# Overview of the structure tensor model

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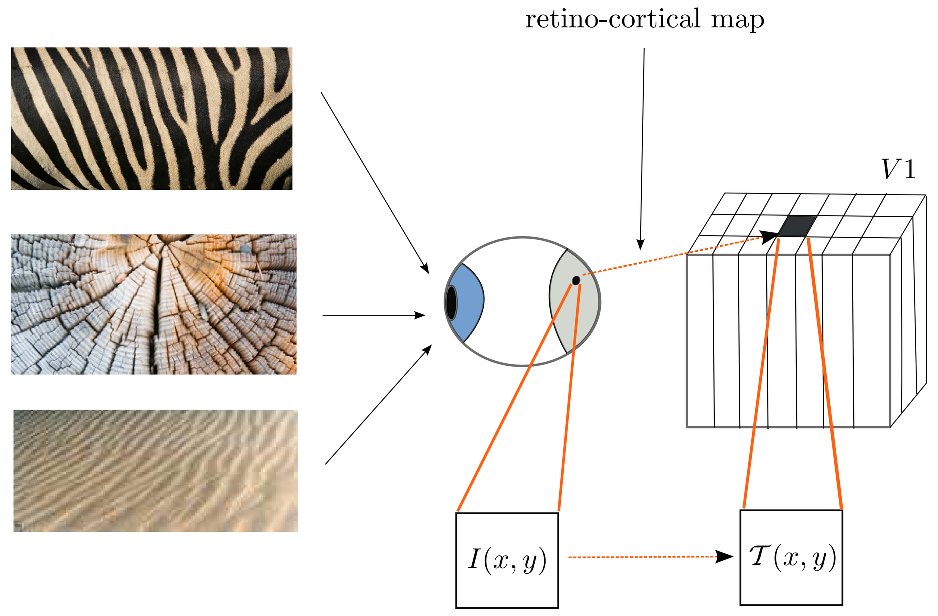
**In this poster, we present an overview of the structure tensor model. We first recall the general ideas developped in this framework: model of V1 seen as a set of hypercolumns which encode texture via the structure tensor. We present and analyse analog of Wilson-Cowan equations written in the feature space of 2x2 symmetric definite positive matrices. We show that this model englobes the well-known ring model of orientation. We present some results about two classes of specific stationary solutions of the system.**


## Hypercolumn of texture



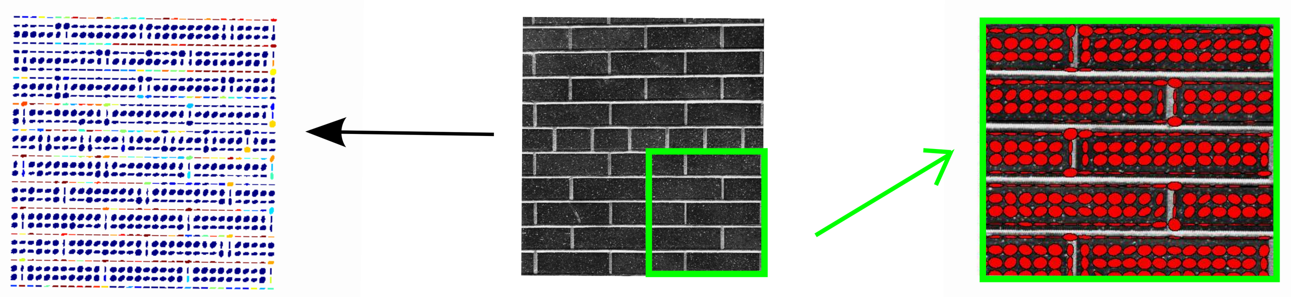

Populations of neurons in each so-called "hypercolumn" of the visual cortex layer V1 are sensitive to informations carried by the structure tensor of the image.

## Structure tensor

Local direction:

$$\nabla I^{\sigma_1}(x,y) = \nabla I \star g_{\sigma_1}(x,y) \quad \text{where} \quad g_\sigma(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

Structure tensor: $\mathcal{T}(x,y) = \left(\nabla I^{\sigma_1}(\nabla I^{\sigma_1})^{\mathbf{T}}\right) \star g_{\sigma_2}(x,y) \in \mathrm{SPD}(2)$



