Computation of some Hilbert functions related to Schubert Calculus

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(Nice)

In this paper, we describe the computation of Hilbert functions for an important class of geometric varieties namely the opposite big cells of Schubert varieties, and the connection with combinatories.

Let $X = (X_i)$ be a set of variables, $\mathbb{k}$ be a field of characteristic zero, $\mathbb{N}$ be the set of non negative integers and $\mathfrak{m}$ the ideal generated by the (X_i) in the polynomial ring $R = \mathbb{k}[X]$. For a homogeneous ideal I in R, the Hilbert function H of the quotient ring R/I is

$$H(v) = \dim_{\mathbb{k}} (R_v / I_v) \quad , \quad v \in \mathbb{N} .$$

For v large enough, say $v \geq v_0$, $H(v)$ is a polynomial.

There has been a renewed interest in the study of this minimal value v_0 ,in connection with the study of the complexity of the algorithm for constructing standard basis and the effective Hilbert theorem. See for instance [2], [4], [6], [8], [11].

Also, S.S.Abhyankar [1] has computed the Hilbert function of a class of polynomial rings with $v_0 = 0$ (see §1), we sketch his proof in §3. From the work of V.Lakshmibai and C.S.Seshadri [13] we recognize the varieties defined by these rings to be open sets (opposite big cells) of Schubert varieties (§4).

This computation is done combinatorialy.

The number $H(v)$ counts certain set of plane partitions or lattice paths studied by I.M.Gessel who obtained formulas for them. Relating the two arguments ,we give a quick proof that $H(v)$ is a polynomial (§6,§7). The explicit expression obtained (§8) is different of that of Abhyankar. These formulas generalize one shown by R.P.Stanley in [16].

In the last section (§9) we compute explicitly an example .

1. Abhyankar's result.

Let us consider the matrix $X = (X_{i,j})$ $1 \leq i \leq m$; $1 \leq j \leq n$, $m \leq n$; with entries mn variables, and denote the determinant of the $p \times p$ minor formed by the rows corresponding to the indices $\alpha_1, \ldots, \alpha_p$ and the columns of X corresponding to the indices $\beta_1, \ldots, \beta_p$, by a bi-vector $(\alpha \mid \beta)$ of length p:

$$(\alpha \mid \beta) = (\alpha_p, \ldots, \alpha_1 \mid \beta_1, \ldots, \beta_p) \; ; \; lg(\alpha \mid \beta) = p;$$

$$\alpha_1 < \ldots < \alpha_p \quad ; \quad \beta_1 < \ldots < \beta_p .$$

We define a partial order on the bi-vectors, namely:

$$(\alpha' \mid \beta') \leq (\alpha'' \mid \beta'') \iff lg(\alpha' \mid \beta') \leq lg(\alpha'' \mid \beta'')$$

$$\&$$

$$\alpha'_k \geq \alpha''_k \, , \; \beta'_k \geq \beta''_k \quad \text{for} \quad 1 \leq k \leq lg(\alpha' \mid \beta').$$

<u>Theorem</u> 1: (S. S. Alhyankar [1]) For any bi-vector $(\alpha \mid \beta)$ of length p, let $I(\alpha \mid \beta)$ be the ideal in $\mathbf{k}[X]$ generated by all the determinants corresponding to $(\alpha' \mid \beta')$ such that $(\alpha' \mid \beta') \not\leq (\alpha \mid \beta)$ and denote by $H(v)$ the Hilbert function of the quotient algebra $R/I(\alpha \mid \beta)$.

Then $H(v)$ is the following polynomial in v:

$$H(v) = \sum_{d \in \mathbb{N}} (-1)^d \binom{M-1-d+v}{v} f_d$$

where $\bar{\alpha}_i = m - \alpha_i$, $\bar{\beta} = n - \beta$ and $M = \sum_{i=1}^{p} (\bar{\alpha}_i + \bar{\beta}_i + 1)$

and $f_d = \sum_{e \in \mathbb{N}} \binom{e}{d} g_e$ and $g_e = \sum_{e_1 + \cdots + e_p = e} g_{e_1, \ldots, e_p}$

and $$g_{e_1, \ldots, e_p} = \det \left[\binom{\bar{\alpha}_i + i - j}{e_i + i - j} \cdot \binom{\bar{\beta}_j + j - i}{e_i} \right]_{1 \leq i,j \leq p} .$$

2. Standard monomials and bi-tableau

Definition 2: The product $\mathcal{M}$ of an ordered sequence of bi-vectors:

$$(\alpha^{(1)}|\beta^{(1)}) \geqslant (\alpha^{(2)}|\beta^{(2)}) \geqslant \dots (\alpha^{(q)}|\beta^{(q)}) \quad ; \quad p_j = lg(\alpha^{(j)}|\beta^{(j)});$$

is called a standard monomial of degree $= p_1 + \dots p_q$.
It corresponds to a standard Young bi-tableau (also denoted by $\mathcal{M}$).

