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Térence Bayen, Francis Mairet, Matthieu Sebbah

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Abstract

This paper is devoted to the study of a one-dimensional optimal control problem of Lagrange type under periodic constraints. The system is governed by a Kolmogorov equation including a control term. We show that the set of initial conditions for which there exists a control such that the associated trajectory is periodic is an interval. Thanks to Pontryagin Maximum Principle, we characterize optimal controls whenever the dynamics has two autonomous phases. Finally, we provide numerical simulation of the problem for a system describing a periodic photobioreactor.

1 Introduction

Bioprocesses have been widely used in various fields, like energy production, wastewater treatment, pharmaceutical production, food industry and more [9]. To support their development, many tools from control theory have been proposed for the analysis, control and optimization of such processes [3]. For continuous bioreactor, the objective is generally to stabilize the process at its steady-state optimum.

Nevertheless, for some bioprocesses, a periodic forcing (due to an external factor) prevents work at steady state as it is usually done. As examples, we can mention microalgae culture grown under solar light (see [2]), or a biological wastewater treatment which faces the daily fluctuation of the input [1]. On the other hand, periodic operation of a bioreactor is also a topic of interest. Actually, theoretical and experimental studies have shown that the performance (for instance micro-algae or bio-gas production) of some optimal steady-state continuous bioreactors can be improved by a periodic modulation of an input such as dilution rate or airflow. For more details on forced periodic operation of bioreactors, the reader is referred to a recent review of Silveston et al [4]. In both cases (whether it is in a forced manner or voluntarily), it is desirable to find a (preferably optimal) control which allows to obtain a periodic solution. Thus, the control can be repeated at each period in order to optimize long run operation.

This framework has motivated the present study, which is devoted to the analysis of an optimal control problem where the dynamics is controlled by a *Kolmogorov equation* (see e.g. [15], [14], [8]). More precisely, our objective is to optimize a cost function over one period under periodic constraints on the system and bounded controls. The essential feature of this work relies on the fact that the system has two autonomous phases. We also assume that there exists a unique minimum of the Lagrangian with respect to the state and which is independent of the time (this occurs in some bioprocesses such as maximal biomass production in the chemostat model, see section 4). In order to find an issue to this optimal control problem, we apply Pontryagin Maximum Principle (PMP). In particular, we show that this minimum corresponds to the unique *singular arc* of the problem meaning that the corresponding control is not extremal. The interesting fact is that this singular arc is non-necessarily controllable on a period. This will lead to different optimal strategies depending both on the periodic constraints on the system and on the controllability of this singular arc. More

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precisely, when the singular arc is always (resp. never) controllable, the structure of the optimal control is singular (resp. bang-bang). Moreover, whenever the singular arc is controllable only on one phase, the optimal control is of type bang-singular-bang. In the latter, the characterization of the optimal control corresponds to a *turnpike* periodic solution with MRAP (most rapid approach) strategy, see [5].

The paper is organized as follows. In the second section, we investigate the question of the existence of periodic solutions of the system. Based on the results of [15], we introduce Kolmogorov controlled equations, and we characterize the set of initial conditions for which there exists a periodic control such that the associated solution is periodic. The third section is devoted to the study of the optimal control problem via the PMP when the dynamics has two autonomous phases. This allows us to describe the structure of optimal trajectories for different cases depending on the controllability of the singular arc. In the last section, we provide numerical simulations of optimal strategies corresponding to two different cases. The first one is concerned with a chemostat model with two phases where the objective consists in maximizing the production of biomass during a period. It illustrates the case where the singular arc is controllable only on one of the two phases which characterizes bang-singular-bang strategy. The second one illustrates the case where the strategy is bang-bang.

2 Kolmogorov equation with control in dimension 1

In this section, we extend a result of [15] on periodic solutions of Kolmogorov equations to the control setting. For sake of completeness, we will recall some results which can be found in [15], and that we have used in order to characterize the set of initial conditions for which there exists an admissible control such that the associated solution is periodic.

2.1 Preliminaries on periodic ODEs

Given a Caratheodory function $g : (t, x) \in \mathbb{R}^2 \to \mathbb{R}$, local Lipschitz continuous with respect to x, we denote by $x(\cdot, t_0, x_0)$ the unique solution of the Cauchy problem

$$\begin{cases} \dot{x} = g(t, x), \\ x(t_0) = x_0, \end{cases} \tag{2.1}$$

defined on some maximal interval $[t_0, t_f[$ for some $t_f > t_0$. Likewise, let $a, b \in \mathbb{R}$ with $a < b$ and consider

$$\begin{array}{rccc} f: & \mathbb{R}^2 \times [a, b] & \to & \mathbb{R} \\ & (t, x, u) & \mapsto & f(t, x, u), \end{array}$$

a measurable function with respect to the pair (t, u) and local Lipschitz continuous with respect to x. Given $u : \mathbb{R} \to [a, b]$ a measurable control function, we denote by $x_u(\cdot, t_0, x_0)$ the unique solution of the Cauchy problem

$$\begin{cases} \dot{x} = f(t, x, u(t)), \\ x(t_0) = x_0, \end{cases} \tag{2.2}$$

which is defined on some maximal interval $[t_0, t_f[$ for some $t_f > t_0$. A solution $x(\cdot) := x(\cdot, t_0, x_0)$ (resp. $x(\cdot) := x_u(\cdot, t_0, x_0)$), will be mentioned as T-periodic for some $T > 0$, if it is defined on $[t_0, +\infty[$ and satisfies

$$x(t + T) = x(t), \quad \forall t \in [t_0, +\infty[.$$

We now recall a result about periodic solutions of ODEs given in [15].

Theorem 2.1. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a Caratheodory function, T-periodic with respect to the first variable t and local Lipschitz continuous with respect to the second variable x. Suppose that there exist T-periodic functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ integrable on $[0, T]$ and a real number $d > 0$ such that:*

(i) $\int_0^T \alpha(t)dt > 0 > \int_0^T \beta(t)dt$;

(ii) for all $t \in \mathbb{R}$,

$$\begin{cases} \alpha(t) \leq g(t, x), \quad \forall x \leq -d, \\ \beta(t) \geq g(t, x), \quad \forall x \geq d. \end{cases}$$

Then there exist two T-periodic solutions $x_*(\cdot)$ and $x^*(\cdot)$ of

$$\dot{x} = g(t, x), \tag{2.3}$$

such that every T-periodic solutions $\tilde{x}(\cdot)$ of (2.3) satisfies, for all $t \in \mathbb{R}$,

$$x_*(t) \leq \tilde{x}(t) \leq x^*(t). \tag{2.4}$$

Moreover, there exists $r > 0$ such that for all $(t_0, x_0) \in \mathbb{R}^2$,

$$x_0 \leq -r \quad \Rightarrow \quad x(t_0 + T, t_0, x_0) > x_0; \tag{2.5}$$

$$x_0 \geq r \quad \Rightarrow \quad x(t_0 + T, t_0, x_0) < x_0. \tag{2.6}$$

Finally, for all $(t_0, x_0) \in \mathbb{R}^2$, there exists a T-periodic solution $\tilde{x}$ such that

$$\lim_{t \to \infty} |\tilde{x}(t) - x(t, t_0, x_0)| = 0.$$

First, notice that the result of [15] is given for a continuous function g instead of a Caratheodory function. However it is mentioned that the result is still true in this context. The functions α and β are also taken continuous in [15] only to have α and β integrable property. Finally, (2.5) and (2.6) are not explicitly stated in the result of [15] even if they are part of the proof. Given $G : \Omega \to \mathbb{R}$ a function such that for all $t \in \mathbb{R}$, $x \longmapsto G(t, x)$ is bounded in a neighborhood of zero, we say that an ODE is characterized as *Kolmogorov equation* if it is of the form:

$$\dot{x} = xG(t, x). \tag{2.7}$$

Remark 2.2. Notice that $\mathbb{R}_+^*$ and $\mathbb{R}_-^*$ are invariant with respect to (2.7).

The next result is an application of Theorem 2.1 in the context of Kolmogorov ODEs.

