Line Localization From Single catadioptric Image

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Abstract Indoor environments often contain several line segments. The 3D reconstruction of such environments can thus be reduced to the localization of lines in the 3D space. Multi-view reconstruction requires the solution of the correspondence problem. The use of a single image to localize space lines is attractive, since the correspondence problem can be avoided. However, using a central camera the line localization from single image is an ill-posed problem, because there are infinitely many lines sharing the same image.

In this work we relaxed the constraint on single viewpoint imaging and considered a wide class of non-central catadioptric cameras, constituted by an axial symmetric mirror and a perspective camera placed at a generic relative position. In the paper we report the results of our study on line localization for such cameras, reporting the conditions that allow a line to be localized from a single image. We show how the analysis can be extended to other classes of non-central devices sharing a similar imaging model. We also present a brief overview of the main algorithms for line localization from single image that have been proposed.

Keywords Non-central catadioptric cameras · line localization · reconstruction from single image

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1 Introduction

Line localization from a single image is an ill-posed problem, as the classical central projection model introduces an ambiguity: the *viewing surface*, *i.e.* the union set of the projection rays associated to the line image is a planar surface that contains infinitely many lines crossing all the viewing rays. Thus, all these lines share the same image. The ambiguity can be solved by exploiting additional information of the scene [28], planar [45] or rigidity [52] constraints, or by employing stereopsis, which, on the other hand, introduces a non-trivial correspondence problem.

The problem of line localization from the projection rays of a single image can be seen as a particular case of a more general problem of computational geometry: given a collection of lines in space, find the line(s) that crosses them all. This is a widely studied problem that can find application, *e.g.*, in computer graphics for ray tracing and visibility computation [50]. It is an established result [7, 31] that, given a generic set of lines in space, there at most two other lines crossing them all, unless the set of lines lies on degenerate surfaces such as a plane: in this case there are infinitely many lines crossing the given lines.

As a consequence, in order to localize a line from single images, we have to employ optical systems whose image formation is not a central projection or, equivalently, whose projection rays do not meet at the same point. Grossberg and Nayar in [26] proposed a general and theoretical imaging model in which the camera is modeled as a set of pixels that capture the light travelling along rays in 3D. The camera is thus fully described by the mapping between each pixel and the corresponding 3D ray, expressed in any suitable reference frame. The mapping identifies the nature (*i.e.*

the image formation model) of the device and in the most general case it can be completely unconstrained, being simply a look-up table between each pixel and its associated 3D direction in space. Different formalisms have been proposed in order to characterize the various classes of cameras [39, 42, 46, 54]. Such general models can describe many special imaging devices recently introduced in the computer vision community, such as camera clusters [41,48], catadioptric cameras [3,29,38], oblique cameras [39], and other special acquisition devices such as rotating cameras [33,40,44], cross-slit cameras [16] or the so-called *compound cameras* [17,18] that emulate insect eyes. The standard perspective camera is then just a particular case in which all rays are constrained to meet at a common point in space, the camera viewpoint.

In this work we focused our attention on a particular subset of devices, the so called non-central catadioptric cameras: these devices are constituted by a standard perspective camera placed in front of a curved, axially symmetric mirror. Such cameras have become popular in robotic vision and video-surveillance applications as the mirror extends the field of view of the camera, providing a 360° view of the scene. Since the projection rays are reflected by the mirror, the image formation model is altered so that, in general, the mapping between pixels and rays cannot be described by a central projection. In order to be able to localize a line from a single image, we needed to study the geometrical properties of the rays associated to the image of a line. In particular we studied the configuration of rays that may prevent localization. We showed that there are only two different types of such configurations: the rays may lie on a plane, as in the perspective case, or on a quadric ruled surface, which contains a one-dimensional set of other lines crossing them all. We also stated some sufficient and general conditions for line localization that hold for a wide range of devices.

The aim of this paper is to collect and summarize all of the major results of this study with a more geometrical insight and a more intuitive understanding. We will also discuss some practical implications of our study concerning robotic applications and some possible extensions of the work to other non-central cameras. For the sake of compactness and readability, we will refer the motivated reader to the relevant references for a more detailed treatment and for all proofs.

The main contributions of the paper are:

- We present our study on the critical configurations preventing line localization from single image in non-central cameras, giving an intuitive geometrical insight of the results.
- We show how these results can be extended to other devices that are commonly used for omnidirectional imaging, such as rotating cameras and catadioptric cameras based on multi-part mirrors.
- We review the main algorithms for line localization from projection rays that have been developed so far.
- We introduce some guidelines for extending the localization problem from single images to other geometric entities (*e.g.* circles) and/or other non-central devices.

The paper is organized as follows. In Section 2 we recall some preliminary notions and basic properties regarding catadioptric cameras. In Section 3 we study the degenerate configurations that may occur in catadioptric cameras and we present sufficient conditions for line localization. We discuss and extend these results to other non-central devices in Section 4, while in Section 5 we briefly review some methods for line localization from single image that have been proposed. Section 6 concludes the paper with some final remarks and possible extensions of the work.

2 Preliminary Notions

In this section we briefly review the state of the art of catadioptric cameras and then we introduce the two geometrical entities that are the core of our study, the planar viewing surfaces and the ruled (quadric) viewing surfaces.

2.1 Catadioptric cameras

A catadioptric camera is obtained by placing a mirror between the scene and a standard perspective camera (see Figure 1). Thus, light coming from a scene point $\mathbf{P}$ is reflected by the mirror at a reflection point $\mathbf{B}$, before it goes through the camera viewpoint $\mathbf{O}$ and crosses the image plane at point $\mathbf{P}'$. We use the concept of viewing ray to indicate the straight path followed by the light coming from $\mathbf{P}$ before it is reflected by the mirror surface:

Definition 1 (Viewing Ray) A *viewing ray* through a scene point $\mathbf{P}$ is the semi-line through $\mathbf{P}$, ending at its reflection point $\mathbf{B}$ on the mirror surface.

Note that, according to the general camera model [26], the catadioptric camera is fully described by the mapping between each pixel of the perspective camera and the corresponding viewing ray.