$$\begin{array}{cccc|cccc}
\alpha^{(1)}_{p_1} & \dots & & \alpha^{(1)}_1 & \beta^{(1)}_1 & & \dots & \beta^{(1)}_{p_1} \\
 & \alpha^{(2)}_{p_2} & \dots & \alpha^{(2)}_1 & \beta^{(2)}_1 & \dots & \beta^{(2)}_{p_2} & \\
 & & & & & & & \\
 & \alpha^{(q)}_{p_q} & \dots & \alpha^{(q)}_1 & \beta^{(q)}_1 & \dots & \beta^{(q)}_{p_q} &
\end{array}$$

We call $\mu_i = \# \{ j : p_j \geqslant i \}$, $1 \leqslant i \leqslant p_1$
the shape of $\mathcal{M}$ and we have $v = \mu_1 + \dots + \mu_{p_1}$

We set $\mathcal{M} \leqslant (\alpha|\beta) \iff (\alpha^{(1)} | \beta^{(1)}) \leqslant (\alpha|\beta)$.

Remark If we consider the $m \times (m+n)$ - matrix :

$$\bar{X} = \left(\begin{array}{c|ccc} X & & & 1 \\ & & 0 & \\ & & \cdot^{\cdot^{\cdot}} & \\ & & & 0 \\ & 1 & & \end{array} \right)$$

the determinant of the minor of rank m formed by the columns

$$(\beta_1, \dots, \beta_p, \beta^*_{p+1}, \dots \beta^*_m) \qquad \text{(written in increasing order)}$$

is equal (up to the sign) to the $p \times p$ -determinant $(\alpha|\beta)$ of X where $\beta^*_j = m+n+1-\alpha^*_{m+1-j}$, $j = 1+p$ to m and

$(\alpha^*_1, \dots, \alpha^*_{m-p})$ is the ordered complement of $\{\alpha_1, \dots, \alpha_p\}$ in $\{1, \dots, m\}$. Note that $\beta_j \leqslant n$ but $\beta^*_{p+j} > n$.
This bijection respects the order.

Then a standard monomial can be written like a standard rectangular 1-tableau:

$$\left|\begin{array}{ccc|ccc}
\beta^{(1)}_1 & \dots & \beta^{(1)}_{p_1} & \beta^{*(1)}_{p_1+1} & \dots & \beta^{*(1)}_m \\
 & & \vdots & & & \\
 & & \vdots & & & \\
\beta^{(q)}_1 & \dots & \beta^{(q)}_{p_q} & \beta^{*(q)}_{p_q+1} & \dots & \beta^{*(q)}_m
\end{array}\right|$$

This remark allows to derive the straightening formula of Doubillet-Roba-Stein [7] "any product of determinants of minors of X is a linear combination over $\mathbb{Z}$ of standard monomials" from the corresponding statement on the maximal rank minors of $\bar{X}$. (cf. [5] p 143, [16] p 255).

In fact the standard monomials form a (homogeneous free basis of R. ([7] , [5] or [1])

3. Sketch of the proof of theorem 1 (following [1]).

Step 1: (Obvious)

$$I(\alpha|\beta) = \{ \sum a_{\mathcal{M}} \cdot \mathcal{M} \quad : \mathcal{M} \text{ standard monomial } \not\leq (\alpha|\beta) \}$$

then $H(\nu) = \# \{$ standard bi-tableaux $\leq (\alpha|\beta)$ of degree $\nu \}$

so $$H(\nu) = \sum_{u_1+\dots+u_p=\nu} \Psi(\alpha\,;u_1,\dots,u_p) \cdot \Psi(\beta\,;u_1,\dots,u_p)$$

where $\Psi(\alpha;u_1,\dots,u_p) = \#\{$standard 1-tableau $\leq \alpha$ of shape $(u_1,\dots,u_p)\}$.

Step 2: (by induction , see § 6).

$$\Psi(\alpha;u_1,\dots,u_p) = \det \left[\begin{bmatrix} m-\alpha_i \\ u_j+i-j \end{bmatrix} \right]_{1\leq i,j\leq p}$$

where we used the notation:

$$S \in \mathbb{Z},\ T \in \mathbb{Z} \qquad \begin{bmatrix} S \\ T \end{bmatrix} = \binom{S+T}{T}$$

i.e.

$$\begin{aligned} &= 0 && \text{if } S<0 \\ &= 1 && \text{if } S=0 \\ &= \frac{(T+1)\dots(T+S)}{S!} && \text{if } S>0 . \end{aligned}$$

Step 3 (expand and simplify):