Theorem 2.3. *Let $G : \mathbb{R} \times [0, +\infty[\to \mathbb{R}$ be a Caratheodory function, T-periodic with respect to the first variable t, and local Lipschitz continuous with respect to the second variable x. Suppose that there exist $M > 0$ and $s_0, s_1 \in [0, T]$, $s_0 < s_1$ such that*

(i) $\int_0^T G(t, 0)dt > 0 > \int_0^T G(t, M)dt$;

(ii) for all $t \in [0, T]$, $G(t, \cdot)$ is nonincreasing;

(iii) for all $t \in [s_0, s_1]$, $G(t, \cdot)$ is (stricly) decreasing.

Then (2.7) admits a unique positive T-periodic solution $\tilde{x}(\cdot)$, and given $t_0 \in \mathbb{R}$, we have for all $x_0 \in]0, +\infty[$,

$$x_0 < \tilde{x}(t_0) \quad \Rightarrow \quad x(t_0 + T, t_0, x_0) > x_0; \tag{2.8}$$

$$x_0 > \tilde{x}(t_0) \quad \Rightarrow \quad x(t_0 + T, t_0, x_0) < x_0. \tag{2.9}$$

Moreover for all $(t_0, x_0) \in \mathbb{R} \times]0 + \infty[$, we have

$$\lim_{t \to \infty} |x(t, t_0, x_0) - \tilde{x}(t)| = 0. \tag{2.10}$$

This result is a slight modification of Proposition 5 in [15] in order to fit in the setting of a Caratheodory dynamics instead of a continuous one. In particular, we have assumed that $G(t, \cdot)$ is decreasing for all $s \in [s_0, s_1]$ whereas this assumption holds in [15] at a single point $\hat{t}$. This modification is to take into account the fact that the dynamics is only measurable with respect to t in our setting. We now investigate how Theorem 2.3 can be generalized in the control setting.

2.2 Controlled Kolmogorov equations

In this part, we add a control variable to the Kolmogorov equation (2.7) and we characterize the set of initial conditions from which there exists a periodic and positive solution. Let $F : \mathbb{R} \times [0, +\infty[\times[a, b] \to \mathbb{R}$ and consider the Kolmogorov equation with control

$$\dot{x} = xF(t, x, u). \tag{2.11}$$

Moreover, we denote by $\mathcal{U}_T^{a,b}$ the set of admissible controls defined by:

$$\mathcal{U}_T^{a,b} := \{u : \mathbb{R} \to [a, b] \mid u \text{ meas. and } \text{ T-periodic}\},$$

where $0 < a < b$. Whenever u is fixed in the set $\mathcal{U}_T^{a,b}$, the function $t \mapsto F(t, x, u(t))$ is also T-periodic. So, the equation

$$\dot{x} = xG_u(t, x) \tag{2.12}$$

where $G_u(t, x) := F(t, x, u(t))$, admits a unique periodic and positive solution denoted $\tilde{x}_u$, provided that G_u satisfies the assumptions of Theorem 2.3. Our aim now is to characterize the set of initial conditions $\mathcal{C}(t_0)$ for which there exists a control such that the solution of (2.12) associated to this control and this initial condition is periodic and positive:

$$\mathcal{C}(t_0) := \{x_0 > 0 \mid \exists u \in \mathcal{U}_T^{a,b} \text{ s.t. } x_u(\cdot, t_0, x_0) \text{ is T-periodic and positive}\}.$$

The next theorem gives a characterization of $\mathcal{C}(t_0)$.

Theorem 2.4. *Let $F : \mathbb{R} \times [0, +\infty[\times[a, b] \to \mathbb{R}$ be a function measurable with respect to the pair (t, u), local Lipschitz continuous with respect to x, T-periodic with respect to t and non-increasing with respect to u. Suppose that there exists $M > 0$ and $s_0, s_1 \in [0, T]$ with $s_0 < s_1$ such that:*

(i) $\int_0^T F(t, 0, b)dt > 0 > \int_0^T F(t, M, a)dt$;

(ii) $\forall t \in [0, T]$, $\forall u \in [a, b]$, $F(t, \cdot, u)$ *is nonincreasing;*

(iii) $\forall t \in [s_0, s_1]$, $\forall u \in [a, b]$, $F(t, \cdot, u)$ *is strictly decreasing.*

Then, for all $u \in \mathcal{U}_T^{a,b}$, there exists a unique T-periodic positive solution of (2.12) *denoted $\tilde{x}_u$ and for all $(t_0, x_0) \in \mathbb{R}\times]0, +\infty[$, we have*

$$\lim_{t\to\infty} |x_u(t, t_0, x_0) - \tilde{x}_u(t)| = 0. \tag{2.13}$$

Moreover for all $t_0 \in \mathbb{R}$,

$$\mathcal{C}(t_0) = [\tilde{x}_b(t_0), \tilde{x}_a(t_0)], \tag{2.14}$$

where $\tilde{x}_a(\cdot)$ (resp. $\tilde{x}_b(\cdot)$) denotes the unique T-periodic positive solution of (2.12) *with a constant control $u = a$ (resp. $u = b$).*

The proof of Theorem 2.4 relies on the following lemma.

Lemma 2.5. *Under the hypotheses of Theorem 2.4, let $t_0, t_1 \in \mathbb{R}$ with $t_0 < t_1$, $x_0 > 0$ and $u_1, u_2 \in \mathcal{U}_T^{a,b}$. If $u_1(t) \leq u_2(t)$, for all $t \in [t_0, t_1]$, then*

a) $x_{u_2}(t, t_0, x_0) \leq x_{u_1}(t, t_0, x_0)$, *for all $t \in [t_0, t_1]$,*

b) $x_{u_1}(t, t_1, x_0) \leq x_{u_2}(t, t_1, x_0)$, *for all $t \in [t_0, t_1]$.*

Proof of the Lemma. We only give the proof of a) since the proof of b) is quite similar. For $i \in \{1, 2\}$ define $x_i(t) = x_{u_i}(t, t_0, x_0)$, for all $t \in [t_0, +\infty[$. Assume that there exists $t_2 \in]t_0, t_1]$ such that $x_2(t_2) > x_1(t_2)$. Without loss of generality, we can assume that $x_2(t_2) = x_1(t_2)$ and $x_2(t) > x_1(t)$, for all $t \in]t_2, t_1]$. Setting $z := x_2 - x_1$, we have $z(t_2) = 0$ and for all $t \in]t_2, t_1]$, $z(t) > 0$. Now, we can write for almost every $t \in [t_2, t_1]$,

$$\dot{z}(t) = z(t)F(t, x_2(t), u_2(t)) + x_1(t)\varphi(t), \tag{2.15}$$

where

$$\varphi(t) := F(t, x_2(t), u_2(t)) - F(t, x_1(t), u_2(t)) + F(t, x_1(t), u_2(t)) - F(t, x_1(t), u_1(t)).$$

Combining the monotonicity property of F together with $x_1 \geq 0$ yields to:

$$\dot{z}(t) \leq z(t)F(t, x_2(t), u_2(t)).$$

By Gronwall's Lemma, we obtain $z \leq 0$ on the interval $[t_2, t_1]$, which is a contradiction. $\square$