## Geometry of SPD(2)

- The distance between two tensors is: $d(\mathcal{T}_1,\mathcal{T}_2) = \left\|\log\left(\mathcal{T}_1^{-\frac{1}{2}}\mathcal{T}_2\mathcal{T}_1^{-\frac{1}{2}}\right)\right\|_F$
- (SPD(2),d) is a Riemannian symmetric space of noncompact type with negative sectional curvature (hyperbolic geometry).
- $\mathrm{SPD}(2) = \mathbb{D} \times \mathbb{R}_*^+$ with $\mathbb{D}$ hyperbolic disk.
- $\mathcal{T} = (z,\Delta) \quad d(z_1,\Delta_1|z_2,\Delta_2) = \sqrt{2\log\left(\frac{\Delta_1}{\Delta_2}\right)^2 + \left(\operatorname{atanh}\frac{|z_1-z_2|}{|1-\bar{z}_1 z_2|}\right)^2}$
- Isometry group: $\mathrm{GL}(2,\mathbb{R}) \simeq \mathrm{U}(1,1) \times \mathbb{R}_*^+$

## Wilson-Cowan equation

The average membrane potential is function of the structure tensor and its time evolution is governed by the following equation:

$$\frac{\partial V}{\partial t}(\mathcal{T},t) = -\alpha V(\mathcal{T},t) + \int_{SPD(2)} w(\mathcal{T},\mathcal{T}')S(V(\mathcal{T}',t))\,d\mathcal{T}' + I(\mathcal{T},t) \quad (1)$$

### Existence-Uniqueness:

THEOREM: If the connectivity function and the external input are sufficiently regular then there exists a unique bounded solution to the above equation.

## Recovering RM of orientation

- $\sigma_2 \to 0$ in the definition of the structure tensor:

$$T = \left(\nabla I^{\sigma_1}(\nabla I^{\sigma_1})^{\mathbf{T}}\right) \star g_{\sigma_2} \underset{\sigma_2\to 0}{\longrightarrow} \nabla I^{\sigma_1}(\nabla I^{\sigma_1})^{\mathbf{T}}$$

- For all structure tensor we have the decomposition:

$$T = \lambda_1 \mathbf{e}_1\mathbf{e}_1^{\mathbf{T}} + \lambda_2 \mathbf{e}_2\mathbf{e}_2^{\mathbf{T}} = (\lambda_1-\lambda_2)\mathbf{e}_1\mathbf{e}_1^{\mathbf{T}} + \lambda_2 \mathbf{I}_2$$

if $\lambda_1 - \lambda_2 \gg \lambda_2$ straight edge along $\mathbf{e}_2$

$$\mathbb{P}: \begin{cases} \mathrm{SPD}(2) \to \mathrm{S}^1 \\ \quad T \quad \mapsto \theta = \arg(\mathbf{e}_2) \end{cases}$$

We can write the projection of equation (1) by $\mathbb{P}$ and recover the ring model of orientation.

## Hyperbolic bumps

Simplified equation:

$$\partial_t V(z,t) = -\alpha V(z,t) + \int_{\mathbb{D}} e^{-\frac{d(z,z')}{b}} H(V(z',t)-\kappa)\mathrm{dm}(z') + \mathcal{I}e^{-\frac{d(0,z)^2}{2\sigma^2}}$$

Stationary solution: $0 = -\alpha V(r) + \mathcal{M}(r,\omega) + I_{\mathrm{ext}}(r)$ with $r = d(0,z)$

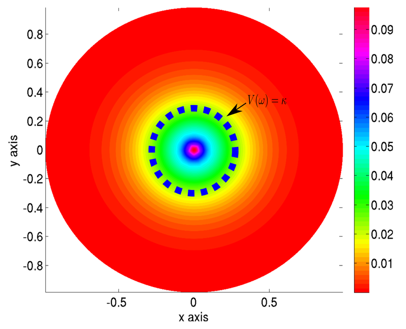
Boundary conditions:

$$\begin{cases} V(r) > \kappa, & r \in [0,\omega[, \quad V(\omega) = \kappa \\ V(r) < \kappa, & r \in ]\omega,\infty[, \quad V(\infty) = 0 \end{cases}$$

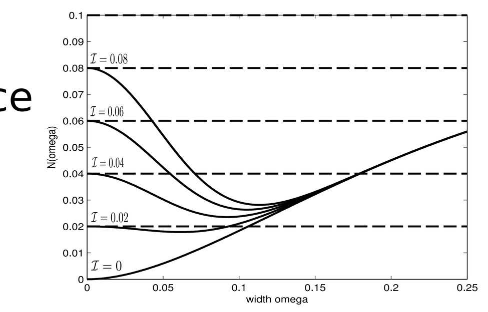
Self-consistency condition: $\alpha\kappa = \mathcal{M}(\omega,\omega) + I_{\mathrm{ext}}(\omega) = N(\omega)$

Stability result: when N'(w)<0 then the hyperbolic bump is stable.

Bump corresponding to the identity tensor.



Existence curves.

