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Abstract: A multi-phase optimization problem in terms of consumption of an electric vehicle involved in the European Shell Eco-marathon race is formulated. An open-loop optimal driving strategy is derived. Next, a time-varying Model Predictive real-time controller is developed to track the optimal solution and to achieve the minimum consumption. The stability and the convergence of the time-varying Model Predictive controller is proved. The convergence is guaranteed despite the variation of the MPC constraints in time. An example emulating an actual race illustrates the effectiveness of the approach.

1. INTRODUCTION

With the aim of promoting the research and innovation around the solution of the low consumption problem and the reduction of harmful emissions by the use of alternative sources of energy, the EcoMotionTeam of the École Supérieure des Sciences et Technologies de l'Ingénieur de Nancy (ESSTIN), has participated in the European Shell Eco-Marathon race since the year 2000. Over 200 teams, from high schools and universities coming from all over Europe, participate in this academic competition that involves ecological and economical vehicles. The goal of this contest is to drive a fixed number of kilometres in a limited range of time and with the lower consumption. Several categories are distinguished according to the energy source: fuel cell (hydrogen), battery, solar energy.

In the field of transport, the research of energy efficiency has been carried out for few decades in the industry (diesel-electric locomotives). Achieving a low consumption requires three central works. First, it must be obtained a valuable model of the vehicle. Secondly, and of special interest, is the problem of how using different energy sources, if any, so that the energy efficiency can be maximized and/or how the vehicle must be driven so that the minimum quantity of fuel is used, the real time implementation constraints being taken into account. This is the *driving strategy* Sciarretta and Guzzella (2007); Santin (2007); Bordons et al. (2010). The reference driving trajectory must be derived in terms of expected position and velocity all along the circuit. Both tasks can be performed off-line. Finally, a powerful tracking strategy must be designed and implemented so that it can works in real-time Sciarretta and Guzzella (2007); Manrique et al. (2012).

Initially, the driving strategy proposed by the EcoMotionTeam was so far a stop-and-go one, that merely reduced to manually turn on or off the engine by the pilot according to whether the vehicle was going up or down, or to drive around an average value, the efficiency of the converter/motor being better when working at full regime.

The objective of this paper is to design a control law which guarantees low energy consumption and satisfies the constraints inherent to both the dynamics and the path. The problem of finding the control law is formulated as two successive optimization problems, where the dynamics of the vehicle and the conditions of the race, such as the maximum final time to complete the task and the road shape (total length, slope, curves), are the main constraints to take into account.

An optimal reference driving strategy is first designed for the nonlinear dynamics. The main issue for achieving this purpose is that the related optimal control is given by a solution of a non-convex optimization problem in the continuous-time domain. Such a problem cannot be solved on-line, but it should be computed off-line using the information on the road path, the system dynamics and the relation between the energy consumption and the system variables. The solution of the optimal control problem results in an open-loop control law that would not be able to compensate the unavoidable model mismatches, disturbances, etc. Therefore, as a second step, the problem of tracking is solved. The knowledge of the current state, available in real-time, is used to compensate such perturbations by means of an additional closed-loop control. Additionally, as will be shown in section 4.1, the formulation of the tracking task implies time-varying input constraints that must be properly included in the loop without compromising the performance of the controller. As it turns out, Model Predictive Control (MPC) is not only known to have an appealing ability to ensure robustness despite the uncertainties in the model Bordons et al. (2010); Koot et al. (2005), but also allow us to include the time-varying constraints properly in the loop. Hence, this is the control we will consider here for achieving the tracking.

Fig. 1. The *Vir'Volt* prototype in the Shell Eco-Marathon 2011.

This paper is organized as follows. In Section 2, the dynamics of the low consumption electric vehicle of the EcoMotionTeam named *Vir'Volt* is briefly presented. In Section 3, the low consumption problem is formulated as a multi-phase optimization problem in terms of the characteristics of the path profile. In Section 4, the time-varying MPC algorithm that will achieve the tracking of the driving strategy is detailed and a study of the stability and the convergence is carried out. Finally, in Section 5, an example emulating an actual race is presented to illustrate the computation of the optimal solution and the tracking performed by the controller. Section 6 is devoted to a conclusion and sketches future works.

2. THE LOW CONSUMPTION ELECTRIC VEHICLE

The EcoMotionTeam has developed successive vehicles over the past 12 years. The prototype named *Vir'Volt* is the fifth generation. This prototype, shown in Fig. 1, has been ranked in 2011 2nd in the Plug-in (battery) category among 12 others vehicles and 7th among the 100 participants of the European competition with a result of 532km/kWh (equivalent to 4732km with one litre of fuel). The prototype is a three wheels vehicle that can reach the speed of 35km/h, the direction is controlled by the front wheel and the propulsion by one of the two rear wheels. The power-train configuration motor-torque coupler-wheel, is composed by a Kypom 22.2V battery that feeds a DC Maxon 200Watt motor which develops 0,4Nm. The torque of the motor is transmitted to one of the rear wheels by a torque coupler. The total weight of the car is 40kg, the pilot needs to weight at least 50kg according to the Shell Eco-Marathon rules. Thus, the total mass of the vehicle is 90kg.