Proof of Theorem 2.4. First, we show that for every fixed $u \in \mathcal{U}_T^{a,b}$, there exists a unique T-periodic positive solution of (2.12). Hereafter, we set $G_u(t, x) := F(t, x, u(t))$. Using that F is nonincreasing with respect to u, we obtain that for all $t \in [0, T]$,

$$G_u(t, 0) = F(t, 0, u(t)) \geq F(t, 0, b).$$

Therefore, assumption (i) implies:

$$\int_0^T G_u(t, 0)dt \geq \int_0^T F(t, 0, b)dt > 0.$$

Likewise we deduce that $\int_0^T G_u(t, M) < 0$. Moreover, it is straightforward from assumptions (ii) and (iii) that, for all $t \in [0, T]$, $G_u(t, \cdot)$ is nonincreasing and that, for all $t \in [s_0, s_1]$, $G_u(t, \cdot)$ is decreasing. We can now apply Theorem 2.3 to get the existence and the uniqueness of a T-periodic positive solution $\tilde{x}_u(\cdot)$ of (2.12). We now show that given $t_0 \in \mathbb{R}$, $\mathcal{C}(t_0) = [\tilde{x}_b(t_0), \tilde{x}_a(t_0)]$. First, we prove that $\tilde{x}_b(t_0) \leq \tilde{x}_a(t_0)$. To do so, we claim that given any control $u \in \mathcal{U}_T^{a,b}$, we have $\tilde{x}_u(t_0) \leq \tilde{x}_a(t_0)$ and $\tilde{x}_u(t_0) \geq \tilde{x}_b(t_0)$, which in particular will show that $\tilde{x}_b(t_0) \leq \tilde{x}_a(t_0)$, but also that $\mathcal{C}(t_0) \subseteq [\tilde{x}_b(t_0), \tilde{x}_a(t_0)]$. Let $u \in \mathcal{U}_T^{a,b}$. We only show that $\tilde{x}_u(t_0) \leq \tilde{x}_a(t_0)$ (the proof of the other inequality is analogous). As $u \geq a$, we obtain from Lemma 2.5

$$x_u(t_0 + T, t_0, \tilde{x}_a(t_0)) \leq x_a(t_0 + T, t_0, \tilde{x}_a(t_0)) = \tilde{x}_a(t_0), \tag{2.16}$$

as $x_a(\cdot, t_0, \tilde{x}_a(t_0)) = \tilde{x}_a(\cdot)$, which is T-periodic. This implies that $\tilde{x}_u(t_0) \leq \tilde{x}_a(t_0)$ (otherwise we would have $\tilde{x}_u(t_0) > \tilde{x}_a(t_0)$ which from Theorem 2.3 implies that $\tilde{x}_a(t_0) < x_u(t_0 + T, t_0, \tilde{x}_a(t_0))$ in contradiction with (2.16)). Thus, the claim is proved. It only remains to show that $[\tilde{x}_b(t_0), \tilde{x}_a(t_0)] \subseteq \mathcal{C}(t_0)$. Let us take $x_0 \in [\tilde{x}_b(t_0), \tilde{x}_a(t_0)]$. If $x_0 = \tilde{x}_b(t_0)$ or $x_0 = \tilde{x}_a(t_0)$ then $x_0 \in \mathcal{C}(t_0)$, so we may assume that $x_0 \in (\tilde{x}_b(t_0), \tilde{x}_a(t_0))$. To simplify the writing, we put

$$x_a(\cdot) := x_a(\cdot, t_0, x_0), \ x_b(\cdot) := x_b(\cdot, t_0 + T, x_0).$$

By considering $x_b(\cdot)$ backward time on $[t_0, t_0 + T]$ and the fact that $x_0 \in]\tilde{x}_b(t_0), \tilde{x}_a(t_0)[$, we obtain from (2.8)-(2.9)

$$x_0 = x_a(t_0) < x_a(t_0 + T), \ x_b(t_0) > x_b(t_0 + T) = x_0.$$

It follows that there exists $t_1 \in]t_0, t_0 + T[$ such that $x_a(t_1) = x_b(t_1)$. Considering the T-periodic control

$$u(t) = \begin{cases} b \text{ if } t \in [t_0, t_1[, \\ a \text{ if } t \in [t_1, t_0 + T[, \end{cases}$$

we have $x_u(\cdot, t_0, x_0) = x_a(\cdot)$ on $[t_0, t_1[$, and $x_u(\cdot, t_0, x_0) = x_b(\cdot)$ on $[t_1, t_0 + T]$. Consequently, $x_u(\cdot, t_0, x_0)$ is T-periodic which means that $x_0 \in \mathcal{C}(t_0)$, hence $[\tilde{x}_b(t_0), \tilde{x}_a(t_0)] \subseteq \mathcal{C}(t_0)$. Finally, the proof of (2.13) is a direct consequence of Theorem 2.3. $\square$

3 Study of an optimal control problem in a two phases environment

In this section, we formalize the concept of day/night environment in the Kolmogorov setting. Finally, we study an optimal periodic control problem in Lagrange's form, associated to such a dynamics. We then address in the last subsection optimality results in the different cases depending on the parameters of the system.

3.1 Statement of the problem

In this section, we consider a Kolmogorov controlled equation with two phases. More precisely, let us consider two functions $f_1, f_2 : [0, +\infty[\times[a, b] \to \mathbb{R}$, and for $u \in \mathcal{U}_T^{a,b}$ let us define the dynamics by:

$$\dot{x}(t) = f(t, x, u(t)), \tag{3.17}$$

where $f : \mathbb{R} \times [0, +\infty[\times[a, b] \to \mathbb{R}$ is given by $f(t, x, u) = xF(t, x, u)$, and F is the T-periodic function defined by:

$$\begin{array}{rccl} F: & \mathbb{R} \times [0, +\infty[\times[a, b] & \to & \mathbb{R} \\ & (t, x, u) & \mapsto & \begin{cases} f_1(x, u) \text{ if } t \in [0, \overline{T}[, \\ f_2(x, u) \text{ if } t \in [\overline{T}, T[. \end{cases} \end{array} \tag{3.18}$$

Given $\ell : [0, T] \times [0, +\infty[\to \mathbb{R}$, the optimal control problem that we consider can be stated as follows;

$$\min_{u \in \mathcal{U}_T^{a,b}} J(u) := \int_0^T \ell(t, \tilde{x}_u(t))dt, \tag{3.19}$$

where $\tilde{x}_u$ is the unique T-periodic positive solution of (3.17) associated to the given control u. In the following, we assume that the system satisfies the following hypotheses.

H 1. The functions f_1, f_2 are continuous on $[0, +\infty[\times[a, b]$ and continuously differentiable with respect to x.

H 2. The functions f_1, f_2 are (strictly) decreasing with respect to x and u.

H 3. The following inequality is fulfilled:

$$\overline{T}(f_1(0, b) - f_2(0, b)) + Tf_2(0, b) > 0, \tag{3.20}$$

H 4. For each $i \in \{1, 2\}$, there exists $x_i^- \in [0, +\infty[$ such that

$$f_i(x_i^-, a) < 0. \tag{3.21}$$

H 5. The lagrangian ℓ is continuous on $[0, \overline{T}[\times[0, +\infty[$ and on $[\overline{T}, T[\times[0, +\infty[$ and can be extended continuously on $[0, \overline{T}] \times [0, +\infty[$ and on $[\overline{T}, T] \times [0, +\infty[$, and it is continuously differentiable with respect to x.

H 6. We suppose that there exists a unique $x_\sigma \in [0, +\infty[$ such that for all $t \in [0, T]$ and all $x \in [0, +\infty[$,

$$\ell(t, x_\sigma) \leq \ell(t, x), \quad \begin{cases} x > x_\sigma \Rightarrow \frac{\partial \ell}{\partial x}(t, x) > 0, \\ x < x_\sigma \Rightarrow \frac{\partial \ell}{\partial x}(t, x) < 0, \end{cases} \tag{3.22}$$

Remark 3.1. (i) The first four hypotheses contains the hypotheses of Theorem 2.4 traduced in the day/night environment setting, in order to guarantee the existence and uniqueness of a T-periodic solution of (3.17), given $u \in \mathcal{U}_T^{a,b}$.

(ii) We note here that Hypothesis 2 is not equivalent to the monotonicity assumptions (ii) and (iii) on F in Theorem 2.4. Indeed, Hypothesis 2 implies that F is strictly decreasing with respect to x for all t (and not only on some time interval $[s_0, s_1]$). This monotonicity property is fundamental in Definition 3.2 that will be used to characterize an optimal control. Moreover, we donot need F to be stritcly decreasing with respect to u to apply Theorem 2.4. However this implies different solutions for different controls which is often the case in a practical point of view.

(iii) Hypotheses 5 together with the continuous differentiability of f_1, f_2 is needed to ensure the existence of an optimal control. Indeed, applying Theorem 2.4 in the day/night environment yields that T-periodic trajectories solution of (3.17) are uniformly bounded. We can then easily conclude on the existence of an optimal control for problem (3.19) by applying [6] (notice that the discontinuity of the dynamics at time $\overline{T}$ can be removed by embedding (3.17) into a two-dimensional system).