Perform successively two changes of coordinates in

$w_m = u_j + p-j$ then $u_j = w_j-p+j$ to obtain

$$H(v) = \frac{1}{p!} \sum_{u_k \in \mathbb{Z};\, u_1+\cdots+u_p=v} \det\left[\begin{bmatrix} m-\alpha_i \\ u_j+i-j \end{bmatrix} \right] \cdot \det\left[\begin{bmatrix} n-\beta_i \\ u_j+i-j \end{bmatrix} \right]$$

then

$$H(v) = \sum_{u_1+\cdots+u_p=v} \sum_{\sigma \in S_p} (-)^{sg(\sigma)} \prod_{i=1}^{p} \begin{bmatrix} \bar{\alpha}_i \\ u_i \end{bmatrix} \cdot \begin{bmatrix} \bar{\beta}_{\sigma(i)} \\ u_i - K_i \end{bmatrix}$$

where

$$\bar{\alpha}_i = m-\alpha_i \,, \quad \bar{\beta}_i = n-\beta_i \,, \quad K_i = i-\sigma(i) \,.$$

Step 4 (u will appear only in one factor):

Some identities:

$$\sum_{P+Q=S} \begin{bmatrix} v \\ P \end{bmatrix} \cdot \begin{bmatrix} w \\ Q \end{bmatrix} = \begin{bmatrix} v+w+1 \\ S \end{bmatrix}$$

$$\begin{bmatrix} A \\ u \end{bmatrix} \cdot \begin{bmatrix} B \\ u-K \end{bmatrix} = \sum_{d \in \mathbb{Z}} \sum_{e \in \mathbb{Z}} (-)^d \cdot \begin{bmatrix} A+B-d \\ u \end{bmatrix} \cdot \binom{e+K}{d} \cdot \binom{A+K}{e+K} \binom{B-K}{e} \,.$$

Step 5 (no more u):

$$H(v) = \sum_{\sigma \in S_p} (-)^{sg(\sigma)} \sum_{\substack{d_1,\ldots,d_p \\ e_1,\ldots,e_p}} \sum_{u_1+\cdots+u_p=v} \prod_{i=1}^{p} (-)^{d_1+\cdots+d_p} (\ldots)$$

$$= \sum_{d} (-)^d \cdot \sum_{e} \sum_{\substack{d_1+\cdots+d_p=d \\ e_1+\cdots+e_p=e}} \sum_{\sigma \in S_p} (-)^{sg(\sigma)} \prod_{i=1}^{p} \binom{e_i+K_i}{d_i} \binom{\bar{\alpha}_i+K_i}{e_i+K_i} \binom{\bar{\beta}_{\sigma(i)}-K_i}{e_i} \mathcal{A}$$

where $\mathcal{A} = \begin{bmatrix} \bar{\alpha}_1+\cdots+\bar{\alpha}_p+\bar{\beta}_1+\cdots+\bar{\beta}_p-d+p-1 \\ v \end{bmatrix}.$

Q.E.D.

4. Geometrical properties

The relation between Schubert varieties and Young tableaux has been studied extensively.

In [13] § 2, V. Lakshmibai and C.S. Seshadri considered the ideals $I(\alpha|\beta)$ and the varieties $D(\alpha|\beta)$ of their zeros in $M_{n,m} = \{ m \times n$ - matrices with entries in $k \}$.

First (p. 12) they identified $M_{n,m}$ with a Zariski open subset of the Grassmanian, that they call the opposite big cell.

Then (p. 15), they characterized $D(\alpha|\beta)$ to be the intersection of the Schubert variety $X(\beta,\beta^*)$ in $G(m,m+n)$ with the opposite big cell ; the notation (β,β^*) has been defined in our § 2.

Therefore $R/I(\alpha|\beta)$ inherits some properties of the Schubert varieties, in particular it is an integral domain and it is Cohen-Macaulay

Let's explain the idea of these identifications ; for more details we refere to [13] and [15] .

Call V a $(m+n)$- dimensional vector space and $e_1, \dots, e_{n+m}$ a basis of V ; we write e_i as a row vector of length $m+n$ (1 in the i^{th} place, 0 elsewhere).

Set $I_{m+n}(m) = \{(i) = (i_1, \dots, i_m) : 1 \leq i_1 < \dots < i_m \leq m+n\}$ and denote by $e_{(i)} = e_{i_1} \wedge \dots \wedge e_{i_m}$, $(i) \in I_{m+n}(m)$ the usual basis of $\Lambda^m V$ and by $\varepsilon_{(i)}$ the dual basis.

Considering the rows of a $m \times (m+n)$- matrix R as elements of V , we obtain a mapping :

$$\begin{aligned} \varphi_1 : M_{n+m,m} &\longrightarrow \Lambda^m V \\ R = (R_1, \dots, R_m) &\longmapsto R_1 \wedge \dots \wedge R_m \end{aligned}$$

and we see that :

$$\Delta_{(i)} = \varepsilon_{(i)} \circ \varphi_1(R) = \begin{cases} \text{determinant of the } m \times m\text{-minor} \\ \text{of } R \text{ corresponding to } (i) \end{cases}$$

The group $H = SL(m+n)$ operates on the projective space $\mathbb{P}(\Lambda^m V)$, we denote by P_m the isotropy subgroup of H at the point of $\mathbb{P}(\Lambda^m V)$ corresponding to $e_1 \wedge \dots \wedge e_m$ then H / P_m can be identified with the Grassmanian $G(m, m+n)$.

