Fast Collaborative Graph Exploration

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Abstract

We study the following scenario of online graph exploration. A team of k agents is initially located at a distinguished vertex r of an undirected graph. At every time step, each agent can traverse an edge of the graph. All vertices have unique identifiers, and upon entering a vertex, an agent obtains the list of identifiers of all its neighbors. We ask how many time steps are required to complete exploration, i.e., to make sure that every vertex has been visited by some agent.

We consider two communication models: one in which all agents have global knowledge of the state of the exploration, and one in which agents may only exchange information when simultaneously located at the same vertex. As our main result, we provide the first strategy which performs exploration of a graph with n vertices at a distance of at most D from r in time $O(D)$, using a team of agents of polynomial size $k = Dn^{1+\epsilon} < n^{2+\epsilon}$, for any $\epsilon > 0$. Our strategy works in the local communication model, without knowledge of global parameters such as n or D.

We also obtain almost-tight bounds on the asymptotic relation between exploration time and team size, for large k. For any constant $c > 1$, we show that in the global communication model, a team of $k = Dn^c$ agents can always complete exploration in $D(1 + \frac{1}{c-1} + o(1))$ time steps, whereas at least $D(1 + \frac{1}{c} - o(1))$ steps are sometimes required. In the local communication model, $D(1 + \frac{2}{c-1} + o(1))$ steps always suffice to complete exploration, and at least $D(1 + \frac{2}{c} - o(1))$ steps are sometimes required. This shows a clear separation between the global and local communication models.

1 Introduction

Exploring an undirected graph-like environment is relatively straight-forward for a single agent. Assuming the agent is able to distinguish which neighboring vertices it has previously visited, there is no better systematic traversal strategy than a simple depth-first search of the graph, which takes $2(n-1)$ moves in total for a graph with n vertices. The situation becomes more interesting if multiple agents want to collectively explore the graph starting from a common location. If arbitrarily many agents may be used, then we can generously send n^D agents through the graph, where D is the distance from the starting vertex to the most distant vertex of the graph. At each step, we spread out the agents located at each node (almost) evenly among all the neighbors of the current vertex, and thus explore the graph in D steps.

While the cases with one agent and arbitrarily many agents are both easy to understand, it is much harder to analyze the spectrum in between these two extremes. Of course, we would like to explore graphs in as few steps as possible (i.e., close to D), while using a team of as few agents as

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possible. In this paper we study this trade-off between exploration time and team size. A trivial lower bound on the number of steps required for exploration with k agents is $\Omega(D + n/k)$: for example, in a tree, some agent has to reach the most distant node from r, and each edge of the tree has to be traversed by some agent. We look at the case of larger groups of agents, for which D is the dominant factor in this lower bound. This complements previous research on the topic for trees [6, 8] and grids [17], which usually focused on the case of small groups of agents (when n/k is dominant).

Another important issue when considering collaborating agents concerns the model that is assumed for the communication between agents. We need to allow communication to a certain degree, as otherwise there is no benefit to using multiple agents for exploration [8]. We may, for example, allow agents to freely communicate with each other, independent of their whereabouts, or we may restrict the exchange of information to agents located at the same location.This paper also studies this tradeoff between global and local communication.

The collaborative online graph exploration problem.

We are given a graph $G = (V, E)$ rooted at some vertex r. The number of vertices of the graph is bounded by n. Initially, a set $\mathcal{A}$ of k agents is located at r. We assume that vertices have unique identifiers that admit a total ordering. In each step, an agent visiting vertex v receives a complete list of the identifiers of the nodes in $N(v)$, where $N(v)$ is the neighborhood of v. Time is discretized into steps, and in each step, an agent can either stay at its current vertex or slide along an edge to a neighboring vertex. Agents have unique identifiers, which allows agents located at the same node and having the same exploration history to differentiate their actions. We do not explicitly bound the memory resources of agents, enabling them in particular to construct a map of the previously visited subgraph, and to remember this information between time steps. An *exploration strategy* for G is a sequence of moves performed independently by the agents. A strategy explores the graph G in t time steps if for all $v \in V$ there exists time step $s \leq t$ and an agent $g \in \mathcal{A}$, such that g is located at v in step s. Our goal is to find an exploration strategy which minimizes the time it takes the explore a graph in the worst case, with respect to the shortest path distance D from r to the vertex furthest from r in the graph.

We distinguish between two communication models. In exploration *with global communication* we assume that, at the end of each step s, all agents have complete knowledge of the explored subgraph. In particular, in step s all agents know the number of edges incident to each vertex of the explored subgraph which lead to unexplored vertices, but they have no information on any subgraph consisting of unexplored vertices. In exploration *with local communication* two agents can exchange information only if they occupy the same vertex. Thus, each agent g has its own view on which vertices were explored so far, constructed based only the knowledge that originates from the agent's own observations and from other agents that it has met.

Our results.

Our main contribution is an exploration strategy for a team of polynomial size to explore graphs in an asymptotically optimal number of steps. More precisely, for any $\epsilon > 0$, the strategy can operate with $Dn^{1+\epsilon} < n^{2+\epsilon}$ agents and takes time $O(D)$. It works even under the local communication model and without prior knowledge of n or D.

We first restrict ourselves to the exploration of trees (Section 2). We show that with global communication trees can be explored in time $D \cdot (1 + 1/(c-1) + o(1))$ for any $c > 1$, using a team of Dn^c agents. Our approach can be adapted to show that with local communication trees can be

Communication Model	*Upper bound*	*Lower bound*
Global communication:	$D \cdot (1 + \frac{1}{c-1} + o(1))$ Thm. 3.3	$D \cdot (1 + \frac{1}{c} - o(1))$ Thm. 4.1
Local communication :	$D \cdot (1 + \frac{2}{c-1} + o(1))$ Thm. 3.3	$D \cdot (1 + \frac{2}{c} - o(1))$ Thm. 4.1

Table 1: Our bounds for the time required to explore general graphs with using Dn^c agents. The same upper and lower bounds hold for trees. The lower bounds use graphs with $D = n^{o(1)}$.

explored in time $D \cdot (1 + 2/(c-1) + o(1))$ for any $c > 1$, using the same number of agents. We then carry the results for trees over to the exploration of general graphs (Section 3). We obtain precisely the same asymptotic bounds for the number of time steps needed to explore graphs with Dn^c agents as for the case of trees, under both communication models.

Finally, we provide lower bounds for collaborative graph exploration that almost match our positive results (Section 4). More precisely, we show that, in the worst case and for any $c > 1$, exploring a graph with Dn^c agents takes at least $D \cdot (1 + 1/c - o(1))$ time steps in the global communication model, and at least $D \cdot (1 + 2/c - o(1))$ time steps in the local communication model. Table 1 summarizes our upper and corresponding lower bounds.

Related work.

Collaborative online graph exploration has been intensively studied for the special case of trees. In [8], a strategy is given which explores any tree with a team of k agents in $O(D + n/\log k)$ time steps, using a communication model with whiteboards at each vertex that can be used to exchange information. This corresponds to a competitive ratio of $O(k/\log k)$ with respect to the optimum exploration time of $\Theta(D+n/k)$ when the graph is known. In [13] authors show that the competitive ratio of the strategy presented in [8] is precisely $k/\log k$. Another DFS-based algorithm, given in [2], has an exploration time of $O(n/k+D^{k-1})$ time steps, which provides an improvement only for graphs of small diameter and small teams of agents, $k = O(\log_D n)$. For a special subclass of trees called sparse trees, [6] introduces online strategies with a competitive ratio of $O(D^{1-1/p})$, where p is the density of the tree as defined in that work. The best currently known lower bound is much lower: in [7], it is shown that any deterministic exploration strategy with $k < \sqrt{n}$ has a competitive ratio of $\Omega(\log k/\log\log k)$, even in the global communication model. A stronger lower bound of $\Omega(k/\log k)$ holds for so-called greedy algorithms [13]. Both for deterministic and randomized strategies, the competitive ratio is known to be at least $2 - 1/k$, when $k < \sqrt{n}$ [8]. None of these lower bounds concern larger teams of agents. In [16] a lower bound of $\Omega(D^{1/(2c+1)})$ on competitive ratio is shown to hold for a team of $k = n^c$ agents, but this lower bound only concerns so-called rebalancing algorithms which keep all agents at the same height in the tree throughout the exploration process.