(iv) The hypothesis 6 is concerned with some monotonicity properties on the lagrangian ℓ and is crucial for the study of the structure of the optimal control problem.

Hypothesis 4 ensures the existence of equilibria of the ODEs

$$\dot{x} = x f_i(x, u),$$

for each $i \in \{1, 2\}$ and $u \in \{a, b\}$.

Definition 3.2. There exists $x_a^1, x_a^2, x_b^1, x_b^2 \in [0, +\infty[$ such that for all $i \in \{1, 2\}$, all $u \in \{a, b\}$ and all $x > 0$:

$$\begin{cases} x < x_u^i \Rightarrow f_i(x, u) > 0, \\ x = x_u^i \Rightarrow f_i(x, u) = 0, \\ x > x_u^i \Rightarrow f_i(x, u) < 0, \end{cases} \tag{3.23}$$

Moreover, one has

$$x_b^1 \leq x_a^1 \quad \text{and} \quad x_b^2 \leq x_a^2. \tag{3.24}$$

Remark 3.3. (i) Given $i \in \{1, 2\}$ and $u \in \{a, b\}$ one has either

$$f_i(x, u) < 0, \quad \forall x \in [0, +\infty[, \tag{3.25}$$

or there exists x_u^i such that (3.23) holds. If (3.25) holds, we set $x_u^i = 0$, to get (3.23).

(ii) The inequality (3.24) is straightforward from Hypothesis 2.

We easily deduce the following proposition.

Proposition 3.4. *(i). If $t_0 \in [0, \overline{T}[$, $u \in \{a, b\}$ and $x_0 \in [0, +\infty[$ then*

$$\begin{cases} x_0 < x_u^1 \Rightarrow x_u(\cdot, t_0, x_0) \text{ is strictly increasing on } [0, \overline{T}[, \\ x_0 > x_u^1 \Rightarrow x_u(\cdot, t_0, x_0) \text{ is strictly decreasing on } [0, \overline{T}[, \\ x_0 = x_u^1 \Rightarrow x_u(\cdot, t_0, x_0) \text{ is constant equal to } x_u^1 \text{ on } [0, \overline{T}[. \end{cases}$$

(ii). If $t_0 \in [\overline{T}, T[$, $u \in \{a, b\}$ and $x_0 \in [0, +\infty[$ then

$$\begin{cases} x_0 < x_u^2 \Rightarrow x_u(\cdot, t_0, x_0) \text{ is strictly increasing on } [0, \overline{T}[, \\ x_0 > x_u^2 \Rightarrow x_u(\cdot, t_0, x_0) \text{ is strictly decreasing on } [0, \overline{T}[, \\ x_0 = x_u^2 \Rightarrow x_u(\cdot, t_0, x_0) \text{ is constant equal to } x_u^2 \text{ on } [0, \overline{T}[. \end{cases}$$

Remark 3.5. Proposition 3.4 implies in particular that

$$\tilde{x}_b(t) \in [\min\{x_b^1, x_b^2\}, \max\{x_b^1, x_b^2\}], \quad \forall t \in [0, T],$$

resp.

$$\tilde{x}_a(t) \in [\min\{x_a^1, x_a^2\}, \max\{x_a^1, x_a^2\}], \quad \forall t \in [0, T],$$

otherwise it would contradict the periodicity of $\tilde{x}_b$, resp. $\tilde{x}_a$.

3.2 Pontryagin maximum principle

In this section, we apply Pontryagin maximum principle on (3.19) in order to obtain necessary conditions on optimal trajectories. The Hamiltonian $H := H(t, x, \lambda, \lambda_0, u)$ associated to the system is given by

$$H := \lambda f(t, x, u) + \lambda_0 \ell(t, x). \tag{3.26}$$

Let u^* be an optimal control of (3.19), and x^* be the T- periodic solution of (3.17) associated to u^*. Then, there exists $\lambda : [0, T] \to \mathbb{R}$ an absolutely continuous function and $\lambda_0 \leq 0$ such that $(\lambda(\cdot), \lambda_0) \neq 0$, with $\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x^*(t), \lambda(t), \lambda_0, u^*(t))$, i.e.,

$$\dot{\lambda}(t) = -\lambda(t) \frac{\partial f}{\partial x}(t, x^*(t), u^*(t)) - \lambda_0 \frac{\partial \ell}{\partial x}(t, x^*(t)), \tag{3.27}$$

and we have the maximality condition:

$$H(t, x^*(t), u^*(t), \lambda(t), \lambda_0) = \max_{v \in [a,b]} H(t, x^*(t), v, \lambda(t), \lambda_0), \quad \text{for a.e. } t \in [0, T], \tag{3.28}$$

together with the transversality condition on the adjoint vector (recall the periodicity of $x^*(\cdot)$):

$$\lambda(0) = \lambda(T). \tag{3.29}$$

An *extremal trajectory* is a quadruplet $(x^*, \lambda, \lambda_0, u^*)$ satisfying (3.17)-(3.27)-(3.28)-(3.29). If $\lambda_0 \neq 0$ (resp. $\lambda_0 = 0$), we say that the extremal is a *normal* (resp. an *abnormal*). As ℓ does not depend on u and $u \mapsto f(t, x, u)$ is nonincreasing, the function λ is the *switching function*. We obtain by (3.28) that for almost every $t \in [0, T]$:

$$\begin{cases} \lambda(t) > 0 \Rightarrow u^*(t) = a, \\ \lambda(t) < 0 \Rightarrow u^*(t) = b. \end{cases} \tag{3.30}$$

When $\lambda = 0$ on some time interval $I := [t_1, t_2]$, we say that the trajectory contains a *singular arc* on I. From (3.27), a singular arc is characterized by $x(t) = x_\sigma$ for all $t \in I$. We say that the singular arc is *controllable* provided that there exists $u_\sigma \in [a, b]$ such that the trajectory remains constant equal to x_σ on I. This is equivalent to the existence of $u^i_\sigma \in [a, b]$ such that

$$f_i(x_\sigma, u^i_\sigma) = 0. \tag{3.31}$$

It is worth noting that, in our context, by Hypothesis 2, equation (3.31) is equivalent to

$$\begin{cases} x_\sigma f_i(x_\sigma, a) \geq x_\sigma f_i(x_\sigma, x_\sigma) = 0, \\ x_\sigma f_i(x_\sigma, b) \leq x_\sigma f_i(x_\sigma, x_\sigma) = 0. \end{cases}$$

Therefore, by Definition 3.2, Equation (3.31) becomes

$$x_\sigma \in [x^i_b, x^i_a]. \tag{3.32}$$

As the controllability of the singular arc will be the crucial point in the study of optimal solutions in the next section, we introduce the following definition.

Definition 3.6. (i) We will say that the singular arc is always controllable when

$$x_\sigma \in [x^1_b, x^1_a] \cap [x^2_b, x^2_a].$$

(ii) We will say that the singular arc is only controllable on the first phase, resp. on the second phase, when

$$(x_\sigma \in [x^1_b, x^1_a] \text{ and } x_\sigma < x^2_b) \quad \text{or} \quad (x_\sigma \in [x^1_b, x^1_a] \text{ and } x_\sigma > x^2_a),$$

resp.

$$(x_\sigma \in [x^2_b, x^2_a] \text{ and } x_\sigma < x^1_b) \quad \text{or} \quad (x_\sigma \in [x^2_b, x^2_a] \text{ and } x_\sigma > x^1_a).$$

(iii) We will say that the singular arc is never controllable when

$$(x_\sigma > x^1_a \text{ or } x_\sigma < x^1_b) \quad \text{and} \quad (x_\sigma > x^2_a \text{ or } x_\sigma < x^2_b).$$

Finally, integrating (3.27) gives for $t_0, t \in [0, T]$,

$$\lambda(t) = \lambda(t_0) e^{-\int_{t_0}^t \frac{\partial f}{\partial x}(s, x^*(s), u^*(s)) ds} - \lambda_0 \int_{t_0}^t \frac{\partial \ell}{\partial x}(s, x^*(s)) e^{-\int_s^t \frac{\partial f}{\partial x}(\tau, x^*(\tau), u^*(\tau)) d\tau} ds. \tag{3.33}$$

If $\lambda_0 = 0$, we must have $\lambda(t) \neq 0$, for all $t \in [0, T]$ (otherwise, we would have $(\lambda(\cdot), \lambda_0) = 0$ which is not possible). Hence, either $\lambda(\cdot)$ is positive or negative, which, by (3.30), implies that either $u^*(t) = a$ and $x^*(t) = \tilde{x}_a(t)$ for all t, or $u^*(t) = b$ and $x^*(t) = \tilde{x}_b(t)$, for all t. In the following, we consider only the case where $\lambda_0 \neq 0$, and by homogeneity we take $\lambda_0 = -1$.

