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TWO PROPERTIES OF TWO-VELOCITY TWO-PRESSURE MODELS FOR TWO-PHASE FLOWS

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Abstract. We study a class of models of compressible two-phase flows. This class, which includes the Baer–Nunziato model, is based on the assumption that each phase is described by its own pressure, velocity and temperature and on the use of void fractions obtained from averaging process. These models are nonconservative and non-strictly hyperbolic. We prove that the mixture entropy is non-strictly convex and that the system admits a symmetric form.

Key-words. Two-phase flows, entropy, symmetrizable system.
AMS subject classifications. 76T05, 35L60, 35F55.

1. Introduction

The modeling of compressible two-phase flows is a challenging task in Thermo-hydraulics. It is a crucial issue for many applications, for instance for water flows in some components of nuclear power plants such as the pressurized water reactors or the steam generators, especially in some specific situations, as the departure from nucleate boiling or the loss of coolant accident. When dealing with highly heterogeneous and disturbed flows, it is now commonly accepted that averaged models have to be considered. However, there is no consensus on the "good" model to use, especially when focusing on the two-fluid approach, where it is assumed that state variables within each phase (namely pressure, velocity and temperature) should not be confused. When restricting to the latter two-fluid framework, one can distinguish the Baer–Nunziato model [1] among the huge literature on the modeling of two-phase flows, both from the mathematical and physical point (see for instance [2, 5, 8, 9, 16] and references therein). Indeed, this system almost has the expected structure: if we only consider its convective part (i.e. the first order differential terms), its eigenvalues are always real and the associated eigenvectors are linearly independent except for some sonic cases (this is the resonance phenomenon). Even more, according to some choices of closure laws, it is possible to provide a wave-by-wave study, in spite of its nonconservative structure, and obtain some positive results on the solution of the associated Riemann problem [5, 3, 7].

In this work, we propose to investigate two properties which are crucial in the theory of nonlinear hyperbolic partial differential equations: the convexity of the entropy and the existence of a symmetric form. While such properties are very well understood for systems of conservation laws since Godunov [11] and Mock [15], it remains an open question for nonconservative and non-strictly hyperbolic systems, such as the Baer–Nunziato model. Equipped with these properties, it is possible to pursue our study of the Baer–Nunziato models towards the Cauchy problem, which will be the subject of forthcoming works.

Actually we will discuss these two properties, not restricting to the exact Baer-

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Nunziato model, and we will consider a larger framework of two-phase flow models, introduced in [7], that contains the former model [1]. Note that such properties have been investigated in [6] for a subclass of the systems we are interested in.

2. The two-velocity two-pressure models

In the models we study, each phase is described by its own density, velocity and pressure (or equivalently any other thermodynamical variable, such as the temperature). They are obtained after averaging Euler equations or Navier–Stokes equations and this averaging process introduces the notion of void fraction $\alpha(t,x)$ which represents the probability of presence (in the case of statistical average) of phase 1 at time t in position x. The resulting system is composed by a transport equation for the void fraction α with velocity v_i, two equations of mass, momentum and total energy, and momentum and total energy equations include nonconservative terms due to the variations of the void fraction α. The main alternative class of models is based on the assumption of the equality of the two pressures, which replaces the transport equation on α. The main inconvenience is that these models are conditionally hyperbolic and their eigen-structure is very difficult to compute. As it is recalled in Proposition 2.1, the structure of the two-pressure models is more appealing, see for instance [7], and our aim is to go further into the understanding of the structure of these models.

The first-order part of the models we focus on can be written under the form

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) + \mathbf{c}(\mathbf{u})\partial_x \alpha = 0 \tag{2.1}$$

where

$$\mathbf{u}^T = (\alpha, \alpha \mathbf{u}_1^T, (1-\alpha)\mathbf{u}_2^T), \qquad \mathbf{f}(\mathbf{u})^T = (0, \alpha \mathbf{f}_1(\mathbf{u}_1)^T, (1-\alpha)\mathbf{f}_2(\mathbf{u}_2)^T),$$
$$\mathbf{u}_k^T = (\rho_k, \rho_k u_k, \rho_k E_k), \qquad \mathbf{f}_k(\mathbf{u}_k)^T = (\rho_k u_k, \rho_k u_k^2/2 + p_k, u_k(\rho_k E_k + p_k)),$$
$$\mathbf{c}(\mathbf{u})^T = (v_i, 0, -p_i, -v_i p_i, 0, p_i, v_i p_i),$$