Denote by $M^m_{m+n,m}$ the subset of $M_{m+n,m}$ formed by matrices of rank m.

As φ_1 is $GL(m+n)$ - equivariant, it induces a surjective morphism

$$\varphi_2 : M^m_{m+n,m} \longrightarrow H/P_m$$

and in fact the Grassmanian H/P_m is the orbit space of $M^m_{m+n,m}$ under the action of $GL(m)$.

Let B be the Borel subgroup of H formed by the upper triangular matrices in H ; let W be the Weyl group of H formed by the permutation matrices,
$W \simeq S_{m+n} = \{(a_1,\dots,a_{m+n}) \ : \ 1 \leqslant a_k \neq a_\ell \leqslant m+n\}$
and W_m be the Weyl group of P_m which is the isotropy subgroup of W at $e_1 \wedge \dots \wedge e_m$. One can see that we obtain a canonical identification:

$$W/W_m \xrightarrow{\ \sim\ } I_{m+n}(m)$$

$(a_1,\dots,a_{m+n}) \longmapsto (a_1,\dots,a_m)$ arranged in the increasing order.

Recall the Bruhat decomposition $H = BWB$ and define a Schubert cell to be $B w e_{P_m}$, $w \in W/W_m$ in H/P_m and a Schubert variety to be $X(w, H/P_m) =$ the Zariski closure of $B w e_{P_m}$.

Now one can prove, see [13] and [15], that

$(i) \leqslant (j)$ (in $W/W_m \simeq I_{n+m}(m)$) $\Leftrightarrow X((i), H/P_m) \subset X((j), H/P_m)$;

that we can define so-called Plucker coordinates $\tilde{\Delta}_{(i)}$ on H/P_m s.t.

$$\tilde{\Delta}_{(j)} \Big|_{X((i), H/P_m)} \neq 0 \iff (j) \leqslant (i) ;$$

that the big cell is $\{ x \in H/P_m : \tilde{\Delta}_{(1,\ldots,m)}(x) \neq 0 \}$; similarly <u>the opposite big cell is</u> defined to be the open set $\{ x \in H/P_m : \tilde{\Delta}_{(n+1,\ldots,n+m)}(x) \neq 0 \}$.

Then, it is easy to check that the inverse image by φ_2 of the opposite big cell in H/P_m is $\{ R \in M^m_{m+n,m} : \Delta_{(n+1,\ldots,n+m)}(R) \neq 0 \}$ and that the restriction of φ_2 to the image of the mapping

$$\begin{array}{ccc} M_{n,m} & \longrightarrow & M_{n+m,m} \\ R & \longmapsto & (R \quad J) \end{array} \quad \text{where } J = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$$

is an isomorphism on the opposite big cell of H/P_m .

5. Relation with combinatorics

In [16] R.P. Stanley considered the rings $R/I(\alpha|\beta)$ of § 1 when length $(\alpha|\beta) = m$ which correspond to Schubert varieties in usual affine coordinates (theorem 5.1, p. 252). He related the Hilbert functions to the numbers of some plane partitions and (p. 253) asked for an explicit formula. A special case of theorem 1 provides an answer.

Let's explain this material.

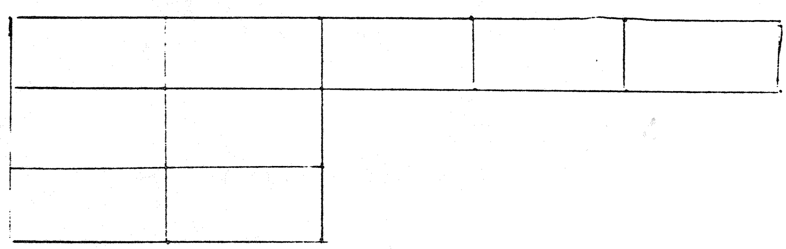
Fix a bi-vector $(\alpha|\beta)$. As we have seen in § 2, a standard monomial $\mathcal{M}$ of degree v such that $\mathcal{M} \leqslant (\alpha|\beta)$ can be viewed as a standard rectangular 1-tableau $\mathcal{T} \leqslant (\beta,\beta^*)$ where v is the number of elements not greater than n in that array.

We transform the array $\mathcal{T}$ in the following way :

(1) substract the entries of the $j^{\underline{th}}$ column from $n+j$

(2) replace each column C by its conjugate partition C'

i.e. if $C = (5, 2, 2, 0)$, draw



and read vertically $C' = (3, 3, 1, 1, 1)$;

Note that the number of elements of the $j^{\underline{th}}$ column of $\mathcal{T}$ which are not greater than n , can be red in the corresponding conjugate partition.

(3) remove entries (which will always equal zero) from the bottom of each column so that the $j^{\underline{th}}$ column will have $(n + j - \beta_j)$ entries.

Then, we obtain a so-called plane partition π of shape $(n+1-\beta_1, \dots, n+m-\beta^*_m)$. The sum of the main diagonal elements of π is called the trace of π .