3.3 Optimal synthesis of the problem

As already mentioned in the previous subsection, the crucial point in this study is the controllability of the singular arc which is traduced by the inclusion (3.32). Therefore, the relative position between x_a^1, x_b^1, x_a^2 and x_b^2 is relevant to know wether or not (3.32) can be fulfilled. For instance if $x_b^2 \leq x_a^2 < x_\sigma < x_b^1 \leq x_a^1$, then the singular arc is never controllable, whereas if $x_b^1 \leq x_b^2 < x_\sigma < x_a^2 \leq x_a^1$, then the singular arc is always controllable. Thus, as $x_b^i \leq x_a^i$ for $i = 1, 2$, we have 6 different cases possible:

(A) $x_2^b \leq x_1^b \leq x_2^a \leq x_1^a$;

(B) $x_2^b \leq x_2^a \leq x_1^b \leq x_1^a$;

(C) $x_1^b \leq x_2^b \leq x_2^a \leq x_1^a$;

(D) $x_1^b \leq x_2^b \leq x_1^a \leq x_2^a$;

(E) $x_1^b \leq x_1^a \leq x_2^b \leq x_2^a$;

(F) $x_2^b \leq x_1^b \leq x_1^a \leq x_2^a$,

and each of the 6 cases, one has 5 subcases possible depending on the position of x_σ, for instance for case (B):

(B1) $x_\sigma \leq x_2^b \leq x_2^a \leq x_1^b \leq x_1^a$;

(B2) $x_2^b \leq x_\sigma \leq x_2^a \leq x_1^b \leq x_1^a$;

(B3) $x_2^b \leq x_2^a \leq x_\sigma \leq x_1^b \leq x_1^a$;

(B4) $x_2^b \leq x_2^a \leq x_1^b \leq x_\sigma \leq x_1^a$;

(B5) $x_2^b \leq x_2^a \leq x_1^b \leq x_1^a \leq x_\sigma$.

Nevertheless, the study of those 30 subcases can be simplify adding two hypotheses without loss of generality. Indeed, it can be easily observed that, by translating the problem on $[\overline{T}, T+\overline{T}]$ instead of $[0, T]$, the case (D), resp. (E) and (F), is analogous to the case (A), resp. (B) and (C). Consequently, without loss of generality, we can assume that

$$x_a^2 \leq x_a^1. \tag{h1}$$

On the other hand, as showed by Theorem 2.4, one has that all the periodic solutions remain in the set

$$\mathcal{B} := [m_1, m_2], \quad \text{with} \quad m_1 := \min_{t\in[0,T]} \tilde{x}_b(t) \quad \text{and} \quad m_2 := \max_{t\in[0,T]} \tilde{x}_a(t).$$

Therefore, if $x_\sigma \notin \mathcal{B}$, it seems rather intuitive that the optimal solution x^* would be the periodic solution the closest to x_σ, that is $\tilde{x}_a$ if $x_\sigma > m_2$, or $\tilde{x}_b$ if $x_\sigma < m_1$. The next theorem formalize this intuition and leads to another hypothesis on x_σ.

Theorem 3.7. *(1) If $x_\sigma \geq m_2$, then the constant control $u_a : t \mapsto a$ is optimal.*

(2) If $x_\sigma \leq m_1$, then the constant control $u_b : t \mapsto b$ is optimal.

Proof. (1): Suppose $x_\sigma \geq m_2$. We have by Theorem 2.4

$$x_\sigma \geq \tilde{x}_a(t) \geq \tilde{x}_u(t), \quad \forall u \in \mathcal{U}_T^{a,b}, \forall t \in [0, T]. \tag{3.34}$$

By Hypothesis 6, we obtain

$$\ell(t, \tilde{x}_u(t)) \leq \ell(t, \tilde{x}_a(t)), \quad \forall u \in \mathcal{U}_T, \forall t \in [0, T],$$

and thus $J(a) = \min_{u\in\mathcal{U}_T^{a,b}} J(u)$ which gives the first result. The proof of (2) is similar. □

This leads to the following hypothesis on x_σ. We assume that

$$x_\sigma \in]m_1, m_2[. \tag{h2}$$

Remark 3.8. Note that, under hypothesis h1 together with Proposition 3.4, one always has $m_2 = \tilde{x}_a(\overline{T})$.

Thus, using hypotheses (h1) and (h2), one has reduced the number of subcases which can now be classify according to the controllability of x_σ (cf. Definition 3.6).

3.3.1 Singular arc always controllable and singular strategy

We begin our study by the simplest case, that is, when the singular arc is always controllable.

Theorem 3.9. *Suppose that the singular arc is always controllable, then there exists $u_\sigma^1, u_\sigma^2 \in [a,b]$ such that $f_i(x_\sigma, u_\sigma^i) = 0$ for each $i = 1, 2$ and the T-periodic control*

$$u^*(t) := \begin{cases} u_1 \text{ if } t \in [0, \overline{T}[, \\ u_2 \text{ if } t \in [\overline{T}, T[, \end{cases} \tag{3.35}$$

is optimal with

$$x^*(t) = x_\sigma, \quad \forall t \in \mathbb{R}. \tag{3.36}$$

Proof. As $x_\sigma \in [x_b^1, x_a^1] \cap [x_b^2, x_a^2]$, one has by Definition 3.2 that for each $i = 1, 2$, $x_\sigma f_i(x_\sigma, b) \leq 0$ and $x_\sigma f_i(x_\sigma, a) \geq 0$. Thus, as $f_i(x_\sigma, \cdot)$ is continuous on $[a, b]$, one has the existence of $u_\sigma^i \in [a, b]$ such that $f_i(x_\sigma, u_\sigma^i) = 0$. Setting $u^*(\cdot)$ as in (3.35), it is obvious that $x^*(t) = x_\sigma$, for all $t \in \mathbb{R}$. Then Hypothesis 6 implies that

$$\ell(t, x^*(t)) \leq \ell(t, \tilde{x}_u(t)), \quad \forall u \in \mathcal{U}_T, \forall t \in [0, T],$$

which gives that $u^*(\cdot)$ is optimal and proves the result. □

Therefore, when the arc singular is always controllable, the optimal strategy consists in a singular strategy by choosing the singular controls which allow to stay on the singular arc x_σ.

3.3.2 Singular arc controllable only on one phase and bang-singular-bang strategy

In this subsection, we will assume that the singular arc is controllable only on one phase, on the first one or on the second one. By Definition 3.2 and under the hypotheses (h1) and (h2), one has to study three different cases:

$$x_\sigma \in [x_b^1, x_a^1] \text{ and } x_\sigma < x_b^2; \tag{case I}$$

$$x_\sigma \in [x_b^1, x_a^1] \text{ and } x_\sigma > x_a^2; \tag{case II}$$

$$x_\sigma \in [x_b^2, x_a^2] \text{ and } x_\sigma < x_b^1. \tag{case III}$$

Equations (case I) and (case II) mean that the singular arc is controllable only on the first phase, whereas (case III) means that it is controllable only on the second phase. The next three theorems give the structure of the optimal control for each of the three cases (case I), (case II) and (case III). The proof of the theorems will be found in a next section.

Theorem 3.10. *Suppose* (case I) *holds. Then $x^*(0) \geq x_\sigma$ and $x_b(\overline{T}, T, x^*(0)) < x_\sigma$, and defining $t_1 \in [0, \overline{T}[$, resp. $t_0 \in [0, t_1]$, such that $x_b(t_1, T, x^*(0)) = x_\sigma$, resp. $x_b(t_0, 0, x^*(0)) = x_\sigma$, one has*

$$u_{b,\sigma,b}(t) = \begin{cases} b, \text{ if } t \in [0, t_0[, \\ u_\sigma^1 \text{ if } t \in [t_0, t_1[, \\ b, \text{ if } t \in [t_1, T], \end{cases} \tag{3.37}$$

is optimal.