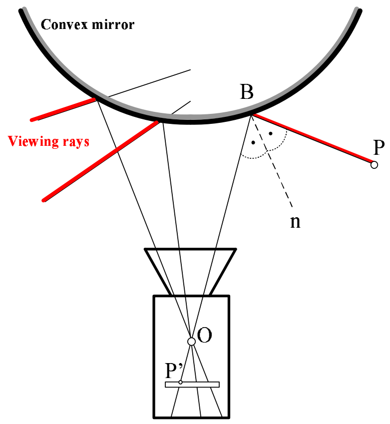


Fig. 1 In a catadioptric camera, light coming from a scene point **P** is reflected by the mirror at a reflection point **B**, before it goes through the camera viewpoint **O** and crosses the image plane at point **P′**. The semi-line through **P** and ending at the reflection point **B** is called viewing ray.

A scene point is *visible* if there is a viewing ray through it, *i.e.* if there is a light ray from the scene point that is reflected by the mirror and passes through the camera viewpoint. Thus, the 3D visible points are all those points that are not occluded by the mirror from the viewpoint **O**. As a consequence, a point **B** on the mirror surface is *visible* if it is a reflection point, *i.e.* if there is a straight line segment directly connecting it to **O**, and the normal surface is *defined* at **B**. The surface normal is *defined* at a point **B** if the surface is differentiable at **B**: *e.g.*, the surface normal is not defined at the apex of a conical surface.

There are two classes of catadioptric cameras: central and non-central. Central catadioptric cameras preserve the single viewpoint constraint: the most common example is a camera facing a planar mirror, where all the viewing rays meet at the virtual viewpoint behind the mirror. Baker and Nayar [2] derived the complete set of central catadioptric cameras preserving the single viewpoint constraint: they can be obtained by placing the camera viewpoint in one of the foci of a mirror based on a quadric of revolution. The most common ones are the *para-catadioptric* cameras, constituted of an orthographic camera placed in front of a paraboloidal mirror [4, 24], and the *hyper-catadioptric* cameras, constituted of a standard perspective camera placed in front of an hyperboloidal mirror [43]. The latter may not be easy to manufacture as it requires a precise alignment of the camera viewpoint, which is also difficult to check. Since they preserve a central projection, central catadioptric cameras cannot be used for single image localization of lines.

Non-central catadioptric cameras [3], on the other hand, are a more general class of cameras whose viewing rays do not meet at the same point but are in general skew: they rather form a locus of viewpoints that can be modeled using *caustics* [47]. This feature makes them suitable for single image line localization.

This class of cameras allows a larger degree of freedom in designing the device, both for the shape of the mirror and the position of the camera, as it can be placed in a unconstrained position w.r.t. the mirror. Thus, exotic devices can be designed with different mirror shapes in order to achieve specific vision tasks. For example non-central catadioptric cameras can be obtained using spherical [5, 32, 35] or conical mirrors [13, 53, 55], which are easy to set up. Other non-central catadioptric systems have been developed by employing ad-hoc mirrors in order to achieve specific features in the resulting image, such as preserving ratios of elevations of points from a ground plane [14], rectifying planes perpendicular to the optical axis [30] or multi-part mirrors that allow different areas of the ground to be monitored [37].

In our work we restricted our attention to non-central catadioptric cameras based on *axial symmetric mirrors* with *convex* profiles. Such mirrors are surfaces of revolution obtained by rotating a planar monotonic curve, the "profile", about the symmetry axis; they are easy and relatively cheap to manufacture with common lathes. The convex mirror allows the field of view of the camera to be extended, so that an ominidirectional image can be produced, which is the main advantage for a catadioptric camera.

Moreover, the mirror convexity guarantees that the camera is "single-image", *i.e.* for any visible 3D point **P** there is only a single corresponding image point **P′**. Indeed, a concave mirror may reflect a point multiple times depending on the relative position of the point and the camera: the light emitted by a 3D point is reflected into converging rays and if the camera is placed where (at least) two of these reflected rays meet, then the camera captures two rays for the same 3D point, thus producing a double image. On the contrary, convex mirrors reflect the light emitted by a 3D point into diverging rays, which can only intersect in the region "behind" the mirror: hence, if the camera is placed anywhere in front of the mirror, it can capture only one reflected ray for any visible point [6].

In order to ease our analysis, we conveniently classify catadioptric cameras in two different types according to their geometrical properties:

– *Axial cameras*, composed of an axial-symmetric mirror plus a perspective camera whose viewpoint is placed along the mirror axis.

– *Off-axis cameras*, composed of an axial-symmetric mirror and a perspective camera placed at generic relative positions.

Note that this classification is based only on the position of the camera viewpoint w.r.t. the mirror, while the orientation of the camera is unconstrained[1]: in axial cameras it is not necessary that the optical axis of the camera is aligned with the mirror axis.

In the next sections we introduce the geometric entities that are the main objects of our analysis.

2.2 Planar viewing surfaces

We have introduced, in Section 1, the notion of a *viewing surface* as the union set of viewing rays associated to the line image. In order to study the conditions preventing line localization, we need to consider all those viewing surfaces that contains infinitely many lines crossing all the viewing rays. The most obvious one is the plane.

Definition 2 (Planar Viewing Surface) Given a catadioptric camera, a *planar viewing surface* (PVS) is the subset of visible points of a plane π whose viewing rays are fully contained in π.

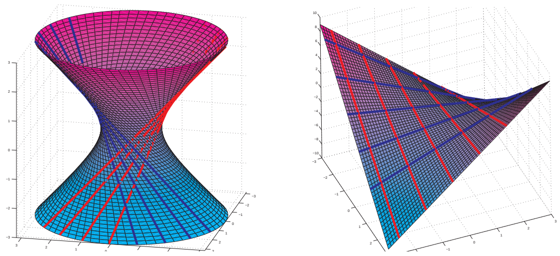
The plane π is called the *supporting plane* of the PVS. All of the lines contained in a PVS share the same image curve: therefore such line cannot be localized from their images. Moreover, the intersection between a PVS and the mirror surface is a planar curve.

