{T}$, there exists a subset $\mathcal{E}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma\in\mathcal{E}_K} \overline{\sigma}$. Furthermore, $\mathcal{E} = \bigcup_{K\in\mathcal{T}} \mathcal{E}_K$;*
4. *for any $(K,L) \in \mathcal{T}^2$ with $K \ne L$, either the "length"(i.e. the $(d-1)$ Lebesgue measure) of $\overline{K} \cap \overline{L}$ is 0 or $\overline{K} \cap \overline{L} = \overline{\sigma}$ for some $\sigma \in \mathcal{E}$. In the latter case, we write $\sigma = K|L$, and*
   - $\mathcal{E}_i = \{\sigma \in \mathcal{E},\ \exists (K,L) \in \mathcal{T}_i^2,\ \sigma = K|L\}$, $\mathcal{E}_{\text{int}} = \mathcal{E}_1 \cup \mathcal{E}_2$, $\mathcal{E}_{K,\text{int}} = \mathcal{E}_K \cap \mathcal{E}_{\text{int}}$,
   - $\mathcal{E}_{\text{ext}} = \{\sigma \in \mathcal{E},\ \sigma \subset \partial\Omega\}$, $\mathcal{E}_{K,\text{ext}} = \mathcal{E}_K \cap \mathcal{E}_{\text{ext}}$,
   - $\mathcal{E}_\Gamma = \{\sigma \in \mathcal{E},\ \exists (K,L) \in \mathcal{T}_1 \times \mathcal{T}_2,\ \sigma = K|L\}$, $\mathcal{E}_{K,\Gamma} = \mathcal{E}_K \cap \mathcal{E}_\Gamma$;
5. *The family of points $(\mathbf{x}_K)_{K\in\mathcal{T}}$ is such that $\mathbf{x}_K \in K$ (for all $K \in \mathcal{T}$) and, if $\sigma = K|L$, it is assumed that the straight line $(\mathbf{x}_K, \mathbf{x}_L)$ is orthogonal to σ.*

For all $\sigma \in \mathcal{E}$, we denote by $m(\sigma)$ the $(d-1)$-Lebesgue measure of σ. If $\sigma \in \mathcal{E}_K$, we note $d_{K,\sigma} = d(\mathbf{x}_K, \sigma)$, and we denote by $\tau_{K,\sigma}$ the transmissibility of K through σ, defined by $\tau_{K,\sigma} = \frac{m(\sigma)}{d_{K,\sigma}}$. If $\sigma = K|L$, we note $d_{K,L} = d(\mathbf{x}_K, \mathbf{x}_L)$ and $\tau_{KL} = \frac{m(\sigma)}{d_{K,L}}$. The size of the mesh is defined by:

$$size(\mathcal{T}) = \max_{K\in\mathcal{T}} diam(K),$$

and a geometrical factor, connected with the regularity of the mesh, is defined by

$$reg(\mathcal{T}) = \max_{K\in\mathcal{T}} \left(\sum_{\sigma=K|L\in\mathcal{E}_{K,\text{int}}} \frac{m(\sigma) d_{K,L}}{m(K)} \right).$$

Remark 2.1 *One can see the spatial discretization introduced above is an admissible mesh in the sense of [26]. In addition we assume that it resolve the interface Γ. We illustrate this definition thanks to Figure 2.*

Definition 2.2 *A uniform time discretization of $(0,T)$ is given by an integer value N and a sequence of real values $(t^n)_{n\in\{0,\dots,N\}}$. We define $\delta t = \frac{T}{N+1}$ and, $\forall n \in \{0,\dots,N\}$, $t^n = n\delta t$. Thus we have $t^0 = 0$ and $t^{N+1} = T$.*

Remark 2.2 *We can easily prove all the results of this paper for a general time discretization, but for the sake of simplicity, we choose to only consider uniform time discretizations.*

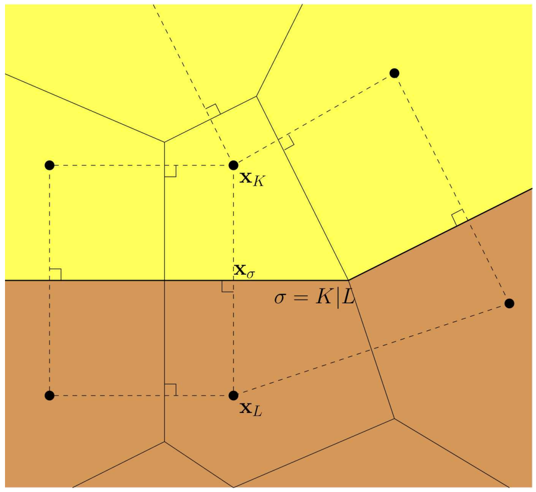

Figure 2: In Definition 2.1, we assume both a classical orthogonality solution in the sense of [26] and the fact that the interface Γ is made of a union of edges.

Definition 2.3 *A finite volume discretization $\mathcal{D}$ of Q_T is a family*

$$\mathcal{D} = (\mathcal{T}, \mathcal{E}, (\mathbf{x}_K)_{K\in\mathcal{T}}, N, (t^n)_{n\in\{0,\dots,N\}}),$$

where $(\mathcal{T}, \mathcal{E}, (\mathbf{x}_K)_{K\in\mathcal{T}})$ is an admissible mesh of Ω in the sense of definition 2.1 and $(N, (t^n)_{n\in\{0,\dots,N\}})$ is a discretization of $(0, T)$ in the sense of definition 2.2. For a given mesh $\mathcal{D}$, one defines:

$$size(\mathcal{D}) = max(size(\mathcal{T}), \delta t), \qquad and\ reg(\mathcal{D}) = reg(\mathcal{T}).$$

2.2 Definition of the scheme and main result

For $K \in \mathcal{T}_i$, we denote by $g_K(s) = g_i(s)$ for all function g whose definition depends on the subdomain Ω_i, as for example $\phi_i, \varphi_i, M_i, f_i, W_i, \dots$. For a function $f : \mathbb{R} \to \mathbb{R}$ and for $(a, b) \in \mathbb{R}^2$ we denote by $\mathcal{R}(f; a, b)$ the Godunov flux

$$\mathcal{R}(f; a, b) = \begin{cases} \min_{c\in[a,b]} f(c) & \text{if } a \le b, \\ \max_{c\in[b,a]} f(c) & \text{if } b \le a. \end{cases} \tag{31}$$

The total flux balance equation is discretized by

$$\sum_{\sigma\in\mathcal{E}_K} m(\sigma) Q^{n+1}_{K,\sigma} = 0, \qquad \forall n \in \{0, \dots, N\}, \forall K \in \mathcal{T}, \tag{32}$$

with

$$Q^n_{K,\sigma} = \begin{cases} \frac{M_{K,L}(s^n_K, s^n_L)}{d_{K,L}} (P^n_K - P^n_L) + Z^n_{K,\sigma} & \text{if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ \frac{M_K(s^n_K)}{d_{K,\sigma}} \left(P^n_K - P^n_{K,\sigma}\right) + Z^n_{K,\sigma} & \text{if } \sigma \in \mathcal{E}_{K,\Gamma}, \\ 0 & \text{if } \sigma \in \mathcal{E}_{K,\text{ext}}, \end{cases} \tag{33}$$

where $M_{K,L}(s^{n+1}_K, s^{n+1}_L) = M_{L,K}(s^{n+1}_L, s^{n+1}_K)$ is a mean value between $M_K(s^{n+1}_K)$ and $M_L(s^{n+1}_L)$. For example, we can consider, as in [35], the harmonic mean

$$M_{K,L}(s^{n+1}_K, s^{n+1}_L) = \frac{M_K(s^{n+1}_K) M_K(s^{n+1}_L) d_{K,L}}{d_{L,\sigma} M_K(s^{n+1}_K) + d_{K,\sigma} M_K(s^{n+1}_L)}. \tag{34}$$

The quantity $Z^n_{K,\sigma}$ is an approximation of $\zeta_K(s)\mathbf{g} \cdot \mathbf{n}_{K,\sigma}$ at the interface σ. We set

$$Z^n_{K,\sigma} = \begin{cases} \dfrac{\zeta_K(s^n_K) d_{L,\sigma} + \zeta_K(s^n_L) d_{K,\sigma}}{d_{K,L}} \mathbf{g} \cdot \mathbf{n}_{K,\sigma} & \text{if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ \zeta_K(s^n_K) \mathbf{g} \cdot \mathbf{n}_{K,\sigma} & \text{if } \sigma \in \mathcal{E}_{K,\Gamma}, \end{cases}$$

where $\mathbf{n}_{K,\sigma}$ denotes the outward normal to σ with respect to K.

Remark 2.3 *Let us briefly justify the choice of the definition* (33) *of* $Q^n_{K,\sigma}$*, in particular the discretization of* M*. For the sake of simplicity, we neglect the gravity, despite our purpose can be extended to the full problem. Assume that all the fluxes are discretized by following the formula*

$$Q^n_{K,\sigma} = \frac{M_K(s^n_K)}{d_{K,\sigma}} \left(P^n_K - P^n_{K,\sigma}\right),$$

with the continuity condition $P^n_{K,\sigma} = P^n_{L,\sigma}$ *for* $\sigma = K|L \in \mathcal{E}_i$*. Then, prescribing the conservativity of the scheme, i.e.,*

$$Q^n_{K,\sigma} + Q^n_{L,\sigma} = 0,$$

we recover the formula

$$Q^n_{K,\sigma} = \frac{M_{K,L}(s^n_K, s^n_L)}{d_{K,L}} \left(P^n_K - P^n_L\right),$$

where $M_{K,L}(s^n_K, s^n_L)$ *is given by the formula* (34).

The oil-flux balance equation is discretized as follows:

$$\phi_K \frac{s^{n+1}_K - s^n_K}{\delta t} m(K) + \sum_{\sigma\in\mathcal{E}_K} m(\sigma) F^{n+1}_{K,\sigma} = 0, \tag{35}$$

with

$$F^n_{K,\sigma} = \begin{cases} Q^n_{K,\sigma}\ f_K(\overline{s}^n_{K,\sigma}) + \mathcal{R}(G_{K,\sigma}; s^n_K, s^n_L) + \dfrac{\varphi_K(s^n_K) - \varphi_K(s^n_L)}{d_{K,L}} \text{ if } \sigma = K|L \in \mathcal{E}_{K,i}, \\ Q^n_{K,\sigma}\ f_K(\overline{s}^n_{K,\sigma}) + \mathcal{R}(G_{K,\sigma}; s^n_K, s^n_{K,\sigma}) + \dfrac{\varphi_K(s^n_K) - \varphi_K(s^n_{K,\sigma})}{d_{K,\sigma}} \text{ if } \sigma \in \mathcal{E}_{K,\Gamma}, \\ 0 \text{ if } \sigma \in \mathcal{E}_{K,\text{ext}}, \end{cases} \tag{36}$$

where $G_{K,\sigma}(s) = \gamma_K(s)\mathbf{g}\cdot\mathbf{n}_{K,\sigma}$ and $\overline{s}^{n+1}_{K,\sigma}$ is the upwind value defined by

$$\overline{s}^{n+1}_{K,\sigma} = \begin{cases} s^{n+1}_K & \text{if } Q^{n+1}_{K,\sigma} \geq 0, \\ s^{n+1}_L & \text{if } Q^{n+1}_{K,\sigma} < 0 \text{ and } \sigma = K|L \in \mathcal{E}_{K,i}, \\ s^{n+1}_{K,\sigma} & \text{if } Q^{n+1}_{K,\sigma} < 0 \text{ and } \sigma \in \mathcal{E}_{K,\Gamma}. \end{cases} \tag{37}$$

The interface values $\left(s^{n+1}_{K,\sigma}, s^{n+1}_{L,\sigma}, P^{n+1}_{K,\sigma}, P^{n+1}_{L,\sigma}\right)$ for $\sigma = K|L \in \mathcal{E}_\Gamma$ are defined by the following nonlinear system. For all $\sigma = K|L \in \mathcal{E}_\Gamma$, for all $n \in \{0,\dots,N\}$, there exists $\pi^{n+1}_\sigma \in \mathbb{R}$ such that

$$\pi^{n+1}_\sigma \in \tilde{\pi}_K(s^{n+1}_{K,\sigma}) \cap \tilde{\pi}_L(s^{n+1}_{L,\sigma}), \tag{38}$$

$$P^{n+1}_{K,\sigma} - W_K\left(\pi^{n+1}_\sigma\right) = P^{n+1}_{L,\sigma} - W_L\left(\pi^{n+1}_\sigma\right), \tag{39}$$

$$Q^{n+1}_{K,\sigma} + Q^{n+1}_{L,\sigma} = 0, \tag{40}$$

$$F^{n+1}_{K,\sigma} + F^{n+1}_{L,\sigma} = 0. \tag{41}$$

In view of relations (13) and (38), given a value of π^{n+1}_σ, the values of the interface saturation $s^{n+1}_{K,\sigma}$ and $s^{n+1}_{L,\sigma}$ are given by

$$s^{n+1}_{K,\sigma} = \theta_K(\pi^{n+1}_\sigma), \quad s^{n+1}_{L,\sigma} = \theta_L(\pi^{n+1}_\sigma). \tag{42}$$

We illustrate the localization of the unknowns of the unknowns on figure 3.