If one assumes that the path profile has no elevation (flat path) or that the effect of the slope on the dynamics is negligible, then the dynamics of the vehicle can be described by the following nonlinear state-space dynamics involving the mass $m[\text{kg}]$ of the vehicle and its acceleration $\frac{dx_2(t)}{dt}[\text{m/s}^2]$ Guzzella and Sciarretta (2005):

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = \frac{u(t)C_{mot} + (1-u(t))C_{min} - C_{pivot}}{mr_w} - N_r g - \frac{1}{2}\frac{\rho C_x S}{m}x_2(t)^2, \quad (1)$$

with $x_1(t)$ the position in [m] and $x_2[\text{m/s}]$ the velocity (in magnitude) of the vehicle. $C_{mot}[\text{Nm}]$ is the torque obtained over the wheel due to the motor and after the torque coupler. Instead, if the engine is not working, a torque of $C_{min}[\text{Nm}]$ is obtained. C_{mot} is assumed to be constant since the maximum torque of the motor does not significantly change in the range of low velocities in consideration. Since in the electric motor the motor current is proportional to the motor torque, the input $u(t) \in [0,1]$ is introduced, where $u(t)$ is the duty cycle of the driving period during which the engine is on, and $(1-u(t))$ is the portion of the period during which the motor is off. Therefore $u(t)C_{mot} + (1-u(t))C_{min}$ is the average torque over the wheel. $C_{pivot}[\text{Nm}]$ is the bearing resistance in the connecting rod of the wheels, and $r_w[\text{m}]$ is the radius of the wheel. The frontal area $S[\text{m}^2]$

of the vehicle, the aerodynamic drag coefficient C_x and the air density $\rho[\text{kg/m}^3]$, represent the aerodynamic frictions. N_r is the rolling resistance and $g[\text{m/s}^2]$ is the gravitation acceleration. The rotational inertias present in the power-train, such as the rotor inertia, are neglected by being smaller than the vehicle mass. For more details on the dynamics of the *Vir'Volt* vehicle, the reader should refer to Manrique et al. (2012).

3. OPEN-LOOP OPTIMAL NONLINEAR CONTROL

In this section, we consider the off-line computation of the reference driving strategy which must achieve the minimization of the energy consumption despite some constraints due to the dynamics and to the race path. During a race, the road profile (slope, curves, etc.) and the constraints in terms of maximum velocity allowed at each curve, maximum time of the race and total number of kilometres, are the main factors to be considered. For a prescribed circuit, the driving strategy which achieves minimal consumption is defined by the triplet $(x_1^*(t), x_2^*(t), u^*(t))$ where $x_2^*(t)$ corresponds to the required velocity assigned to the position $x_1^*(t)$ in the circuit at time t and $u^*(t)$ is the required input to achieve $x_1^*(t)$ and $x_2^*(t)$.

It turns out that, since the current of the battery is proportional to the torque of the motor and this last one can be written in terms of the input $u(t)$ as $u(t)C_{mot}$, then the minimum consumption is achieved if the minimum control signal $u^*(t)$ is used to perform the driving task. Defining the state space vector $x(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, this optimization problem can be written as:

$$\min_{u(t),x(t),t_f} \int_0^{t_f} u(t)\,dt$$

$$s.t. \quad (1)$$
$$t_f \leq t_{fmax},$$
$$x(0) = 0,$$
$$x_1(t_f) = x_{1total}, \quad (2)$$
$$x_2(t) \leq x_{2max}, \forall t \leq t_f$$
$$x_{2curve} \leq \sqrt{F_t r_c/m},$$
$$u(t) \in [0,1], \forall t \leq t_f,$$

with t_{fmax} the maximum time allowed to complete the race, x_{1total} the total distance to cover, and x_{2max} the vehicle maximum allowable speed due to the maximum velocity allowed for the motor. In a curve, the centrifugal force over the car $F_c = m(x_2)^2/r_c$, with $r_c[\text{m}]$ the radius of the curve, must not be larger than the total frictions forces wheel-road $F_t[\text{N}]$ to prevent the car slipping over, and therefore $x_{2curve} \leq \sqrt{F_t r_c/m}$ Santin (2007). The optimal values of the nominal control input $u^*(t)$ and the related optimal trajectory $x^*(t)$ are solution of the problem (2).