Observation 1 *Notice that a PVS within a plane π may omit some visible points on π: these points may be crossed by viewing rays not contained in π.*

A planar viewing surface $\mathcal{P} \subseteq \pi$ constituted by all the visible points of π, is called a *complete* planar viewing surface. In a complete PVS, any visible point of its supporting plane π is crossed by a viewing ray contained in π.

Planar viewing surfaces are important entities, in that they prevent line localization. On the other hand they are difficult to study, since they are "local" entities. The key idea that allows us to simplify the study of planar viewing surfaces is the adoption of a continuity hypothesis: exploiting C^∞ continuity of the mirror surface, we relate local properties to global properties, most of which can be studied by elementary geometry. The main mechanism that relates local aspects to global ones is the Taylor expansion: the value of a C^∞ function at any point can be determined by the value of the function, together with the value of all its derivatives, at a given point.

[1] The only (obvious) constraint is that the mirror be (at least partially) in the field of view of the perspective camera.



(a) Hyperboloid of one sheet (b) Hyperbolic paraboloid

Fig. 2 Doubly ruled quadrics are surfaces composed by two families of lines so that each line of one family (*e.g.* the blue ones) crosses all the lines of the other family (*e.g.* the red ones) and the lines of the same family are mutually skew. If the viewing rays lie on one of the family, then there are infinitely many lines crossing them and there is no unique solution to the line localization problem.

2.3 Ruled quadrics viewing surfaces

The plane is just a particular case of a class of surfaces containing lines: the ruled surfaces.

Definition 3 (Ruled Surface) A surface S is *ruled* if through every point of S there is a straight line that lies on S [31].

Beside the plane, the most common ruled surfaces are the cylinder and the cone. In general, a ruled surface can be obtained by sweeping a line in space: *e.g.* the cylinder can be obtained by rotating a line about an axis parallel to the line itself.

We are interested in a particular sub-class of ruled surfaces containing two different sets of intersecting lines.

Definition 4 (Doubly Ruled Surface) A ruled surface S is *doubly-ruled* if through every one of its points there are two distinct lines that lie on S [31].

Again, the plane is a degenerate doubly ruled surface since for any of its points there are infinitely many lines passing through it. There are only two non-degenerate doubly ruled surfaces, the hyperbolic paraboloid and the hyperboloid of one sheet (see Figure 2), both of which are quadric surfaces. For these the following property can be proved:

Property 1 A non-degenerate doubly ruled surface contains two distinct families of straight lines, such that each line of one family crosses (only once) all the lines of the other family, while any two lines of the same family are either identical or skew [34] (see Figure 2).

Property 2 Any three skew lines define a unique doubly ruled quadric, either a hyperboloid of one sheet or a hyperbolic paraboloid [31].

In particular, three pairwise skew lines always define a hyperboloid of one-sheet, except in the case where they are all parallel to a single plane but not to each other. In this case, they determine a hyperbolic paraboloid [31].

If a viewing surface is a doubly ruled surface, *i.e.* the viewing rays constitute one of the two families of lines, then line localization is prevented: all the other lines belonging to the other family of lines cross the viewing rays, thus sharing the same image. Therefore doubly ruled viewing surfaces are another type of surface that we need to take into consideration.

Definition 5 (Ruled Quadric Viewing Surface) Given a catadioptric camera, a *(doubly) ruled quadric viewing surface* (RQVS) is the subset of visible points of a doubly ruled quadric S whose viewing rays are fully contained in S.

According to Property 2, if we consider the viewing surface of a line L which is neither a PVS nor a RQVS, then any three skew viewing rays of the surface define a doubly ruled quadric S, with L part of the the second family of lines crossing all the three viewing rays. Since we are supposing that the viewing surface is not a RQVS, there is at least one other viewing ray that does not belong to the ruled quadric S. This viewing ray intersects S in two points, which may be identical in the case that it is tangent to S[2]. For each of these intersection points on S there are two lines passing through it and fully contained in S: one belongs to the same family as the three skew viewing rays generating S, whereas the other belongs to the other family and crosses the three viewing rays (see Property 1). Hence, depending on the number of intersection points, there can be one or at most two lines crossing the four viewing rays, one of which is L. Repeating the same reasoning for the other viewing rays, it appears that if the viewing surface associated to a line is neither a PVS nor a RQVS, then there are at most two distinct lines crossing all the viewing rays. Hence PVS and RQVS are the only surfaces preventing line localization that we must take into consideration.

Moreover, the previous argument is generalized by a classical theorem of geometry.

Result 1 *[31] [7, Theorem 1] Given a set of $n > 3$ arbitrary lines, the relevant set of line crossing them all consists of either at most two lines or else of infinitely many lines.*

[2] Since the considered viewing ray must intersect L, it cannot be part of the second family of lines of S because those lines are mutually skew (Property 1). Hence it must be a line not lying on S and intersecting it in one or two points: indeed a line can intersect a quadric in at most two points [31].

As a consequence, given the viewing rays associated to a line in space, the line localization will be *univocal* if there is only one line crossing them all, otherwise we need to disambiguate between two different solutions. We will see in the next section how the geometry of the catadioptric device helps to determine the actual line. Finally, we will see in Section 5 how this theorem can be exploited for line localization from a single image.

3 Line Localization from single images

In this section we report our main results, trying to give the reader an insight into the results rather than a complete proof of the results. We refer the interested reader to [9, 10, 22] for a more detailed treatment and for all the proofs.

In the following we consider the two degenerate configurations that may occur in non-central catadioptric cameras, PVS and RQVS. For each of them we report the conditions under which they prevent line localization in the two types of cameras we are considering (see Section 2.1), the axial cameras and the off-axis cameras.

We recall the main assumptions we are making:

(i) The mirror surface is a convex surface of revolution;
(ii) The mirror surface is infinitely differentiable C^∞ everywhere (unless otherwise specified).
(iii) The camera viewpoint $\mathbf{O}$ lies "below" the mirror apex[3].