The same model for online exploration is studied in [17], where a strategy is proposed for exploring graphs which can be represented as a $D \times D$ grid with a certain number of disjoint rectangular holes. The authors show that such graphs can be explored with a team of k agents in time $O(D\log^2 D + n\log D/k)$, i.e., with a competitive ratio of $O(\log^2 D)$. By adapting the approach for trees from [7], they also show lower bounds on the competitive ratio in this class of graphs of $\Omega(\log k/\log\log k)$ for deterministic strategies and $\Omega(\sqrt{\log k}/\log\log k)$ for randomized strategies. These lower bounds also hold in the global communication model.

Collaborative exploration has also been studied with different optimization objectives. An ex-

ploration strategy for trees with global communication is given in [7], achieving a competitive ratio of $(4-2/k)$ for the objective of minimizing the maximum number of edges traversed by an agent. In [5] a corresponding lower bound of 3/2 is provided.

Our problem can be seen as an online version of the k Traveling Salesmen Problem (k-TSP) [9]. Online variants of TSP (for a single agent) have been studied in various contexts. For example, the geometric setting of exploring grid graphs with and without holes is considered by [10, 11, 14, 15, 17], where a variety of competitive algorithms with constant competitive ratios is provided. A related setting is studied in [4], where an agent has to explore a graph while being attached to the starting point by a rope of restricted length. A similar setting is considered in [1], in which each agent has to return regularly to the starting point, for example for refueling. Online exploration of polygons is considered in [3, 12].

2 Tree exploration

We start our considerations by designing exploration strategies for the special case when the explored graph is a tree T rooted at a vertex r.

For any exploration strategy, the set of all encountered vertices (i.e., all visited vertices and their neighbors) at the beginning of step $s = 1, 2, 3, \ldots$ forms a connected subtree of T, rooted at r and denoted by $T^{(s)}$. In particular, $T^{(1)}$ is the vertex r together with its children, which have not yet been visited. For $v \in V(T)$ we write $T^{(s)}(v)$ to denote the subtree of $T^{(s)}$ rooted at v. We denote by $L(T^{(s)}, v)$ the number of leaves of the tree $T^{(s)}(v)$. Note that $L(T^{(s)}, v) \leq L(T^{(s+1)}, v)$ because each leaf in $T^{(s)}(v)$ is either a leaf of the tree $T^{(s+1)}$ or the root of a subtree containing at least one vertex. If v is an unencountered vertex at the beginning of step s, i.e., its parent was not yet visited, we define $L(T^{(s)}, v) = 1$.

2.1 Tree exploration with global communication

We are ready to give the procedure `TEG` (*Tree Exploration with Global Communication*). The pseudocode uses the command "move$^{(s)}$", describing the move to be performed by each agent, specifying the destination at which the agent appears at the start of time step $s+1$. Since the agents can communicate globally, the procedure can centrally coordinate the movements of each agent. For simplicity we assume that x agents spawn in r in each time step, for some given value of x. Then, the total number of agents used after l steps is simply lx.

Procedure `TEG` (tree T with root r, integer x) **at time step** s:
  Place x new agents at r.
  for each $v \in V(T^{(s)})$ which is not a leaf **do**: { determine moves of the agents located at v }
    Let $\mathcal{A}_v^{(s)}$ be the set of agents currently located at v.
    Denote by $v_1, v_2, \ldots, v_d$ the set of children of v.
    Let $i^* := \arg\max_i \{L(T^{(s)}, v_i)\}$. { v_{i^*} is the child of v with the largest value of L }
    Partition $\mathcal{A}_v^{(s)}$ into disjoint sets $\mathcal{A}_{v_1}, \mathcal{A}_{v_2}, \ldots, \mathcal{A}_{v_d}$, such that:
    (*i*) $|\mathcal{A}_{v_i}| = \left\lfloor \frac{|\mathcal{A}_v^{(s)}| \cdot L(T^{(s)}, v_i)}{L(T^{(s)}, v)} \right\rfloor$, for $i \in \{1, 2, \ldots, d\} \setminus \{i^*\}$,
    (*ii*) $|\mathcal{A}_{v_{i^*}}| = |\mathcal{A}_v^{(s)}| - \sum_{i \in \{1,2,\ldots,d\} \setminus \{i^*\}} |\mathcal{A}_{v_i}|$.
    for each $i \in \{1, 2, \ldots, d\}$ **do for each** agent $g \in \mathcal{A}_{v_i}$ **do move**$^{(s)}$ g to vertex v_i.
  end for
end procedure `TEG`.

The following lemma provides a characterization of the tradeoff between exploration time and

the number of agents x released at every round in procedure TEG. In the following, all logarithms are with base 2 unless a different base is explicitly given.

Lemma 2.1. *In the global communication model, procedure* TEG *with parameter x explores any rooted tree T in at most $D \cdot (1 + \frac{1}{\log_n x - 1 - \log_n(2\log x)})$ time steps, for $x > 6(n \log n + 1)$.*

Proof. Fix any leaf f of the tree T. We want to prove that procedure TEG visits the leaf f after at most $D \cdot (1 + \frac{1}{\log_n x - 1 - \log_n(2\log x)})$ time steps. Take the path $\mathcal{F} = (f_0, f_1, f_2, \ldots, f_{D_f})$ from r to f in T, where $r = f_0, f = f_{D_f}$, and $D_f \leq D$. We define the *wave* of agents w_s starting from r at time s and traversing the path $\mathcal{F}$ as the maximum sequence of the non-empty sets of agents which leave the root in step s and traverse edges of $\mathcal{F}$ in successive time steps, i.e., $w_s = (\mathcal{A}_{f_0}^{(s)}, \mathcal{A}_{f_1}^{(s+1)}, \ldots)$, where we use the notation from procedure TEG. The size of wave w_s in step $s + t$ is defined to be $|\mathcal{A}_{f_t}^{(s)}|$, i.e., the number of exploring agents located at vertex f_t at the beginning of time step $s + t$; initially, every wave has size $|\mathcal{A}_{f_0}^{(s)}| = x$. Note that each agent in $\mathcal{A}_{f_i}^{(s+i)}$, $0 \leq i < D_f$, is located at r at the start of time step s. We denote the number of leaves in the subtree of $T^{(i)}$ rooted at f_j by $\lambda_j^{(i)} = L(T^{(i)}, f_j)$. Recall that if f_j is not yet discovered in step i, by definition of the function L, we have $\lambda_j^{(i)} = 1$. In general, $1 \leq \lambda_j^{(i)} \leq n$. We define

$$\alpha_i = \frac{x}{2} \frac{\lambda_1^{(i)}}{\lambda_0^{(i)}} \frac{\lambda_2^{(i+1)}}{\lambda_1^{(i+1)}} \cdots \frac{\lambda_{D_f}^{(i+D_f-1)}}{\lambda_{D_f-1}^{(i+D_f-1)}}.$$

(See Figure 1 in the Appendix for an illustration of the construction.) We define the value α_i^* as the number of agents of the i-th wave that reach the leaf f, i.e., the size of the i-th wave in step $i + D_f$. If $\alpha_1^* = \alpha_2^* = \cdots = \alpha_{i-1}^* = 0$ and $\alpha_i^* \geq 1$ for some time step i, then we say that leaf f is explored by the i-th wave. Before we proceed with the analysis, we show the following auxiliary claim.