Moreover, we impose the discrete counterpart of the equation (25), that is, for all $n \in \{0,\dots,N\}$,

$$\sum_{K\in\mathcal{T}_1} m(K) P^{n+1}_K = 0. \tag{43}$$

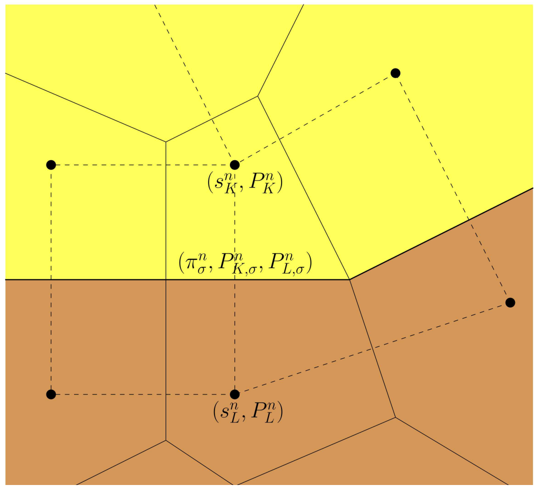

Figure 3: In the scheme, we use cell unknowns (s_K^n, P_K^n) corresponding to the saturation and the global pressure as well as interface unknowns $(\pi_\sigma^n, P_{K,\sigma}^n, P_{L,\sigma}^n)$ corresponding to the capillary pressure and the one-sided global pressures. From the capillary pressure π_σ^n, we reconstruct one-sided saturations thanks to (42). As it will be noticed in the sequel, the interface global pressures $P_{K,\sigma}^n$ and $P_{L,\sigma}^n$ can be eliminated thanks to the linear system (39)–(40).

We will show below in Section 2.3 that the system (38)-(41) possesses a solution. We denote by $\mathcal{X}(\mathcal{D}, i)$ the finite dimensional space of piecewise constant functions $u_\mathcal{D}$ defined almost everywhere in $Q_{i,T}$ having a trace on the interface Γ, i.e.

$$\mathcal{X}(\mathcal{D}, i) := \Big\{ u_{\mathcal{D},i} : Q_{i,T} \to \mathbb{R} \text{ s.t. for all } (K, \sigma, n) \in \mathcal{T} \times \mathcal{E}_\Gamma \times \{0, \dots, N\}, \\ u_{\mathcal{D},i} \text{ is constant on } K \times (t^n, t^{n+1}], u_{\mathcal{D},i} \text{ is constant on } \sigma \times (t^n; t^{n+1}) \Big\},$$

and by $\mathcal{X}(\mathcal{D})$ the space of the functions $u_\mathcal{D}$ whose restriction $(u_\mathcal{D})_{|_{\overline{Q}_{i,T}}}$ belongs to $\mathcal{X}(\mathcal{D}, i)$. We define the solution $(s_\mathcal{D}, P_\mathcal{D}) \in \mathcal{X}(\mathcal{D})^2$ of the scheme by

$$s_\mathcal{D}(\mathbf{x}, t) = s_K^{n+1}, \quad P_\mathcal{D}(\mathbf{x}, t) = P_K^{n+1} \quad \text{if } (\mathbf{x}, t) \in K \times (t^n, t^{n+1}],$$

and, for $\mathbf{x} \in \sigma = K|L \subset \Gamma$ for some $K \in \mathcal{T}_1$, $L \in \mathcal{T}_2$, for $t \in (t^n, t^{n+1})$, the traces

$$s_{\mathcal{D}_{|_{\Gamma,1}}}(\mathbf{x}, t) = s_{K,\sigma}^{n+1}, \qquad s_{\mathcal{D}_{|_{\Gamma,2}}}(\mathbf{x}, t) = s_{L,\sigma}^{n+1}.$$

In this paper we prove the following convergence result.

Theorem 1 *Assume that Assumptions 1 and 2 hold. Let $(\mathcal{D}_m)_m$ be a sequence of admissible discretizations of Q_T in the sense of Definition 2.3, then for all $m \in \mathbb{N}$, there exists a discrete solution $(s_{\mathcal{D}_m}, P_{\mathcal{D}_m}) \in \mathcal{X}(\mathcal{D}_m)^2$ to the scheme. Moreover, if $\lim_{m\to\infty} size(\mathcal{D}_m) = 0$, and if there exists $\zeta > 0$ such that, for all m, $reg(\mathcal{D}_m) \le \zeta$, then up to a subsequence, $s_{\mathcal{D}_m}$ converges, towards $s \in L^\infty(Q_T; [0, 1])$ in the $L^p(Q_T)$ topology for all $p \in [1, \infty)$, $P_{\mathcal{D}_m}$ converges to P weakly in $L^2(Q_T)$, where (s, P) is a weak solution of Problem (18)-(25) in the sense of Definition 1.1.*

2.3 The interface conditions system

Define, for all $\sigma = K|L \in \mathcal{E}_\Gamma$, for all $n \in \{0, \dots, N\}$,

$$P_\sigma^{n+1}(\pi_\sigma^{n+1}) := P_{K,\sigma}^{n+1} - W_K\left(\pi_\sigma^{n+1}\right) = P_{L,\sigma}^{n+1} - W_L(\pi_\sigma^{n+1}), \tag{44}$$

and

$$Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1}) := \alpha_K^{n+1}\left(P_K^{n+1} - P_\sigma^{n+1}(\pi_\sigma^{n+1}) - W_K(\pi_\sigma^{n+1})\right) + Z_{K,\sigma}^n, \tag{45}$$

where $\alpha_K^{n+1} = \frac{M_K(s_K^{n+1})}{d_{K,\sigma}}$. Then, the balance of the fluxes on the interface (40)–(41) can be rewritten as

$$Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1}) + Q_{L,\sigma}^{n+1}(\pi_\sigma^{n+1}) = 0 \tag{46}$$

$$\begin{aligned} & Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1}) f_K\left(\overline{s}_{K,\sigma}^{n+1}(\pi_\sigma^{n+1})\right) + Q_{L,\sigma}^{n+1}(\pi_\sigma^{n+1}) f_L\left(\overline{s}_{L,\sigma}^{n+1}(\pi_\sigma^{n+1})\right) \\ & \quad + \mathcal{R}(G_{K,\sigma}; s_K^{n+1}, \theta_K(\pi_\sigma^{n+1})) + \mathcal{R}(G_{L,\sigma}; s_L^{n+1}, \theta_L(\pi_\sigma^{n+1})) \\ & \quad + \frac{\varphi_K(s_K^{n+1}) - \varphi_K \circ \theta_K(\pi_\sigma^{n+1})}{d_{K,\sigma}} + \frac{\varphi_L(s_L^{n+1}) - \varphi_L \circ \theta_L(\pi_\sigma^{n+1})}{d_{L,\sigma}} = 0, \end{aligned} \tag{47}$$

where

$$\overline{s}_{K,\sigma}^{n+1}(p) = \begin{cases} s_K^{n+1} & \text{if} \quad Q_{K,\sigma}^{n+1}(p) \geq 0, \\ \theta_K(p) & \text{if} \quad Q_{K,\sigma}^{n+1}(p) < 0. \end{cases} \tag{48}$$

We deduce from (46) that

$$\begin{aligned} P_\sigma^{n+1} &= \frac{\alpha_K^{n+1}(P_K^{n+1} - W_K(\pi_\sigma^{n+1})) + \alpha_L^{n+1}(P_L^{n+1} - W_L(\pi_\sigma^{n+1}))}{\alpha_K^{n+1} + \alpha_L^{n+1}} \\ & \quad + \frac{Z_{K,\sigma}^n + Z_{L,\sigma}^n}{\alpha_K^{n+1} + \alpha_L^{n+1}} \end{aligned} \tag{49}$$

and thus that

$$\begin{aligned} Q_{K,\sigma}^{n+1}(\pi_\sigma^{n+1}) &= \frac{\alpha_K^{n+1}\alpha_L^{n+1}}{\alpha_K^{n+1} + \alpha_L^{n+1}} \left(P_K^{n+1} - P_L^{n+1} - W_K(\pi_\sigma^{n+1}) + W_L(\pi_\sigma^{n+1})\right) \\ & \quad + \frac{\alpha_L^{n+1} Z_{K,\sigma}^n) - \alpha_K^{n+1} Z_{L,\sigma}^n}{\alpha_K^{n+1} + \alpha_L^{n+1}}. \end{aligned} \tag{50}$$

As a direct consequence of Lemma 1.2, $Q_{K,\sigma}^{n+1}$ belong to $\mathcal{C}^1(\mathbb{R};\mathbb{R})$ and admits finite limits as $p \to \pm\infty$.

Denote by

$$\begin{aligned} \Psi_\sigma^{n+1}(p) &:= Q_{K,\sigma}^{n+1}(p)\left(f_K(\overline{s}_{K,\sigma}(p)) - f_L(\overline{s}_{L,\sigma}(p))\right) \\ & \quad + \mathcal{R}(G_{K,\sigma}; s_K^{n+1}, \theta_K(p)) + \mathcal{R}(G_{L,\sigma}; s_L^{n+1}, \theta_L(p)) \\ & \quad + \frac{\varphi_K(s_K^{n+1}) - \varphi_K \circ \pi_K^{-1}(p)}{d_{K,\sigma}} + \frac{\varphi_L(s_L^{n+1}) - \varphi_L \circ \pi_L^{-1}(p)}{d_{L,\sigma}}, \end{aligned}$$

then Ψ_σ is continuous on $\mathbb{R}$.

Lemma 2.1 *Let* $(s_K^{n+1}, s_L^{n+1}) \in [0,1]^2$, *there exists* $\pi_\sigma^{n+1} \in [\min_i \pi_i(0), \max_i \pi_i(1)]$ *such that* $\Psi_\sigma^{n+1}(\pi_\sigma^{n+1}) = 0$.

Proof: From the definition (48) of $\overline{s}_{K,\sigma}^{n+1}(p)$, since $\lim_{p\to\min_i \pi_i(0)} \theta_K(p) = 0$, and since $Q_{K,\sigma}^{n+1}(p)$ admits a limit as $p \to \min_i \pi_i(0)$, one has

$$\lim_{p\to\min_i \pi_i(0)} Q_{K,\sigma}^{n+1}(p)\left(f_K(\overline{s}_{K,\sigma}^{n+1}(p)) - f_L(\overline{s}_{L,\sigma}^{n+1}(p))\right) \geq 0$$

and also

$$\lim_{p\to\min_i \pi_i(0)} \mathcal{R}(G_{M,\sigma}; s_M^{n+1}, \theta_M(p)) = \max_{s\in[0,s_M]} G_{M,\sigma}(s) \geq 0, \qquad \text{with} \qquad M \in \{K, L\}.$$

This yields that

$$\lim_{p\to\min_i \pi_i(0)} \Psi_\sigma^{n+1}(p) \geq \frac{\varphi_K(s_K^{n+1})}{d_{K,\sigma}} + \frac{\varphi_L(s_L^{n+1})}{d_{L,\sigma}} \geq 0.$$

One obtains similarly that $\lim_{p\to\max_i \pi_i(1)} \Psi_\sigma^{n+1}(p) \leq 0$. One conclude thanks to the continuity of Ψ_σ^{n+1}. □

Proposition 2.2 *Let $\sigma = K|L \in \mathcal{E}_\Gamma$, and let $\left(s_K^{n+1}, s_L^{n+1}, P_K^{n+1}, P_L^{n+1}\right) \in \mathbb{R}^4$, then there exists a solution $\left(\pi_\sigma^{n+1}, s_{K,\sigma}^{n+1}, s_{L,\sigma}^{n+1}, P_{K,\sigma}^{n+1}, P_{L,\sigma}^{n+1}\right) \in [\min_i \pi_i(0), \max_i \pi_i(1)] \times [0,1]^2 \times \mathbb{R}^2$ to the nonlinear system* (38)–(41).

Proof: Let $\pi_\sigma^{n+1} \in \overline{\mathbb{R}}$ be a solution of the equation $\Psi_\sigma^{n+1}(\pi_\sigma^{n+1}) = 0$, whose existence has been claimed in Lemma 2.1. Firstly, defining $s_{K,\sigma}^{n+1} := \pi_K^{-1}(\pi_\sigma^{n+1})$ and $s_{L,\sigma}^{n+1} := \pi_L^{-1}(\pi_\sigma^{n+1})$, one has directly that

$$\pi_\sigma^{n+1} \in \tilde{\pi}_K(s_{K,\sigma}^{n+1}) \cap \tilde{\pi}_L(s_{L,\sigma}^{n+1}).$$

As it was noticed in Lemma 1.2, the function $p \mapsto W_K(p) - W_L(p)$ is uniformly bounded. In view of (44) and (50) the values $P_{K,\sigma}^{n+1}$ and $P_{L,\sigma}^{n+1}$ are also finite. It is now easy to check that $\left(\pi_\sigma^{n+1}, s_{K,\sigma}^{n+1}, s_{L,\sigma}^{n+1}, P_{K,\sigma}^{n+1}, P_{L,\sigma}^{n+1}\right)$ is a solution to the system (38)–(41) thanks to the analysis carried out above. □

3 *A priori* estimates and existence of a discrete solution

3.1 $\mathbf{L^\infty(Q_T)}$ estimate on the saturation

Proposition 3.1 *Let $(s_\mathcal{D}, P_\mathcal{D})$ be a solution to the scheme* (32)–(43)*, then*

$$0 \le s_\mathcal{D} \le 1 \quad \text{a.e. in } Q_T. \tag{51}$$

Proof: We will prove that for all $K \in \mathcal{T}$, for all $n \in \{0, \dots, N\}$,

$$s_K^{n+1} \le 1.$$

The proof for obtaining $s_K^{n+1} \ge 0$ is similar.