These conditions are general and apply to wide spectrum of devices that are actually used in many applications. We will also see that in some cases it is possible to relax condition (ii) and extend some of the results to, *e.g.*, conical mirrors.

3.1 Planar Viewing Surfaces

3.1.1 Axial Cameras

The case of axial cameras was discussed in [9]. In axial cameras the viewpoint of the perspective camera is placed on the mirror axis. Thus, the whole system is axial-symmetric w.r.t. the mirror axis and any viewing

[3] Without loss of generality, if we fix the reference frame such that the z-axis coincides with the mirror axis and is directed towards the internal part of the mirror (*e.g.* Figure 3), then the lowest point of the convex mirror, the *apex*, is on the z-axis and the camera viewpoint $\mathbf{O}$ has a lower value of the z-coordinate than the apex. Vice-versa, if the z-axis points towards the outer part of the mirror, then the camera viewpoint must be "above" the apex, *i.e.* it must have a higher value of the z-coordinate than the apex.

ray is coplanar with the mirror axis. Also, the line supporting any viewing ray intersects the mirror axis: thus, the mirror axis is contained in any viewing surface and it is always a solution for the localization problem. In this case the two solutions for the line localization problem is univocal as the two solution of (Result 1) can be easily disambiguated.

From this simple geometrical intuition we can note that any line coplanar with the mirror axis cannot be localized from a single image. In fact all the viewing rays lie on the plane defined by the mirror axis and the line itself. We call "axial planes" the span of planes containing the mirror axis: they constitute a one-dimensional set of PVS for any axial camera. Figure 3 shows an example of axial planes for an axial camera.

There is also another planar viewing surface. A viewing ray is *horizontal* if it is perpendicular to the mirror axis. By symmetry the set of horizontal rays is contained in a *horizontal* plane, *i.e.* a plane perpendicular to the mirror axis (see Figure 4). This plane is a planar viewing surface. Given the assumption of convexity and monotonicity of the mirror, there is at most one such horizontal planar surface in an axial camera. Knowing the shape of the mirror and the position of the camera on its axis, it is straightforward to determine this plane by applying the law of reflection.

The following theorem summarizes these results and proves that in an axial-symmetric, non-central camera, there is only a one-dimensional set of planar viewing surfaces [9]:

Theorem 1 *In a axial camera based on a convex mirror there are no planar viewing surfaces other than the axial planes and the horizontal plane.*

3.1.2 Off-axis Cameras

The case of off-axis cameras was discussed in [10]. In off-axis cameras the camera viewpoint $\mathbf{O}$ is placed in a generic position w.r.t. the mirror, hence the resulting catadioptric system is not axial-symmetric. However, the plane through $\mathbf{O}$ and containing the mirror axis is a symmetry plane for the system (see Figure 5). This symmetry plane π_o is a PVS for the system and no line lying on this plane can be localized.

As in the case of axial cameras, we consider other possible PVSs generated by horizontal viewing rays. To this end, we consider the *horizontal curve* h, defined as the set of points on the mirror surface crossed by horizontal viewing rays. In axial cameras the horizontal curve is a planar curve, *i.e.* a circumference centered on the mirror axis (see Figure 4). In off-axis cameras, instead, it is not a planar curve. This means that, in

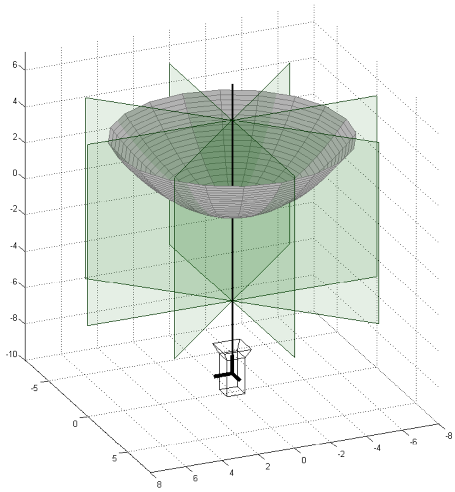


Fig. 3 In an axial camera there is a one-dimensional set of planar viewing surfaces, the axial planes containing the mirror axis (light green).

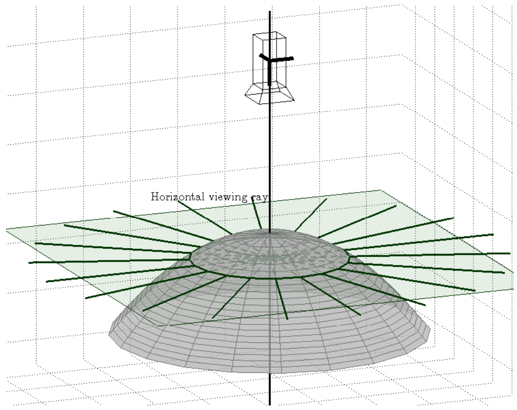


Fig. 4 In an axial camera there is (at most) one horizontal planar viewing surface (light green) composed of the set of horizontal viewing rays (dark green), *i.e.* the viewing rays that are perpendicular to the mirror axis. The intersection between the horizontal plane and the mirror surface is called the *horizontal curve* (circumference in dark green).

general, the horizontal viewing rays do not lie on the same plane but they are, in general, skew. Therefore, no PVS can have a horizontal supporting plane. Moreover, the following Lemma can be proved [10]:

Lemma 1 *Under conditions* (i)-(iii)*, any PVS through* $\mathbf{O}$ *must coincide with the symmetry plane* π_o*.*

A further result establishes three fundamental properties for any PVS not coincident with the symmetry plane π_o:

Lemma 2 *Under conditions* (i)-(iii)*, any PVS not coincident with the symmetry plane* π_o*:*

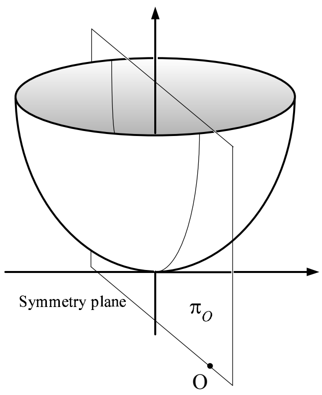


Fig. 5 The geometry of an off-axis catadioptric camera: the plane π_o through the viewpoint O that contains the mirror axis, is a symmetry plane for the catadioptric camera.

a. is perpendicular to π_o*;*
b. crosses the horizontal curve h *at least twice;*
c. contains two colinear horizontal viewing rays.