Claim (*). Let i be a time step for which $\alpha_i \geq \log x$. Then, $\alpha_i^* \geq \alpha_i$, and thus α_i is a lower bound on the number of agents reaching f in step $i + D_f$.

Proof (of the claim). We define $c_j = \lambda_{j+1}^{(i+j)} / \lambda_j^{(i+j)}$ for $j = 0, \ldots, D_f - 1$. For $i \geq 1$ we have

$$\alpha_i = x/2 \prod_{j=0}^{D_f-1} c_j.$$

Since $c_j \leq 1$ for all j and since $\alpha_i \geq \log x$, there exist at most $\log x$ different j such that $c_j \leq 1/2$. Denote the set of all such j by $\mathcal{J}$, with $|\mathcal{J}| \leq \log x$. Also, denote the size of wave w_i in step $i + s$ by a_s (for $s = 0, 1, 2, \ldots$), in particular $a_0 = x$.

Consider some index s for which $c_s > 1/2$. We have $\lambda_{s+1}^{(i+s)} / \lambda_s^{(i+s)} > 1/2$, thus more than half of all leaves of the tree $T^{(i+s)}(f_s)$ also belong to the tree $T^{(i+s)}(f_{s+1})$. But then, in time step $i + s + 1$, agents are sent from f_s to f_{s+1} according to the definition in expression (ii) in procedure TEG. Thus, we can lower-bound the size of wave w_i in step $i + s + 1$ by $a_{s+1} \geq a_s c_s$. Otherwise, if $c_s \leq 1/2$ (i.e., if $s \in \mathcal{J}$), then agents are sent according the definition in expression (i) in procedure TEG, and hence $a_{s+1} \geq \lfloor a_s c_s \rfloor$. Note that these bounds also hold if there are no agents left in the wave, i.e., $a_s = a_{s+1} = 0$. Thus, we have:

$$a_{s+1} \geq a_s c_s - \delta_s, \quad \text{where } \delta_s = \begin{cases} 1, & \text{if } s \in \mathcal{J}, \\ 0, & \text{otherwise.} \end{cases}$$

In this way we expand the expression for $\alpha_i^* = a_{D_f}$:

$$\alpha_i^* = a_{D_f} \geq a_{D_f-1}c_{D_f-1} - \delta_{D_f-1} \geq \ldots \geq (\ldots((a_0c_0 - \delta_0)c_1 - \delta_1)c_2 - \ldots)c_{D_f-1} - \delta_{D_f-1} =$$

$$= x\prod_{j=0}^{D_f-1} c_j - \sum_{j=0}^{D_f-1}\left(\delta_j \prod_{p=j+1}^{D_f-1} c_j\right) \geq 2\alpha_i - \sum_{j=0}^{D_f-1}\delta_j \geq 2\alpha_i - |\mathcal{J}| \geq 2\alpha_i - \log x.$$

Since by assumption $\alpha_i \geq \log x$, we obtain $\alpha_i^* \geq 2\alpha_i - \log x \geq \alpha_i$, which completes the proof of the claim.

We now show that if the number of waves a in the execution of the procedure is sufficiently large, then there exists an index $i \leq a$, such that $\alpha_i \geq \log x$. Thus, taking into account Claim (*), leaf f is explored at the latest by the a-th wave.

Take a waves and consider the product $\prod_{i=1}^{a}\alpha_i$. Note that $\lambda_{D_f}^{(s)} = 1$ for every s. Thus, simplifying the product of all α_i by shortening repeating terms in numerators and denominators, and using $1 \leq \lambda_j^{(i)} \leq n$, we get

$$\prod_{i=1}^{a}\alpha_i = (x/2)^a \prod_{i=1}^{a}\prod_{j=0}^{D_f-1}\frac{\lambda_{j+1}^{(i+j)}}{\lambda_j^{(i+j)}} = (x/2)^a \frac{\prod_{i=1}^{a}\prod_{j=0}^{D_f-1}\lambda_{j+1}^{(i+j)}}{\prod_{i=1}^{a}\prod_{j=0}^{D_f-1}\lambda_j^{(i+j)}} = (x/2)^a \frac{\prod_{i'=0}^{a-1}\prod_{j'=1}^{D_f}\lambda_{j'}^{(i'+j')}}{\prod_{i=1}^{a}\prod_{j=0}^{D_f-1}\lambda_j^{(i+j)}} =$$

$$= (x/2)^a \frac{\left(\prod_{j'=1}^{D_f}\lambda_{j'}^{(j')}\right)\left(\prod_{i'=1}^{a-1}\prod_{j'=1}^{D_f-1}\lambda_{j'}^{(i'+j')}\right)\left(\prod_{i'=1}^{a-1}\lambda_{D_f}^{(i'+D_f)}\right)}{\left(\prod_{i=1}^{a}\lambda_0^{(i)}\right)\left(\prod_{i=1}^{a-1}\prod_{j=1}^{D_f-1}\lambda_j^{(i+j)}\right)\left(\prod_{j=1}^{D_f-1}\lambda_j^{(a+j)}\right)} \geq \frac{(x/2)^a}{n^a n^{D_f-1}} \geq \frac{(x/2)^a}{n^{a+D}}. \quad (1)$$

We want to find a, such that

$$\prod_{i=1}^{a}\alpha_i \geq (\log x)^a.$$

Taking into account (1), it is sufficient to find a satisfying

$$\frac{(x/2)^a}{n^{a+D}} \geq (\log x)^a,$$

which for sufficiently large x (we take $x > 6(n\log n + 1)$) can be equivalently transformed by taking logarithms and elementary arithmetic to the form

$$a \geq \frac{D}{\log_n x - 1 - \log_n(2\log x)}.$$

Hence, for $a = \lceil \frac{D}{\log_n x - 1 - \log_n(2\log x)} \rceil$, we have that there exists some i such that $\alpha_i \geq \log x$. For the same i we have $\alpha_i^* \geq \log x$, by Claim (*). Thus, a waves are sufficient to explore the path $\mathcal{F}$. This analysis can be done for any leaf f, thus it is enough to send a waves in order to explore the graph G. Considering that a wave w_i is completed by the end of step $D + i - 1$, the exploration takes at most $D + a - 1$ time steps in total. Thus, the exploration takes at most $D \cdot (1 + \frac{1}{\log_n x - 1 - \log_n(2\log x)})$ time steps. □

We remark that in the above Lemma, the total number of agents used throughout all steps of procedure `TEG` is $x \cdot D \cdot (1 + \frac{1}{\log_n x - 1 - \log_n(2\log x)})$. For any $c > 1$, by appropriately setting $x = \Theta(n^c)$, we directly obtain the following theorem.

Theorem 2.2. *For any fixed $c > 1$ and known n, the online tree exploration problem with global communication can be solved in at most $D \cdot \left(1 + \frac{1}{c-1} + o(1)\right)$ time steps using a team of $k \geq Dn^c$ agents.*

2.2 Tree exploration with local communication

In this section we propose a strategy for tree exploration under the local communication model. In the implementation of the algorithm we assume that whenever two agents meet, they exchange all information they possess about the tree. Thus, after the meeting, the knowledge about the explored vertices and their neighborhoods, is a union of the knowledge of the two agents before the meeting. Since agents exchange information only if they occupy the same vertex, at any time s, the explored tree $T^{(s)}$ may only partially be known to each agent, with different agents possibly knowing different subtrees of $T^{(s)}$.

In order to obtain a procedure for the local communication model, we modify procedure `TEG` from the previous section. Observe that in procedure `TEG`, agents never move towards the root of the tree, hence, in the local communication model, agents cannot exchange information with other agents located closer to the root. The new strategy is given by the procedure `TEL` (*Tree Exploration with Local Communication*).