Theorem 3.11. *Suppose* (case II) *holds. Then $x^*(0) \leq x_\sigma$ and $x_a(\overline{T}, T, x^*(0)) > x_\sigma$, and definig $t_1 \in [0, \overline{T}[$, resp. $t_0 \in [0, t_1]$, such that $x_a(t_1, T, x^*(0)) = x_\sigma$, resp. $x_a(t_0, 0, x^*(0)) = x_\sigma$, one has*

$$u_{a,\sigma,a}(t) = \begin{cases} a, \text{ if } t \in [0, t_0[, \\ u_\sigma^1 \text{ if } t \in [t_0, t_1[, \\ a, \text{ if } t \in [t_1, T], \end{cases} \tag{3.38}$$

is optimal.

Theorem 3.12. *Suppose* (case III) *holds. Then* $x^*(0) \leq x_\sigma$ *and* $x_b(\overline{T}, 0, x^*(0)) \geq x_\sigma$*, defining* $t_0 \in [\overline{T}, T[$*, resp.* $t_1 \in [t_0, T]$*, such that* $x_b(t_0, T, x^*(0)) = x_\sigma$*, resp.* $x_b(t_1, T, x^*(0)) = x_\sigma$*, one has*

$$u_{b,\sigma,b}(t) = \begin{cases} b, & \text{if } t \in [0, t_0[, \\ u_\sigma^2 & \text{if } t \in [t_0, t_1[, \\ b, & \text{if } t \in [t_1, T], \end{cases} \tag{3.39}$$

is optimal.

Remark 3.13. In Theorems 3.10, 3.11 and 3.12, one can have that $t_0 = t_1$, implying that $u_b : t \mapsto b$, resp. $u_a : t \mapsto a$, is optimal for Theorems 3.10 and 3.12, resp. for Theorem 3.11.

3.3.3 Singular arc never controllable and bang-bang strategy

In this subsection, we will assume that the singular arc is never controllable. By Definition 3.2 and under the hypotheses (h1) and (h2), we only have to treat the case:

$$x_a^2 < x_\sigma < x_b^1. \tag{case IV}$$

As in the previous subsection, the proof of the following theorem will be found in a next section..

Theorem 3.14. *If* (case IV) *then* $x^*(0) \leq x_\sigma$*.*

(1) If $x^*(0) = x_\sigma$*, then* $\lambda(0) > 0$ *and* $u_a : t \mapsto a$ *is optimal.*

(2) If $x^*(0) < x_\sigma$ *with* $\lambda(0) = 0$*, then* $x_b(\overline{T}, 0, x^*(0)) > x_\sigma$ *and* $x_a(\overline{T}, T, x^*(0)) > x_\sigma$*, and defining* t_0 *as the unique time* $t \in [0, T]$ *such that* $x_b(t, 0, x^*(0)) = x_a(t, T, x^*(0))$*, one has*

$$u_{b,a}(t) := \begin{cases} b, & \text{if } t \in [0, t_0[, \\ a, & \text{if } t \in [t_0, T], \end{cases} \tag{3.40}$$

optimal.

(3) If $x^*(0) < x_\sigma$ *with* $\lambda(0) > 0$ *then* $x_a(\overline{T}, T, x^*(0)) > x_\sigma$*, and either*

- $u_a : t \mapsto a$ *is optimal, or,*
- *there exists* $x_0 \in]x^*(0), x_\sigma[$ *such that* $x_b(T, 0, x_0) < x^*(0)$ *and* $x_b(\overline{T}, 0, x_0) > x_\sigma$*, and defining* t_0*, resp.* t_1*, as the unique time* $t \in [0, T]$ *such that*

$$x_a(t, 0, x^*(0)) = x_b(t, 0, x_0) < x_\sigma, \quad \text{and} \quad x_a(t, T, x^*(0)) = x_b(t, 0, x_0) > x_\sigma, \tag{3.41}$$

one has

$$u_{a,b,a}(t) := \begin{cases} a, & \text{if } t \in [0, t_0[, \\ b, & \text{if } t \in [t_0, t_1[, \\ a, & \text{if } t \in [t_1, T], \end{cases} \tag{3.42}$$

optimal.

(4) If $x^*(0) < x_\sigma$ *with* $\lambda(0) < 0$ *then* $x_b(\overline{T}, 0, x^*(0)) \geq x_\sigma$*, and either*

- $x_b(\overline{T}, 0, x^*(0)) = x_\sigma$ *and* $u_b : t \mapsto b$ *is optimal, or,*
- $x_b(\overline{T}, 0, x^*(0)) > x_\sigma$ *and there exists* $x_0 \in]x^*(0), x_\sigma[$ *such that* $x_a(0, T, x_0) < x^*(0)$ *and* $x_a(\overline{T}, T, x^*(0)) > x_\sigma$*, and defining* t_0*, resp.* t_1*, as the unique time* $t \in [0, T]$ *such that*

$$x_b(t, 0, x^*(0)) = x_a(t, T, x_0) < x_\sigma, \quad \text{and} \quad x_b(t, T, x^*(0)) = x_a(t, T, x_0) > x_\sigma, \tag{3.43}$$

one has

$$u_{b,a,b}(t) := \begin{cases} b, & \text{if } t \in [0, t_0[, \\ a, & \text{if } t \in [t_0, t_1[, \\ b, & \text{if } t \in [t_1, T] \end{cases} \tag{3.44}$$

is optimal.

This concludes our study.

3.3.4 Discussion

In this subsection, we discuss the previous results. Let us start with Table 1 that summarizes the different cases. We recall that

$$m_1 := \min_t \tilde{x}_b(t), \quad \text{and,} \quad m_2 := \max_t \tilde{x}_a(t).$$

Table 1: Structure of the optimal control.

position of x_σ	controllability of x_σ	cases	initial condition	structure of the optimal control
$x_\sigma \notin]m_1, m_2[$		$x_\sigma \geq \max_t \tilde{x}_a(t)$	$x^*(0) = \tilde{x}_a(0)$	a
		$x_\sigma \leq \min_t tx_b(t)$	$x^*(0) = \tilde{x}_b(0)$	b
$x_\sigma \in]m_1, m_2[$	always contr.	$x_\sigma \in [x_b^1, x_a^1] \cap [x_b^2, x_a^1]$	$x^*(0) = x_\sigma$	$u_\sigma^1 - u_\sigma^2$ (sing.)
	controllable only on one phase	(case I)	$x^*(0) \geq x_\sigma$	$b - (u_\sigma^1) - b$ (b-s-b)
		(case II)	$x^*(0) \leq x_\sigma$	$a - (u_\sigma^1) - a$ (b-s-b)
		(case III)	$x^*(0) < x_\sigma$	$b - (u_\sigma^2) - b$ (b-s-b)
	never contr.	(case IV)	$x^*(0) \leq x_\sigma$	$b - a$ (b-b) or $b - (a) - b$ (b-b) or $a - (b) - a$ (b-b)

Remark 3.15. In Table 1, the parenthesis in the structure of optimal control mean that the control in the parenthesis may not appear. For instance, $b - (u_\sigma^1) - b$ means that the optimal control is either constant equal to b or of the form $b - u_\sigma^1 - b$.

We can make 3 observations on the results.

(1) The structure of the optimal control depends on the controllability of the singular arc. Indeed, when the singular arc is always controllable, the optimal strategy is singular, when the singular arc is controllable only on one phase, the structure is bang-singular-bang, and when the singular arc is never controllable, the strategy is bang-bang.

(2) If x_σ is controllable only on one phase, we set $m := \min\{x_b^1, x_b^2\}$ and $M := \max\{x_b^1, x_b^2\}$, and one has either

$$x_\sigma \in [x_a^2, x_a^1] \quad \text{and} \quad x_\sigma \notin [m, M], \quad \text{in (case II)}, \tag{3.45}$$

or

$$x_\sigma \in [x_a^2, x_a^1] \quad \text{and} \quad x_\sigma \notin [m, M], \quad \text{in (case I) and (case III)}. \tag{3.46}$$

We remark that, when (3.45) holds, the structure of the optimal control is of the form $a - u_\sigma - a$, whereas if (3.46) holds, the structure is of the form $b - u_\sigma - b$. Therefore, we obtain the following proposition.