In other words, these results states that any PVS other than π_o is perpendicular to π_o and it intersects the horizontal curve in two points: the viewing rays associated to these points are horizontal and they lie on the same line, *i.e.* they are colinear. Moreover, due to the continuity and the convexity assumptions for the mirror, it can be proved that there is only one pair of colinear horizontal rays.

Since there is a unique pair of horizontal colinear rays, and any PVS $\neq \pi_o$ must contain the pair of rays, then the possible planar viewing surface may vary with at most one degree of freedom. This would lead to a one-dimensional set of planar viewing surfaces other than π_o, *i.e.* all the planes containing the the pair of rays and not passing through **O**. We proved that, instead, the possible planar viewing surfaces may vary within only a discrete set[4] [10].

Theorem 2 *Under conditions* (i)-(iii)*, in a off-axis camera based on a convex mirror there is only a discrete set of PVS other than the symmetry axis* π_o*.*

The actual number of PVSs depends on the particular shape of the mirror, as we showed in [10].

Notice that this result does not apply to an off-axis camera based on a conical mirror: since the mirror surface is discontinuous at the cone apex, there may exist an infinite number of PVSs going through the cone apex, and this is actually the case.

We extended this result also to multi-part mirrors [22].

[4] A discrete set is a set containing either a finite or a *countably infinite* number of elements: the set of integers $\mathbb{N}$, *e.g.*, is a discrete subset of the real numbers $\mathbb{R}$.

Corollary 1 *Let a mirror be constituted by (a finite number of) parts, each one symmetric about its own axis and satisfying the above conditions* (i)-(iii)*. If the boundary curve between any two neighbouring parts is non-planar, then there is only a discrete set of PVS.*

If we consider a multi-part mirror consisting of at least three parts having symmetry planes with no line in common, then there exists no common horizontal line for all the parts. Hence, there will be no PVSs for any camera based on such multipart mirror. Clearly, junctions between neighbouring C^∞ parts must be designed in such a way that the image of a line is continuous across junctions: *i.e.*, the mirror surface must be C^1 even at the junctions.

3.2 Ruled Quadric Viewing Surfaces

The study of the number of ruled quadrics in a generic catadioptric camera is more difficult than the study of planar viewing surfaces, due to the growth in the number of the involved parameters. Rather than determining the number of RQVS, we defined sufficient conditions for a line to be localized, *i.e.* sufficient conditions for a line not to form a RQVS.

3.2.1 Axial Cameras

The study of RQVS for axial cameras has been proposed in [9]. In the following the concept of *viewing cone* will be useful.

Definition 6 (Viewing cone) In an axial camera, the viewing rays having the same slope w.r.t. the mirror axis constitute a cone whose axis is the mirror axis, called a *viewing cone*.

According to the definition, the horizontal plane is a (degenerate) viewing cone.

In the case of axial cameras, two sufficient conditions for line localization from a single image can be stated. We refer the reader to [9] for the relevant proofs.

Theorem 3 *Under conditions* (i)-(iii)*, in an axial camera any line* L *not contained in a PVS and crossing the horizontal plane at a visible point can be localized univocally.*

The second condition for line localization is the following:

Theorem 4 *Under conditions* (i)-(iii)*, in an axial camera any line* L *intersecting a viewing cone at two distinct points can be localized univocally.*

As discussed in Section 3.1.1, the solution is univocal since one of the two solution is always the mirror axis.

According to the two theorems, a broad set of lines can be univocally localized. Let A be the set of lines that can be localized by Theorem 3 and B the set of lines that can be localized by Theorem 4, it can be shown that $A \cap B \neq \emptyset$: in fact a line that crosses a viewing cone at two distinct points can also cross the horizontal plane at a visible point.

On the other hand, there are lines parallel to the horizontal plane (but not contained in it) crossing a viewing cone at two distinct points. Hence this yields to $B \setminus A \neq \emptyset$.

Moreover, a line can cross the viewing cone only once and it also can cross the horizontal plane at visible point. This yields to $A \setminus B \neq \emptyset$.

3.2.2 Off-axis Cameras

The study of RQVS for off-axis cameras has been proposed in [10].

A first sufficient condition for the localization of a line is stated by the following theorem.

Theorem 5 *In an off-axis camera, under conditions* (i)-(iii)*, if the two extremal points of a line, not lying on a PVS, are both visible, then the line can be localized up to two distinct solutions.*

The two "extremal points" of a line correspond to the points with limiting abscissae $+\infty$ and $-\infty$. Although in projective geometry the two extremal points collapse into the same point, the viewing ray through these two points may be different (although parallel).

A further sufficient condition for line localization holds:

Theorem 6 *In an off-axis camera, under conditions* (i)-(iii) *above, any line L that crosses the symmetry plane π_o at a visible point $\mathbf{P}_o$ can be localized (up to two distinct solutions).*

In order to understand the meaning of these two results, we recall that the set of visible points is given by the set-difference between the whole 3D space and the part of space that is occluded by the mirror. Then, from Theorem 6 all the lines crossing the plane π_o in its visible part can be localized. This seems to exclude all the lines that are parallel to π_o. However, according to Theorem 5, if the extrema of a line are both visible, then the line can be still localized.

Notice that all the derived properties apply to real mirrors, whose surfaces contains (even a small) part of a surface satisfying the conditions (i)-(iii). In particular, the properties apply to multi-part mirrors, whose component parts are C^∞ continuous.

4 Discussions and extensions

In this section we briefly discuss the results for both types of cameras, and we show how the results can be extended to other devices.