with $k = 1, 2$. The notations are classical: α is the void fraction of phase 1 (and $1-\alpha$ the void fraction of phase 2), ρ_k the density, u_k the velocity, p_k the pressure and E_k the specific total energy of the phase k, with $k = 1, 2$. Besides, v_i and p_i, usually called the interfacial velocity and the interfacial pressure, are given functions of $\mathbf{u}$. The total energies are defined by

$$E_k = \varepsilon_k + \frac{u_k^2}{2},$$

where ε_k denotes the specific internal energy of the phase k. We assume in the sequel that each phase admits an entropy s_k, complying with

$$T_k ds_k = d\varepsilon_k + p_k d\tau_k, \tag{2.2}$$

noting T_k the temperature and $\tau_k = 1/\rho_k$ the specific volume of the phase k. The knowledge of these entropies enables us to deduce the temperature and the pressure of each phase:

$$\frac{\partial s_k}{\partial \tau_k}(\tau_k, \varepsilon_k) = \frac{p_k}{T_k}(\tau_k, \varepsilon_k) \quad \text{and} \quad \frac{\partial s_k}{\partial \varepsilon_k}(\tau_k, \varepsilon_k) = \frac{1}{T_k}(\tau_k, \varepsilon_k).$$

Besides, we assume that *each entropy s_k is a strictly concave function of τ_k and ε_k*. Moreover, one can define the sound speeds c_k by

$$\rho_k(c_k)^2 = \left(\frac{p_k}{\rho_k} - \rho_k(\partial_{\rho_k}\varepsilon_k)(\rho_k, p_k)\right)\left((\partial_{p_k}\varepsilon_k)(\rho_k, p_k)\right)^{-1}.$$

Let us recall the hyperbolicity property of system (2.1) for solutions in the set of admissible states

$$\Omega = \{\mathbf{u} \in \mathbb{R}^7; \alpha \in (0,1), \rho_k > 0, \varepsilon_k > 0, k = 1,2\}. \qquad (2.3)$$

PROPOSITION 2.1. *System* (2.1) *admits seven real eigenvalues on* Ω*:* v_i*,* u_k *and* $u_k \pm c_k$*,* $k = 1,2$*. The corresponding eigenvectors form a basis of* $\mathbb{R}^7$ *as soon as*

$$(u_k - v_i)^2 \neq (c_k)^2, \quad k = 1,2. \qquad (2.4)$$

This is called the non-resonance condition.

3. Non strict convexity of the mixture entropy

Let us introduce the entropy pair (S_k, F_k) defined by

$$S_k(\mathbf{u}_k) = -\rho_k s_k \quad \text{and} \quad F_k'(\mathbf{u}_k) = (S_k'(\mathbf{u}_k))^T \mathbf{f}_k'(\mathbf{u}_k), \qquad (3.1)$$

associated with the Euler systems $\partial_t \mathbf{u}_k + \partial_x \mathbf{f}_k(\mathbf{u}_k) = 0$.

It is classical to define the entropy of the two-phase model (2.1) by

DEFINITION 3.1. *The* mixture entropy *for system* (2.1) *is*

$$S(\mathbf{u}) = \alpha S_1(\mathbf{u}_1) + (1 - \alpha)S_2(\mathbf{u}_2), \qquad (3.2)$$

and the associated mixture entropy flux

$$F(\mathbf{u}) = \alpha F_1(\mathbf{u}_1) + (1 - \alpha)F_2(\mathbf{u}_2). \qquad (3.3)$$

These definitions are classical and may lead to a *conservative* entropy inequality under some assumptions on p_i, see [3, 7].