Using the definition (36) of $F_{K,\sigma}^{n+1}$, one can rewrite (35) under the form

$$H_K\left(s_K^{n+1}, s_K^n, \left(s_L^{n+1}\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) = 0, \tag{52}$$

where H_K is non increasing with respect to $s_K^n, \left(s_L^{n+1}\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}$. Making use of the notations $a \top b = \max(a,b)$, we obtain that

$$H_K\left(s_K^{n+1}, s_K^n \top 1, \left(s_L^{n+1} \top 1\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1} \top 1\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) \le 0.$$

We remark that for all $K \in \mathcal{T}$ and for all $s \in [0,1]$ one has

$$\sum_{\sigma\in\mathcal{E}_K} m(\sigma) G_{K,\sigma}(s) = 0. \tag{53}$$

Combining (53) and (32) we have

$$H_K\left(1, 1, (1)_{L\in\mathcal{N}_K}, (1)_{\sigma\in\mathcal{E}_{K,i}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) = 0.$$

Hence, using once again the monotonicity of H_K, one obtains

$$H_K\left(1, s_K^n \top 1, \left(s_L^{n+1} \top 1\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1} \top 1\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) \le 0.$$

Since $a \top b$ is either equal to a or to b, one has

$$H_K\left(s_K^{n+1} \top 1, s_K^n \top 1, \left(s_L^{n+1} \top 1\right)_{L\in\mathcal{N}_K}, \left(s_{K,\sigma}^{n+1} \top 1\right)_{\sigma\in\mathcal{E}_{K,\Gamma}}, \left(Q_{K,\sigma}^{n+1}\right)_{\sigma\in\mathcal{E}_K}\right) \le 0. \tag{54}$$

Next we remark that for any $\sigma = K|L \in \mathcal{E}_\Gamma$, the equation (41) can be written as

$$H_\sigma \left(s_K^{n+1}, s_L^{n+1}, \left(s_{M,\sigma}^{n+1} \right)_{M\in\{K,L\}}, \left(Q_{M,\sigma}^{n+1} \right)_{M\in\{K,L\}} \right) = 0.$$

Thanks to (40) and using $\gamma_i(1) = 0$ for $i \in \{1, 2\}$, one has

$$H_\sigma \left(1, 1, (1)_{M\in\{K,L\}}, \left(Q_{M,\sigma}^{n+1} \right)_{M\in\{K,L\}} \right) = 0.$$

We remark that H_σ is non decreasing with respect to s_K^{n+1}, s_L^{n+1}. Furthermore, since $s_{M,\sigma}^{n+1} = \theta_M(\pi_\sigma^{n+1})$ for $M \in \{K, L\}$, we obtain that $s_{M,\sigma}^{n+1} \in [0, 1]$, implying that $s_{M,\sigma}^{n+1} \top 1 = 1$. Hence, we deduce that

$$H_\sigma \left(s_K^{n+1} \top 1, s_L^{n+1} \top 1, \left(s_{M,\sigma}^{n+1} \top 1 \right)_{M\in\{K,L\}}, \left(Q_{M,\sigma}^{n+1} \right)_{M\in\{K,L\}} \right) \geq 0. \tag{55}$$

Multiplying (54) by δt and summing over $K \in \mathcal{T}$ provides, using (55) and the conservativity of the scheme,

$$\sum_{K\in\mathcal{T}} \phi_K (s_K^{n+1} - 1)^+ m(K) \leq \sum_{K\in\mathcal{T}} \phi_K (s_K^n - 1)^+ m(K).$$

Since $s_0 \in L^\infty(Q_T; [0, 1])$, $s_K^0 \in [0, 1]$ for all $K \in \mathcal{T}$. A straightforward induction allows us to conclude. $\square$

3.2 Energy estimate

Definition 3.1 *We define the discrete $L^2(0, T; H^1(\Omega_i))$ semi-norm of an element $u_\mathcal{D} \in \mathcal{X}(\mathcal{D}, i)$ by*

$$|u_\mathcal{D}|_{\mathcal{D},i}^2 := \sum_n \delta t \sum_{\sigma=K|L\in\mathcal{E}_i} \tau_{KL} \left(u_K^{n+1} - u_L^{n+1} \right)^2 + \sum_n \delta t \sum_{K\in\mathcal{T}_i} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \tau_{K\sigma} \left(u_K^{n+1} - u_{K,\sigma}^{n+1} \right)^2.$$

In what follows we prove the following energy estimate.

Proposition 3.2 *There exists a positive constant C_1, depending only on data, such that*

$$\sum_{i\in\{1,2\}} \left(|P_\mathcal{D}|_{\mathcal{D},i}^2 + |\varphi(s_\mathcal{D})|_{\mathcal{D},i}^2 \right) \leq C_1. \tag{56}$$

Let us first establish some technical results.

Lemma 3.3 *The following inequalities hold:*

- *for all $\sigma = K|L \in \mathcal{E}_{\text{int}}$,*

$$Q_{K,\sigma}^{n+1} f_K \left(\overline{s}_{K,\sigma}^{n+1} \right) \left(\pi_K(s_K^{n+1}) - \pi_L(s_L^{n+1}) \right) \geq Q_{K,\sigma}^{n+1} \left(W_K(\pi_K(s_K^{n+1})) - W_L(\pi_L(s_L^{n+1})) \right); \tag{57}$$

- *for all $\sigma \in \mathcal{E}_{K,\Gamma}$,*

$$Q_{K,\sigma}^{n+1} f_K \left(\overline{s}_{K,\sigma}^{n+1} \right) \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1} \right) \geq Q_{K,\sigma}^{n+1} \left(W_K(\pi_K(s_K^{n+1})) - W_K(\pi_\sigma^{n+1}) \right). \tag{58}$$

Proof: Since $f_K \circ \theta_K$ is a non decreasing function, then function $W_K : p \mapsto \int_0^p f_K \circ \theta_K(a) da$ is convex, so that for all $(a, b) \in \mathbb{R}^2$,

$$f_K \circ \theta_K(a) (b - a) \leq W_K(b) - W_K(a) \leq f_K \circ \theta_K(b) (b - a).$$

The inequalities (57) and (58) follow from the definition (37) of $\overline{s}_{K,\sigma}^{n+1}$, from the property (12) of θ_K, and from the fact that $\pi_K \equiv \pi_L$ and $W_K \equiv W_L$ if $K|L \in \mathcal{E}_{\text{int}}$. $\square$

Lemma 3.4 *Let us define*

$$\mathcal{G}_{K,\sigma}(p) := \int_0^p G_{K,\sigma}\left(\theta_K(\tau)\right) \mathrm{d}\tau \tag{59}$$

for all $K \in \mathcal{T}$ and $\sigma \in \mathcal{E}_K$. Then, the following estimates hold:

- *for all $\sigma = K|L \in \mathcal{E}_{\text{int}}$,*

$$\mathcal{R}(G_{K,\sigma}; s_K^{n+1}, s_L^{n+1}) \left(\pi_K(s_K^{n+1}) - \pi_L(s_L^{n+1})\right) \geq \mathcal{G}_{K,\sigma}(\pi_K(s_K^{n+1})) - \mathcal{G}_{K,\sigma}(\pi_L(s_L^{n+1})) \tag{60}$$

- *for all $\sigma \in \mathcal{E}_{K,\Gamma}$,*

$$\mathcal{R}(G_{K,\sigma}; s_K^{n+1}, s_{K,\sigma}^{n+1}) \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right) \geq \mathcal{G}_{K,\sigma}(\pi_K(s_K^{n+1})) - \mathcal{G}_{K,\sigma}(\pi_\sigma^{n+1}). \tag{61}$$

Proof: For any $a, b \in \mathbb{R}$ one has

$$\mathcal{R}(G_{K,\sigma}; \theta_K(a), \theta_K(b))\,(a-b) = \int_b^a G_{K,\sigma}\left(\theta_K(p)\right) \mathrm{d}p + \int_b^a \mathcal{R}\left(G_{K,\sigma}; \theta_K(a), \theta_K(b)\right) - G_{K,\sigma}\left(\theta_K(p)\right) \mathrm{d}p. \tag{62}$$

We only have to remark that in view of (31) the last term in the right hand side of (62) is positive, and that $\pi_K \equiv \pi_L$ and $\theta_K \equiv \theta_L$ in the case $K|L \in \mathcal{E}_{\text{int}}$. □

Lemma 3.5 *For all $K \in \mathcal{T}$, for all $n \in \{0, \dots, N\}$ and for all $\sigma \in \mathcal{E}_{K,\Gamma}$, one has*

$$\left(\varphi_K(s_K^{n+1}) - \varphi_K(s_{K,\sigma}^{n+1})\right) \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right) \geq \left(\varphi_K(s_K^{n+1}) - \varphi_K(s_{K,\sigma}^{n+1})\right) \left(\pi_K(s_K^{n+1}) - \pi_K(s_{K,\sigma}^{n+1})\right). \tag{63}$$

Proof: Assume that $s_{K,\sigma}^{n+1} \in (0,1)$, then $\tilde{\pi}_K(s_{K,\sigma}^{n+1}) = \{\pi_K(s_{K,\sigma}^{n+1})\}$, thus the inequality (63) is in fact an equality (see Figure 1). Assume now that $s_{K,\sigma}^{n+1} = 0$, then $\pi_\sigma^{n+1} \leq \pi_K(s_{K,\sigma}^{n+1}) \leq \pi_K(s_K^{n+1})$, and $\varphi_K(s_{K,\sigma}^{n+1}) \leq \varphi_K(s_K^{n+1})$. The inequality (63) follows. Similarly, if $s_{K,\sigma}^{n+1} = 1$, then $\pi_\sigma^{n+1} \geq \pi_K(s_{K,\sigma}^{n+1}) \geq \pi_K(s_K^{n+1})$, and $\varphi_K(s_{K,\sigma}^{n+1}) \geq \varphi_K(s_K^{n+1})$, leading also to (63). □

Proof of Proposition 3.2: Multiplying the equation (35) by $\delta t \pi_K(s_K^{n+1})$ and summing over $K \in \mathcal{T}$ and $n \in \{0, \dots, N\}$ yield, after reorganizing the sum,

$$A + B = 0, \tag{64}$$

where

$$\begin{aligned} A &= \sum_{n=0}^{N} \sum_{K \in \mathcal{T}} \phi_K \pi_K(s_K^{n+1}) \left(s_K^{n+1} - s_K^n\right) m(K), \\ B &= \sum_{n=0}^{N} \delta t \sum_{\sigma = K|L \in \mathcal{E}_{\text{int}}} m(\sigma) F_{K,\sigma}^{n+1} \left(\pi_K(s_K^{n+1}) - \pi_K(s_L^{n+1})\right) \\ &\quad + \sum_{n=0}^{N} \delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} m(\sigma) F_{K,\sigma}^{n+1} \left(\pi_K(s_K^{n+1}) - \pi_\sigma^{n+1}\right), \end{aligned}$$

where we have used (41). The definition (36) of $F_{K,\sigma}^{n+1}$ gives

$$B = B_1 + B_2 + B_3, \tag{65}$$

where

$$\begin{aligned}
B_1 &= \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)Q_{K,\sigma}^{n+1} f_K(\overline{s}_{K,\sigma}^{n+1})\left(\pi_K(s_K^{n+1})-\pi_K(s_L^{n+1})\right)\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)Q_{K,\sigma}^{n+1} f_K(\overline{s}_{K,\sigma}^{n+1})\left(\pi_K(s_K^{n+1})-\pi_\sigma^{n+1}\right),\\
B_2 &= \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)\mathcal{R}(G_{K,\sigma};s_K^{n+1},s_L^{n+1})\left(\pi_K(s_K^{n+1})-\pi_K(s_L^{n+1})\right)\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)\mathcal{R}(G_{K,\sigma};s_K^{n+1},s_{K,\sigma}^{n+1})\left(\pi_K(s_K^{n+1})-\pi_\sigma^{n+1}\right),\\
B_3 &= \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \tau_{KL}\left(\varphi_K(s_K^{n+1})-\varphi_K(s_L^{n+1})\right)\left(\pi_K(s_K^{n+1})-\pi_K(s_L^{n+1})\right)\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \tau_{K\sigma}\left(\varphi_K(s_K^{n+1})-\varphi_K(s_{K,\sigma}^{n+1})\right)\left(\pi_K(s_K^{n+1})-\pi_\sigma^{n+1}\right).
\end{aligned}$$

It follows from Lemma 3.3 that

$$\begin{aligned}
B_1 &\geq \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)Q_{K,\sigma}^{n+1}\left(W_K(\pi_K(s_K^{n+1}))-W_K(\pi_K(s_L^{n+1}))\right)\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)Q_{K,\sigma}^{n+1}\left(W_K(\pi_K(s_K^{n+1}))-W_K(\pi_\sigma^{n+1})\right).
\end{aligned}$$

Multiplying the equation (32) by $\delta t\left(P_K^{n+1}-W_K(\pi_K(s_K^{n+1}))\right)$ and summing over $K\in\mathcal{T}$ and $n\in\{0,\dots,N\}$ yields, after reorganizing the sum and using (39) and (40),

$$\begin{aligned}
&\sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)Q_{K,\sigma}^{n+1}(P_K^{n+1}-P_L^{n+1})\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)Q_{K,\sigma}^{n+1}(P_K^{n+1}-P_{K,\sigma}^{n+1})\\
&\qquad=\sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma)Q_{K,\sigma}^{n+1}\left(W_K(\pi_K((s_K^{n+1}))-W_K(\pi_K(s_L^{n+1}))\right)\\
&\qquad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)Q_{K,\sigma}^{n+1}\left(W_K(\pi_K(s_{K,}^{n+1}))-W_K(\pi_\sigma^{n+1})\right).
\end{aligned}$$

Therefore, using the definition (33) of $Q_{K,\sigma}^n$, we deduce that

$$B_1 \geq B_4 + B_5, \tag{66}$$

where

$$\begin{aligned}
B_4 &= \sum_{n=0}^{N}\delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \frac{m(\sigma)M_{K,L}}{d_{K,L}}(P_K^{n+1}-P_L^{n+1})^2\\
&\quad+\sum_{n=0}^{N}\delta t \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \frac{m(\sigma)M_K}{d_{K,\sigma}}(P_K^{n+1}-P_{K,\sigma}^{n+1})^2
\end{aligned}$$

and

$$
\begin{aligned}
B_5 &= \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma) Z^n_{K,\sigma}(P^{n+1}_K - P^{n+1}_L) \\
&+ \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) Z^n_{K,\sigma}(P^{n+1}_K - P^{n+1}_{K,\sigma}).
\end{aligned} \tag{67}
$$

Using (14), i.e. the fact that for all $s \in \mathbb{R}$, $M_i(s) \geq \alpha_M > 0$ we obtain

$$
B_4 \geq \alpha_M \sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i}\,. \tag{68}
$$