It can be observed that the constraint on the velocity in curves $x_{2curve} \leq \sqrt{F_t r_c/m} \leq x_{2max}$ is concerned exclusively to the position x_1 where there is precisely a curve, otherwise the constraint is only $x_2 \leq x_{2max}$. To include appropriately these constraints in the solution of (2), the space state $x_1(t) \in [0, x_{1total}]$ is divided into consecutive phases depending if there is a curve or not, and for every phase, the curve constraint is applied or not. To illustrate this, the example lap road of Fig. 2 to be run counter clockwise from $x_1(0) = x_{1Start}$ to $x_1(t_f) = x_{1Finish}$ is considered. This road has four $\pi/2$ curves with radius $r_{ci}, i = 2,4,6,8$ and four straight paths of length $l_j, j = 1,3,5,7$ each one. As a result, for this road, the state x_1 is divided in eight consecutive phases e.g. straight path number 1, curve number 2, second straight path number 3, curve number 4 and so on. The end of
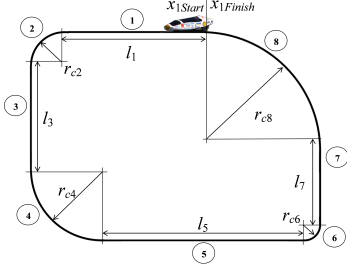
Fig. 2. Example road with straight lines and curves. This road has eight phases indicated by the circles.

a phase is the beginning of the next one. Only for phases 2,4,6 and 8, the constraint for curves $x_{2curve} \leq \sqrt{F_t r_c / m} \leq x_{2max}$ is applied. This is important to stress that the problem (2) and the algorithm presented here, can be formulated and solved for any road with any quantity of curves and straight paths, where the angle of the curves can be different from $\pi/2$ and also sequences of curve-curve can be considered.

The solution of the above multi-phase optimization problem is computed using the General Pseudospectral Optimal control Software (GPOPS) Rao et al. (2010); Garg et al. (2011a). This software uses the orthogonal Radau Pseudospectral Method Garg et al. (2011b), to solve optimal control problems of multiple phases where for each phase, the next functions need to be defined: the cost function to be minimized, the dynamics constraints and its relation with the cost function, the boundary conditions at the beginning and at the end of each phase, the inequality path constraints of the solution from the beginning until the end of each phase, and finally the linkage constraints between phases. Since the open-loop optimization problem described in (2) has a multiphase nature that requires the constraints between phases to be properly defined, the GPOPS tool is well suitable for the solution of this problem. An example of the multiphase nature of the optimization problem will be given in Section 5 devoted to results.

4. MODEL PREDICTIVE CONTROL

This section is devoted to the controller design, which is intended to be embedded on-board during the race, and must guarantee the tracking of the driving strategy despite unpredictable events like emergency braking, wind, etc. and the modelling mismatches. In addition, the formulation of this close-loop problem highlights the existence of time-varying constraints, as will be shown in subsection 4.1, that must be properly included in the control law. As pointed out in the introduction, a Model Predictive Control approach is well suited for those purposes. A discrete-time linear system is used to model the error between the ideal system and the real one. Assuming that the nonlinear dynamics is close to the real one, such errors can be considered to be small enough and then the discretization and linearization error might be tolerable.

From (1), the linearized discrete-time state space dynamics is obtained:

$$\hat{x}(k+1) = \mathbf{A}\hat{x}(k) + \mathbf{B}\hat{u}(k), \tag{3}$$

where $\hat{x}(k) = [x_1(k) + x_{2e}T_s, x_2(k) - x_{2e}]^T \in \mathbb{R}^2$, $\hat{u}(k) = u(k) - u_e$ and

$$\mathbf{A} = \begin{bmatrix} 1 & T_s \\ 0 & 1 - \dfrac{\rho C_x S T_s}{m} x_{2e} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{(C_{mot} - C_{min})}{m r_w} T_s \end{bmatrix},$$

with T_s the sample time. The linearization point is given by the average velocity x_{2e} that together with the input $u_e \in [0,1]$ fulfill the steady-state condition $\dot{x}_2 = 0$ in (1), i.e.:

$$x_{2e} = \sqrt{2 \cdot \frac{(C_{mot} - C_{min})u_e + C_{min} - C_{pivot} - N_r g m r_w}{\rho C_x S r_w}}, \tag{4}$$

Notice that, as expected, the steady-state values for the nonlinear continuous-time systems, i.e. x_{2e} and u_e, are also steady-state for the discrete-time linear one. The full state is assumed to be accessible, which means that both the position and the velocity are measured.