Procedure `TEL` (tree T with root r, integer x) **at time step** s:
  Place x new agents at r in state "exploring".
  for each $v \in V(T^{(s)})$ which is not a leaf **do**: { determine moves of the agents located at v }
    if $v \neq r$ **then for each** agent g at v in state "notifying" **do move**$^{(s)}$ g to the parent of v.
    if v contains at least two agents in state "exploring" **and** agents at v do not have information of any agent which visited v before step s **then**:
      { send two new notifying agents back to the root from newly explored vertex v }
      Select two agents g^*, g^{**} at v in state "exploring".
      Change state to "notifying" for agents g^* and g^{**}.
      move$^{(s)}$ g^* to the parent of v. { g^{**} will move to the parent one step later }
    end if
    Let $\mathcal{A}_v^{(s)}$ be the set of all remaining agents in state "exploring" located at v.
    Denote by $v_1, v_2, \ldots, v_d$ all children of v, and by δ the distance from r to v.
    $s' := \lfloor \frac{\delta+s}{2} \rfloor$. { s' is a time in the past such that $T^{(s')}(v)$ is known to the agents at v }
    Let $i^* := \arg\max_i \{L(T^{(s')}, v_i)\}$. { v_{i^*} is the child of v with the largest value of L }
    Partition $\mathcal{A}_v^{(s)}$ into disjoint sets $\mathcal{A}_{v_1}, \mathcal{A}_{v_2}, \ldots, \mathcal{A}_{v_d}$, such that:
    (i) $|\mathcal{A}_{v_i}| = \left\lfloor \frac{|\mathcal{A}_v^{(s)}| \cdot L(T^{(s')}, v_i)}{L(T^{(s')}, v)} \right\rfloor$, for $i \in \{1, 2, \ldots, d\} \setminus \{i^*\}$,
    (ii) $|\mathcal{A}_{v_{i^*}}| = |\mathcal{A}_v^{(s)}| - \sum_{i \in \{1,2,\ldots,d\} \setminus \{i^*\}} |\mathcal{A}_{v_i}|$.
    for each $i \in \{1, 2, \ldots, d\}$ **do if** $|\mathcal{A}_{v_i}| \geq 2$ **then for each** agent $g \in \mathcal{A}_{v_i}$ **do move**$^{(s)}$ g to v_i.
    for each $i \in \{1, 2, \ldots, d\}$ **do if** $|\mathcal{A}_{v_i}| = 1$ **then** change state to "discarded" for agent in $\mathcal{A}_{v_i}$.
  end for
  for each $v \in V(T^{(s)})$ which is a leaf **do move**$^{(s)}$ all agents located at v to the parent of v.
end procedure `TEL`.

In procedure `TEL`, all agents are associated with a state flag which may be set either to the value "exploring" or "notifying". Agents in the "exploring" state act similarly as in global exploration, with the requirement that they always move to a vertex in groups of 2 or more agents. Every time a group of "exploring" agents visits a new vertex, it detaches two of its agents, changes their state to "notifying", and sends them back along the path leading back to the root. These agents notify every agent they encounter on their way about the discovery of the new vertices. Although information about the discovery may be delayed, in every step s, all agents at vertex v know the entire subtree $T^{(s')}(v)$ which was explored until some previous time step $s' \leq s$. The state flag

also has a third state, "discarded", which is assigned to agents no longer used in the exploration process.

The formulation of procedure TEL is not given from the perspective of individual agents, however, based on its description, the decision on what move to make in the current step can be made by each individual agent. The correctness of the definition of the procedure relies on the following lemma, which guarantees that for a certain value s' the tree $T^{(s')}(v)$ is known to all agents at v.

Lemma 2.3. *Let T be a tree rooted at some vertex r and let v be a vertex with distance δ to r. After running procedure* TEL *until time step s, all agents which are located at vertex v at the start of time step s know the tree $T^{(s')}(v)$, for $s' = \lfloor \frac{\delta+s}{2} \rfloor$.*

Proof. Suppose the claim of the lemma holds until time step $s-1$, i.e., procedure TEL is well defined until time step $s-1$.

Assume that agents following procedure TEL discover vertex v^* in the subtree of v at distance δ^* from v at the beginning of time step $s^* \leq s$. This means that the parent of v^* is visited at the beginning of s^* and notifying agents sent from the parent of v^* carry knowledge about v^* towards the root. We need to prove that that if $s^* \leq s'$ (i.e., if $v^* \in V(T^{(s')})$), then agents located at v at time s know of v^*. It suffices to show that, by the start of time step s, these agents have met a notifying agent (as defined in procedure TEL) coming from the parent of v^*.

Since the distance from the root to the parent of v^* is $\delta + \delta^* - 1$, we have $s^* \geq \delta + \delta^* - 1$. Thus:

$$\frac{\delta + s}{2} \geq s' \geq s^* \implies s \geq 2s^* - \delta \geq s^* + \delta^* - 1.$$

Since $s \geq s^* + \delta^* - 1$, the first of the notifying agents for v^* (agent g^* sent out from parent of v^* at time s^*) reached vertex v on the path to the root by the start of time step s, and then continued its walk on the path to the root. The second of the corresponding notifying agents, g^{**}, is exactly one step further from the root. Suppose that $g \in \mathcal{A}_v^{(s)} \neq \emptyset$. By the construction of procedure TEL, agent g has been descending along a path from root r to vertex v in consecutive time steps, reaching v at the start of time step s. It follows that g has encountered at some vertex on the path from r to v exactly one of the notifying agents g^*, g^{**} (passing the other on an edge), and so the claim holds. □

Lemma 2.4. *In the local communication model, procedure* TEL *with parameter x explores any rooted tree T in at most $D \cdot (1 + \frac{2+1/\log n}{\log_n x - 1 - \log_n(4 \log x)})$ time steps, for $x > 17(n \log n + 1)$.*

Proof. As in the proof of Lemma 2.1, we consider any leaf f and the path $\mathcal{F} = (f_0, f_1, \ldots, f_{D_f})$ from r to f. As before, we denote the number of leaves in the subtree of $T^{(i)}$ rooted at f_j by $\lambda_j^{(i)} = L(T^{(i)}, f_j)$. Recall that if f_j is not yet discovered in step i, we have $L(T^{(i)}, f_j) = 1$. We adopt the definition of a wave from Lemma 2.1. We define the values α_i differently, however, to take into account the fact that the procedure relies on a delayed exploration tree, and that some waves lose agents as a result of deploying notifying agents:

$$\alpha_i = \frac{x}{4} \frac{\lambda_1^{(\lfloor \frac{i}{2} \rfloor)}}{\lambda_0^{(\lfloor \frac{i}{2} \rfloor)}} \frac{\lambda_2^{(\lfloor \frac{i}{2} \rfloor + 1)}}{\lambda_1^{(\lfloor \frac{i}{2} \rfloor + 1)}} \cdots \frac{\lambda_{D_f}^{(\lfloor \frac{i}{2} \rfloor + D_f - 1)}}{\lambda_{D_f - 1}^{(\lfloor \frac{i}{2} \rfloor + D_f - 1)}}.$$

We call a wave that discovered at least $\lceil \log x \rceil$ new nodes (or equivalently, a wave whose agents were the first to visit at least $\lceil \log x \rceil$ nodes of the tree) a *discovery wave*. Thus, there are at

most $\lfloor \frac{D_f}{\lceil \log x \rceil} \rfloor \leq \lfloor \frac{D}{\log x} \rfloor$ discovery waves along the considered path. Observe that if a wave is not a discovery wave, then the number of notifying agents it sends out is at most $2 \log x$.

We define by α_i^* the number of agents of the i-th wave that reach leaf f. We now prove that the following analogue of Claim (*) from the proof of Lemma 2.1 holds for non-discovery waves.

Claim (**). Let i be a time step for which w_i is not a discovery wave and $\alpha_i \geq \log x$. Then, $\alpha_i^* \geq \alpha_i$, and thus α_i is a lower bound on the number of agents reaching f in step $i + D_f$.