The Cauchy-Schwarz inequality applied to the right hand side of (67) implies

$$
\begin{aligned}
|B_5| &\leq E_{int} \left(\sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} \frac{m(\sigma)}{d_{K,L}} (P^{n+1}_K - P^{n+1}_L)^2 \right)^{\frac{1}{2}} \\
&+ E_\Gamma \left(\sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \frac{m(\sigma)}{d_{K,\sigma}} (P^{n+1}_K - P^{n+1}_{K,\sigma})^2 \right)^{\frac{1}{2}},
\end{aligned}
$$

where

$$
(E_{int})^2 = \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma) d_{K,L} \left(Z^n_{K,\sigma}\right)^2
$$

and

$$
(E_\Gamma)^2 = \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) d_{K,\sigma} \left(Z^n_{K,\sigma}\right)^2.
$$

Therefore we deduce that,

$$
B_5^2 \leq \frac{3}{2} T |\mathbf{g}|^2 d \sum_{i\in\{1,2\}} m(\Omega_i) \|\zeta_i\|^2_{L^\infty((0,1))} \sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i}\,, \tag{69}
$$

where d stands for the dimension of Ω. Combining (66), (68) and (69) one has

$$
B_1 \geq \alpha_M \sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i} - \left(\frac{3}{2} T |\mathbf{g}|^2 d \sum_{i\in\{1,2\}} m(\Omega_i) \|\zeta_i\|^2_{L^\infty((0,1))} \right)^{\frac{1}{2}} \left(\sum_{i\in\{1,2\}} |P_{\mathcal{D}}|^2_{\mathcal{D},i} \right)^{\frac{1}{2}}. \tag{70}
$$

We now will show the estimates on the term B_2. Using Lemma 3.4 we have

$$
\begin{aligned}
B_2 &\geq \sum_{n=0}^{N} \delta t \sum_{\sigma=K|L\in\mathcal{E}_{\rm int}} m(\sigma) \left(\mathcal{G}_{K,\sigma}\left(\pi_K(s^{n+1}_K)\right) - \mathcal{G}_{K,\sigma}\left(\pi_K(s^{n+1}_L)\right) \right) \\
&+ \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) \left(\mathcal{G}_{K,\sigma}\left(\pi_K(s^{n+1}_K)\right) - \mathcal{G}_{K,\sigma}\left(\pi^{n+1}_\sigma\right) \right).
\end{aligned} \tag{71}
$$

Recombining terms we obtain

$$
\begin{aligned}
B_2 &\geq \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\mathcal{E}_{K,\rm int}} m(\sigma) \mathcal{G}_{K,\sigma}\left(\pi_K(s^{n+1}_K)\right) \\
&+ \sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) \left(\mathcal{G}_{K,\sigma}\left(\pi_K(s^{n+1}_K)\right) - \mathcal{G}_{K,\sigma}\left(\pi^{n+1}_\sigma\right) \right),
\end{aligned}
$$

which in view of (59) and (53) implies

$$B_2 \geq -\sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma)\mathcal{G}_{K,\sigma}\left(\pi_\sigma^{n+1}\right).$$

Remark that if $\sigma = K|L \in \mathcal{E}_\Gamma$ then the function $\mathcal{G}_{K,\sigma}(p) + \mathcal{G}_{L,\sigma}(p)$ in general is not equal to zero. However we can write an lower bound for the term B_2. Indeed, comparing the definition (17) of φ_i with the definition (19) of γ_i, and using the fact that $\gamma_i(0) = 0$ and $\gamma_i(1) = 0$ one has

$$\int_0^{\pi_\sigma^n} \gamma_K \circ \theta_K(p)\mathrm{d}p = \int_0^{s_{K,\sigma}^n} \gamma_K(a)\pi_K'(a)\mathrm{d}a = (\rho_o - \rho_w)\varphi_K(s_{K,\sigma}^n)$$

and thus, in view of Proposition 3.1

$$B_2 \geq -|\rho_o - \rho_w||\mathbf{g}| \max_{i\in\{1,2\}} \varphi_i(1)\ m(\Gamma)T.$$

Because of the definition (17) of the function φ_i, then, for all $(a,b)\in[0,1]^2$,

$$(\varphi_i(a) - \varphi_i(b))(\pi_i(a) - \pi_i(b)) \geq \frac{\max(\mu_o,\mu_w)}{K_i}\left(\varphi_i(a) - \varphi_i(b)\right)^2. \tag{72}$$

Then it follows from Lemma 3.5 and for inequality (72) that

$$B_3 \geq \frac{\max(\mu_o,\mu_w)}{\min_{i\in\{1,2\}} K_i} \sum_{i\in\{1,2\}} |\varphi_i(s_\mathcal{D})|^2_{\mathcal{D},i}. \tag{73}$$

We define $\Pi_i(s) = \int_0^s \pi_i(a)\mathrm{d}a$, then Π_i is a continuous convex function. As a consequence, for all $(a,b)\in[0,1]^2$,

$$\pi_i(b)(b-a) \geq \Pi_i(b) - \Pi_i(a).$$

Therefore,

$$\begin{aligned} A &\geq \sum_{n=0}^{N}\sum_{K\in\mathcal{T}} \phi_K\left(\Pi_K(s_K^{n+1}) - \Pi_K(s_K^n)\right) m(K) \\ &= \sum_{K\in\mathcal{T}} \phi_K\left(\Pi_K(s_K^{N+1}) - \Pi_K(s_K^0)\right) m(K). \end{aligned}$$

Using the fact that, for all $(a,b)\in[0,1]^2$, one has

$$\Pi_i(b) - \Pi_i(a) = \int_a^b \pi_i(u)\mathrm{d}u \geq -\int_0^1 |\pi_i(u)|\mathrm{d}u,$$

it follows from Proposition 3.1 that

$$A \geq -\sum_{i\in\{1,2\}} \phi_i m(\Omega_i)\|\pi_i\|_{L^1((0,1))}. \tag{74}$$

Taking (70), (73), (73) and (74) into account in (64) we have.

$$\begin{aligned} \alpha_M \sum_{i\in\{1,2\}} |P_\mathcal{D}|^2_{\mathcal{D},i} \quad &- \left(\frac{3}{2}\frac{T|\mathbf{g}|^2}{d} \sum_{i\in\{1,2\}} m(\Omega_i)\|\zeta_i\|^2_{L^\infty((0,1))}\right)^{\frac{1}{2}} \left(\sum_{i\in\{1,2\}} |P_\mathcal{D}|^2_{\mathcal{D},i}\right)^{\frac{1}{2}} \\ &+ \frac{\max(\mu_o,\mu_w)}{\min_{i\in\{1,2\}} K_i} \sum_{i\in\{1,2\}} |\varphi_i(s_\mathcal{D})|^2_{\mathcal{D},i} \leq C. \end{aligned} \tag{75}$$

Applying Young's inequality to (71) we complete the proof of Proposition 3.2. Indeed,

$$\frac{\alpha_M}{2} \sum_{i\in\{1,2\}} |P_\mathcal{D}|^2_{\mathcal{D},i} + \frac{\max(\mu_o,\mu_w)}{\min_{i\in\{1,2\}} K_i} \sum_{i\in\{1,2\}} |\varphi_i(s_\mathcal{D})|^2_{\mathcal{D},i} \leq C.$$

□

3.3 Existence of a discrete solution

Proposition 3.6 *There exists (at least) a solution to the scheme* (35)-(43)*.*

Proof: The proof is based on a topological degree argument (see for example [22]). For $\nu \in [0,1]$, we introduce the functions

- $f_i^\nu(s) = \nu f_i(s) + (1-\nu)s$,
- $\zeta_i^\nu(s) = \nu\zeta_i(s), \quad \gamma_i^\nu(s) = \nu\gamma_i(s)$
- $M_i^\nu(s) = \nu M_i(s) + (1-\nu)\alpha_M$,
- $\lambda_i^\nu(s) = \nu\lambda_i(s) + (1-\nu)\alpha_M s(1-s)$,
- $\pi_i^\nu(s) = \nu\pi_i(s) + (1-\nu)\pi_1(s)$,
- $\varphi_i^\nu(s) = \displaystyle\int_0^s \lambda_i^\nu(a)\left(\pi_i^\nu\right)'(a)\mathrm{d}a$,
- $W_i^\nu(s) = \displaystyle\int_{s^\star}^s f_i^\nu(a)\left(\pi_i^\nu\right)'(a)\mathrm{d}a$.

We denote by $(s_\mathcal{D}^\nu, P_\mathcal{D}^\nu)$ the solution to the modified scheme. For $\nu = 0$, the problem becomes homogeneous, corresponding to the equations

$$\begin{cases} \partial_t s^0 - \operatorname{div}\left(s^0\nabla P^0 - \nabla\varphi^0(s^0)\right) = 0, \\ -\alpha_M \Delta P^0 = 0. \end{cases} \tag{76}$$

The pressure equation provides a classical linear Finite Volume scheme which is completely uncoupled from the saturation equation. The transmission conditions (40),(39) turn to

$$P_{K,\sigma}^{n+1,0} = P_{L,\sigma}^{n+1,0} = \frac{\tau_{K\sigma}P_K^{n+1,0} + \tau_{L\sigma}P_L^{n+1,0}}{\tau_{K\sigma} + \tau_{L\sigma}},$$

and thus

$$Q_{K,\sigma}^{n+1,0} = \tau_{KL}\left(P_K^{n+1,0} - P_L^{n+1,0}\right).$$

Note that the *a priori* estimates (51) and (56) still hold for $(s_\mathcal{D}^\nu, P_\mathcal{D}^\nu)$ instead of $(s_\mathcal{D}, P_\mathcal{D})$. We introduce now a new parameter $\eta \in [0,1]$, and we approximate the problem

$$\begin{cases} \partial_t s^{0,\eta} - \eta \operatorname{div}\left(s^{0,\eta}\nabla P^0 - \nabla\varphi^0(s^{0,\eta})\right) = 0, \\ -\alpha_M \Delta P^0 = 0. \end{cases}$$

The corresponding discrete solution $s_\mathcal{D}^{0,\eta}$ satisfies

$$0 \le s_\mathcal{D}^{0,\eta} \le 1, \quad \forall \eta \in [0,1]. \tag{77}$$

We introduce the compact set

$$\mathcal{K} = \left\{ (u_\mathcal{D}, v_\mathcal{D}) \in (\mathcal{X}(\mathcal{D}))^2 \;\middle|\; \|u_\mathcal{D}\|_\infty \le 2 \text{ and } |v_\mathcal{D}|_\mathcal{D} \le 2C_1 \right\},$$

where C_1 is the quantity introduced in Proposition 3.2. Since, for $\nu = \eta = 0$, the problem turns to an invertible linear problem, we can claim that the corresponding topological degree is equal to +1 (since the determinant of the underlying matrix is positive). One can let first η go to 1, and thanks to (56),(77), $\left(s_\mathcal{D}^{0,\eta}, P_\mathcal{D}^0\right)$ never belongs to the boundary $\partial\mathcal{K}$ of $\mathcal{K}$. Hence, the topological degree is constant for $\eta \in [0,1]$, and, for $\eta = 1$, the discrete counterpart of (76) admits at least a solution. Letting then ν tend to 1 provides thanks to similar arguments the existence of a solution to the scheme (35)-(38). □

4 Convergence analysis of the scheme

In order to prove the convergence of the scheme, we will use the method presented in [26] to derive the relative compactness of the sequencies $(s_{\mathcal{D}_m})_{m\in\mathbb{N}}$ and $(P_{\mathcal{D}_m})_{m\in\mathbb{N}}$, where $(\mathcal{D}_m)_{m\in\mathbb{N}}$ is a sequence of admissible discretizations of $\Omega \times (0,T)$ in the sense of Definition 2.3, for which the discretization parameter $h_m :=$ size$(\mathcal{D}_m)$ tends to 0 as $m \to \infty$, while the regularity parameter reg$(\mathcal{D}_m)$ remains bounded.

Firstly, since $0 \le s_{\mathcal{D}_m} \le 1$ almost everywhere in Q_T, we can claim that there exists $s \in L^\infty(Q_T;[0,1])$, such that, up to a subsequence,

$$s_{\mathcal{D}_m} \rightharpoonup s \text{ in the } L^\infty(Q_T) \text{ weak-}\star \text{ sense as } m \to \infty.$$

This is of course not sufficient to pass to the limit, so that we seek for additional compactness on the family of approximate solutions $(s_{\mathcal{D}_m}, P_{\mathcal{D}_m})_m$.

The compactness arguments used for the quantities defined in $Q_{i,T}$ are fairly standard (see [26]). For the sake of completeness and clarity, they are briefly recalled in Section 4.1. But in our problem, we have to focus on the convergence of the traces on the interface. Up to our knowledge, the convergence of the traces for piecewise constant functions with bounded discrete $L^2(H^1)$ semi-norms has not been proved before. This will be done in Section 4.2.

4.1 Estimates on differences of space and time translates

In this section we show that there exists a subsequence of $(\mathcal{D}_m)_{m\in\mathbb{N}}$ (which we will denote again by $(\mathcal{D}_m)_{m\in\mathbb{N}}$), such that $\varphi_i(s_{\mathcal{D}_m}) \to \varphi_i(s)$ strongly in $L^p(Q_{i,T})$ while $s_{\mathcal{D}_m} \to s$ strongly $L^p(Q_{i,T})$ for any $p \in [1,\infty)$. Let us firs recall here two lemmas adapted from [26].

Lemma 4.1 (Internal space translates (Lemma 4.2 of [26])) *Let $u_\mathcal{D}$ be an element of $\mathcal{X}(\mathcal{D})$, then for all $\xi \in \mathbb{R}^d$,*

$$\int_0^T \int_{\Omega_{i,\xi}} (u_\mathcal{D}(\mathbf{x}+\xi,t) - u_\mathcal{D}(\mathbf{x},t))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \le |u_\mathcal{D}|^2_{\mathcal{D},i} |\xi| \left(|\xi| + 2\mathrm{size}(\mathcal{D})\right),$$

where $\Omega_{i,\xi} = \{x \in \Omega_i \mid [\mathbf{x}, \mathbf{x}+\xi] \subset \Omega_i\}$.