We state now the crucial property of the mixture entropy:

THEOREM 3.2. *The mixture entropy* S *is a convex, but not strictly convex, function of* $\mathbf{u}$*. Moreover, for any* $\mathbf{u} = (\alpha, \alpha\mathbf{u}_1, (1 - \alpha)\mathbf{u}_2) \in \Omega$*, the degeneracy manifold of* $S''(\mathbf{u})$ *is only due to variations with respect to the void fraction:*

$$\mathbf{v} \in \mathscr{D}(\mathbf{u}) = \{\mathbf{v} \in \Omega; \mathbf{v} = (\beta, \beta\mathbf{u}_1, (1 - \beta)\mathbf{u}_2)), \beta \in (0,1) \setminus \{\alpha\}\}$$
$$\iff S(\mathbf{v}) - S(\mathbf{u}) = S'(\mathbf{u})^T(\mathbf{v} - \mathbf{u}).$$

Proof. With a slight abuse of notation, let us rewrite the mixture entropy S as a function of $(\alpha, \alpha\mathbf{u}_1, (1 - \alpha)\mathbf{u}_2)$:

$$S(\alpha, \alpha\mathbf{u}_1, (1 - \alpha)\mathbf{u}_2) = \alpha S_1\left(\frac{\alpha\mathbf{u}_1}{\alpha}\right) + (1 - \alpha)S_2\left(\frac{(1 - \alpha)\mathbf{u}_2}{1 - \alpha}\right).$$

Then, the Hessian matrix of S with respect to $(\alpha, \alpha u_1, (1 - \alpha)u_2)$ has the form

$$S''(\mathbf{u}) = \begin{pmatrix} A & B^T & C^T \\ B & \frac{1}{\alpha}S_1''(\mathbf{u}_1) & 0 \\ C & 0 & \frac{1}{1-\alpha}S_2''(\mathbf{u}_2) \end{pmatrix}$$

with

$$A = \frac{1}{\alpha}\mathbf{u}_1^T S_1''(\mathbf{u}_1)\mathbf{u}_1 + \frac{1}{1-\alpha}\mathbf{u}_2^T S_2''(\mathbf{u}_2)\mathbf{u}_2,$$

$$B = -\frac{1}{\alpha}S_1''(\mathbf{u}_1)\mathbf{u}_1 \quad \text{and} \quad C = \frac{1}{1-\alpha}S_2''(\mathbf{u}_2)\mathbf{u}_2.$$

3

Let (a, b^T, c^T) be a non-null vector of $\mathbb{R}^7$. Let us check that the Hessian S'' is positive as soon as S_1'' and S_2'' are positive. We have

$$(a, b^T, c^T)\; S''(\mathbf{u}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a^2 A + a B^T b + a C^T c$$
$$+ \, ab^T B + \frac{1}{\alpha} b^T S_1''(\mathbf{u}_1) b + a c^T C + \frac{1}{1-\alpha} c^T S_2''(\mathbf{u}_2) c.$$

Using the definitions of A, B and C we obtain

$$(a, b^T, c^T)\; S''(\mathbf{u}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \frac{1}{\alpha}(b - a\mathbf{u}_1)^T S_1''(\mathbf{u}_1)(b - a\mathbf{u}_1)$$
$$+ \frac{1}{1-\alpha}(c + a\mathbf{u}_2)^T S_2''(\mathbf{u}_2)(c + a\mathbf{u}_2).$$

This right-hand side is clearly non-negative since S_1 and S_2 are strictly convex, which proves the non strict convexity of S. The case of degeneracy of $S''(\mathbf{u})$ corresponds to

$$(a, b^T, c^T)\; S''(\mathbf{u}) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \iff (a, b, c) = a(1, \mathbf{u}_1, -\mathbf{u}_2). \tag{3.4}$$

In other words, from a given state $\mathbf{u} = (\alpha, \alpha\mathbf{u}_1, (1 - \alpha)\mathbf{u}_2) \in \Omega$, the degeneracy manifold of $S''(\mathbf{u})$ is given by the set $\{\mathbf{u} + (a, a\mathbf{u}_1, -a\mathbf{u}_2), a \in \mathbb{R} \setminus \{0\}\} \cap \Omega$. Now, introducing $\mathbf{w} = (a, a\mathbf{u}_1, -a\mathbf{u}_2)$, let us remark that $\mathbf{u} + \mathbf{w} = (\alpha + a, (\alpha + a)\mathbf{u}_1, (1 - \alpha - a)\mathbf{u}_2)$ and setting $\beta = \alpha + a$, we recover the set $\mathscr{D}(\mathbf{u})$. $\quad\square$