4.1 Time-varying Model Predictive Control

This subsection addresses the problem of driving the vehicle in real-time in order to guarantee that the state $x(k) = [x_1(k), x_2(k)]^T \in \mathbb{R}^2$ of the vehicle remains as close as possible to the optimal one, as defined in Section 3, for every sample time i.e. $x^*(k) = [x_1^*(k), x_2^*(k)]^T$.

Remark 1. We consider as constant values the nominal optimal velocity x_2^* and input u^* within the prediction horizon. That is, $u_2^*(k+i) = u_2^*(k)$, $x_2^*(k+i) = x_2^*(k)$ and $x_1^*(k+i) = x_1^*(k) + iT_s x_2^*(k)$ for all $i = 0, \ldots, N_p$ with N_p prediction and control horizon. This assumption introduces some additional uncertainty, but it allows to prove the stability of the linear time-varying MPC, see Section 4.2.

Define $\Delta u(k) = u(k) - u^*(k)$ as the difference between the predicted input $u(k)$ and the target control $u^*(k)$ at the instant k. Furthermore, define $\Delta x(k) = x(k) - x^*(k)$ as the difference between the predicted state $x(k)$ and the target state $x^*(k)$ at the instant k. By Remark 1, in $\Delta u(k)$ and $\Delta x(k)$, the optimal profiles u^* and x_2^* remain constant during the prediction horizon. Notice that, from the accessibility of $x(k)$ and the availability of $x^*(k)$, so it is for $\Delta x(k)$. Defining the error $\Delta \hat{u}(k)$ and $\Delta \hat{x}(k)$ for the linear system (3) as $\Delta \hat{u}(k) = \hat{u}(k) - (u^*(k) - u_e) = \Delta u(k)$ and $\Delta \hat{x}(k) = \hat{x}(k) - (x^*(k) + [x_{2e}T_s; -x_{2e}]^T) = \Delta x(k)$ at every instant k, then the dynamics of the error $\Delta u(k)$ and $\Delta x(k)$ is given by:

$$\Delta x(k+1) = \mathbf{A}\Delta x(k) + \mathbf{B}\Delta u(k), \tag{5}$$

As for the constraints on the tracking error state Δx, we impose bounds on their value. In particular we pose

$$\Delta x_{min} \leq \Delta x(k) \leq \Delta x_{max}, \tag{6}$$

where $\Delta x_{max} \in \mathbb{R}^n$ and $\Delta x_{min} \in \mathbb{R}^n$ are respectively the desired upper and lower bounds. The constraints on the input $u(k)$ are

$$0 \leq u(k) \leq 1, \tag{7}$$

recalling that the input is the duty cycle of the driving strategy. We have that the input of the error system (5) has to satisfy, for every k, the parameter-dependent constraints

$$\Delta u_{min}(k) \leq \Delta u(k) \leq \Delta u_{max}(k), \tag{8}$$

with

$$\begin{aligned} \Delta u_{min}(k) &= -u^*(k), \\ \Delta u_{max}(k) &= 1 - u^*(k), \end{aligned} \tag{9}$$

In plain words, the constraints depend on the value of the nominal optimal control input $u^*(k)$ that can change at any instant. Notice that the formulation of the MPC problem allows us to include the time-varying constraints (9) in a suitable way. The MPC problem to solve on-line is then the following:

$$\min_{\Delta u(k),\ldots,\Delta u(k+N_p-1)} \sum_{i=0}^{N_p-1} \left(\Delta x^T(k+j)Q\Delta x(k+j) + \Delta u^T(k+j)R\Delta u(k+j) \right) + \\ + \Delta x^T(k+N_p)P\Delta x(k+N_p),$$

$$\begin{aligned}
s.t. \quad & \Delta x(k+i+1) = \mathbf{A}\Delta x(k+i) + \mathbf{B}\Delta u(k+i), && \forall i = 0,\ldots,N_p-1, \\
& \Delta x(k+i) \in X, && \forall i = 0,\ldots,N_p-1, \\
& \Delta u(k+i) \in U(k), && \forall i = 0,\ldots,N_p-1, \\
& \Delta x(k+N_p) \in \bar{\Omega}(k),
\end{aligned}$$

$$(10)$$

which is the classical MPC problem for linear systems, see Scokaert and Rawlings (1998); Mayne et al. (2000); Camacho and Bordons (2004), except the fact that the constraints on the input and on the terminal predicted state depend on the nominal optimal input $u^*(k)$ at time k. This means that the control input and the terminal predicted state are constrained within sets which might be changing in time. As a matter of fact, such bounding sets will be denoted hereafter and with a slight abuse of notation, $U(k) = U(u^*(k))$ and $\bar{\Omega}(k) = \bar{\Omega}(u^*(k))$. Considerations on how to obtain the set $\bar{\Omega}(u^*(k))$ such that it ensures stability for constant nominal inputs and such that it is easily computable on-line are the purposes of Section 4.2. The positive definite matrix P is obtained by solving the LQR problem for the system (5) with weighting matrices Q and R. Indeed, P is involved in the terminal cost, that is in the term corresponding to the cost to infinity. As a result, it means that implicitly, we suppose that after N_p steps, the infinite horizon LQR control gain is applied, as usual.