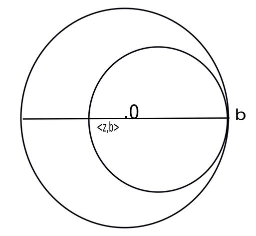


## Bifurcation of H-planforms

Reduced problem: $\partial_t V(z,t) = -\alpha V(z,t) + \int_{\mathbb{D}} W(d(z,z'))S(V(z',t))dz'$ with $S(0) = 0$, $\mu = S'(0)$

Linearized equation: $\partial_t V(z,t) = -\alpha V(z,t) + \mu\int_{\mathbb{D}} W(d(z,z'))V(z',t)dz'$

### Linear stability analysis:



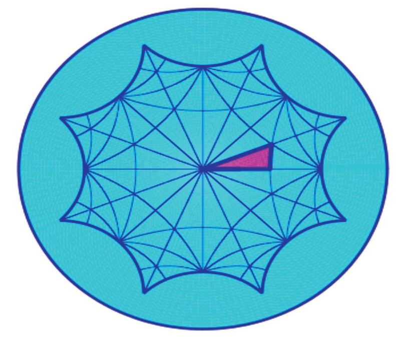
- $e_{\rho,b}(z) = e^{(i\rho+\frac{1}{2})<z,b>}$ : eigenfunctions of the Laplace-Beltrami operator in $\mathbb{D}$ .
- look for stability against perturbations of the form: $V(z,t) = e_{\rho,b}(z)e^{\sigma t}$
- with a "Mexican-hat" connectivty (difference of Gaussians), there exists critical value $\mu = \mu_c$ such that V=0 becomes unstable when $\mu > \mu_c$ .

**Problem:** The spectrum associated to the linearized equation is continuous due to the equivariance of the equation.

**Idea:** Use the technics developped in the Euclidean plane for solutions of equivariant systems: restriction to solution which are periodic under a lattice subgroup of the group of deplacements of the plane E(2). We have applied these ideas to our hyperbolic problem in $\mathbb{D}$, in the case of an octagonal lattice.

### Octagonal lattice:

The octagonal lattice group is generated by four hyperbolic translations. The group of symmetry of the octagon has 96 elements and is generated by reflections through sides of the purple triangle.

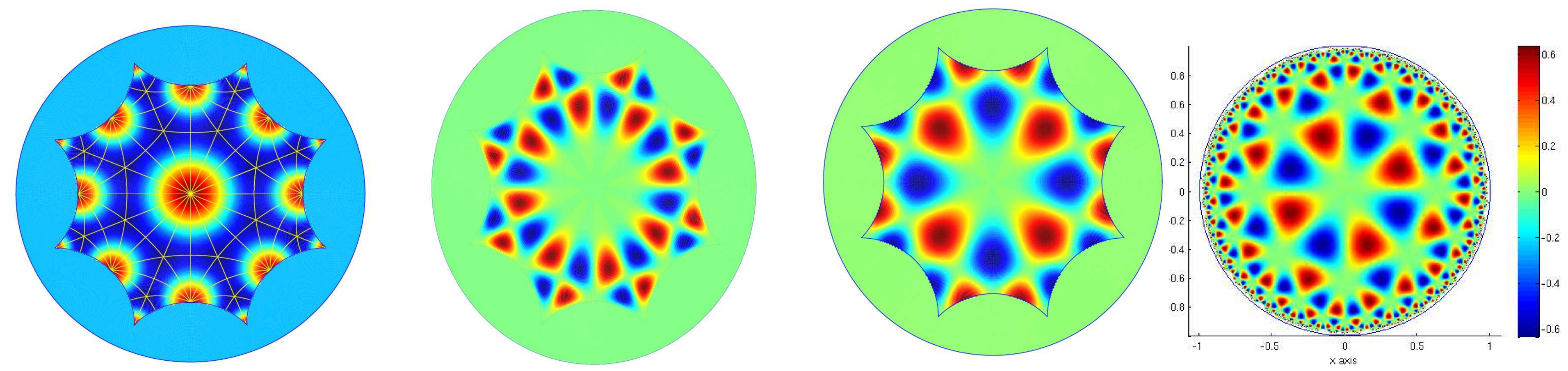


### Bifurcation analysis:

Bifurcation machinery: we have applied center manifold technics and equivariant bifurcation theory to obtain generic bifurcation diagrams for all 13 irreductible representations. In the four dimensional case, there is a heteroclinic behavior for some range of the parameters which implies multistability of the system in this case.

### Computation of octagonal H-planform:

H-planforms are eigenfunctions of the Laplace-Beltrami operator in the octagon wich are invariant under a subgroup of symmetry of the octagon. The computation are performed with finite element methods.



Dim 1   Dim 2   Dim 3

## Current work:

- Spatialization of the model: to connect hypercolumns.
- Validation of the structure tensor model (from biology and computer science).
- Observabilty of some patterns that have been predicted by theory.
- From a field of structure tensors, produce possible image.
- Revisit neurogeometric models introduced by Petitot-Citti-Sarti for orientation.

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