The value $H(v)$ is equal to the number of the plane partitions of this given shape and whose trace is v .

In an unpublished work I.M. Gessel has transformed these plane partitions into n-uples of non-intersecting lattice paths, this enables him to answer the previous question.

In the next section we adapt Abhyankar's argument to recover I.M. Gessel's result.

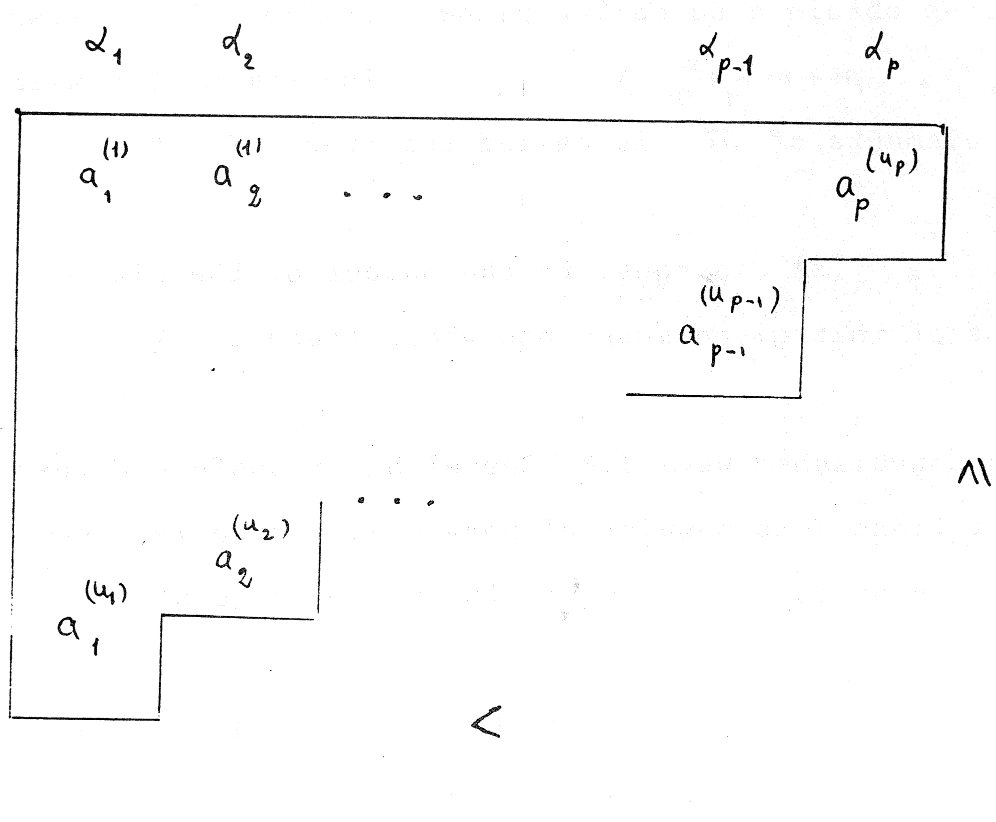
6. Weighted tableau

Our aim is to prove a stronger version of step 2 of § 3

A 1-tableau $\mathcal{A} = (a_i^{(j)}) \leqslant \alpha = (\alpha_1, \dots, \alpha_p)$

of shape $u_1 \geqslant \dots \geqslant u_p$ with $a_i^{(j)} \leqslant m$

is a plane partition filled as follow :



if $u_p \neq 0$.

We first consider the 1-tableau $\mathcal{C} = (c_i^{(j)})$ obtained by setting $c_i^{(j)} = m - a_i^{(j)}$ and $\bar{\alpha}_i = m - \alpha_i$ then we associate to it the monomial in the variables $Z_0, \dots, Z_{m-1}$

$$w(\mathcal{C}) = \prod_{i,j} Z_{c_i^{(j)}}$$ "the weight of $\mathcal{C}$",

and we would like to compute the sum

$$\mathcal{Y}(\bar{\alpha}; u_1, \dots, u_p; Z) = \sum w(\mathcal{C})$$

over all $\mathcal{C}$ defined as above.

(When we plug $Z_0 = \dots = Z_{m-1} = 1$ in $\mathcal{Y}$ we recover Ψ of § 2).

In the case $p = 1$; $u = u_1$; $\gamma = \bar{\alpha}_1$ call:

$$h(\gamma; u; Z) = \mathcal{Y}(\gamma; u; Z) = \sum_{0 \leq a_u \leq \dots \leq a_1 \leq \gamma} Z_{a_1} \dots Z_{a_u} ;$$

then $h(\gamma; u; 1) = \begin{bmatrix} \gamma \\ u \end{bmatrix} = \binom{\gamma + u}{u}$,

set $h(\gamma; u; Z) = 0$ if $u < 0$, and $h(\gamma; 0; Z) = 1$.