4.2 MPC stability

The MPC problem under consideration here is the regulation of a linear system with time-varying input constraints. In the time-invariant case, the standard method to prove stability is based on the existence of the invariant set as terminal constraint, see Mayne et al. (2000). The greatest one is usually pre-computed (since hard to be obtained on-line) and then used on-line. Here, since it depends on the constraints, we cannot compute the greatest one on-line. To circumvent the problem, we propose here a method for obtaining a parameter dependent invariant set. The result is a smaller invariant set but the on-line computation required is very low, almost negligible.

Assumption 1. The constraint sets $X \in \mathbb{R}^n$ and $U(k) \in \mathbb{R}^m$ are assumed to be convex, compact and to contain the origin in their interior.

The Assumption 1, supposed to hold in the following, permits us to use properties of convex, compact sets. The case of unbounded sets X and U would deserve specific considerations. Nevertheless, from the practical point of view, there is no loss of generality in assuming their boundedness since both the state and the input are bounded in actual applications.

Definition 1. Given the value of $u^* \in \mathbb{R}^m$, define the sets

$$\begin{aligned}
U(u^*) &= \{u \in \mathbb{R}^m : -u^* \le u \le 1-u^*\}, \\
\hat{U} &= \{u \in \mathbb{R}^m : -1 \le u \le 1\}, \\
\bar{U}(u^*) &= \{u \in \mathbb{R}^m : -\alpha(u^*) \le u \le \alpha(u^*)\},
\end{aligned}$$

$$(11)$$

with

$$\alpha(u^*) = \min_{i \in \mathbb{N}_m} \min\{|u_i^*|, |1-u_i^*|\} = \min\{\|u^*\|_\infty, \|1-u^*\|_\infty\}.$$

The set $\bar{U}(u^*)$ provides an inner bound of the input dependent constraint set $U(u^*(k))$.

Proposition 1. Given $u^* \in \mathbb{R}^m$, one have that $0 \in \bar{U}(u^*)$, $0 \le \alpha(u^*) \le 0.5$ and

$$\bar{U}(u^*) = \alpha(u^*)\hat{U} \subseteq U(u^*), \qquad (12)$$

for all u^* such that $0 \le u^* \le 1$. Moreover if $0 < u^* < 1$ then $0 \in \text{int}(\bar{U}(u^*))$ and $0 < \alpha(u^*) < 0.5$.

Proof: The facts that $0 \in \bar{U}(u^*)$ and $\bar{U}(u^*) = \alpha(u^*)\hat{U}$ follow directly from the Definition 1 and from the non-negativity of $\alpha(u^*)$ for all $u^* \in \mathbb{R}^m$. Since $u_i^* \ge 0$, which implies $u_i^* = |u_i^*|$, we have

$$\begin{aligned}
-u_i^* &\le \max\{-|u_i^*|, -|1-u_i^*|\} = \\
&= -\min\{|u_i^*|, |1-u_i^*|\} \le -\alpha(u^*),
\end{aligned}$$

for all $i \in \mathbb{N}_m$. Analogously, from $1 - u_i^* \ge 0$, it follows that

$$\alpha(u^*) \le \min\{|u_i^*|, |1-u_i^*|\} \le 1-u_i^*,$$

for every $i \in \mathbb{N}_m$ and thus $u \in U(u^*)$ if $u \in \bar{U}(u^*)$. Moreover $\alpha(u^*) \ge 0$ and $\alpha(u^*) \le 0.5$ since $\min\{u_i^*, 1-u_i^*\} \le 0.5$ for all $i \in \mathbb{N}_m$. If $0 < u^* < 1$ then $u_i^* > 0$ and $1 - u_i^* > 0$ for all $i \in \mathbb{N}_m$ which implies that $\alpha(u_i) > 0$. Hence the origin is in the interior of $\bar{U}(u^*)$. Finally $0 < \alpha(u^*) < 0.5$ is due to the fact that $0 < \min\{u_i^*, 1-u_i^*\} < 0.5$ for all $i \in \mathbb{N}_m$, for every $0 < u^* < 1$. ∎