4. The system is symmetrizable out of resonance

DEFINITION 4.1. *The system (2.1) is said to be* symmetrizable *if there exists a $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}^7$ to $\mathbb{R}^7$ $\varphi\colon \mathbf{u} \mapsto \mathbf{y}$, a symmetric positive definite matrix $P(\mathbf{y}) \in \mathbb{R}^{7\times 7}$ and a symmetric matrix $Q(\mathbf{y}) \in \mathbb{R}^{7\times 7}$ such that the smooth solutions of system (2.1) satisfy*

$$P(\mathbf{y})\partial_t \mathbf{y} + Q(\mathbf{y})\partial_x \mathbf{y} = 0. \tag{4.1}$$

THEOREM 4.2. *System (2.1) is symmetrizable if and only if the non-resonance condition (2.4) holds.*

Proof. Let us define $\mathbf{y} = \varphi(\mathbf{u}) := (\alpha_2, u_2, p_2, s_2, u_1, p_1, s_1)^t$ and introduce the partial masses $m_k = \alpha_k \rho_k$. One may check by classical manipulations that the smooth solutions of system (2.1) satisfy

$$\partial_t \mathbf{y} + M(\mathbf{y})\partial_x \mathbf{y} = 0$$

where

$$M = \begin{pmatrix} v_i & 0 & 0 \\ M_{2\alpha} & M_2 & 0 \\ M_{1\alpha} & 0 & M_1 \end{pmatrix}, \quad M_k = \begin{pmatrix} u_k & \tau_k & 0 \\ \rho_k(c_k)^2 & u_k & 0 \\ 0 & 0 & u_k \end{pmatrix}$$

and

$$M_{k\alpha}^T = \left((-1)^k \frac{p_k - p_i}{m_k}, M_{k\alpha}^{(2)}, M_{k\alpha}^{(3)} \right),$$

$$M_{k\alpha}^{(2)} = (-1)^k \frac{u_k - v_i}{\alpha_k} \left(\rho_k (c_k)^2 + \frac{p_i - p_k}{\rho_k} (\partial_{p_k} \varepsilon_k)^{-1} \right),$$

$$M_{k\alpha}^{(3)} = (-1)^k (u_k - v_i) \frac{p_i - p_k}{m_k T_k}.$$

The proof we provide here is constructive for the sake of understanding. We seek for a matrix of symmetrization $P(\mathbf{y})$ of the form

$$P = \begin{pmatrix} P_\alpha & P_{2\alpha}^T & P_{1\alpha}^T \\ P_{2\alpha} & P_2 & 0 \\ P_{1\alpha} & 0 & P_1 \end{pmatrix}, \quad P_k = \begin{pmatrix} (\rho_k c_k)^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the associated symmetric convection matrix is $Q(\mathbf{y}) = P(\mathbf{y}) M(\mathbf{y})$. We have to find P such that it is positive definite and that Q is symmetric. Let us first focus on this latter condition. We have

$$Q = \begin{pmatrix} P_\alpha v_i + P_{2\alpha}^T M_{2\alpha} + P_{1\alpha}^T M_{1\alpha} & P_{2\alpha}^T M_2 & P_{1\alpha}^T M_1 \\ v_i P_{2\alpha} + P_2 M_{2\alpha} & P_2 M_2 & 0 \\ v_i P_{1\alpha} + P_1 M_{1\alpha} & 0 & P_1 M_1 \end{pmatrix}.$$

This matrix is symmetric if and only if we have for $k = 1, 2$

$$(M_k^T - v_i \mathbb{I}) P_{k\alpha} = P_k M_{k\alpha}, \tag{4.2}$$

where $\mathbb{I}$ is the 3×3 identity matrix. Assume now that

$$\delta_k := (u_k - v_i)^2 - (c_k)^2 \neq 0,$$

i.e. inequality (2.4) holds. As a consequence, the first two equations of system (4.2) can be solved:

$$\begin{pmatrix} P_{k\alpha}^{(1)} \\ P_{k\alpha}^{(2)} \end{pmatrix} = \frac{1}{\delta_k} \begin{pmatrix} u_k - v_i & -\rho_k (c_k)^2 \\ -\tau_k & u_k - v_i \end{pmatrix} \begin{pmatrix} (-1)^k \rho_k (c_k)^2 \frac{p_k - p_i}{\alpha_k} \\ M_{k\alpha}^{(2)} \end{pmatrix}. \tag{4.3}$$