Proposition Let $\bar{\mathcal{Y}}(\bar{\alpha}; u_1, \dots, u_p; Z) = \det\left[h(\bar{\alpha}_i; u_j + i - j; Z)\right]_{(1 \leq i,j \leq p)}$

then $\mathcal{Y} = \bar{\mathcal{Y}}$.

Proof: By induction on p and u_p. Note that, for $p = 1$, $\mathcal{Y} = \bar{\mathcal{Y}}$.

If $\operatorname{tronc}(\bar{\alpha})$ denote the vector $(\bar{\alpha}_1, \dots, \alpha_{p-1})$ then obviously

$$\mathcal{Y}(\bar{\alpha}; u_1, \dots, u_{p-1}, 0; Z) = \mathcal{Y}(\operatorname{tronc}(\bar{\alpha}); u_1, \dots, u_{p-1}; Z) .$$

Because the last column of the determinant is $(0, \ldots, 0, 1)$, this equality is also satisfied by $\bar{\varphi}$.

Now we remark that, if we fix p,

$$\varphi(\bar{\alpha}; u_1+1, \ldots, u_p+1; Z) = \sum_{\delta_1 \leqslant \alpha_1, \ldots, \delta_p \leqslant \alpha_p} \varphi(\delta; u_1, \ldots, u_p; Z) \cdot Z_{\delta_1} \cdot \ldots \cdot Z_{\delta_p} .$$

It is enough to prove that $\bar{\varphi}$ satisfies the same induction formula.

Lemma: For $r \leqslant s$,

$$h(s; u+1; Z) - h(r; u+1, Z) = \sum_{r \leqslant w \leqslant s} Z_w \cdot h(w; u; Z).$$

Proof:

$$h(s; u+1; Z) = \sum_{0 \leqslant a_1 \leqslant s} Z_{a_1} \cdot \sum_{0 \leqslant a_u \leqslant \ldots \leqslant a_2 \leqslant a_1} Z_{a_2} \cdot \ldots \cdot Z_{a_u} \qquad \blacksquare$$

We have the inequalities :

$$\begin{array}{ccccccc} \alpha_1 & \geqslant & \ldots & \geqslant \alpha_{p-1} & \geqslant & \alpha_p \\ \text{IV} & & \ldots & \text{IV} & & \text{IV} \\ \delta_1 & \geqslant & \ldots & \geqslant \delta_{p-1} & \geqslant & \delta_p \end{array}$$

then

$$\mathcal{D} = \sum_{\delta_1 \leq \bar{\alpha}_1, \ldots, \delta_p \leq \bar{\alpha}_p} \det \left[Z_{\delta_i} \cdot h(\delta_i ; u_j + i - j ; Z) \right] \quad (1 \leq i,j \leq p)$$

$$\mathcal{D} = \sum_{0 \leq \delta_p \leq \bar{\alpha}_p} \; \sum_{\delta_p \leq \delta_{p-1} \leq \bar{\alpha}_{p-1}} \cdots \sum_{\delta_2 \leq \delta_1 \leq \bar{\alpha}_1} \det \left[Z_{\delta_i} \cdot h(\delta_i ; u_j + i - j ; Z) \right]$$

Now, in the first sum only the first row changes. By applying the lemma we obtain the difference of two determinants: the first line of the first is $h(\bar{\alpha}_1 ; u_j + 1 + 1 - j ; Z)$ and the first two. lines of the second are proportional . The other sums simplify the same way and we obtain:

$$\mathcal{D} = \det \left[h(\bar{\alpha}_i ; u_j + 1 + i - j ; Z) \right] \quad (1 \leq i,j \leq p) .$$

Q.E.D.

In other words, if we consider the matrix

$$A(\bar{\alpha} ; Z) = \Big(h(\bar{\alpha}_i ; \ell + i - p ; Z) \Big)_{\substack{1 \leq i \leq p \\ 0 \leq \ell \leq v+p-1}}$$

then

$\mathcal{S}(\bar{\alpha} ; u_1, \ldots, u_p ; Z)$ = determinant of the maximal minor formed by the columns $u_1 - 1 + p ,\ u_2 - 2 + p ,\ \ldots ,\ u_p$.

7. Computation of the Hilbert series (after [9]).

By step 1 § 2, $H(v) = \sum_{u_1+\cdots+u_p=v} \mathcal{S}(\bar{\alpha}; u_1,\ldots,u_p; 1)\, \mathcal{S}(\bar{\beta}; u_1,\ldots,u_p; 1)$.