Hence from Proposition 1 and (8), we have that $\bar{U}(u^*(k)) \subseteq U(u^*)$, which means that $\Delta u \in \bar{U}(u^*)$ is a sufficient condition for the constraints (8) to hold. Such condition will be implicitly imposed in the MPC in place of the condition (8) for Δu_k with $k \ge N_p$. This, if on one hand introduces some conservativeness, on the other permits us to design an input dependent invariant set which can be determined on-line with no significant computational effort. Notice that the computation of an invariant set whose elements satisfy some constraints, useful to prove the convergence of the MPC, requires a computational effort which has usually to be performed off-line, Gilbert and Tan (1991); Kolmanovsky and Gilbert (1998); Blanchini and Miani (2008). Since in our case, the constraints on the input (and then on the state, once a local feedback law is determined) change in time, such invariant computation should be repeated at any change of u^*, which is not admissible. Then, the aim is to parametrize the invariant set $\bar{\Omega}(k)$ in terms of $\alpha(u^*)$, to have at every instant k that the invariant set is $\bar{\Omega}(k) = \alpha(u^*(k))\hat{\Omega}$, with $\hat{\Omega}$ precomputed.

Given a stabilizing feedback control law $\Delta u = K\Delta x$ we design off-line an invariant set for the closed-loop system

$$\Delta x^+ = (\mathbf{A} + \mathbf{B}K)\Delta x = \hat{\mathbf{A}}\Delta x. \qquad (13)$$

We consider as K the solution of the LQR problem for system (5) with weighting matrices R and Q and P the definite positive matrix determining the quadratic optimal cost. We recall the definition of invariant set for a linear deterministic discrete-time system, see Blanchini and Miani (2008).

Definition 2. A set $\Omega \subseteq \mathbb{R}^n$ is an invariant set for the system (13) and constraints $\Delta x \in \mathscr{X}$ if $\Omega \subseteq \mathscr{X}$ and $\hat{\mathbf{A}}\Delta x \in \Omega$, for all $\Delta x \in \Omega$.

Notice that in our case, the state constraint $\mathscr{X}$ is time-varying and nominal input dependent since given by the constraints on the input Δu. In fact, from (6) and (8), we have that the system has to satisfy the input and state constraints $\Delta u = K\Delta x \in U(u^*)$ and $\Delta x \in X$, for a properly defined X. This is equivalent to $\Delta x \in \mathscr{X}(u^*)$ with

$$\mathscr{X}(u^*) = \{x \in \mathbb{R}^n : x \in X, Kx \in U(u^*)\} = X \cap X_u(U(u^*)), \qquad (14)$$

where

$$X_u(U) = \{x \in \mathbb{R}^n : Kx \in U\}, \qquad (15)$$

for every $U \subseteq \mathbb{R}^m$. The set $\mathscr{X}(u^*)$ depends on u^* and then it is time-varying in general. A method to design the necessary

parameter dependent invariant set is proposed in what follows, focusing in particular to the low computational requirement for it to be obtained on-line. The lemma below is functional to that purpose.

Lemma 1. Given $U, V \subseteq \mathbb{R}^m$, the set $X_u(\cdot)$ as in (15) is such that $X_u(U) \subseteq X_u(V)$ if $U \subseteq V$. Given moreover $\alpha \in \mathbb{R}$ then $\alpha X_u(U) = X_u(\alpha U)$ if $\alpha > 0$ and $\alpha X_u(U) \subseteq X_u(\alpha U)$ if $\alpha \geq 0$.

Proof: The first part follows directly from definition (15). Concerning the second part, if $\alpha > 0$ we have

$$\alpha X_u(U) = \{\alpha x \in \mathbb{R}^n : Kx \in U\} = \\ = \{y \in \mathbb{R}^n : Ky \in \alpha U\} = X_u(\alpha U).$$

If $\alpha = 0$ we have $\alpha X_u(U) = \{0\}$ and $X_u(\alpha U) = X_u(\{0\}) = \ker(K)$ (ker stands for the kernel) which implies $\alpha X_u(U) \subseteq X_u(\alpha U)$ in this case. ∎

The method is based on the following proposition, which stems from the properties of invariant sets for linear systems Blanchini and Miani (2008).

Proposition 2. Considering the Definition 1, given an invariant set $\hat{\Omega} \subseteq \mathbb{R}^n$ for the system (13) with constraints $\Delta x \in \hat{\mathscr{X}} = 2X \cap X_u(\hat{U})$, then the set $\bar{\Omega}(u^*) = \alpha(u^*)\hat{\Omega}$ is an invariant set for (13) with constraints $\Delta x \in \mathscr{X}(u^*)$, as in (14), for all u^* such that $0 \leq u^* \leq 1$.