Lemma 4.2 (Truncated $\mathbb{R}^d$ space translates (Lemma 4.3 of [26])) *Let $u_\mathcal{D}$ be an element of $\mathcal{X}(\mathcal{D})$, and let $T_i(u_\mathcal{D})$ the function of $L^2(\mathbb{R}^{d+1})$ defined by*

$$T_i(u_\mathcal{D})(\mathbf{x},t) = \begin{cases} u_\mathcal{D}(\mathbf{x},t) & \text{if } (\mathbf{x},t) \in \Omega_i \times (0,T), \\ 0 & \text{otherwise,} \end{cases}$$

then for all $\xi \in \mathbb{R}^d$,

$$\int_0^T \int_{\mathbb{R}^d} (T_i(u_\mathcal{D})(\mathbf{x}+\xi,t) - T_i(u_\mathcal{D})(\mathbf{x},t))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t$$
$$\le |u_\mathcal{D}|^2_{\mathcal{D},i} |\xi| \left(|\xi| + 2\mathrm{size}(\mathcal{D}) + 2m(\partial\Omega_i) \|u_\mathcal{D}\|_\infty\right),$$

where $\Omega_{i,\xi} = \{x \in \Omega_i \mid [\mathbf{x}, \mathbf{x}+\xi] \subset \Omega_i\}$.

The following result is an extension of Lemma 4.6 of [26] (see also Proposition 5.1 in [30]).

Lemma 4.3 *There exists C_3, which does not depend on size$(\mathcal{T})$, δt nor on τ such that for all $\tau \in (0,T)$,*

$$\int_0^{T-\tau} \sum_{i\in\{1,2\}} \int_{\Omega_i} (\varphi_i(s_\mathcal{D})(\mathbf{x},t+\tau) - \varphi_i(s_\mathcal{D})(\mathbf{x},t))^2 \,\mathrm{d}\mathbf{x}\mathrm{d}t \le C_3\tau. \quad (78)$$

Proposition 4.4 *The sequence $(\varphi_i(s_{\mathcal{D}_m}))_m$ converges strongly in $L^2(Q_{i,T})$, up to a subsequence, towards the function $\varphi_i(s) \in L^2(0,T;H^1(\Omega_i))$.*

Proof: First recall that, by Proposition 3.1, $(\varphi_i(s_{\mathcal{D}_m}))_m$ is bounded in $L^\infty(Q_{i,T})$ for $i \in \{1,2\}$ and that by Proposition 3.2 the sequence $(|\varphi_i(s_{\mathcal{D}_m})|_{\mathcal{D}_m,i})_m$ is bounded. Thanks to the lemmas 4.2 and 4.3 and the Kolmogorov compactness criterion (see e.g. [9] or [26, Theorem 3.9]), it follows that $(T_i(\varphi_i(s_{\mathcal{D}_m})))_m$ is relatively compact in $L^2(\mathbb{R}^{d+1})$ for $i \in \{1,2\}$. Thus we can extract a subsequence, still denoted by $(T_i(\varphi_i(s_{\mathcal{D}_m})))_m$, such that both $T_1(\phi_1(s_{\mathcal{D}_m}))$ and $T_2(\phi_2(s_{\mathcal{D}_m}))$ converge to their limit strongly in $L^2(Q_{1,T})$ and $L^2(Q_{2,T})$ respectively. As a direct consequence, $(\varphi_i(s_{\mathcal{D}_m}))_m$ converges in $L^2(Q_{i,T})$ for $i \in \{1,2\}$ towards a function ϕ, which satisfies, thanks to Lemma 4.1,

$$\int_0^T \int_{\Omega_{i,\xi}} (\phi(\mathbf{x}+\xi,t) - \phi(\mathbf{x},t))^2 \mathrm{d}\mathbf{x}\mathrm{d}t \le C|\xi|^2, \qquad \forall \xi \in \mathbb{R}^d.$$

This implies (see [9]) that $\phi \in L^2(0,T;H^1(\Omega_i))$. It remains to identify ϕ as $\varphi_i(s)$, $i \in \{1,2\}$. This can be done using Minty's lemma (see e.g. [28, Theorem 4.1]). □

Corollary 4.5 *Up to a subsequence, $(s_{\mathcal{D}_m})_m$ converges towards s strongly in $L^p(Q_T)$ for all $p \in [1,\infty)$.*

Proof: Since $(\varphi_i(s_{\mathcal{D}_m}))_m$ converges in $L^2(Q_T)$ towards $\varphi_i(s)$, it converges (up to a new subsequence) almost everywhere in Q_T. Since φ_i^{-1} is continuous, $s_{\mathcal{D}_m}$ tends to s almost everywhere. The result then follows from the uniform bound on $(s_{\mathcal{D}_m})_m$ stated in Proposition 3.1. □

Using the discrete Poincaré-Wirtinger inequality [31] and the energy estimates given by Proposition 3.2, one can obtain the following convergence result.

Lemma 4.6 *There exists $\mathcal{P} \in L^2(0,T;H^1(\Omega_i))$ such that, up to a subsequence,*

$$P_{\mathcal{D}_m} - m_{\Omega_i}(P_{\mathcal{D}_m}) \rightharpoonup \mathcal{P} \text{ weakly in } L^2(Q_{i,T}) \text{ as } m \to \infty.$$

We denote again by $(\mathcal{D}_m)_m$ a subsequence of $(\mathcal{D}_m)_m$ for which the convergence results stated by Proposition 4.4, Corollary 4.5 and Lemma 4.6 hold.

4.2 Convergence of the traces

We denote by $s_{\mathcal{D}_{|\Gamma,i}}$ (resp. $P_{\mathcal{D}_{|\Gamma,i}}$) the trace of $s_\mathcal{D}$ (resp. $P_\mathcal{D}$) on Γ from the side of Ω_i, defined by

$$s_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) = s_{K,\sigma}^{n+1}, \quad P_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) = P_{K,\sigma}^{n+1}, \qquad \forall (\mathbf{x},t) \in \sigma \times (t^n, t^{n+1}],$$

where $\sigma \in \mathcal{E}_{K,\Gamma}$, $K \subset \Omega_i$.

It has been proven in Proposition 4.4 that $\varphi_i(s_{\mathcal{D}_m})$ converges strongly in $L^2(Q_{i,T})$ towards $\varphi_i(s) \in L^2(0,T;H^1(\Omega_i))$. Hence, $\varphi_1(s)$ and $\varphi_2(s)$ admits a trace in the sense of $L^2(\Gamma \times (0,T))$. Since φ_i^{-1} is continuous, s also admits a traces on the interface, denoted by s_1 and s_2. We claim in Corollary 4.10 below that $s_{\mathcal{D}_{m|\Gamma,i}}$ converges *strongly* in $L^p(\Gamma \times (0,T))$ towards s_i for all $p \in [1,\infty)$.

We now introduce another definition of the trace, denoted by $\tilde{u}_{|\Gamma,i}$. For a function u of $\mathcal{X}(\mathcal{D})$ we define

$$\tilde{u}_{|\Gamma,i}(\mathbf{x},t) := u_K^{n+1} \text{ if } (\mathbf{x},t) \in \sigma \times (t^n, t^{n+1}], \sigma \subset \Gamma \cap \partial K, K \subset \Omega_i.$$

Lemma 4.7 *Let $u \in \mathcal{X}(\mathcal{D})$, then*

$$\int_0^T \int_\Gamma |u_{|\Gamma,i} - \tilde{u}_{|\Gamma,i}| \mathrm{d}\mathbf{x}\mathrm{d}t \le |u|_\mathcal{D} \left(T m(\Gamma) \mathrm{size}(\mathcal{D})\right)^{1/2}.$$

Proof: From the definitions of the traces of u,

$$\int_0^T \int_\Gamma |u_{|\Gamma,i} - \tilde{u}_{|\Gamma,i}| \mathrm{d}\mathbf{x}\mathrm{d}t = \sum_{n=0}^N \delta t \sum_{K \in \mathcal{T}_i} \sum_{\sigma \in \mathcal{E}_{K,\Gamma}} m(\sigma) |u_{K,\sigma}^{n+1} - u_K^{n+1}|.$$

Cauchy-Schwarz inequality yields that

$$\int_0^T \int_\Gamma |u_{|\Gamma,i} - \tilde{u}_{|\Gamma,i}| \mathrm{d}\mathbf{x}\mathrm{d}t \leq \left(\sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}_i} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} \tau_{K,\sigma} (u_{K,\sigma}^{n+1} - u_K^{n+1})^2 \right)^{1/2} \times \left(\sum_{n=0}^{N} \delta t \sum_{K\in\mathcal{T}_i} \sum_{\sigma\in\mathcal{E}_{K,\Gamma}} m(\sigma) d_{K,\sigma} \right)^{1/2}.$$

The result follows. □

Since Ω_i is supposed to be polygonal, Γ is made of a finite number of faces $(\Gamma_j)_{1\leq j\leq J}$ contained in affine hyperplanes of $\mathbb{R}^d$. We denote by $\mathbf{n}_{i,j}$ the outward normal to Γ_j with respect to Ω_i. Let $\varepsilon > 0$ and $j \in \{1, \ldots, J\}$, then, following [27], we define the open subset $\omega_{i,j,\varepsilon}$ of Ω_i as the largest cylinder of width ε generate by Γ_j and $n_{i,j}$ included in Ω_i, that is

$$\omega_{i,j,\varepsilon} := \left\{ x - h\mathbf{n}_{i,j} \in Q_{i,T} \; |x \in \Gamma_j, 0 < h < \varepsilon \text{ and } [\mathbf{x}, x - \varepsilon\mathbf{n}_{i,j}] \subset \overline{\Omega}_i \right\}. \tag{79}$$

We refer to Figure 4 for an illustration. We also define the subset $\Gamma_{i,j,\varepsilon} = \partial\omega_{i,j,\varepsilon} \cap \Gamma_j$ of Γ_j, that satisfies

$$m(\Gamma_j \setminus \Gamma_{i,j,\varepsilon}) \leq C\varepsilon, \tag{80}$$

where C only depends on Ω.

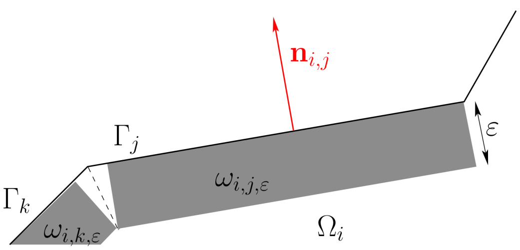

Figure 4: The largest cylinder $\omega_{i,j,\varepsilon}$ of width ε generated by Γ_j included in Ω_i.

Lemma 4.8 *Let $u \in \mathcal{X}(\mathcal{D})$, then for all $j \in \{1, \ldots, J\}$,*

$$\int_0^T \frac{1}{\varepsilon} \int_{\Gamma_{i,j,\varepsilon}} \int_0^\varepsilon \left(\tilde{u}_{|\Gamma,i}(\mathbf{x}, t) - u(\mathbf{x} - h\mathbf{n}_{i,j}, t) \right)^2 \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t \leq |u|_{\mathcal{D}}^2 \left(\varepsilon + \mathrm{size}(\mathcal{D}) \right).$$

Proof: For all $\sigma \in \mathcal{E}_{\mathrm{int}}$, we denote by

$$\chi_\sigma(\mathbf{x}, \mathbf{y}) := \begin{cases} 1 & \text{if } (\mathbf{x}, \mathbf{y}) \cap \sigma \text{ is reduced to a single point,} \\ 0 & \text{otherwise} \end{cases}$$

and we introduce the quantity

$$T_{\mathcal{D}}(\mathbf{x}, h, t) := \left| \tilde{u}_{|\Gamma,i}(\mathbf{x}, t) - u_{\mathcal{D}}(\mathbf{x} - h\mathbf{n}_{i,j}, t) \right|,$$

which satisfies

$$T_{\mathcal{D}}(\mathbf{x}, h, t) \leq \sum_{\sigma=K|L\in\mathcal{E}_i} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \left| u_K^{n+1} - u_L^{n+1} \right|$$

for almost all $\mathbf{x} \in \Gamma_{i,j,\varepsilon}$, almost all $h \in (0, \varepsilon)$ and for all $t \in (t^n, t^{n+1}]$. It follows from the Cauchy-Schwarz inequality that, for $t \in (t^n, t^{n+1}]$,

$$(T_{\mathcal{D}}(\mathbf{x}, h, t))^2 \leq \left(\sum_{\sigma=K|L\in\mathcal{E}_i} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \frac{\left(u_K^{n+1} - u_L^{n+1}\right)^2}{d_{KL}|\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}|} \right) \times \left(\sum_{\sigma=K|L\in\mathcal{E}_i} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) d_{KL} |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}| \right).$$

For almost all $\mathbf{x} \in \Gamma_{i,j,\varepsilon}$, there exists a unique $K_1 \in \mathcal{T}_i$ such that $\mathbf{x} \in \partial K_1$. Moreover, for almost all $h \in (0, \varepsilon)$, there exists a unique $K_2 \in \mathcal{T}_i$ such that $\mathbf{x} - h\mathbf{n}_{i,j}$ belongs to K_2 (possibly K_2 coincides with K_1). Let σ be such that $\chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) = 1$, then we suppose, without loss of generality, that $\sigma = K|L$ where the straight line from $\mathbf{x}$ to $\mathbf{x} - h\mathbf{n}_{i,j}$ crosses the interface $\sigma = K|L$ from K to L. Therefore, the quantity $\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}$ has a constant negative sign. Moreover, using the fact that $\mathbf{x}_L - \mathbf{x}_K = d_{KL}\mathbf{n}_{KL}$, we can claim that

$$\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) d_{KL} |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}| = (\mathbf{x}_{K_1} - \mathbf{x}_{K_2}) \cdot \mathbf{n}_{i,j}$$
$$\leq (\mathbf{x}_{K_1} - \mathbf{x}) \cdot \mathbf{n}_{i,j} + h + |(\mathbf{x}_{K_2} - (\mathbf{x} - h\mathbf{n}_{i,j})) \cdot \mathbf{n}_{i,j}| . \tag{81}$$