Proposition 3.16. *If the singular arc is controllable only on one phase, then the structure of the optimal control is of the form:*

$$v - u_\sigma^i - v, \tag{3.47}$$

where $v \in \{a, b\}$ such that $x_\sigma \in [\min\{x_v^1, x_v^2\}, \max\{x_v^1, x_v^2\}]$ and $i \in \{1, 2\}$ such that $x_\sigma \in [x_b^i, x_a^i]$.

(3) If the singular arc is controllable only on one phase, the optimal strategy is of the turnpike type with most rapid approach (cf. [5]). Indeed in cases (case I), (case II) and (case III), the optimal strategy consists in reaching the singular arc x_σ one the phase where it is controllable, then to stay on the singular arc as long it is possible and finally to quit the arc in order to reach the initial condition satisfying the periodic transversality condition $\lambda(0) = \lambda(T)$. Moreover, we observe that:

- In case (case I), one has x_σ controllable in the first phase, with $x^*(0) \geq x_\sigma$. Therefore by Lemma 2.5, the quickest way to reach x_σ from $x^*(0)$ is with the constant control $u_b : t \mapsto b$.
- In the case (case II), one has x_σ controllable in the first phase with $x^*(0) \leq x_\sigma$, so the most rapid approach is with the constant control $u_a : t \mapsto a$.

- In case (case III), one has that x_σ is controllable in the second phase with $x^*(0) \leq x_\sigma$. As the singular arc is controllable in the second phase, the quickest way to reach x_σ where it is controllable from $x^*(0) = x^*(T)$ is in backward time using $u_b : t \mapsto b$.

Consequently, the optimal strategy when the singular arc is controllable only on one phase is of the turnpike type with most rapid approach.

3.4 Proofs of the theorems

In this subsection w give the proofs of Theorems 3.10, 3.11, 3.12 and 3.14. As the proofs use similar arguments, we decided to write them as a succession of Lemmas.

Lemma 3.17. *Let $t_0, t_1 \in [0,T]$ such that $t_0 < t_1$. Then, the trajectory satisfies the following property:*
(i). If $\lambda(t_0) > 0$ and $x_a(t, t_0, x^*(t_0)) > x_\sigma$, for all $t \in [t_0, t_1[$, then $\lambda(t) > 0$, for all $t \in [t_0, t_1]$..
(ii). If $\lambda(t_0) < 0$ and $x_b(t, t_0, x^*(t_0)) < x_\sigma$, for all $t \in [t_0, t_1[$, then $\lambda(t) < 0$ for all $t \in [t_0, t_1]$.
(iii). If $\lambda(t_1) > 0$ and $x_a(t, t_1, x^*(t_1)) < x_\sigma$, for all $t \in]t_0, t_1]$, then $\lambda(t) > 0$, for all $t \in [t_0, t_1]$.
(iv).If $\lambda(t_1) < 0$, and $x_b(t, t_1, x^*(t_1)) > x_\sigma$, $\forall t \in]t_0, t_1]$, then $\lambda(t) < 0$, $\forall t \in [t_0, t_1]$.

Proof. (i) Suppose there exists $t_2 \in]t_0, t_1]$ such that $\lambda(t_2) \leq 0$. By continuity of $\lambda(\cdot)$, we can assume without loss of generality that $\lambda(t_2) = 0$ and $\lambda(t) > 0$, for all $t \in [t_0, t_2[$, implying by (3.30) that $u^*(t) = a$, for almost every $t \in [t_0, t_2[$. Therefore, one has

$$x^*(t) = x_a(t, t_0, x^*(t_0)) > x_\sigma, \quad \forall t \in [t_0, t_2[,$$

implying by Hypothesis 6 that

$$\frac{\partial \ell}{\partial x}(s, x^*(s)) > 0, \quad \forall s \in [t_0, t_2[.$$

Thus, one obtains by (3.33) that $\lambda(t_2) \neq 0$ which is a contradiction. The proofs of (ii), (iii) and (iv) are similar. □

3.4.1 Singular arc controllable only on one phase and bang-singular-bang strategy

. As the proof of final theorem for each of the three cases follow the same organization but with different mathematical arguments, we decided to write it as a succession of Lemmas in order to avoid the repetition.

Lemma 3.18. *(1) If* (case I)*, then the constant control $u : t \mapsto a$ is not optimal.*

(2) If (case II)*, then the constant control $u : t \mapsto b$ is not optimal.*

(3) If (case III)*, then the constant control $u : t \mapsto a$ is not optimal.*

Proof. (1): We first note that by (case I), one has $x_b^1 < x_b^2$. This implies by Proposition 3.4 that $\min_t \tilde{x}_b(t) = \overline{T}$ and so by (h2) that $x_\sigma > \tilde{x}_b(\overline{T})$. We set $x_0 := x_b(T, \overline{T}, x_\sigma)$. Then, by Proposition 3.4 one has $x_0 > x_\sigma$ and as

$$x_b(\overline{T}, \overline{T}, x_\sigma) = x_\sigma > \tilde{x}_b(\overline{T}),$$

we deduce that

$$x_0 = x_b(T, \overline{T}, x_\sigma) > \tilde{x}_b(T) = \tilde{x}_b(0).$$

Therefore, by (2.9), one has

$$x_b(T, 0, x_0) < x_0 = x_b(T, \overline{T}, x_\sigma).$$

It follows that

$$x_b(\overline{T}, 0, x_0) < x_b(\overline{T}, \overline{T}, x_\sigma) = x_\sigma,$$

and thus, there exists $t_0 \in]0, \overline{T}[$ such that $x_b(t_0, 0, x_0) = x_\sigma$. Moreover, as $x_\sigma \in [x_b^1, x_a^1]$, there exists $u_\sigma^1 \in [a,b]$ such that $f_1(x_\sigma, u_\sigma^1) = 0$. Thus, by setting

$$\overline{u}(t) = \begin{cases} b, & \text{if } t \in [0, t_0[, \\ u_\sigma^1, & \text{if } t \in [t_0, \overline{T}[, \\ b, & \text{if } t \in [\overline{T}, T], \end{cases}$$

we obtain that $x_{\overline{u}}(\cdot, 0, x_0)$ is T-periodic with, by Proposition 3.4,

$$x_\sigma \leq x_{\overline{u}}(t, 0, x_0) < x_b^2, \quad \forall t \in [0, T].$$

This, associated to (case I) and Proposition 3.4, gives

$$x_\sigma \leq x_{\overline{u}}(t, 0, x_0) < \tilde{x}_a(t), \quad \forall t \in [0, T],$$

which by Hypothesis 6 implies that

$$J(\overline{u}) < J(a)$$

and concludes the proof of (1). The proof of (2) is similar than the proof of (1) inverting a and b. (3): One has $x_b^1 < x_b^2$, so (h2) gives $\tilde{x}_b(0) < x_\sigma$. Therefore by (2.9), one has $x_\sigma < x_b(T, 0, x_\sigma)$. Thus calling $t_0 \in]\overline{T}, T[$ such that $x_b(t_0, 0, x_\sigma) = x_\sigma$, one obtains the result considering

$$\overline{u}(t) = \begin{cases} b, \text{ if } t \in [0, t_0[, \\ u_\sigma^2, \text{ if } t \in [t_0, T], \end{cases}$$

with $f_2(x_\sigma, u_\sigma^2) = 0$. □

The following Lemmas give structure results in function of the values of x_σ, $x^*(0)$ the initial condition of the optimal solution and $\lambda(0)$ the initial condition of the adjoint state.