Proof: From properties of invariant sets for linear systems with constraints $\Delta x \in \hat{\mathscr{X}}$, we have that also the set $\alpha\hat{\Omega}$ is an invariant set for the same constraints and for every $\alpha \in [0,1]$. From $0 \leq u^* \leq 1$ and the Proposition 1, it follows that $0 \leq \alpha(u^*) \leq 0.5$ and then also $\bar{\Omega}(u^*)$ is an invariant set for the system (13) with constraint $\Delta x \in \hat{\mathscr{X}}$, which means that $\hat{A}\bar{\Omega}(u^*) \subseteq \bar{\Omega}(u^*) \subseteq \hat{\mathscr{X}}$. From Proposition 1, Lemma 1 and definition (14) we have that

$$\bar{\Omega}(u^*) = \alpha(u^*)\hat{\Omega} \subseteq \alpha(u^*) \left(2X \cap X_u(\hat{U})\right) \subseteq \\ \subseteq X \cap \alpha(u^*)X_u(\hat{U}) \subseteq X \cap X_u(\alpha(u^*)\hat{U}) = \\ = X \cap X_u(\bar{U}(u^*)) \subseteq X \cap X_u(U(u^*)) = \mathscr{X}(u^*),$$

and then $\bar{\Omega}(u^*) \subseteq \mathscr{X}(u^*)$. From this and the fact that $\hat{A}\bar{\Omega}(u^*) \subseteq \bar{\Omega}(u^*)$, we have that $\bar{\Omega}(u^*)$ is an invariant set for (13) with constraints $\Delta x \in \mathscr{X}(u^*)$, by definition. ∎

Then, finally, given a value of the nominal control input $0 \leq u^* \leq 1$, it is sufficient to compute $\alpha(u^*)$ to have an invariant set $\bar{\Omega}(u^*)$ for the system (13) with constraint $\Delta x \in \mathscr{X}(u^*)$, the set $\hat{\Omega}$ being known. Hence, the asymptotic stability and the constraints satisfaction of the MPC can be proved for constant values of u^* using standard results Mayne et al. (2000).

Proposition 3. For every constant $u^* \in \mathbb{R}^m$ such that $0 \leq u^* \leq 1$ if the problem (10) is feasible at k_0, then the system (5) in closed-loop with $\Delta u_k = \Delta u_k^*$ is exponentially stable and the constraints $\Delta u_k = K\Delta x \in U(u^*)$ and $\Delta x_k \in X$ are satisfied for all $k \geq k_0$.

The Proposition 3 ensures convergence of the closed-loop system and recursive constraints satisfaction for every constant value of u^*, with $0 \leq u^* \leq 1$, provided that the problem is feasible at the initial instant. Since in such case, the problem is reduced to a deterministic linear MPC, it is sufficient to have that $\bar{\Omega}(u^*)$ is an invariant set for (13) with constraints $\Delta x \in \mathscr{X}(u^*)$, that is proved in Proposition 2.

Remark 2. The problem could have also been dealt with by means of techniques of tracking MPC Fiacchini et al. (2006); Limón et al. (2008). On the other hand, in our specific case, the

objective is not to steer the state to different constant values, only the constraint on the input changes in time. The stability is then proved provided that an invariant set for the particular constraint set is available.

5. RESULTS

5.1 Parameter estimation

The nonlinear model (1) involves two unknown parameters C_x and N_r. For the estimation of those parameters, multiple deceleration to zero experiments have been performed for the *Vir'Volt* vehicle. It has been done by turning off the motor after accelerating the vehicle until it reaches a maximum velocity Vel_{ini} in a low slope variations road and without curves. For the *Vir'Volt* vehicle and its environment, the physical known parameters are the gravity acceleration $g = 9.81[\text{m/s}^2]$, the wheel radius $R = 0.24\text{m}$, the frontal surface $S = 0.275\text{m}^2$, the total mass $m = 90\text{kg}$, the air density $\rho = 1.225[\text{kg/m}^3]$, $C_{mot} = 6.228[\text{Nm}]$, $C_{min} = 0[\text{Nm}]$, C_{pivot} being neglected. The maximal velocity is $Vel_{ini} = 40\text{km/h}$. The experiment gives $C_x = 0.085$ and $N_r = 0.0029$. The 63% response time of the nonlinear system in $t_r = 20$s. Finally, the linear discrete-time model (3) has been obtained by considering a sampled time $T_s = 0.2$s. The operating point $u_e = 0.5$ considered for the linearization corresponds to the middle of the range $[0, 1]$, being thereby as far as possible from the lower and upper bounds.