Proof (of claim). We define $c_j = \lambda_{j+1}^{(\lfloor \frac{i}{2} \rfloor + j)} / \lambda_j^{(\lfloor \frac{i}{2} \rfloor + j)}$ for $j = 0, \ldots, D_f - 1$. Then

$$\alpha_i = x/4 \prod_{j=1}^{D_f - 1} c_j.$$

Since $c_j \leq 1$ for all j and since $\alpha_i \geq \log x$, there exist at most $\log x$ different j such that $c_j \leq 1/2$. Denote the set of all such j by $\mathcal{J}$, with $|\mathcal{J}| \leq \log x$. Denote by $\mathcal{Q}$ the set of all such indices s that wave w_i sends two notifying agents from vertex f_s. By the assumption of the claim, we have that w_i is not a discovery wave thus $|\mathcal{Q}| \leq \lceil \log x \rceil - 1 \leq \log x$. Also, denote the size (number of agents) of wave w_i in step $i + s$ by a_s ($s = 0, 1, 2, \ldots$), where $a_0 = x$. Finally, let $\mathcal{R}$ be the set of indices s such that $a_s \geq 2$ and $a_{s+1} = 0$; note that $\mathcal{R}$ has at most one element.

Consider an index $s \notin \mathcal{R}$ for which $c_s > 1/2$ and assume that wave w_i does not send notifying agents from vertex f_s (i.e. $s \notin \mathcal{Q}$). We have $\lambda_{s+1}^{(i+s)} / \lambda_s^{(i+s)} > 1/2$, thus more than half of all leafs of the tree $T^{(i+s)}(f_s)$ also belong to the tree $T^{(i+s)}(f_{s+1})$. But then, in time step $i + s + 1$, agents are sent from f_s to f_{s+1} according to the definition in expression (ii) in the pseudocode of procedure `TEL`. Thus, we can lower-bound the size of wave w_i in step $i + s + 1$ as: $a_{s+1} \geq a_s c_s$. Otherwise, if $s \notin \mathcal{R} \cup \mathcal{Q}$ and $c_s \leq 1/2$ (i.e., if $s \in \mathcal{J}$), then agents are sent according the definition in expression (i) in the pseudocode, and then $a_{s+1} \geq \lfloor a_s c_s \rfloor$. Finally, if $s \in \mathcal{Q}$ then in vertex f_s wave w_i reduces by 2 notifying agents, while if $s \in \mathcal{R}$ then the wave may be reduced by one more agent ($a_{s+1} = 0$ instead of $a_{s+1} = 1$, since agents are always deployed in groups of two or more), and after that we can perform a similar analysis. Eventually, depending on which of the sets $\mathcal{J}, \mathcal{Q}, \mathcal{R}$ node s belongs to, we obtain:

$$a_{s+1} \geq a_s c_s - \delta_s, \quad \text{where } \delta_s = \delta_{sj} + \delta_{sq} + \delta_{sr},$$

and

$$\delta_{sj} = \begin{cases} 0, & \text{if } s \notin \mathcal{J} \\ 1, & \text{if } s \in \mathcal{J} \end{cases}, \quad \delta_{sq} = \begin{cases} 0, & \text{if } s \notin \mathcal{Q} \\ 2, & \text{if } s \in \mathcal{Q} \end{cases}, \quad \delta_{sr} = \begin{cases} 0, & \text{if } s \notin \mathcal{R} \\ 1, & \text{if } s \in \mathcal{R}. \end{cases}$$

In this way we expand the expression for $\alpha_i^* = a_{D_f}$:

$$\alpha_i^* = a_{D_f} \geq a_{D_f - 1} c_{D_f - 1} - \delta_{D_f - 1} \geq \ldots \geq (\ldots((a_0 c_0 - \delta_0) c_1 - \delta_1) c_2 - \ldots) c_{D_f - 1} - \delta_{D_f - 1} =$$

$$= x \prod_{j=0}^{D_f - 1} c_j - \sum_{j=0}^{D_f - 1} \left(\delta_j \prod_{p=j+1}^{D_f - 1} c_j \right) \geq 4\alpha_i - \sum_{j=0}^{D_f - 1} \delta_j \geq 4\alpha_i - |\mathcal{J}| - 2|\mathcal{Q}| - |\mathcal{R}| \geq$$

$$\geq 4\alpha_i - 3 \log x - |\mathcal{R}|.$$

Since by assumption $\alpha_i \geq \log x$, we obtain $\alpha_i^* \geq 4\alpha_i - 3 \log x - |\mathcal{R}| \geq \alpha_i - |\mathcal{R}|$. Since $|\mathcal{R}| \leq 1$, it follows that $\alpha_i^* \geq \log x - 1 \geq 2$, hence $\mathcal{R} = \emptyset$. So, we have $\alpha_i^* \geq \alpha_i$, which completes the proof of the claim.

It is left to prove that if the number of waves a in the execution of the procedure is sufficiently large, there exists an index $i \leq a$, such that wave w_i is not a discovery wave and $\alpha_i \geq \log x$. We again consider the product

$$\prod_{i=1}^{a} \alpha_i = (x/4)^a \prod_{i=1}^{a} \prod_{j=0}^{D_f-1} \frac{\lambda_{j+1}^{(\lfloor \frac{i}{2} \rfloor + j)}}{\lambda_j^{(\lfloor \frac{i}{2} \rfloor + j)}} =$$

$$= (x/4)^a \frac{\prod_{i=1}^{a} \prod_{j=0}^{D_f-1} \lambda_{j+1}^{(\lfloor \frac{i}{2} \rfloor + j)}}{\prod_{i=1}^{a} \prod_{j=0}^{D_f-1} \lambda_j^{(\lfloor \frac{i}{2} \rfloor + j)}} = (x/4)^a \frac{\prod_{i'=-1}^{a-2} \prod_{j'=1}^{D_f} \lambda_{j'}^{(\lfloor \frac{i'}{2} \rfloor + j')}}{\prod_{i=1}^{a} \prod_{j=0}^{D_f-1} \lambda_j^{(\lfloor \frac{i}{2} \rfloor + j)}} =$$

$$= (x/4)^a \frac{\left(\prod_{j'=1}^{D_f} \lambda_{j'}^{(j'-1)}\right) \left(\prod_{j'=1}^{D_f} \lambda_{j'}^{(j')}\right) \left(\prod_{i'=1}^{a-2} \prod_{j'=1}^{D_f-1} \lambda_{j'}^{(i'+j')}\right) \left(\prod_{i'=1}^{a-2} \lambda_{D_f}^{(\lfloor \frac{i'}{2} \rfloor + D_f)}\right)}{\left(\prod_{i=1}^{a} \lambda_0^{(\lfloor \frac{i'}{2} \rfloor)}\right) \left(\prod_{i=1}^{a-2} \prod_{j=1}^{D_f-1} \lambda_j^{(i+j)}\right) \left(\prod_{j=1}^{D_f-1} \lambda_j^{(\lfloor \frac{a-1}{2} \rfloor + j)}\right) \left(\prod_{j=1}^{D_f-1} \lambda_j^{(\lfloor \frac{a}{2} \rfloor + j)}\right)} \geq$$

$$\geq (x/4)^a \frac{1^{(D_f-1)\cdot(a-1)}}{n^{a+2D_f-2}} \geq \frac{(x/4)^a}{n^{a+2D}}. \tag{2}$$

We now choose a so as to guarantee that there exists at least one non-discovery wave $\alpha_i \geq \log x$. Since there are at most $\lfloor \frac{D}{\log x} \rfloor$ discovery waves, we require that the $\left(\lfloor \frac{D}{\log x} \rfloor + 1\right)$-st biggest value α_i is at least $\log x$. Observe that since we have $\alpha_i \leq x$, it suffices to choose a so that:

$$\prod_{i=1}^{a} \alpha_i \geq x^{\frac{D}{\log x}} (\log x)^a.$$

Taking into account (2), it is sufficient to find a satisfying

$$\frac{(x/4)^a}{n^{a+2D}} \geq x^{\frac{D}{\log x}} (\log x)^a,$$

which holds for sufficiently large x (we assume that $x > 17(n \log n + 1)$) for $a = \lceil \frac{2D + D/\log n}{\log_n x - 1 - \log_n (4 \log x)} \rceil$. Now, we have that there exists some index $i \leq a$ such that $\alpha_i \geq \log x$ and wave w_i is not a discovery wave. For the same i we have $\alpha_i^* \geq \log x$, by Claim (**). Thus, a waves are sufficient to explore the path $\mathcal{F}$. This analysis can be done for any leaf f, thus it is enough to send a waves in order to explore the graph G. We obtain that exploration takes at most $D + a - 1 \leq D \cdot (1 + \frac{2 + 1/\log n}{\log_n x - 1 - \log_n (4 \log x)})$ time steps. □

Acting as in the previous Subsection, from Lemma 2.4 we obtain a strategy for online exploration of trees in the model with local communication.