By the Cauchy-Binet theorem:

$$\det\left(A(\bar{\alpha}; Z).A(\bar{\beta}; Z')^T\right) = \sum_{v \geq u_1 \geq \cdots \geq u_p \geq 0} \mathcal{S}(\bar{\alpha}; u_1,\ldots,u_p; Z)\, \mathcal{S}(\bar{\beta}; u_1,\ldots,u_p; Z')$$

we set $Z_0 = \ldots = Z_{m-1} = t \quad ; \quad Z'_0 = \ldots = Z'_{m-1} = 1$; so

$h(\bar{\alpha}_i ; \ell-(p-i); t) = \left[\begin{matrix}\bar{\alpha}_i \\ \ell-(p-i)\end{matrix}\right] . t^{\ell-(p-i)}$ then

$$H(v) = \text{coef of } t^v \text{ in } \det\left[\sum_{\ell=0}^{v+p-1} \left[\begin{matrix}\bar{\alpha}_i \\ \ell-(p-i)\end{matrix}\right] \cdot \left[\begin{matrix}\bar{\beta}_j \\ \ell-(p-j)\end{matrix}\right] t^{\ell-(p-i)}\right]$$

$$(1 \leq i,j \leq p)$$

then, the Hilbert series is

$$\sum_{v=0}^{\infty} H(v)\, t^v = \det\left[\, m_{i,j} \,\right] \qquad (1 \leq i,j \leq p)$$

with

$$m_{i,j} = \sum_{\ell=0}^{\infty} \binom{\ell-(p-i)+\bar{\alpha}_i}{\ell-(p-i)} \binom{\ell-(p-j)+\bar{\beta}_j}{\ell-(p-j)}\, t^{\ell-(p-i)}$$

We recall the following identity :

$$m_{i,j} = (1-t)^{-(\bar{\alpha}_i+\bar{\beta}_j+1)} \sum_{\ell=0}^{\infty} \binom{(p-i)+\bar{\beta}_j-(p-j)}{(p-i)+\bar{\beta}_j-\ell} \cdot \binom{(p-j)+\bar{\alpha}_i-(p-i)}{(p-j)+\bar{\alpha}_i-\ell} t^{\ell-(p-i)}$$

which is a form of Saalschutz formula ([3],[10]).

Now we can state the result as a theorem.

8. Theorem 2

$H(v)$ being the function defined in § 1. We set the following notations ;

$$\bar{\alpha}_i = m - \alpha_i \;, \quad \bar{\beta}_j = n - \beta_j \;, \quad M = \sum_{i=1}^{p} (\bar{\alpha}_i + \bar{\beta}_i + 1) \;,$$

$$\Delta_{i,j} = \sum_{\ell=p-i}^{\bar{\beta}_j+(p-i)} \binom{\bar{\beta}_j + (j-i)}{\bar{\beta}_j + (p-i) - \ell} \binom{\bar{\alpha}_i + (i-j)}{\bar{\alpha}_i + (p-j) - \ell} t^{\ell-(p-i)} .$$

Then the Hilbert series of the quotient algebra $R/I(\alpha|\beta)$ is

$$\sum_{v=0}^{\infty} H(v)\, t^v = (1-t)^M \cdot \det [\, \Delta_{i,j} \,] \qquad (1 \leq i,j \leq p) .$$

Morever, let $N = \inf\left(\sum_{i=1}^{p} \bar{\alpha}_i , \sum_{i=1}^{p} \bar{\beta}_i\right)$ and denote by

$$Q(t) = q_N (1-t)^N + \cdots + q_1 (1-t) + q_0$$

an expression of the polynomial $Q = \det(\Delta_{i,j})$,

we have :

$$\sum_{v=0}^{\infty} H(v)\, t^v = \frac{q_0}{(1-t)^M} + \cdots + \frac{q_N}{(1-t)^{M-N}}$$

then

$$H(v) = \sum_{k=M-N}^{M} q_{M-k} \frac{(v+1)\cdots(v+k-1)}{(k-1)!}$$

which is obviously a polynomial in v of degree $M-1$.

Thus, the variety $D(\alpha|\beta)$ has dimension $M-1$ and degree

$$q_0 = Q(1) = \det\left[\sum_{\substack{i \leq \lambda \leq \bar{\beta}_j + j \\ j \leq \lambda \leq \bar{\alpha}_i + i}} \binom{\bar{\beta}_j + j - i}{\lambda - i} \cdot \binom{\bar{\alpha}_i + i - j}{\lambda - j}\right] \quad (1 \leq i, j \leq p).$$

9. Example

Let $m = 3$, $n = 4$, $p = 2$

$\alpha_1 = 1$, $\alpha_2 = 3$, $\beta_1 = 2$, $\beta_2 = 3$

Then $(\alpha \mid \beta) = (3 \quad 1 \mid 2 \quad 3)$

$\alpha_1^* = 3$, $\beta_3^* = 3+4+1-3 = 5$, $(\beta, \beta^*) = (2 \quad 3 \quad 5)$,

$\bar{\alpha}_1 = 2$, $\bar{\alpha}_2 = 0$, $\bar{\beta}_1 = 2$, $\bar{\beta}_2 = 1$, $M = 7$, $N = 2$.