Since $\mathbf{x} - h\mathbf{n}_{i,j}$ belongs to K_2, we have

$$|(\mathbf{x}_{K_2} - (\mathbf{x} - h\mathbf{n}_{i,j})) \cdot \mathbf{n}_{i,j}| \leq \text{size}(\mathcal{D}),$$

and since $\mathbf{x}$ belongs to Γ_i, $(\mathbf{x}_{K_1} - \mathbf{x}) \cdot \mathbf{n}_{i,j} \leq 0$. Then we obtain

$$\sum_{\sigma=K|L \in \mathcal{E}_{\text{int}}} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) d_{KL} |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}| \leq \varepsilon + \text{size}(\mathcal{D}). \tag{82}$$

For all $\sigma \in \mathcal{E}_{\text{int}}$ with $\sigma \cap \omega_{i,j,\varepsilon} = \emptyset$ and all $h \in (0, \varepsilon)$, one has

$$\int_{\Gamma_i^\varepsilon} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \mathrm{d}\mathbf{x} = 0.$$

For all $\sigma \in \mathcal{E}_{i,j,\varepsilon} := \{\sigma \in \mathcal{E}_i \mid \sigma \cap \omega_{i,j,\varepsilon} \neq \emptyset\}$, one has

$$\forall h \in (0, \varepsilon), \qquad \int_{\Gamma_{i,j,\varepsilon}} \chi_\sigma(\mathbf{x}, \mathbf{x} - h\mathbf{n}_{i,j}) \mathrm{d}\mathbf{x} \leq m(\sigma) |\mathbf{n}_{i,j} \cdot \mathbf{n}_{KL}|. \tag{83}$$

We obtain from (82) and (83) that for all $t \in (t^n, t^{n+1}]$, for all $h \in (0, \varepsilon)$,

$$\int_{\Gamma_{i,j,\varepsilon}} (T_{\mathcal{D}}(\mathbf{x}, h, t))^2 \, \mathrm{d}\mathbf{x} \leq (\varepsilon + \text{size}(\mathcal{D})) \sum_{\sigma=K|L \in \mathcal{E}_{i,j,\varepsilon}} \tau_{KL} \left(u_K^{n+1} - u_L^{n+1}\right)^2,$$

which complete the proof. □

Proposition 4.9 *Up to a subsequence, the sequence* $\left(\varphi_i(s_{\mathcal{D}_{m|\Gamma,i}})\right)_m$ *converges towards* $\varphi_i(s_i)$ *strongly in* $L^1(\Gamma \times (0,T))$ *as* $m \to \infty$.

Proof: For notation convenience, we remove the subscripts m in the proof. Denote by

$$A_{i,j,\mathcal{D}} := \int_0^T \int_{\Gamma_j} \left|\varphi_i(s_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t, \tag{84}$$

then in view of Lemma 4.7 and Proposition 3.2, there exists C not depending on $\mathcal{D}$ such that

$$A_{i,j,\mathcal{D}} = \int_0^T \int_{\Gamma_j} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t + C\text{size}(\mathcal{D})^{1/2}. \tag{85}$$

By (80), for any $\varepsilon > 0$, one has

$$\int_0^T \int_{\Gamma_j} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t \leq \int_0^T \int_{\Gamma_{i,j,\varepsilon}} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t + \varphi_i(1)C\varepsilon. \tag{86}$$

Next we apply the triangle inequality to deduce that

$$\int_0^T \int_{\Gamma_{i,j,\varepsilon}} \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}}) - \varphi_i(s_i)\right| \mathrm{d}\mathbf{x}\mathrm{d}t \leq B_{1,\mathcal{D},\varepsilon} + B_{2,\mathcal{D},\varepsilon} + B_{3,\varepsilon}, \tag{87}$$

where

$$\begin{aligned}
B_{1,\mathcal{D},\varepsilon} &= \frac{1}{\varepsilon}\int_0^T\int_{\Gamma_{i,j,\varepsilon}}\int_0^\varepsilon \left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}})(\mathbf{x},t)-\varphi_i(s_\mathcal{D})(\mathbf{x}-h\mathbf{n}_{i,j},t)\right| \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,\\
B_{2,\mathcal{D},\varepsilon} &= \frac{1}{\varepsilon}\int_0^T\int_{\omega_{i,j,\varepsilon}} |\varphi_i(s_\mathcal{D})-\varphi_i(s)|\,\mathrm{d}\mathbf{x}\mathrm{d}t,\\
B_{3,\varepsilon} &= \frac{1}{\varepsilon}\int_0^T\int_{\Gamma_{i,j,\varepsilon}}\int_0^\varepsilon |\varphi_i(s_i)(\mathbf{x},t)-\varphi_i(s)(\mathbf{x}-h\mathbf{n}_{i,j},t)|\,\mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,
\end{aligned}$$

where we have used (79). From Cauchy-Schwarz inequality, one has

$$(B_{1,\mathcal{D},\varepsilon})^2 \le m(\Gamma_{i,j,\varepsilon})T\int_0^T\int_{\Gamma_{i,j,\varepsilon}}\frac{1}{\varepsilon}\int_0^\varepsilon(\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}})(\mathbf{x},t)-\varphi_i(s_\mathcal{D})(\mathbf{x}-h\mathbf{n}_{i,j},t))^2\mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,$$

and then, from Proposition 3.2 and Lemma 4.8, one has

$$|B_{1,\mathcal{D},\varepsilon}| \le (C_1(\mathrm{size}(\mathcal{D})+\varepsilon)m(\Gamma_i)T)^{1/2}. \tag{88}$$

We can now let size($\mathcal{D}$) tend to 0 in (87). Thanks to Proposition 4.4, we can claim that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} B_{2,\mathcal{D},\varepsilon}=0.$$

Then it follows from (86) and (88) that

$$\limsup_{\mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_{\Gamma_j}\left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}})-\varphi_i(s_i)\right|\mathrm{d}\mathbf{x}\mathrm{d}t \le C(\varepsilon+\sqrt{\varepsilon})+B_{3,\varepsilon}. \tag{89}$$

Since $\varphi_i(s_i)$ is the trace of $\varphi_i(s)$ on Γ, $\lim_{\varepsilon\to 0} B_{3,\varepsilon}=0$. Therefore, letting ε tend to 0 in (89) implies that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_{\Gamma_j}\left|\varphi_i(\tilde{s}_{\mathcal{D}_{|\Gamma,i}})-\varphi_i(s_i)\right|\mathrm{d}\mathbf{x}\mathrm{d}t=0.$$

Then the result follows from (84) and (4.2). □

Corollary 4.10 *Up to a subsequence, the sequence* $\left(s_{\mathcal{D}_{m|\Gamma,i}}\right)_m$ *converges towards* s_i *strongly in* $L^p(\Gamma\times(0,T))$ *for all* $p\in[1,\infty)$.

Proof: This corollary is just a consequence from the fact that $\varphi_i(s_{\mathcal{D}_{m|\Gamma,i}})$ converges, up to a subsequence, almost everywhere on $\Gamma\times(0,T)$, from the fact that φ_i^{-1} is continuous and from the fact that $s_{\mathcal{D}_{m|\Gamma,i}}$ is essentially uniformly bounded between 0 and 1. □

Lemma 4.11 *Up to a subsequence, the sequence* $\left((P_{\mathcal{D}_m})_{|\Gamma,i}-m_{\Omega_i}(P_\mathcal{D})\right)_m$ *converges towards* $\mathcal{P}_i$ *weakly in* $L^2(\Gamma\times(0,T))$.

Proof: Let $\psi\in\mathcal{D}(\Gamma_i\times(0,T))$, then, there exists $\varepsilon_\star$ depending on ψ such that, for any ε in $(0,\varepsilon_\star)$ one has $\mathrm{supp}(\psi)\subset\Gamma_{i,j,\varepsilon}\times(0,T)$. We aim to prove that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_{\Gamma_j}\left(P_{\mathcal{D}_{|\Gamma,i}}-m_{\Omega_i}(P_\mathcal{D})-\mathcal{P}_i\right)\psi\mathrm{d}\mathbf{x}\mathrm{d}t=0. \tag{90}$$

Thanks to Lemma 4.7 and to Proposition 3.2, it is sufficient to show that

$$\lim_{\mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_{\Gamma_j}\left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}}-m_{\Omega_i}(P_\mathcal{D})-\mathcal{P}_i\right)\psi\mathrm{d}\mathbf{x}\mathrm{d}t=0.$$

Let $\varepsilon\in(0,\varepsilon_\star)$, then one has

$$\int_0^T\int_{\Gamma_j}\left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}}-m_{\Omega_i}(P_{\mathcal{D}_m})-\mathcal{P}_i\right)\psi\mathrm{d}\mathbf{x}\mathrm{d}t=E_{1,\mathcal{D},\varepsilon}+E_{2,\mathcal{D},\varepsilon}+E_{3,\varepsilon},$$

where

$$\begin{aligned}
E_{1,\mathcal{D},\varepsilon} &= \int_0^T \frac{1}{\varepsilon}\int_{\Gamma_{i,j,\varepsilon}}\int_0^\varepsilon \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) - P_{\mathcal{D}}(\mathbf{x}-h\mathbf{n}_{i,j},t)\right)\psi(\mathbf{x},t)\mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,\\
E_{2,\mathcal{D},\varepsilon} &= \int_0^T \frac{1}{\varepsilon}\int_{\Gamma_{i,j,\varepsilon}}\int_0^\varepsilon (P_{\mathcal{D}}(\mathbf{x}-h\mathbf{n}_{i,j},t) - m_{\Omega_i}(P_{\mathcal{D}}) - \mathcal{P}(\mathbf{x}-h\mathbf{n}_{i,j},t))\,\psi(\mathbf{x},t)\mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t,\\
E_{3,\varepsilon} &= \int_0^T \frac{1}{\varepsilon}\int_{\Gamma_{i,j,\varepsilon}}\int_0^\varepsilon (\mathcal{P}(\mathbf{x}-h\mathbf{n}_{i,j},t) - \mathcal{P}_i)\,\psi(\mathbf{x},t)\mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned}$$

The Cauchy-Schwarz inequality gives that

$$\begin{aligned}
(E_{1,\mathcal{D},\varepsilon})^2 &\le \int_0^T \frac{1}{\varepsilon}\int_{\Gamma_{i,j,\varepsilon}}\int_0^\varepsilon \left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}}(\mathbf{x},t) - P_{\mathcal{D}}(\mathbf{x}-h\mathbf{n}_{i,j},t)\right)^2 \mathrm{d}h\mathrm{d}\mathbf{x}\mathrm{d}t\\
&\quad\times \int_0^T\int_{\Gamma_j} (\psi(\mathbf{x},t))^2\,\mathrm{d}\mathbf{x}\mathrm{d}t.
\end{aligned}$$

Using Proposition 3.2 and Lemma 4.8 yields

$$|E_{1,\mathcal{D},\varepsilon}| \le \|\psi\|_{L^2(\Gamma_j\times(0,T))}\,(C_1(\varepsilon+\mathrm{size}(\mathcal{D})))^{1/2}.$$

It has been stated in Lemma 4.6 that $P_{\mathcal{D}} - m_{\Omega_i}(P_{\mathcal{D}})$ tends to $\mathcal{P}$ weakly in $L^2(Q_{i,T})$ as size($\mathcal{D}$) tends to 0, then

$$\lim_{\mathrm{size}(\mathcal{D})\to 0} E_{2,\mathcal{D},\varepsilon} = 0.$$

Therefore,

$$\limsup_{\mathrm{size}(\mathcal{D})\to 0}\left|\int_0^T\int_{\Gamma_j}\left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - P_i\right)\psi\mathrm{d}\mathbf{x}\mathrm{d}t\right| \le C_\psi\sqrt{\varepsilon} + |E_{3,\varepsilon}|.$$

Since P_i is the trace on Γ of P from the side of Ω_i, one has

$$\lim_{\varepsilon\to 0} E_{3,\varepsilon} = 0.$$

Thus, letting $\varepsilon \to 0$, one obtains that for all $\psi \in \mathcal{D}(\Gamma_j \times (0,T))$,

$$\lim_{\mathrm{size}(\mathcal{D})\to 0}\int_0^T\int_{\Gamma_j}\left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}}) - \mathcal{P}_i\right)\psi\mathrm{d}\mathbf{x}\mathrm{d}t = 0. \tag{91}$$

A straightforward generalization of [26, Lemma 3.10] allows us to claim, using Proposition 3.2 and the discrete Poincaré-Wirtinger inequality [31], that $\left(\tilde{P}_{\mathcal{D}_{|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}})\right)_{\mathcal{D}}$ is uniformly bounded in $L^2(\Gamma\times(0,T))$. Then, we conclude, using a classical density argument, that (91) holds for all $\psi \in L^2(\Gamma_j \times (0,T))$. □

Proposition 4.12 *There exists $P \in L^2(0,T;H^1(\Omega_i))$ such that, up to a subsequence, $P_{\mathcal{D}_m}$ tends to P weakly in $L^2(Q_T)$ as $m \to \infty$, and such that $\left(P_{\mathcal{D}_{m|\Gamma,i}}\right)_m$ converges weakly in $L^2(\Gamma\times(0,T))$ towards P_i.*

Proof: Firstly, since we have enforced $m_{\Omega_1}(P_{\mathcal{D}_m}) = 0$, we can set $P := \mathcal{P}$ in $Q_{1,T}$. Next we search for a uniform bound on $\|P_{\mathcal{D}_m}\|_{L^2(Q_{2,T})}$. In view of the discrete Poincaré-Wirtinger inequality

$$\|P_{\mathcal{D}_m}\|^2_{L^2(Q_{2,T})} \le (m_{\Omega_2}(P_{\mathcal{D}_m}))^2 + C, \tag{92}$$

it only remains to check that $m_{\Omega_2}(P_{\mathcal{D}_m})$ is uniformly bounded w.r.t. m. This is a consequence of the fact that, almost everywhere on $\Gamma\times(0,T)$, one has

$$m_{\Omega_2}(P_{\mathcal{D}_m}) = P_{\mathcal{D}_{m|\Gamma,1}} - \left(P_{\mathcal{D}_{m|\Gamma,2}} - m_{\Omega_2}(P_{\mathcal{D}_m})\right) - (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m})).$$