Theorem 2.5. *For any fixed $c > 1$, the online tree exploration problem can be solved in the model with local communication and knowledge of n using a team of $k \geq Dn^c$ agents in at most $D\left(1 + \frac{2}{c-1} + o(1)\right)$ time steps.*

3 General graph exploration

In this section we develop strategies for exploration of general graphs, both with global communication and with local communication. These algorithms are obtained by modifying the tree-exploration procedures given in the previous section.

Given a graph $G = (V, E)$ with root vertex r, we call $P = (v_0, v_1, v_2, \ldots, v_m)$ with $r = v_0$, $v_i \in V$, and $\{v_i, v_{i+1}\} \in E$ a *walk* of length $\ell(P) = m$. Note that a walk may contain a vertex more than once. We introduce the notation $P[j]$ to denote v_j, i.e., the j-th vertex of P after the root, and $P[0, j]$ to denote the walk $(v_0, v_1, \ldots, v_j)$, for $j \leq m$. The last vertex of path P is denoted by $end(P) = P[\ell(P)]$. The concatenation of a vertex u to path P, where $u \in N(end(P))$ is defined as the path $P' \equiv P + u$ of length $\ell(P) + 1$ with $P'[0, \ell(P)] = P$ and $end(P') = u$.

Let $\mathcal{P}$ be the set of walks P in G having length $0 \leq \ell(P) < n$. We introduce a linear order on walks in $\mathcal{P}$ such that for two walks P_1 and P_2, we say that $P_1 < P_2$ if $\ell(P_1) < \ell(P_2)$, or $\ell(P_1) = \ell(P_2)$ and there exists an index $j < \ell(P_1)$ such that $P_1([0, j]) = P_2([0, j])$ and $P_1([j + 1]) < P_2([j + 1])$. The comparison of vertices from V is understood as comparison of their identifiers in G.

We now define the tree T with vertex set $\mathcal{P}$, root $(r) \in \mathcal{P}$, such that vertex P' is a child of vertex P if and only if $P' = P + u$, for some $u \in N(end(P))$. We first show that agents can simulate the exploration of T while in fact moving around graph G. Intuitively, while an agent is following a path from the root to the leaves of T, its location in T corresponds to the walk taken by this agent in G.

Lemma 3.1. *A team of agents can simulate the virtual exploration of tree T starting from root (r), while physically moving around graph G starting from vertex r. The simulation satisfies the following conditions:*

(1) *An agent virtually occupying a vertex P of T is physically located at a vertex $end(P)$ in G.*

(2) *Upon entering a vertex P of T in the virtual exploration, the agent obtains the identifiers of all children of P in T.*

(3) *A virtual move along an edge of T can be performed in a single time step, by moving the agent to an adjacent location in G.*

(4) *Agents occupying the same virtual location P in T can communicate locally, i.e., they are physically located at the same vertex of G.*

Proof. We define the simulation so that claims (1-4) hold for all time steps. Initially, claim (1) is trivially true since $end((r)) = r$. Suppose that at the start of some step s, an agent occupies some virtual location P in T, and its corresponding physical location is $end(P)$. Claim (2) holds for this step, since the set of children of P in T is given as $\{P + u \in \mathcal{P} : u \in N(end(P))\}$, P is stored in the agents memory (as the identifier of its location in T), and the neighborhood of $end(P)$ in G is accessible to the agent by definition. When required to move to a virtual location P' adjacent to P in T, the agent performs a move to vertex $end(P') \in V$. Note that if P' is the child of P in T, then $end(P') \in N(end(P))$ by definition of T, whereas if P' is the parent of P in T, then $end(P') = P[\ell(P) - 1] \in N(end(P))$ from the definition of walk P. After such a move, claim (1) is immediately satisfied, and claims (2-3) follow by induction on time. Claim (4) is a trivial consequence of claim (1). □

We remark that the number of vertices of tree T is exponential in n. Hence, our goal is to perform the simulation with only a subset of the vertices of T. For a vertex $v \in V$, let $P_{\min}(v) \in \mathcal{P}$ be the minimum (with respect to the linear order on $\mathcal{P}$) walk ending at v. We observe that, by property (1) in Lemma 3.1, if, for all $v \in V$, the vertex $P_{\min}(v)$ of T has been visited by at least one agent in the virtual exploration of T, the physical exploration of G is completed. We define $\mathcal{P}_{\min} = \{P_{\min}(v) : v \in V\}$, and show that all vertices of $\mathcal{P}_{\min}$ are visited relatively quickly if we employ the procedure `TEG` (or `TEL`) for T, subject to a simple modification. In the original algorithm, we divided the agents descending to the children of the vertex according to the number of leaves of the discovered subtrees. We introduce an alternate definition of the function $L(T^{(s)}, v)$, so as to take into account only the number of vertices in $T^{(s)}$ corresponding to walks which are smallest among all walks in $T^{(s)}$ sharing the same end-vertex. (See Figure 2 in the Appendix for an example.)

Lemma 3.2. *Let $T^{(s)} \subseteq T$ be a subtree of T rooted at (r). For $P \in V(T^{(s)})$, let $L(T^{(s)}, P)$ be the number of vertices v of G, for which the subtree of $T^{(s)}$ rooted at P contains a vertex representing the smallest among all walks contained in $T^{(s)}$ which end at v:*

$$L(T^{(s)}, P) = \left| V(T^{(s)}(P)) \cap \bigcup_{v \in V} \left\{ \min\{P' \in V(T^{(s)}) : end(P') = v\} \right\} \right|,$$

and for $P \in \mathcal{P} \setminus V(T^{(s)})$, let $L(T^{(s)}, P) = 1$. Subject to this definition of L, procedure `TEG` with parameter $x > 6(n \log n + 1)$ (procedure `TEL` with parameter $x > 17(n \log n + 1)$) applied to tree T starting from root (r) visits all vertices from $\mathcal{P}_{\min}$ within $D \cdot (1 + \frac{1}{\log_n x - 1 - \log_n(2 \log x)})$ (respectively, $D \cdot (1 + \frac{2 + 1/\log n}{\log_n x - 1 - \log_n(4 \log x)})$) time steps.

Proof. The set $\mathcal{P}_{\min}$ spans a subtree $T_{\min} = T[\mathcal{P}_{\min}]$ in T, rooted at (r). We can perform an analysis analogous to that used in the Proofs of Lemmas 2.1 and 2.4, evaluating sizes of waves of agents along paths in the subtree $T_{\min}$. We observe that for any $P \in \mathcal{P}_{\min}$ which is not a leaf in $T_{\min}$, we always have $L(T^{(s)}, P) \geq 1$. Moreover, we have $L(T^{(s)}, P) \leq |V(T^{(s)}(P))|$, and so $L(T^{(s)}, P) \leq n$. Since these two bounds were the only required properties of the functions L in the Proofs of Lemmas 2.1 and 2.4, the analysis from these proofs applies within the tree $T_{\min}$ without any changes. It follows that each vertex of $\mathcal{P}_{\min}$ is reached by the exploration algorithm within $D \cdot (1 + \frac{1}{\log_n x - 1 - \log_n(2 \log x)})$ time steps in case of global communication, and within $D \cdot (1 + \frac{2 + 1/\log n}{\log_n x - 1 - \log_n(4 \log x)})$ time steps in case of local communication. □

We recall that by Lemma 3.1, one step of exploration of tree T can be simulated by a single step of an agent running on graph G. Thus, appropriately choosing $x = \Theta(n^c)$ in Lemma 3.2, we obtain our main theorem for general graphs.