5.2 Open-loop optimal solution

Let us consider the track shown in Fig. 2 characterized by the straight paths with lengths $l_1 = 150\text{m}$, $l_3 = 180\text{m}$, $l_5 = 330\text{m}$, $l_7 = 100\text{m}$ and the curves with radius $r_{c2} = 150\text{m}$, $r_{c4} = 70\text{m}$, $r_{c6} = 100\text{m}$, $r_{c8} = 200\text{m}$, with a total distance of $x_{1Total} = 1.5768\text{km}$, a maximum final time of $t_{fmax} = 1$h and $F_t = 2.5428[\text{N}]$. The solution of the multi-phase optimization problem (2), where the end of each phase is the beginning of the next one, is obtained by using GPOPS and is depicted in Fig. 3. The solution found has a final time $t_f = 737.61$s, the final position reached is $x_1(737.61\text{s}) = 1.5768\text{km}$, the velocity in curves is not bigger than the allowed velocity $x_{2curvei}$ for the curve with radius r_{ci}. This fact can be appreciated in the flat regions obtained in the velocity solution x_2^*, where the velocity remains constant all along the curve. Besides, the input is not bigger than 1 or smaller than 0, accomplishing in this way all the constraints imposed in (2) for the minimization of the consumption. The MPC controller must guarantee the tracking of the obtained optimal profile $x_2^*(k)$ respecting the constant constraints Δx_{min} and Δx_{max}, and assuring that the difference $\Delta u(k)$ remains between the time-varying constraints Δu_{min} and Δu_{max} given by (9) and shown in Fig. 3.

5.3 MPC controller

The MPC controller has been tested in simulation for the tracking of the low consumption strategy depicted in Fig. 3 with the MPC state restrictions set as $\Delta x_{min} = -5\text{km/h}$ and $\Delta x_{max} = 5\text{km/h}$ (see (6)). The prediction horizon is $N_p = 10$ and the weighting matrices Q and R in (10) are identity matrices. An error of $\pm 10\%$ has been added to the estimation of the parameters C_x and N_r. In Fig. 4 the tracking differences ($x_2 - x_2^*$) and ($u - u^*$) are depicted for a variation in the vehicle mass m of 10%, 20% and 50%. Notice that the constraints Δx_{min} and
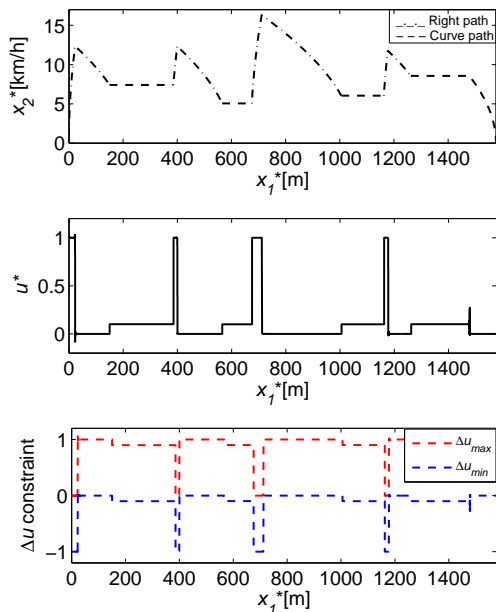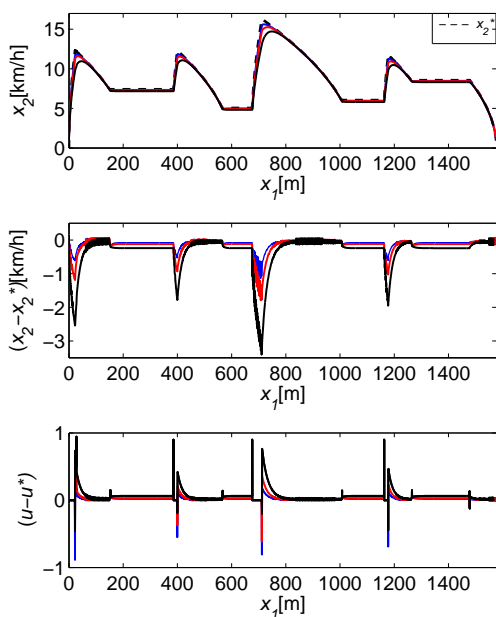
Fig. 3. Solution of the optimization problem.

Fig. 4. Tracking simulation experiment for a mass variation of
10% (blue), 20% (red) and 50% (black).

Δx_{max} are accomplished, and also that Δu remains between the boundaries Δu_{min} and Δu_{max}. The results shown in Fig. 4 point out a good tracking of the driving strategy, which guarantees a low consumption near the optimal one.

6. CONCLUSION AND FUTURE WORKS

The issue of achieving the minimum consumption of an electric vehicle involved in the European Shell Eco-marathon race has been formulated as a multi-phase optimization problem. A model of the vehicle has been obtained and used to design a time-varying Model Predictive Control tracking strategy. The controller achieves good tracking performances, despite the estimation error introduced in the loop. Also, stability and convergence has been proved for the time-varying MPC law. As future work, the controller will be tested on board for the actual race.

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