It remains to solve the third equation of system (4.2), which writes

$$(u_k - v_i) P_{k\alpha}^{(3)} = (-1)^k (u_k - v_i) \frac{p_i - p_k}{m_k T_k}. \tag{4.4}$$

Clearly, this equation admits a unique solution if $u_k \neq v_i$ and infinitely many solutions if $u_k = v_i$. Therefore, if $P_{k\alpha}$ is defined by (4.3) and (4.4), the matrices P and Q are symmetric.

Let us now check that P is a positive definite matrix, that is to say for all non-null vector (a, b_2^T, b_1^T) of $\mathbb{R}^7$, we have

$$(a, b_2^T, b_1^T) \, P \begin{pmatrix} a \\ b_2 \\ b_1 \end{pmatrix} = a^2 P_\alpha + 2a(P_{2\alpha}^T b_2 + P_{1\alpha}^T b_1) + b_2^T P_2 b_2 + b_1^T P_1 b_1 > 0.$$

This is a polynomial of degree 2 in a and its discriminant is

$$\Delta = 4\Big[|P_{2\alpha}^T b_2 + P_{1\alpha}^T b_1|^2 - P_\alpha(b_2^T P_2 b_2 + b_1^T P_1 b_1)\Big]$$

$$= 4\Big[|(P_2^{-1/2}P_{2\alpha})^T \bar b_2 + (P_1^{-1/2}P_{1\alpha})^T \bar b_1|^2 - P_\alpha(|\bar b_2|^2 + |\bar b_1|^2)\Big]$$

$$= 4\Big[(|P_2^{-1/2}P_{2\alpha}|^2 + |P_1^{-1/2}P_{1\alpha}|^2 - P_\alpha)(|\bar b_2|^2 + |\bar b_1|^2)$$

$$- \big|(P_2^{-1/2}P_{2\alpha})^T \bar b_1 - (P_1^{-1/2}P_{1\alpha})^T \bar b_2\big|^2\Big]$$

where $\bar b_k = P_k^{1/2} b_k$ (as usual, $P_k^{1/2}$ is the symmetric positive definite matrix such that $P_k^{1/2}P_k^{1/2} = P_k$ and $P_k^{-1/2}$ is its inverse). The discriminant Δ is negative if

$$P_\alpha > |P_2^{-1/2}P_{2\alpha}|^2 + |P_1^{-1/2}P_{1\alpha}|^2, \tag{4.5}$$

which is realizable under the condition of non-resonance (2.4). □

5. Consequences and further works

First of all, it is worth noting that Theorems 3.2 and 4.2 have been obtained for system (2.1) without any assumption on the definitions of v_i and p_i. Moreover, these results have been obtained for any admissible equations of state within each phase. It is then straightforward to extend these results to similar models (such as the barotropic Baer-Nunziato model, or the extended model introduced in [4]) and it also seems to be a reasonable assumption for the model discussed in [8]. Moreover, we have restricted ourselves to the one-dimensional case, but since this model is invariant under frame rotation (under some natural conditions on v_i and p_i), these properties are still verified in the multidimensional setting.

It is well-known for hyperbolic symmetric systems that the L^∞ norm of the spatial derivative of the solution may blow up in finite time. Here, the symmetric form (4.1) is admissible far from resonance. As a consequence, starting from a non-resonant solution, there exists a local-in-time smooth solution to the Cauchy problem. The final time corresponds to the smaller time for which the L^∞ norm of the spatial derivative of the solution blows up or for which the solution becomes resonant. This is a direct application of Kato's theorem [14]. Let us mention that the resonance phenomenon prevents us from proving a well-posedness result in a weaker setting (as entropy weak solutions with small total variation), since the Riemann problem is known to admit up to three solutions [13, 10].