Theorem 3.3. *For any $c > 1$, the online graph exploration problem with knowledge of n can be solved using a team of $k \geq Dn^c$ agents:*

- *in at most $D \cdot \left(1 + \frac{1}{c-1} + o(1)\right)$ time steps in the global communication model.*
- *in at most $D \cdot \left(1 + \frac{2}{c-1} + o(1)\right)$ time steps in the local communication model.*

For the case when we do not assume knowledge of (an upper bound on) n, we provide a variant of the above theorem which also completes exploration in $O(D)$ steps, with a slightly larger multiplicative constant.

Theorem 3.4. *For any $c > 1$, there exists an algorithm for the local communication model, which explores a rooted graph of unknown order n and unknown diameter D using a team of k agents, such that its exploration time is $O(D)$ if $k \geq Dn^c$.*

Proof. Let $c' = \frac{c+1}{2}$, $1 < c' < c$. For a graph G, the algorithm proceeds by assuming geometrically increasing upper bounds $\bar{D} = 1, 2, 4, \ldots$, on the value of D. For a fixed value of $\bar{D}$, we set $\bar{n} = \lfloor (k/\bar{D})^{1/c'} \rfloor$, and perform exploration of the graph using the algorithm from Theorem 3.3 with parameter c', assuming that the explored graph has at most $\bar{n}$ vertices, and using $\bar{D}\bar{n}^{c'} \leq k$ agents. After at most $\bar{D} \cdot \left(1 + \frac{2}{c'-1} + o(1)\right)$ time steps (where the asymptotic $o(1)$ value follows from Theorem 3.3) exploration is interrupted, and all agents return to the root vertex in at most $O(\bar{D})$ steps. If exploration of G has been completed, then the algorithm stops. This can be detected since the agents are aware which vertices still have unexplored neighbors. If the exploration has not been completed, we continue for a doubled value of $\bar{D}$, until the bound $\bar{n} = 0$ is reached. Finally, if exploration has been unsuccessful so far, we perform an arbitrary valid exploration algorithm, e.g. Depth First Search (DFS) with a single agent.

The algorithm always completes exploration successfully in finite time. Observe that if in the stage with $\bar{D} = 2^{\lceil \log_2 D \rceil}$ and $\bar{n} = \lfloor (k/\bar{D})^{1/c'} \rfloor$ we have $\bar{n} \geq n$, then exploration is completed successfully in this stage, and the total time of all exploration stages is $O(D)$. Observe that we have $\bar{D} < 2D$ and $k \geq Dn^c$, and so it suffices that $\lfloor (n^c/2)^{c'} \rfloor \geq n$. This holds for sufficiently large n. If the condition $k \geq Dn^c$ does not hold or n is too small, then the algorithm reaches the final phase in which DFS is executed, resulting in a correct exploration of the graph in finite time. □

We remark that by choosing $x = \Theta(n \log n)$ in Lemma 2.3, we can also explore a graph using $k = \Theta(Dn \log n)$ agents in time $\Theta(D \log n)$, with local communication. This bound is the limit of our approach in terms of the smallest allowed team of agents.

4 Lower bounds

In this section, we show lower bounds for exploration with Dn^c agents, complementary to the positive results given by Theorem 3.3. The graphs that produce the lower bound are a special class of trees. The same class of trees appeared in the lower bound from [8] for the competitive ratio of tree exploration algorithms with small teams of agents. In our scenario, we obtain different lower bounds depending on whether communication is local or global.

Theorem 4.1. *For all $n > 1$ and for every increasing function f, such that $\log f(n) = o(\log n)$, and every constant $c > 0$, there exists a family of trees $\mathcal{T}_{n,D}$, each with n vertices and height $D = \Theta(f(n))$, such that*

(i) for every exploration strategy with global communication that uses Dn^c agents there exists a tree in $\mathcal{T}_{n,D}$ such that number of time steps required for its exploration is at least $D\left(1 + \frac{1}{c} - o(1)\right)$,

(ii) for every exploration strategy with local communication that uses Dn^c agents there exists a tree in $\mathcal{T}_{n,D}$ such that number of time steps required for exploration is at least $D\left(1 + \frac{2}{c} - o(1)\right)$.

Proof. We prove the theorem assuming that the number of agents is n^c, rather than Dn^c. The asymptotic form of the bounds in claims (i) and (ii) remains unchanged since $D = n^{o(1)}$ by assumption, and $Dn^c = n^{c+o(1)}$. In previous sections we assumed that certain number of agents

spawned in the root r every round. Here we assume that all n^c agents are available in the first round.

(i) First we define the family of trees $\mathcal{T}_{n,D}$. It is possible to find $D = \Theta(f(n))$ such that for any n there exist integers Δ and κ such that $n = D\Delta + \kappa + 1$ and $0 \leq \kappa \leq D-1$. Note that $\Delta = \frac{n-(\kappa+1)}{D}$. Given a vector $\boldsymbol{q} = (q_1, \ldots, q_D) \in \{1, \ldots, \Delta\}^D$, we define $T(\boldsymbol{q})$ as the tree rooted at r with vertex set

$$V(T(\boldsymbol{q})) = \{r\} \cup \bigcup_{i=1}^{D} V_i \cup W,$$

where $V_i = \{v_1^i, \ldots, v_\Delta^i\}$ is the set of nodes at distance i from the root r and $W = \{w_1, w_2, \ldots, w_\kappa\}$ is the set of additional nodes attached to the root. For convenience, we set $v_{q_0}^0 = r$, and we define the edge set by

$$E(T(\boldsymbol{q})) = \bigcup_{i=1}^{D} \left\{ \{v_{q_{i-1}}^{i-1}, v_j^i\} \mid j = 1, \ldots, \Delta \right\} \cup \left\{ \{r, w_j\} \mid j = 1, 2, \ldots \kappa \right\},$$

which means that one specific vertex $v_{q_{i-1}}^{i-1}$ from level $i-1$ is connected to all vertices on level i. We set $\mathcal{T}_{n,D} = \{T(\boldsymbol{q}) \mid q \in \{1, \ldots, \Delta\}^D\}$. Since we are interested in lower bounds we will not consider vertices from W, we assume that exploration is finished when all vertices from V_i sets are explored.

We prove that each exploration strategy that uses at most n^c agents needs at least $D(1+\frac{1}{c}-o(1))$ steps to explore some tree in $\mathcal{T}_{n,D}$.

Let $\mathcal{S}$ be any exploration strategy that uses at most n^c agents. We select a tree from $\mathcal{T}_{n,D}$ based on the behavior of $\mathcal{S}$ in the class of trees $\mathcal{T}_{n,D}$. More precisely, let $T \in \mathcal{T}_{n,D}$ be such that, for each $i = 1, \ldots, D-1$, if s is the first step in which a vertex in V_i is visited, then one of the vertices in V_i holding the minimum number of agents in step s is the one having the vertices in V_{i+1} as children. In the following we bound the number of steps of $\mathcal{S}$ while exploring T. We say that $\mathcal{S}$ *makes progress* in time step s if for some $i \in \{1, \ldots, D\}$, some vertex in V_i is not visited in step $s-1$ and all vertices in V_i are visited at the start of step s. If only a strict non-empty subset of vertices of V_i are visited in some step s, then, by choosing T appropriately, the adversary may act so that the vertex in V_i that has Δ children is among those not visited in step s. We have n^c agents in $v_{q_0}^0$ in step 1. In step 2 at most n^c/Δ agents reach $v_{q_1}^1$. In steps 3 at most n^c/Δ^2 agents reach $v_{q_2}^2$, and so on. Thus $\mathcal{S}$ exploring tree T can make progress in at most $\lfloor \log_\Delta n^c \rfloor$ consecutive time steps. This is due to the choice of T.