Then, integrating on $\Gamma\times(0,T)$ and using Lemma 1.2 provides

$$|m_{\Omega_2}(P_{\mathcal{D}_m})| \le \frac{1}{m(\Gamma)T}\sum_{i\in\{1,2\}}\left\|P_{\mathcal{D}_{m|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}_m})\right\|_{L^1(\Gamma\times(0,T))} + \|W_1 - W_2\|_\infty.$$

For all $i \in \{1,2\}$ the quantities $\left\| P_{\mathcal{D}_{m|\Gamma,i}} - m_{\Omega_i}(P_{\mathcal{D}_m}) \right\|_{L^1(\Gamma\times(0,T))}$ are bounded by the proof of Lemma 4.11. Hence, in view of (92), $(P_{\mathcal{D}_m})_m$ converges towards some function P weakly in $L^2(Q_{i,T})$. From the analysis performed in the proof of Lemma 4.6, we deduce that $P \in L^2(0,T;H^1(\Omega_i))$, and from the analysis of Lemma 4.11, we deduce the weak convergence of the traces. □

Lemma 4.13 *Let $s_1, s_2 \in L^\infty(\Gamma\times(0,T))$ be the respective limits of $\left(s_{\mathcal{D}_{m|\Gamma,1}}\right)_m$ and $\left(s_{\mathcal{D}_{m|\Gamma,2}}\right)_m$, then,*

$$\tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset \qquad a.e.\ on\ \Gamma\times(0,T). \tag{93}$$

Proof: For all $m \in \mathcal{N}$, one has

$$\tilde{\pi}_1(s_{\mathcal{D}_{m|\Gamma,1}}) \cap \tilde{\pi}_2(s_{\mathcal{D}_{m|\Gamma,2}}) \neq \emptyset.$$

Since the set $F = \{(a,b) \in [0,1]^2 \mid \tilde{\pi}_1(a) \cap \tilde{\pi}_2(b) \neq \emptyset\}$ is closed in $[0,1]^2$, we conclude that (93) holds. □

We now focus on the last technical difficulty for proving Theorem 1, that is the convergence of the sequence $(\pi_{\mathcal{D}_m})_m$. This is done by following the same path as in [16].

In the sequel, we denote by $T_{[A,B]}$, the truncature operator defined by

$$T_{[A,B]}(s) = \begin{cases} s & \text{if } s \in [A,B], \\ A & \text{if } s \le A, \\ B & \text{if } s \ge B, \end{cases}$$

and by

$$\mathcal{U} = \{(\mathbf{x},t) \in \Gamma\times(0,T) \mid \{s_1,s_2\} = \{0,1\}\}, \qquad \mathcal{V} = \mathcal{U}^c,$$

so that

$$(\mathbf{x},t) \in \mathcal{U} \quad \text{iff} \quad \{s_1(\mathbf{x},t) = 0 \text{ and } s_2(\mathbf{x},t) = 1\} \text{ or } \{s_1(\mathbf{x},t) = 1 \text{ and } s_2(\mathbf{x},t) = 0\}.$$

Note that, thanks to Lemma 4.13, the set $\mathcal{U}$ is empty if $\min_i \pi_i(1) > \max_i \pi_i(0)$.

Lemma 4.14 *There exists a measurable function π defined on $\mathcal{V}$ with values in $\overline{\mathbb{R}}$, such that, up to a subsequence,*

$$\pi_{\mathcal{D}_m} \to \pi \qquad a.e.\ in\ \mathcal{V}.$$

Proof: We define the functions $\tilde{\varphi}_i$ by

$$\tilde{\varphi}_i : p \mapsto \int_{\pi_i(0)}^{p} K_i \frac{k_{o,i}(\theta_i(a))k_{w,i}(\theta_i(a))}{\mu_w k_{o,i}(\theta_i(a)) + \mu_o k_{w,i}(\theta_i(a))} \mathrm{d}a.$$

In view of Assumptions 1 and 1, $\tilde{\varphi}_i$ satisfy

$$\pi \in \tilde{\pi}_i(s) \implies \tilde{\varphi}_i(\pi) = \tilde{\varphi}_i(\pi_i(s)) = \varphi_i(s), \tag{94}$$

moreover

$$\text{its restriction } (\tilde{\varphi}_i)_{|[\pi_i(0),\pi_i(1)]} \text{ admits a continuous inverse function.} \tag{95}$$

Thanks to Proposition 4.9 and to (94), we can claim that, up to a subsequence, $\tilde{\varphi}_i(\pi_{\mathcal{D}_m})$ converges almost everywhere on $\Gamma\times(0,T)$ towards $\tilde{\varphi}_i(\pi_i(s_i))$. For a.e. $(\mathbf{x},t) \in \mathcal{V}$, the set $\tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2)$ is reduced to the singleton $\{\pi_{i_0}(s_{i_0})\}$ for some $i_0 \in \{1,2\}$. Thanks to (95), we can identify the limit π of $\pi_{\mathcal{D}_m}$ as $\pi_{i_0}(s_{i_0})$. □

Lemma 4.15 *Assume that $[\min_i \pi_i(1), \max_i \pi_i(0)] \neq \emptyset$, then there exists*

$$\pi \in L^\infty(\mathcal{U}; [\min_i \pi_i(1), \max \pi_i(0)]),$$

such that, for all bounded interval $\mathcal{J} \subset \mathbb{R}$ such that $[\min_i \pi_i(1), \max_i \pi_i(0)] \subset \mathring{\mathcal{J}}$,

$$T_{\mathcal{J}}(\pi_{\mathcal{D}_m}) \to \pi \qquad in\ the\ L^\infty(\mathcal{U})\ weak\text{-}\star\ sense.$$

Proof: For the sake of simplicity, we assume, without loss of generality, that $\pi_1(1) \le \pi_2(0)$, then thanks to Lemma 4.13, almost everywhere in $\mathcal{U}$, $s_1 = 1$ and $s_2 = 0$.

The sequence $(T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))_m$ is bounded in $L^\infty(\mathcal{U})$, thus, up to a subsequence, it converges towards a function $\pi_{\mathcal{J}}$ in the $L^\infty(\mathcal{U})$ weak-$\star$ sense. Let us now show that $\pi_{\mathcal{J}}$ does not depend on the choice of the bounded interval $\mathcal{J}$. Because of Lemma 4.13, one has, for a.e. $(\mathbf{x}, t) \in \mathcal{U}$,

$$\liminf_m \pi_{\mathcal{D}_m} \ge \pi_1(1), \qquad \limsup_m \pi_{\mathcal{D}_m} \le \pi_2(0). \tag{96}$$

Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two bounded intervals such that $[\pi_1(1), \pi_2(0)] \subset \overset{\circ}{\mathcal{J}}_k$ ($k \in \{1,2\}$). Then, it follows from (96) that, for a.e. $(\mathbf{x}, t) \in \mathcal{U}$, for m large enough (depending on $(\mathbf{x}, t)$),

$$T_{\mathcal{J}_1}(\pi_{\mathcal{D}_m}(\mathbf{x}, t)) - T_{\mathcal{J}_2}(\pi_{\mathcal{D}_m}(\mathbf{x}, t)) = 0.$$

As a consequence, the sequence $(T_{\mathcal{J}_1}(\pi_{\mathcal{D}_m}) - T_{\mathcal{J}_2}(\pi_{\mathcal{D}_m}))_m$ converges almost everywhere to 0 on $\mathcal{U}$, and is uniformly bounded in $L^\infty(\mathcal{U})$. The dominated convergence theorem yields that for all $\psi \in L^1(\mathcal{U})$,

$$\iint_{\mathcal{U}} (T_{\mathcal{J}_1}(\pi_{\mathcal{D}_m}) - T_{\mathcal{J}_2}(\pi_{\mathcal{D}_m}))\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t \to 0 = \iint_{\mathcal{U}} (\pi_{\mathcal{J}_1} - \pi_{\mathcal{J}_2})\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Choosing $\psi = (\pi_{\mathcal{J}_1} - \pi_{\mathcal{J}_2})$ provides that $\pi_{\mathcal{J}_1} = \pi_{\mathcal{J}_2} = \pi$ almost everywhere in $\mathcal{U}$. □

Lemma 4.16 *Assume that* $[\min_i \pi_i(1), \max_i \pi_i(0)] \ne \emptyset$, *then there exists* $\pi \in L^\infty(\mathcal{U})$ *such that, for all bounded interval* $\mathcal{J} \subset \mathbb{R}$ *such that* $[\min_i \pi_i(1), \max_i \pi_i(0)] \subset \overset{\circ}{\mathcal{J}}$, *the sequence* $(W_i(T_{\mathcal{J}}(\pi_{\mathcal{D}_m})))_m$ *converges towards* $W_i(\pi)$ *in the* $L^\infty(\mathcal{U})$ *weak-*$\star$ *sense.*

Proof: We suppose, without loss of generality, that $\pi_1(1) \le \pi_2(0)$. Then on $\mathcal{U}$, $s_2 = 0$ and $s_1 = 1$. One has

$$W_2(T_{\mathcal{J}}(\pi_{\mathcal{D}_m})) \;=\; \int_0^{\pi_2(0)} f_2 \circ \pi_2^{-1}(p)\mathrm{d}p + \int_{\pi_2(0)}^{\pi_{\mathcal{D}_m}} f_2 \circ \pi_2^{-1}(p)\mathrm{d}p.$$

Since for almost every $(\mathbf{x}, t) \in \mathcal{U}$,

$$\limsup_m \pi_{\mathcal{D}_m}(\mathbf{x}, t) \le \pi_2(0),$$

and since $f_2 \circ \pi_2^{-1}(p) = 0$ for all $p \le \pi_2(0)$, then for almost every $(\mathbf{x}, t) \in \mathcal{U}$,

$$\int_{\pi_2(0)}^{\pi_{\mathcal{D}_m}(\mathbf{x},t)} f_2 \circ \pi_2^{-1}(p)\mathrm{d}p \to 0 \text{ as } m \to \infty.$$

Since the function $W_2 \circ T_{\mathcal{J}}$ is uniformly bounded on $\mathbb{R}$, the dominated convergence theorem yields that, for all $\psi \in L^1(\mathcal{U})$,

$$\lim_{m\to\infty} \int_{\mathcal{U}} W_2(T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))\psi \mathrm{d}\mathbf{x}\mathrm{d}t \to \iint_{\mathcal{U}} W_2(\pi_2(0))\psi \mathrm{d}\mathbf{x}\mathrm{d}t = \iint_{\mathcal{U}} W_2(\pi)\psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Similarly, we obtain that

$$\iint_{\mathcal{U}} (W_1(T_{\mathcal{J}}(\pi_{\mathcal{D}_m})) - T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t \to \iint_{\mathcal{U}} (W_1(\pi_1(1)) - \pi_1(1))\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Since, thanks to Lemma 4.15, $T_{\mathcal{J}}(\pi_{\mathcal{D}_m})$ tends to π in the $L^\infty(\mathcal{U})$ weak-$\star$ sense, one has

$$\begin{aligned}\lim_{m\to\infty} \iint_{\mathcal{U}} W_1(T_{\mathcal{J}}(\pi_{\mathcal{D}_m}))\psi \mathrm{d}\mathbf{x}\mathrm{d}t \;&=\; \iint_{\mathcal{U}} (W_1(\pi_1(1)) + \pi - \pi_1(1))\, \psi \mathrm{d}\mathbf{x}\mathrm{d}t \\ &=\; \iint_{\mathcal{U}} W_1(\pi)\psi \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

□

Proposition 4.17 *There exists a measurable function π on $\Gamma \times (0,T)$, with $\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2)$ a.e. on $\Gamma \times (0,T)$, with value in $[\min_i(\pi_i(0)), \max_i(\pi_i(1))]$ such that,*

$$W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}) \to W_1(\pi) - W_2(\pi) \text{ in the } L^\infty(\Gamma \times (0,T)) \text{ weak-}\star \text{ sense as } n \to \infty.$$

Proof: We know, from Lemma 1.2, that $W_1(p) - W_2(p)$ is uniformly bounded on $[\min_i \pi_i(0), \max_i \pi_i(1)]$. Hence, the sequence $(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}))_m$ converges in the $L^\infty(\Gamma\times(0,T))$ weak-$\star$ sense towards a function $\mathcal{Z}$. Let $\psi \in L^1(\Gamma \times (0,T))$, then

$$\begin{aligned}\int_0^T \int_\Gamma (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}))\,\psi \mathrm{d}\mathbf{x}\mathrm{d}t &= \iint_{\mathcal{U}} (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}))\,\psi \mathrm{d}\mathbf{x}\mathrm{d}t \\ &+ \iint_{\mathcal{V}} (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}))\,\psi \mathrm{d}\mathbf{x}\mathrm{d}t.\end{aligned}$$

Thanks to Lemma 4.14, $\pi_{\mathcal{D}_m}$ tends almost everywhere to π on $\mathcal{V}$, then for almost every $(\mathbf{x},t) \in \mathcal{V}$, we can identify $\mathcal{Z}(\mathbf{x},t)$ as $W_1(\pi(\mathbf{x},t)) - W_2(\pi(\mathbf{x},t))$. Thus

$$\lim_{m\to\infty} \iint_{\mathcal{V}} (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}))\,\psi \mathrm{d}\mathbf{x}\mathrm{d}t = \iint_{\mathcal{V}} (W_1(\pi) - W_2(\pi))\,\psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