Here is an example of a standard monomial $\mathcal{M} \leqslant (\alpha \mid \beta)$

$$\begin{array}{cc|cc} 3 & 1 & 2 & 3 \\ \hline 3 & 2 & 2 & 4 \\ & 2 & 3 & \end{array}$$

$$\mathcal{M} = \det \begin{bmatrix} X_{22} & X_{24} \\ X_{32} & X_{34} \end{bmatrix} \times X_{23} \quad , \quad \text{degree}(\mathcal{M}) = 3 ,$$

Shape $(\mathcal{M}) = (2, 1)$

Other representation

$$\begin{array}{ccc} 2 & 3 & 5 \\ \hline 2 & 4 & 7 \\ 3 & 5 & 7 \end{array}$$

Transformation of this array to obtain a plane partition

(1) $\begin{array}{ccc} 3 & 3 & 2 \\ \hline 3 & 2 & 0 \\ 2 & 1 & 0 \end{array}$ (2) $\begin{array}{ccc} 3 & 3 & 2 \\ \hline 2 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & \boxed{0} \end{array}$ (3) - - -

Computation of $H(v)$ by use of theorem 1 :

$$H(v) = \sum_d (-)^d \binom{6-d-v}{v} f_d \quad , \quad f_d = \sum_e \binom{e}{d} g_e$$

$$g_e = \sum_{e_1+e_2=e} g_{e_1,e_2} \quad , \quad g_{e_1,e_2} = \det \begin{bmatrix} \binom{2}{e_1}\binom{2}{e_1} & \binom{1}{e_1-1}\binom{2}{e_1} \\ \binom{1}{e_2+1}\binom{1}{e_2} & \binom{0}{e_2}\binom{0}{e_2} \end{bmatrix}$$

$g_{0,0} = 1$,

$g_{1,0} = 2$,

$g_{2,0} = 0$,

The others are automatically equal to zero. Then

$$f_0 = 3 \quad ; \quad f_1 = 2 \quad ; \quad f_2 = \cdots = 0 .$$

Thus $$H(v) = 3 \frac{(v+1)\ldots(v+6)}{6!} - 2 \frac{(v+1)\cdots(v+5)}{5!} .$$

Computation of $H(v)$ by use of theorem 2:

$$Q = \det \begin{bmatrix} \sum_{\ell=1}^{3} \binom{2}{3-\ell}\binom{2}{3-\ell} t^{\ell-1} & \sum_{\ell=1}^{3} \binom{1}{2-\ell}\binom{2}{2-\ell} t^{\ell-1} \\ \sum_{\ell=1}^{1} \binom{1}{2-\ell}\binom{1}{1-\ell} t^{\ell} & \sum_{\ell=0}^{0} \binom{0}{-\ell}\binom{0}{-\ell} t^{\ell} \end{bmatrix}$$

$$Q = 1 + 2t = -2(1-t) + 3 \quad . \quad q_0 = 3 .$$

REFERENCES

1 ABHYANKAR S.S. : " Combinatoire des tableaux de Young, varietés déterminantielles et calcul de fonctions de Hilbert. " Preprint Université de Nice (1983)

2 ANGENIOL B. : " Résidus et effectivité." Preprint(1984). Ecole Polytechnique.

3 BAILEY W.N. : " Generalized Hypergeometric Series " Hafner New York (1972).

4 BAYER D. : Ph. D. Harvard (1982).

5 DE CONCINI, EISENBUD, PROCESI : " Young diagrams and Determinantal varieties " Inv. Math. 56, 129, 165, (1980)

6 DEMAZURE M. : Manuscript (1983).

7 DOUBILLET P., ROTA G.C., STEIN J. : " Fundation of combinatorics IX" Studies in Appl. Math 53 (1974), p. 185-216.

8 GALLIGO A. :"Algorithmes de calcul de bases standard" Preprint n° 9 Université de NIce (1983)

9 GESSEL I.M. : personal communication. (1984).

10 GESSEL I.M., STANTON D. : " Short proofs of Saalschutz's and Dixon's theorems." To be published in J. Combin. Theory Ser. A.

11 GIUSTI M. : " Some effective problems in polynomial ideal theory" Preprint (1983). Ecole Polytechnique.

12 HOCHSTER M. : " Grassmanians and their Schubert subvarieties are arithmetically Cohen-Macaulay " J. of Algebra 25, (1973)p.40-57.

13 LAKSHMIBAI V. , SESHADRI C.S. : " Geometry of G/P - II " Proc Indian Acad Sci A, vol 87A ≠ 2 (1978) pp 1-54.

14 MUSILI C. : J. Indian Soc 38, 131, (1974).

15 SESHADRI C.S. :" Geometry of G/P I " in C.P. Ramanujan : A tribute, 207-239, (Springer 1978) Tata Inst. Studies in Math. n° 8 .

16 STANLEY R.I. : " Some combinatorial aspect of the Schubert Calculus " Combinatoire et représentation du groupe symétrique, Strasbourg, (1976) Springer Lecture Notes n° 579 .

UNIVERSITE DE NICE

INSTITUT DE MATHEMATIQUES
ET SCIENCES PHYSIQUES

PARC VALROSE
06034 NICE CEDEX
TEL. (93) 51.91.00 (FRANCE)