One of the most interesting applications for catadioptric cameras is robotic vision, as they can provide an omnidirectional image of the whole environment in just one image-capture. This is a desirable feature in robotic vision as it can help navigation and self-localization tasks. In this kind of application, line localization can be a crucial task, especially in human-like environments[5], where lines constitute the most interesting features to exploit for mapping and self-localization (*e.g.* SLAM [51]). In such a context line localization from a single image can help to speed-up the algorithms without using, *e.g.*, stereopsis.

Axial cameras are attractive as they are easy to setup and they provide a radially symmetric image. However, our results showed that there are some restrictions on the number of lines that can actually be localized. According to Theorem 1, all the lines coplanar with the mirror axis cannot be localized as well as the lines lying on the horizontal plane: this prevents the localization of vertical lines (such as those belonging to doors or windows) or of a very small subset of horizontal lines. On the other hand, there is still a wide set of lines that can be localized according to Theorems 3 and 4, including all the lines parallel to but not lying on the horizontal plane: these lines are interesting because they are also frequent in human environments and their images can be easily detected with standard techniques [20] since they are arcs of circumference. These lines can be exploited for the localization of the vertical lines: for example, one can realistically assume that the robot is operating in a Manhattan world [21], *i.e.* in a world made of planar surfaces with three dominant directions, such as walls, floors and ceilings. Under this assumption, it is then relatively easy to cluster the localized lines into planes, and intersect these planes with the PVSs containing the vertical lines in order to localize them.

Off-axis cameras are easier to set-up since they do not require any accurate alignment of the perspective camera w.r.t. the mirror. From Theorem 2 we can note that there is a discrete set of PVSs preventing line localization. Such PVSs can be determined during the design of the device, knowing the parameters of the mirror surface and the position of the camera [10]; then the horizontal curve and the two colinear horizontal viewing rays can be computed and the PVSs determined. The mutual position between the perspective camera and

[5] Sometimes referred to also as *carpentered* environments [25].

the mirror is important in designing an off-axis camera since it determines the visible space of the camera and hence the lines that can be localized according to Theorems 5 and 6.

Off axis camera are interesting since they allow vertical lines to be localized, provided that they do not lie on the symmetry plane π_O. More generally, they guarantee a wider set of lines that can be localized, but they also produce images with a non uniform spatial resolution and a non uniform field of view: this represents a trade-off between the image quality and the number of lines that can be localized, and it has to be tailored to the final application requirements.

Besides robotic applications for mapping and self-localization, line localization from single images is interesting for recovering the shape of a specular surface [12]. Given a camera and an axial symmetric mirror of unknown profile, it is possible to use the images of lines for estimating the shape of the mirror and eventually calibrating the (perspective) camera: by imposing that the viewing rays must cross just one line in space and exploiting the axial symmetry of the mirror, the surface of the mirror can be estimated as well as the intrinsic parameters of the camera.

Finally, the conditions we derived can be extended to other imaging systems that do not satisfy conditions (i)-(iii). In particular, the case of catadioptric cameras based on conical mirrors can be of interest as these devices have many applications in mobile robotics thanks to their relative simplicity [13, 53]. In the following we briefly report some conditions for line localization from single images for catadioptric cameras based on mirrors.

4.1 Extension to conical mirrors

Catadioptric cameras based on conical mirrors do not satisfy condition (ii) since the apex of the cone is a point of discontinuity of the surface. Nevertheless we proved that some conditions for line localization can be derived as well.

Axial case. Due to the axial symmetry, the locus of viewpoints, *i.e.* the caustics, forms a circle centred on the mirror axis [2]. As for the PVSs, the same properties stated before hold for a conical mirror: the only PVSs are the span of axial planes and the horizontal plane. Moreover, we proved that there are no RQVSs, thus yielding to the following result [9, 22]:

Theorem 7 *In an axial symmetric catadioptric camera based on a conical mirror, if a line L is not contained in a PVS, then it can be univocally localized.*

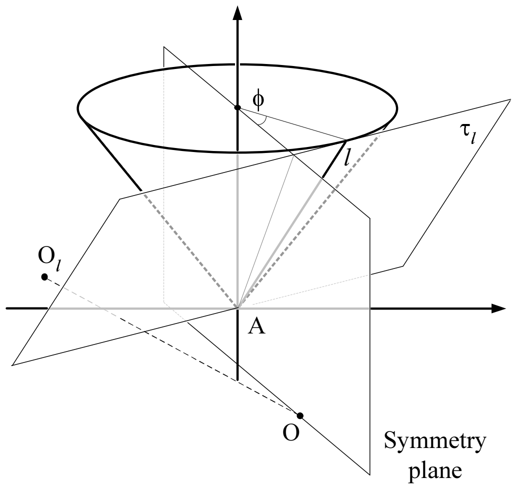


Fig. 6 The plane τ_l tangent to the cone at an axial line l.

Again, the line can be localized univocally since the mirror axis is always one of the two possible solutions.

Off-axis case. If the camera viewpoint is in a general position w.r.t. to the mirror, then again one PVS is the symmetry plane π_o containing the mirror axis and the camera viewpoint $\mathbf{O}$. Let $\mathbf{A}$ be the cone apex. An *axial line* l is a line through $\mathbf{A}$ contained in the cone surface: notice that an axial line is the intersection between an axial plane and the cone surface. For any axial line l there exists a tangent plane τ_l common to all the points on l (see Figure 6). The set of viewing rays through points of an axial line l identify a planar viewing surface π_l. In fact, since all the reflection points on l share the same tangent plane τ_l, they act as a single reflecting plane: the viewing surface thus describes a plane through l and the "reflected" viewpoint $\mathbf{O}_l$, where $\mathbf{O}_l$ and $\mathbf{O}$ are symmetric w.r.t. the tangent plane τ_l (see Figure 7). We proved the following results for PVS [8, 22]:

Theorem 8 *In an off-axis catadioptric camera based on a conical mirror, the only PVSs are the symmetry axis π_o and the span of planes π_l through $\mathbf{O}_l$ and the axial lines.*

Finally we proved the following condition for line localization [8, 22]:

Theorem 9 *In an off-axis catadioptric camera based on a conical mirror, if a line L is not contained in a PVS and it crosses the symmetry plane π_o at a visible point, then it can be localized with at most two solutions.*

4.2 Extension to rotating cameras

The result we derived for conical mirrors can be applied to another class of non-central cameras that shares a

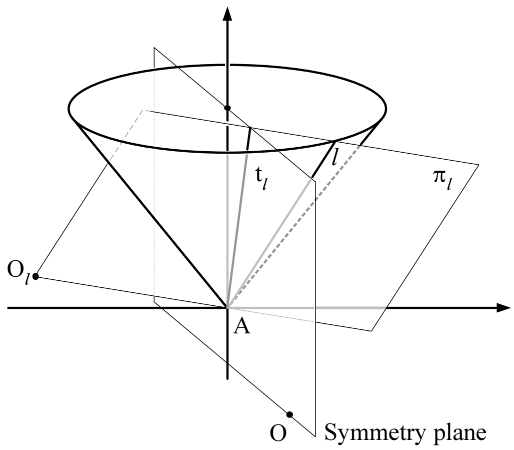


Fig. 7 Planar viewing surface π_l associated with an axial line l.

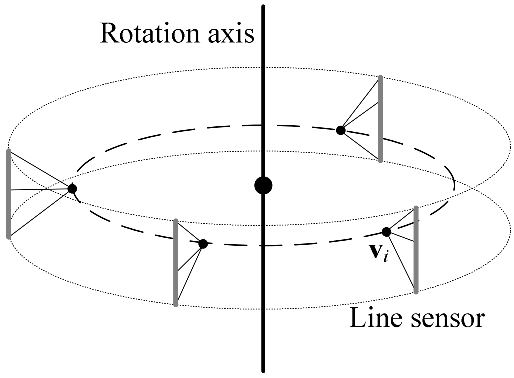


Fig. 8 The model of a rotating, panoramic, line camera: the line camera rotates about the axis and the locus of viewpoints $\mathbf{v}_i$ forms a circular caustic. The plane containing the circle is perperdicular to the axis and it is a (horizontal) PVS for the device.

similar imaging model, as we showed in [22]. Besides using mirrors, omnidirectional images can be also obtained mosaicing images captured by a rotating camera [15,33,40,44]. A line-scan camera, *i.e.* a single camera provided with a single array (line) of pixels, collects (line) images that are then merged into a single, panoramic, cylindrical image. An imaging model of this camera is shown in Figure 8. If the rotation axis goes through the pinhole of the camera, the camera is a central camera. Otherwise, it is a non-central camera whose caustic is a circle and the rotation axis is perpendicular to the plane π containing the caustic, a configuration similar to that of a catadioptric camera based on a conical mirror.

Even if it is a completely different imaging device from an axial catadioptric camera, the same geometrical properties hold when studying the PVSs. As in the case of axial catadioptric cameras, there is a horizontal PVS that corresponds to the plane π containing the caustics. Also for a given viewpoint $\mathbf{v}_i$, the plane containing the line sensor and the viewpoint $\mathbf{v}_i$ is a PVS. The span of these planes and the horizontal plane π are the only PVSs for a rotating camera. We then derived a localization condition similar to Theorem 7:

Theorem 10 *In a non-central, rotating, line camera, if a line L is not contained in a PVS, then L can be univocally localized [22].*

Similarly to the case of axial cameras, the rotation axis is always one of the solutions to the localization problem and can be discarded: thus, the actual line can be univocally localized.

5 Line Localization Algorithms

In this section we briefly review the methods that have been proposed for localizing a line from its image curve c and exploiting the minimum number of viewing rays. In the following the camera is always supposed to be calibrated and we suppose that the image of a line lying neither in a PVS nor in a RQVS has been detected.

As explained in Section 1, the line reconstruction problem can be seen as a particular case of the general problem of finding the lines crossing a given set of lines. A first algorithm to solve such problem was developed by Teller *et al.* [50] for computer graphics applications related to visibility computation. They propose a minimal solution employing only four viewing rays, as stated in Result 1.

They use the Plücker coordinates for representing rays and lines in 3D space, as it is a convenient representation, widely used in computer graphics, computational geometry and computer vision. Given two 3D points $\mathbf{A}$ and $\mathbf{B}$, the line $\mathbf{L}$ joining them can be expressed (up to scale) via the Plücker coordinate vector of length 6,

$$\mathbf{L} = \left[(\mathbf{A} - \mathbf{B})^\mathsf{T} \; (\mathbf{A} \times \mathbf{B})^\mathsf{T}.\right]^\mathsf{T} \tag{1}$$

Given two lines $\mathbf{L}_1$ and $\mathbf{L}_2$, their *side* product is defined as

$$\mathrm{side}\,(\mathbf{L}_1, \mathbf{L}_2) \triangleq \mathbf{L}_1^\mathsf{T} \begin{bmatrix} \mathtt{0} & \mathtt{I} \\ \mathtt{I} & \mathtt{0} \end{bmatrix} \mathbf{L}_2, \tag{2}$$

that is zero if they intersect or are parallel, and nonzero otherwise. Any 6-vector $\mathbf{L}$ corresponds to an actual 3D line if and only if it satisfies the constraint *side* $(\mathbf{L}, \mathbf{L}) = 0$. The *side* relation can conveniently express the constraint that 4 given viewing rays cross a common line in space: since each given viewing ray $\mathbf{R}_i$ crosses the common space line $\mathbf{L}$ (which is, of course, unknown), the relation $side(\mathbf{R}_i, \mathbf{L}) = 0$ must hold for all i, thus leading to the following linear system

$$\begin{bmatrix} \mathbf{R}_1^\top \\ \mathbf{R}_2^\top \\ \mathbf{R}_3^\top \\ \mathbf{R}_4^\top \end{bmatrix} \begin{bmatrix} \mathtt{0} & \mathtt{I} \\ \mathtt{I} & \mathtt{0} \end{bmatrix} \mathbf{L} = \mathtt{M}\,\mathbf{L} = \mathbf{0}. \tag{3}$$