We denote by

$$\begin{aligned}A_m &= \iint_{\mathcal{U}} (W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}) - W_1(\pi) + W_2(\pi))\,\psi \mathrm{d}\mathbf{x}\mathrm{d}t, \\ &= \iint_{\mathcal{U}} \left(\widehat{W}_1(\pi_{\mathcal{D}_m}) - \widehat{W}_1(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t + \iint_{\mathcal{U}} \left(\widehat{W}_2(\pi_{\mathcal{D}_m}) - \widehat{W}_2(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t,\end{aligned}$$

where $\widehat{W}$ is given by (30). Let $R \in \mathbb{R}$ such that $[\min_i \pi_i(0), \max_i \pi_i(1)] \subset [-R, R]$, then

$$A_m = B_{1,m}(R) - B_{2,m}(R) + C_m(R),$$

where

$$B_{i,m}(R) = \iint_{\mathcal{U}} \left(\widehat{W}_i(\pi_{\mathcal{D}_m}) - \widehat{W}_i(T_{[-R,R]}(\pi_{\mathcal{D}_m}))\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t$$

and

$$C_m(R) = \iint_{\mathcal{U}} \left(W_1(T_{[-R,R]}(\pi_{\mathcal{D}_m})) - W_2(T_{[-R,R]}(\pi_{\mathcal{D}_m})) - W_1(\pi) + W_2(\pi)\right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Let $\varepsilon > 0$, then since $\widehat{W}_i$ admits finite limits as $p \to \min_i \pi_i(0)$ and $p \to \max_i \pi_i(1)$, there exists $R_0(\varepsilon) > 0$ such that

$$R > R_0(\varepsilon) \implies \|\widehat{W}_i - \widehat{W}_i \circ T_{[-R,R]}\|_\infty \le \varepsilon.$$

Thus, for $R > R_0(\varepsilon)$ fixed,

$$|B_{i,m}(R)| \le Tm(\Gamma)\varepsilon.$$

Thanks to Lemma 4.16,

$$\lim_{m\to\infty} C_m(R) = 0,$$

then, for all $\varepsilon > 0$,

$$\limsup_{m\to\infty} |A_m| \le 2Tm(\Gamma)\varepsilon.$$

As a consequence, since the above estimate holds for all $\varepsilon > 0$, A_m tends to 0, concluding the proof of Proposition 4.17. □

4.3 End of the proof of Theorem 1

We have proven in the section 4 that, up to a subsequence, the sequence of approximate solutions $(s_{\mathcal{D}_m}, P_{\mathcal{D}_m})_m$ converge towards (s, P) as $m \to \infty$. Moreover, it as been stated in Lemmata 4.14 and 4.15 that $(\pi_{\mathcal{D}_m})_m$ converges in some sense on $\Gamma \times (0,T)$ towards a measurable function π. In order to conclude the proof of Theorem 1, it remains to check that (s, P) satisfy the weak formulations (28) and (29), and that the transmission conditions (21) and (22) are fulfilled. Let us begin by this latter point.

It follows from the construction of the function π carried out in Lemmata 4.14 and 4.15 that, for almost every $(\mathbf{x}, t) \in \Gamma \times (0,T)$,

$$\pi(\mathbf{x}, t) \in \tilde{\pi}_1(s_1(\mathbf{x}, t)) \cap \tilde{\pi}_2(s_2(\mathbf{x}, t)). \tag{97}$$

Let $\psi \in L^2(\Gamma \times (0,T))$, then thanks to (39), one has, for all $\psi \in L^2(\Gamma \times (0,T))$,

$$\int_0^T \int_\Gamma \left(P_{\mathcal{D}_{m_{|\Gamma,1}}} - P_{\mathcal{D}_{m_{|\Gamma,2}}} \right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Gamma \left(W_1(\pi_{\mathcal{D}_m}) - W_2(\pi_{\mathcal{D}_m}) \right) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Letting m tend to ∞ provides, thanks to Propositions 4.12 and 4.17, that

$$\int_0^T \int_\Gamma (P_1 - P_2) \psi \mathrm{d}\mathbf{x}\mathrm{d}t = \int_0^T \int_\Gamma (W_1(\pi) - W_2(\pi)) \psi \mathrm{d}\mathbf{x}\mathrm{d}t.$$

Hence,

$$P_1 - W_1(\pi) = P_2 - W_2(\pi) \quad \text{a.e. on } \Gamma \times (0,T). \tag{98}$$

In order to recover the weak formulations (28) and (29), we can apply to our case the analysis carried out in the proof of Theorem 5.1 in [35].

5 Numerical results

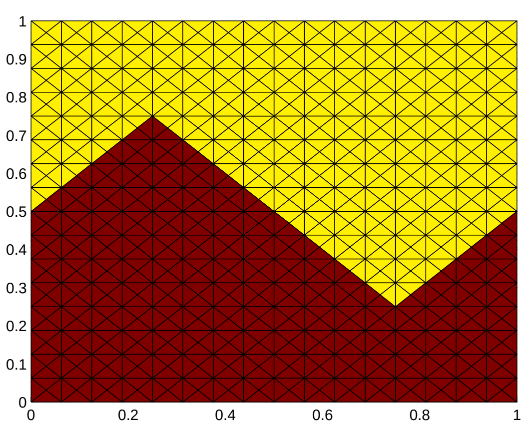

Figure 5: The model porous medium $\Omega_1 \cup \Omega_2$

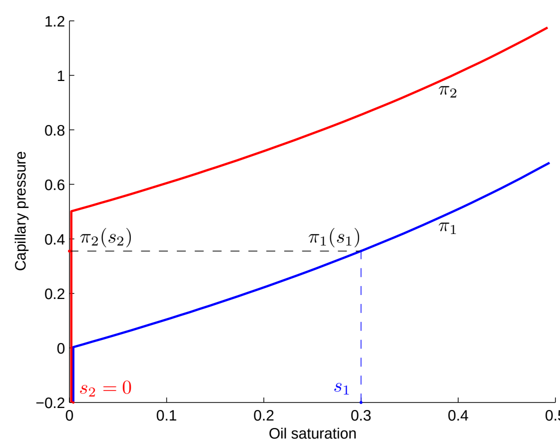

Figure 6: Capillary pressure connection at t = 0

In this Section we consider a model porous medium $\Omega = (0,1)^2$ composed of two layers Ω_1 and Ω_2, which are separated by an "S-shaped" interface Γ (see Fig. 5), and which have different capillary pressure laws. The porosity ϕ is constant and set to $\phi = 1$, and the absolute permeability K is given by $K_1 = 1$ and $K_2 = 0.5$. The oil and water densities are given by $\rho_o = 0.81$, $\rho_w = 1$ respectively, and $\mathbf{g} = -9.81\mathbf{e}_z$. We suppose that the porosity is such that $\phi_i = 1, i \in \{1,2\}$, and we define the oil and water mobilities by

$$\eta_{o,i}(s) = s^2 \quad \text{and} \quad \eta_{w,i} = 3(1-s)^2, \quad i \in \{1,2\}.$$

Moreover we suppose that the capillary pressure curves have the form

$$\pi_1(s) = \ln(1-s) \quad \text{and} \quad \pi_2(s) = 0.5 - \ln(1-s).$$

Test case 1. We suppose that the layer Ω_1 contains some quantity of oil and it is situated below Ω_2, which is saturated with water, that is to say $\Omega_1 = \{(x,z) \in \Omega \mid z < \Gamma(x)\}$ and $\Omega_2 = \{(x,z) \in \Omega \mid z > \Gamma(x)\}$. The initial saturation is given by

$$s_0(\mathbf{x}) = \begin{cases} 0.3 & \text{if } x \in \Omega_1, \\ 0 & \text{otherwise.} \end{cases}$$

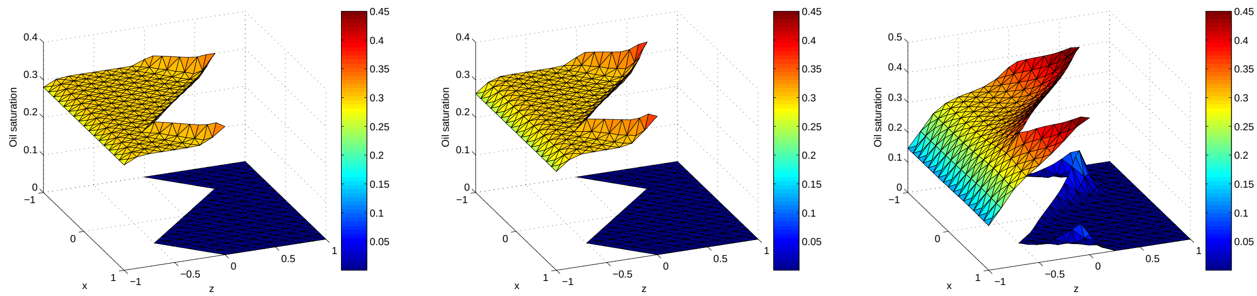

Figure 7: Test case 1, oil saturation for $t = 0.0125$, $t = 0.025$ and $t = 0.2$.

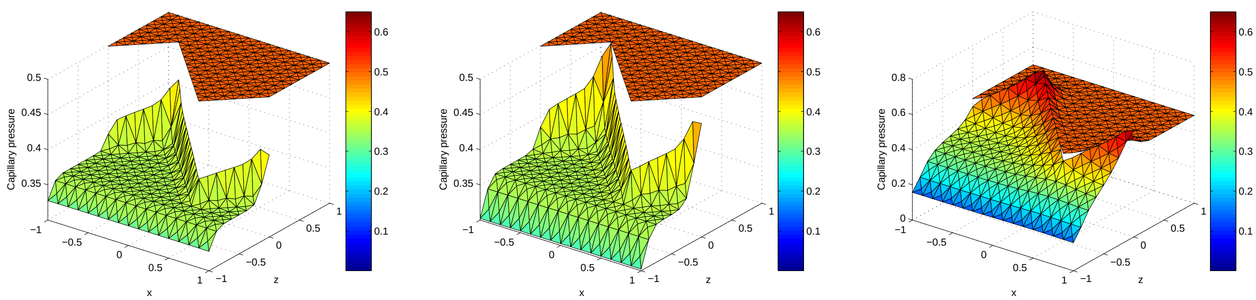

Figure 8: Test case 1, capillary pressure for $t = 0.0125$, $t = 0.025$ and $t = 0.2$.

The flow is driven by buoyancy, making the oil move along $\mathbf{e}_z$ until it reaches the interface Γ. As one can see on the figures 7 and 8, for $t \leq 0.11$, oil can not access the domain Ω_2, since the capillary pressure $\pi_1(s_1)$ is lower than the threshold value $\pi_2(0) = 0.5$, which is called *the entry pressure* (see Fig. 6). Hence the saturation below the interface s_1 increases, as well as the capillary pressure $\pi_1(s_1)$. As soon as the capillary pressure $\pi_1(s_1)$ reaches the entry pressure $\pi_2(0)$, oil starts to penetrate in the domain Ω_2. Nevertheless, as pointed out in [8, 11], a finite quantity of oil remains trapped under the rock discontinuity. This phenomenon is called *oil trapping*. It is worth noting that the solution at $t = 0$ satisfies (21), thus in the absence of gravity the initial distribution of oil-phase would be a steady state solution $s(\mathbf{x}, t) = s_0(\mathbf{x})$.

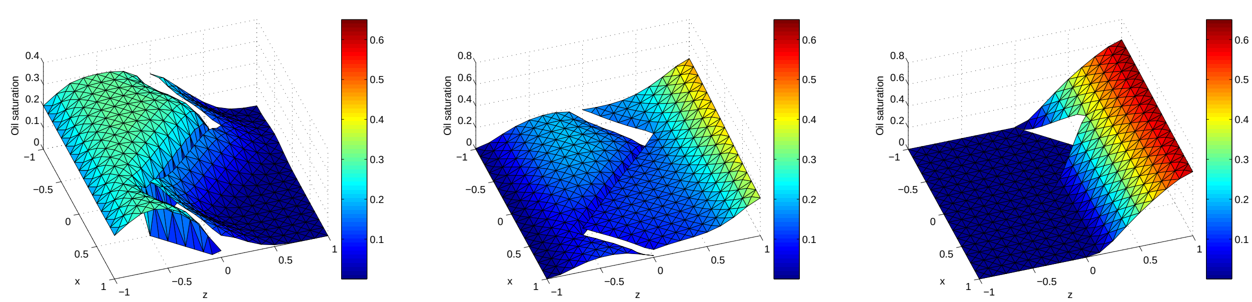

Figure 9: Test case 2, oil saturation for $t = 0.2$, $t = 1.5$ and $t = 2$.

Test case 2. We assume that the oil is initially situated in the rock with a higher *entry pressure* pressure i.e.

$$s_0(\mathbf{x}) = \begin{cases} 0.3 & \text{if } x \in \Omega_2, \\ 0 & \text{otherwise.} \end{cases}$$

where this time $\Omega_1 = \{(x, z) \in \Omega \mid z > \Gamma(x)\}$ and $\Omega_2 = \{(x, z) \in \Omega \mid z < \Gamma(x)\}$. This time the flow is driven not only by a buoyancy, but also by a difference in the capillary pressure potential (the solution at $t = 0$ does not fulfill (21)). As a result the oil-phase can immediately penetrate the domain Ω_1. The figure 9 shows that the oil propagates in the domain Ω_1 with a finite speed. Remark that in this case the capillary pressure and the oil pressure remain discontinuous (see Fig. 10), yet the oil phase may pass through the discontinuity.

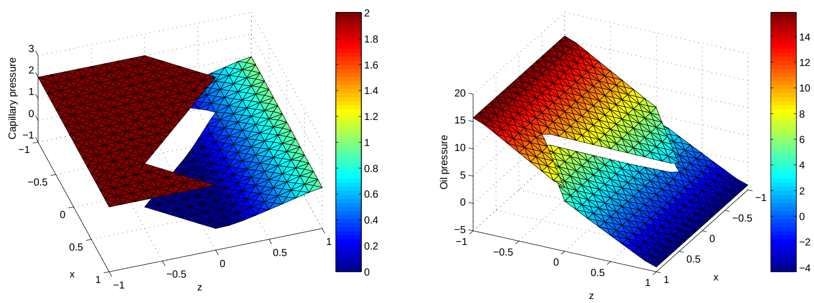

Figure 10: Test case 2, capillary and oil pressure at $t = 5$.

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