Nonetheless, we must recall here that the full version of two-phase models also includes source terms which govern the trend of a flow to converge towards equilibrium: equality of pressures, velocities, temperatures and chemical potential. These respective equilibria correspond to mechanic, kinematic, thermal and thermodynamical equilibrium, and thus they tend to remove the occurrence of the resonance phenomenon, since condition (2.4) is expected to be satisfied when an equilibrium is not far from being reached. Such source terms are entropy-dissipative, that is to say that their contribution to the mixture entropy balance law is non-positive (see for instance [4]). As a consequence, one may wonder if they may help to obtain a global-in-time solution to the Cauchy problem, following [12] and [17]. Even if these two papers are dedicated to systems of conservation laws, the analysis relies on the use of the entropy (and equivalently in the conservation case, upon the symmetric form of the equations). Let us note that the crucial assumptions exhibited in [12] and [17] are the following: entropy dissipative source terms and Kawashima–Shizuta structure.

It is well-known that the aforementioned source terms are entropy dissipative with respect to entropy (3.1). As far as the Kawashima–Shizuta structure is concerned, the situation is not so clear and it is probable that all the relaxation source terms will be needed. Moreover, the degeneracy of the Hessian of the entropy has to be carefully handled to obtain the global-in-time result. This work is under investigation.

Eventually, these properties may also be used for computational purposes.

REFERENCES

[1] M. R. Baer and J. W. Nunziato. A two-phase mixture theory for the deflagration-to-detonation transition (DDT) in reactive granular materials. *Int. J. Multiphase Flow*, 12:861–889, 1986.

[2] J.B. Bdzil, R. Menikoff, S.F. Son, A.K. Kapila, and D.S. Stewart. Two-phase flow modeling of a DDT in granular materials: acritical examination of modeling issues with micro-inertia. *Physics of Fluids*, 11:378–402, 1999.

[3] F. Coquel, T. Gallouët, J.-M. Hérard, and N. Seguin. Closure laws for a two-fluid two-pressure model. *C. R. Math. Acad. Sci. Paris*, 334(10):927–932, 2002.

[4] F. Coquel, J.-M. Hérard, K. Saleh, and N. Seguin. A class of two-fluid two-phase flow models. In *42nd AIAA Fluid Dynamics Conference and Exhibit*, AIAA 2012-3356. American Institute of Aeronautics and Astronautics, http://www.aiaa.org/, 2012.

[5] P. Embid and M. Baer. Mathematical analysis of a two-phase continuum mixture theory. *Contin. Mech. Thermodyn.*, 4(4):279–312, 1992.

[6] A. Forestier and S. Gavrilyuk. Criterion of hyperbolicity for non-conservative quasilinear systems admitting a partially convex conservation law. *Math. Meth. Appl. Sci.*, 32:2148–2158, 2011.

[7] T. Gallouët, J.-M. Hérard, and N. Seguin. Numerical modeling of two-phase flows using the two-fluid two-pressure approach. *Math. Models Methods Appl. Sci.*, 14(5):663–700, 2004.

[8] S. Gavrilyuk and R. Saurel. Mathematical and numerical modelling of two phase compressible flows with micro-inertia. *J. Comp. Phys.*, 175(1):326–360, 2002.

[9] J. Glimm, D. Saltz, and D.H. Sharp. Two-phase flow modeling of a fluid mixing layer. *J. of Fluid Mechanics*, 378:119–143, 1999.

[10] P. Goatin and P. G. LeFloch. The Riemann problem for a class of resonant hyperbolic systems of balance laws. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(6):881–902, 2004.

[11] S. K. Godunov. An interesting class of quasi-linear systems. *Dokl. Akad. Nauk SSSR*, 139:521–523, 1961.

[12] B. Hanouzet and R. Natalini. Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Arch. Ration. Mech. Anal.*, 169(2):89–117, 2003.

[13] E. Isaacson and B. Temple. Convergence of the 2×2 Godunov method for a general resonant nonlinear balance law. *SIAM J. Appl. Math.*, 55(3):625–640, 1995.

[14] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58(3):181–205, 1975.

[15] M. S. Mock. Systems of conservation laws of mixed type. *J. Differential Equations*, 37(1):70–88, 1980.

[16] V. Ransom, and D.L. Hicks. Hyperbolic two-pressure models for two-phase flows. *J. Comp. Phys.*, 53:124–xx, 1984.

[17] W.-A. Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Ration. Mech. Anal.*, 172(2):247–266, 2004.