Let p be the number of maximal sequences of consecutive time steps in which $\mathcal{S}$ makes progres. Let s_i, $i = 1, \ldots, p$, be the length of the i-th such sequence. By the above, we obtain that $s_i \leq \lfloor \log_\Delta(n^c) \rfloor$ for each $i = 1, \ldots, p$. Since the strategy $\mathcal{S}$ explores the entire tree T, the total number of steps in which $\mathcal{S}$ makes progress equals D, the height of the tree T. We obtain $D = \sum_{i=1}^{p} |s_i| \leq p \lfloor \log_\Delta n^c \rfloor$. Thus we can lower bound the value p

$$p \geq \frac{D}{\lfloor c \log_\Delta n \rfloor} \geq D \frac{\log \Delta}{c \log n} = D \frac{\log(n-(\kappa+1)) - \log D}{c \log n} = D\left(\frac{1}{c} - o(1)\right), \tag{3}$$

because $\log D = \Theta(\log f) = o(\log n)$ and $\log(n-(\kappa+1))/\log n = 1 - o(1)$. Each pair of maximal sequences of consecutive time steps in which $\mathcal{S}$ makes progress has to be separated by at least one step in which $\mathcal{S}$ makes no progress in tree T. Thus the are at least $p-1$ steps without progress and at most D steps with progress. Let s' be the first step in which all vertices are visited when executing $\mathcal{S}$ in T. By (3) and by the choice of T,

$$s' \geq p - 1 + D \geq D\left(\frac{1}{c} - o(1)\right) - 1 + D = D\left(1 + \frac{1}{c} - \frac{1}{D} - o(1)\right) \geq D\left(1 + \frac{1}{c} - o(1)\right),$$

where $\frac{1}{D} = o(1)$ because f is increasing. This completes the proof of (i).

(ii) We use the same family of trees $\mathcal{T}$ as in (i). Let $\mathcal{S}$ be any exploration strategy with local communication that uses at most n^c agents.

We select a tree from $\mathcal{T}_{n,D}$ based on the behavior of $\mathcal{S}$ in the class of trees $\mathcal{T}_{n,D}$. If step s is the first step in which a vertex in V_i is visited, then, since communication is local, agents located in vertex $v_{q_{i-1}}^{i-1}$ have no knowledge about the degrees of the vertices in V_i in step s. Since no agent comes back from V_i in step s, agents in $v_{q_{i-1}}^{i-1}$ have the same knowledge in steps $s-1$ and s. We select $T \in \mathcal{T}$ in such a way that a vertex in V_i for which the sum of the number of agents in steps s and $s+1$ is minimized, is the vertex $v_{q_i}^i$. Now, similarly as in (i), we lowerbound the number of steps.

We have n^c agents in $v_{q_0}^0$ in step 1. Together, in steps 2 and 3, in total at most n^c/Δ agents reach $v_{q_1}^1$. In steps 3 and 4 at most n^c/Δ^2 agents reach $v_{q_2}^2$, and so on. Thus in the first $\lfloor \log_\Delta n^c \rfloor + 2$ time steps, there are be two steps in which the algorithm does not make progress in terms of levels explored. Similarly as in previous part of the lemma the number of time steps without progress can be lowerbounded by $D\left(\frac{2}{c} - o(1)\right)$. Thus the exploration takes at least $D\left(1 + \frac{2}{c} - o(1)\right)$ steps. □

When looking at the problem of minimizing the size of the team of agents, our work (Theorem 3.4) shows that it is possible to achieve asymptotically-optimal online exploration time of $O(D)$ using a team of $k \leq Dn^{1+\epsilon}$ agents, for any $\epsilon > 0$. For graphs of small diameter, $D = n^{o(1)}$, we can thus explore the graph in $O(D)$ time steps using $k \leq n^{1+\epsilon}$ agents. This result almost matches the lower bound on team size of $k = \Omega(n^{1-o(1)})$ for the case of graphs of small diameter, which follows from the trivial lower bound $\Omega(D + n/k)$ on exploration time (cf. e.g. [8]). The question of establishing precisely what team size k is necessary and sufficient for performing exploration in $O(D)$ steps in a graph of larger diameter remains open.

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Figures

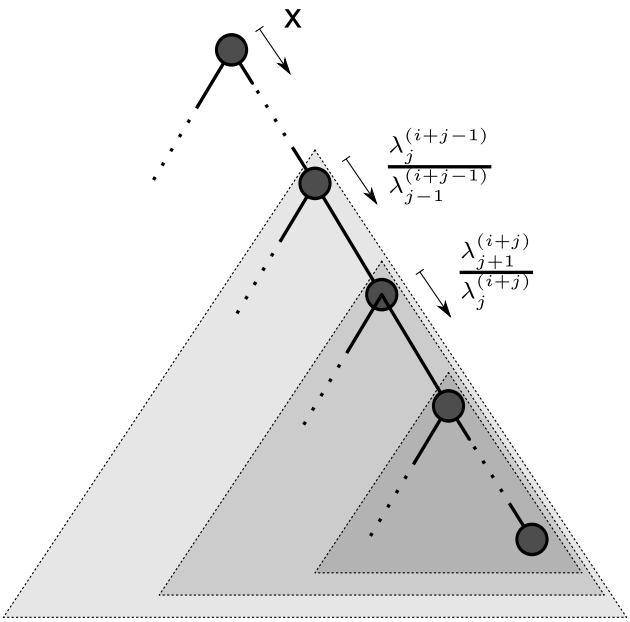

Figure 1: Illustration of proof of Lemma 2.1: computation of value of α_i for a wave of agents descending in tree T

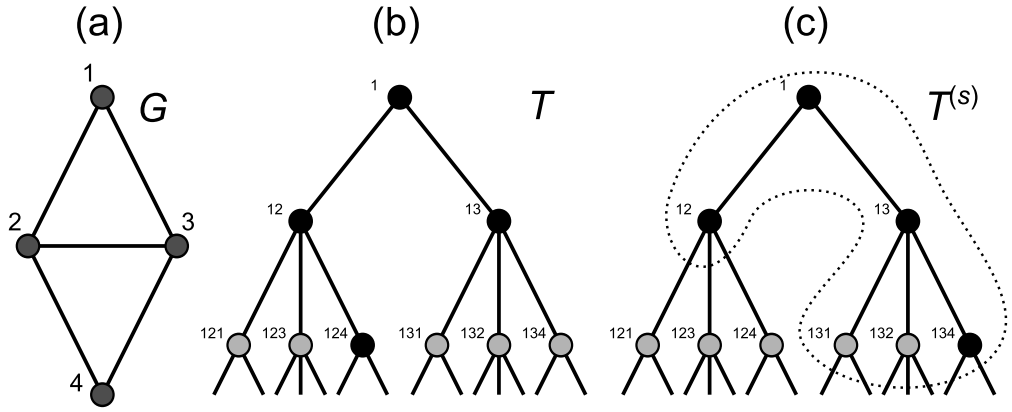

Figure 2: Illustration of exploration of general graphs: (a) The explored graph G, (b) The virtually explored tree of walks T, with highlighted nodes belonging to $\mathcal{P}_{\min}$, (c) An example of a subtree $T^{(s)} \subseteq T$ with highlighted nodes which are counted when computing function L (this tree $T^{(s)}$ does not correspond to a real execution of procedure `TEL` on T).