If the 4 rays are not lying on a PVS or a RQVS, then $\mathtt{M}$ has rank 4 and the linear system can be solved through, *e.g.*, the Singular Value Decomposition (SVD). The null space of $\mathtt{M}$ gives the set of all 6-vectors $\mathbf{L}$ that satisfies (3); in order to find the two "real" 3D lines, the constraint $side\,(\mathbf{L}, \mathbf{L}) = 0$ must be imposed, requiring the solution of a 2nd order polynomial equation. The reader can find more details in [50]. The two solutions are the two lines that cross all the 4 viewing rays. In order to disambiguate between them, the catadioptric set-up must be taken into account. In axial symmetric cameras one of the two lines is always the mirror axis and the disambiguation is straightforward. In off-axis cameras the lines can be still disambiguated using all the other viewing rays: if a significant number of additional viewing rays goes through (or close enough to) just one of the two above determined lines, then this line is kept as unique solution for the localization problem. Also solutions that are not physically possible, *i.e.* the line lies inside the mirror, can be discarded.

This technique has been reprised by Avidan *et al.* [1] for the so called "Trajectory triangulation", *i.e.* the reconstruction of the 3D coordinates of a point moving along a line seen from a monocular camera. Later on, Lanman *et al.* [36] were the first to apply this approach for line localization from a single image taken by a catadioptric camera based on a set of spherical mirrors.

This method is easy to implement and it is computationally efficient. On the other hand, since it is a minimal solution, the obtained solutions are sensitive to noise and to calibration accuracy. In order to refine the minimal solution computed via (3), more viewing rays can be taken into account in the linear system. Moreover, the solution can be used as initial guess in for an optimization step in order to minimize the reprojection error. Another approach that can be taken into consideration in order to avoid outliers and improve the robustness of the localization is to employ the RANSAC technique [19]. The interested reader may refer to [11] for a comparison on the localization accuracy of these optimization methods.

We proposed another minimal solution [11] which avoids the multiplicity of solutions and exploits simple geometrical properties. The method is based on coplanarity relationships among viewing rays. For reconstruction purposes, a pair of coplanar viewing rays and two other additional viewing rays are considered. The procedure is simple and relatively fast. Once a pair of coplanar viewing rays are found, the line is constrained to lie on the plane containing these two rays. The points where the two additional rays cross the plane univocally identify the 3D line. The method is easy to implement as it exploits simple and standard geometrical procedures. A simple procedure in order to robustly choose a pair of viewing rays is also proposed. On the other hand, it is not always applicable in the case of off-axis cameras, since a pair of coplanar viewing rays may not exist for a given 3D line.

Finally, Swaminathan *et al.* [49] proposed another minimal approach for axial cameras using a different representation for 3D lines. They represent a line as the intersection of two planes, $\mathbf{\Pi}_0 = \left[A_0,\, B_0,\, 1\ 0\right]$ passing through the origin and not parallel to the z-axis, and $\mathbf{\Pi}_z = \left[A_z,\, B_z,\, 0\ 1\right]$ parallel to the z-axis and not passing through the origin (where the z-axis is the camera axis, which cannot be localized given such parametrization of the lines). Then, imposing the constraint that a point of the 3D line must lie on a viewing ray and on the two planes, they arrive at a simple linear system in the 4 unknown plane parameters $\left[A_0,\, B_0, A_0,\, B_0\right]$, which, again, requires at least 4 viewing rays to be solved.

6 Conclusions and final remarks

In this paper we showed that employing non-central cameras it is possible to localize a line in space from a single image, which is an ill-posed problem for a central camera. In non-central cameras, indeed, the projection rays are in general skew and the line localization problem from single images reduces to the general problem of finding the line that crosses a given set of other lines. It is a well established result that there can be either at most 2 or infinitely many lines that cross all the given lines. In order to have a finite set of solutions, the lines must not lie on a plane or on a ruled quadric. In order to determine which lines can be localized it is then necessary to study the number of such degenerate configurations and eventually determine sufficient conditions allowing a line to be reconstructed.

In particular we considered the case of non-central catadioptric cameras and we determined the number of planar viewing surfaces and sufficient conditions for line localization. We proved that for axial cameras there are no PVS other than the axial planes and the horizontal plane. On the other hand we showed that for off-axis cameras there is at most a discrete set of planar viewing surfaces for the considered class of cameras. This establishes a qualitative difference w.r.t. axial cameras, for which there is a one-dimensional set of planar viewing surfaces.

We also showed that the only other viewing surfaces preventing line localization are ruled quadrics. We proved some sufficient conditions for the localization of a broad set of space lines, for both axial-symmetric and off-axis cameras. We also extended the above results to catadioptric cameras based on multi-part mirrors, conical mirrors and rotating cameras, which share a similar imaging model.

The study and the analysis presented here are not confined to catadioptric cameras but they can be seen also as a guideline to derive similar properties and conditions for other non-central devices. Given a general class of non-central devices, conditions for line localization must be derived by studying the possible degenerate configurations of the viewing rays, *i.e.* when the viewing rays lie on a PVS or a RQVS. Such geometrical analysis and results can also find applications in more general frameworks, *e.g.* when studying the multiview geometry of general cameras using lines [23]. Generalizing the classical multi-focal tensor for perspective cameras [27] to the case of general cameras, the geometry of projection rays coming from different views and crossing a common line in space has to be studied. Again, PVS and RQVS are degenerate configurations of projection rays that must be avoided in order to properly define the geometrical constraints and then the tensors that conveniently represent the multi-view geometry of the cameras.

Finally, other geometrical features may be worth studying for single image localization. For example, Swaminathan *et al*. [49] proposed some algorithms for the localization of conics in space using non-central cameras, although without proving sufficient conditions for localization or the existence of degenerate configurations. This is not a trivial task, indeed, as it requires the study of more complex geometrical entities: in non-central cameras the viewing surface of a circle or, more generally, of a conic is a so-called "conoid", a ruled surface representing the generalization of a cone. In this case degenerate configurations may occur when the conoidal viewing surface contains more than one conic.

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