Lemma 3.19. *(1) If* (case I) *holds together with $x^*(0) < x_\sigma$ and $\lambda(0) \geq 0$, then $u_a : t \mapsto a$ is optimal.*

(2) If (case II) *holds together with $x^*(0) > x_\sigma$ and $\lambda(0) \leq 0$, then $u_b : t \mapsto b$ is optimal.*

(3) If (case III) *holds together with $x^*(0) > x_\sigma$ and $\lambda(0) \geq 0$, then $u_a : t \mapsto a$ is optimal.*

Proof. (1): Suppose at first that $\lambda(0) > 0$. As $\lambda(0) = \lambda(T)$ and $\lambda(\cdot)$ is continuous, one has the existence of $\eta > 0$ such that $\lambda(t) > 0$, for all $t \in [T - \eta, T[$. So by (3.30), one has $x^*(t) = x_a(t, T, x^*(0))$, for all $t \in [T - \eta, T]$. Moreover, as

$$x^*(0) < x_\sigma < x_b^2 \leq x_a^2, \tag{3.48}$$

we have $x_a(\cdot, T, x^*(0))$ increasing on $[\overline{T}, T]$. And as, $x^*(0) \leq \tilde{x}_a(0)$, we obtain from Theorem 2.3 that

$$x_a(t, T, x^*(0)) < x_\sigma, \quad \forall t \in [0, T]. \tag{3.49}$$

Then it suffices to apply Lemma 3.17 (3) on a point $\bar{t} \in [\max\{T - \eta, \overline{T}\}, T[$, to obtain that $\lambda(t) > 0$, for all $t \in [0, \bar{t}]$, which give the result by (3.30). Suppose now that $\lambda(0) = \lambda(T) = 0$. We obtain by (3.33) that

$$\lambda(t) = -\int_t^T \frac{\partial \ell}{\partial x}(s, x^*(s)) e^{-\int_s^t \frac{\partial f}{\partial x}(\tau, x^*(\tau), u^*(\tau)) d\tau} ds, \quad \forall t \in [0, T]. \tag{3.50}$$

But as $x^*(0) = x^*(T) < x_\sigma$, one has, by continuity of $x^*(\cdot)$, the existence of $\eta > 0$ such that $x^*(t) < x_\sigma$, for all $t \in [T - \eta, T]$. Therefore, using Hypothesis 6 in (3.50), one gets $\lambda(t) > 0$, for all $t \in [T - \eta, T[$ and this case reduces to the previous one. The proofs of (2) and (3) use the same arguments. □

Lemma 3.20. *(1) If* (case I) *holds together with $x^*(0) < x_\sigma$ and $\lambda(0) < 0$, then $u_b : t \mapsto b$ is optimal.*

(2) If (case II) *holds together with $x^*(0) > x_\sigma$ and $\lambda(0) > 0$, then $u_a : t \mapsto a$ is optimal.*

(3) If (case III) *holds together with $x^*(0) > x_\sigma$ and $\lambda(0) < 0$, then $u_b : t \mapsto b$ is optimal.*

Proof. (1): This is a direct consequence of Lemma 3.17 as (case I) with $x^*(0) < x_\sigma$ implies by Proposition 3.4 and inequalities of Theorem 2.3 that $x_b(t, 0, x^*(0)) < x_\sigma$, for all $t \in [0, T]$, which gives the result by (3.30). The proofs of (2) and (3) use the same arguments. □

Lemma 3.21. *(1) If* (case I) *holds with $x^*(0) < x_\sigma$, then $u_b : t \mapsto b$ is not optimal.*

(2) If (case II) *holds with $x^*(0) > x_\sigma$, then $u_a : t \mapsto a$ is not optimal.*

(3) If (case III) *holds with $x^*(0) > x_\sigma$, then $u_b : t \mapsto b$ is not optimal.*

Proof. (1): As $\tilde{x}_b(0) \leq x^*(0) < x_\sigma$, one has by (2.9) that $x_b(0, T, x_\sigma) > x_\sigma$. Also as $x_\sigma < x_b^2$, one has by Proposition 3.4 that $x_b(\overline{T}, T, x_\sigma) < x_\sigma$. This implies, by the monotonicity of $x_b(\cdot, T, x_\sigma)$ that there exists a unique $t_0 \in]0, \overline{T}[$ such that $x_b(t_0, T, x_\sigma) = x_\sigma$. Moreover, as $x_\sigma \in [x_b^1, x_a^1]$, there exists $u_\sigma^1 \in [a, b]$ such that $f_1(x_\sigma, u_\sigma^1) = 0$. Thus, by setting

$$\overline{u}(t) := \begin{cases} u_\sigma^1, \text{ if } t \in [0, t_0[, \\ b, \text{ if } t \in [t_0, T], \end{cases}$$

we obtain $x_{\overline{u}}(\cdot)$ T-periodic with

$$x_\sigma \geq x_{ou}(t) > \tilde{x}_b(t), \quad \forall t \in [0, T]. \tag{3.51}$$

This implies by Hypothesis 6 and (3.33) that $J(\overline{u}) < J(u_b)$ and proves the result. (2): The proof is similar that the proof of (1) considering the control

$$\overline{u}(t) := \begin{cases} u_\sigma^1, \text{ if } t \in [0, t_0[, \\ a, \text{ if } t \in [t_0, T], \end{cases}$$

with t_0 being the time in $t_0 \in]0, \overline{T}[$ such that $x_a(\cdot, T, x_\sigma) = x_\sigma$. (3): Assuming u_b is optimal together with (case III) and $x^*(0) > x_\sigma$ is a direct contradiction to the assumption (h2). □

Lemma 3.22. *(1) If* (case I) *holds together with $x^*(0) \geq x_\sigma$ and $\lambda(0) \geq 0$, then $u_a : t \mapsto a$ is optimal.*

(2) If (case II) *holds together with $x^*(0) \leq x_\sigma$ and $\lambda(0) \leq 0$, then $u_b : t \mapsto b$ is optimal.*

(3) If (case III) *holds together with $x^*(0) \leq x_\sigma$ and $\lambda(0) \geq 0$, then $u_a : t \mapsto a$ is optimal.*

Proof. We have to consider three cases: $\lambda(0) > 0$ and $x^*(0) \geq x_\sigma$ (case a), $\lambda(0) = 0$ and $x^*(0) > x_\sigma$ (case b) and $\lambda(0) = 0$ and $x^*(0) = x_\sigma$. Case a: Suppose that $\lambda(0) > 0$ with $x^*(0) \geq x_\sigma$. Then by continuity of $\lambda(\cdot)$, there exists $\eta > 0$ such that

$$\lambda(t) > 0, \quad \forall t \in]0, \eta[. \tag{3.52}$$

Moreover, since $x^*(0) \geq x_\sigma$ with (case I), one has

$$x_a(t, 0, x^*(0)) > x_\sigma, \quad \forall t \in]0, T[. \tag{3.53}$$

Therefore, we deduce from Lemma 3.17 (1) that $\lambda(t) > 0$ for all $t \in]0, T[$, which give the result by (3.30). Case b: Suppose $\lambda(0) = 0$ and $x^*(0) > x_\sigma$. One has by continuity of $x^*(\cdot)$ that there exists $\eta > 0$ such that

$$x^*(t) > x_\sigma, \quad \forall t \in [0, \eta]. \tag{3.54}$$

So, using $\lambda(0) = 0$ together with Hypothesis 6 in (3.33), one obtains that

$$\lambda(t) > 0, \quad \forall t \in]0, \eta[, \tag{3.55}$$

and this case reduces to case a. Case c: Suppose $\lambda(0) = 0$ and $x^*(0) = x_\sigma$. As $x_\sigma < x_b^2$ with $x^*(0) = x_\sigma$, one has

$$x_b(t, T, x^*(0)) < x_\sigma, \forall t \in [\overline{T}, T[. \tag{3.56}$$

Thus, by Lemma 2.5, we obtain that

$$x^*(t) < x_\sigma, \quad \forall t \in [\overline{T}, T[. \tag{3.57}$$

Therefore, using $\lambda(0) = 0$ together with Hypothesis 6 in (3.33), one gets

$$\lambda(t) > 0, \quad \forall t \in [\overline{T}, T[. \tag{3.58}$$

Moreover, one has by (case I) combined to the inequalities of Theorem 2.3 that

$$x_a(t, T, x^*(0)) = x_a(t, T, x_\sigma) < x_\sigma, \quad \forall t \in]0, T[. \tag{3.59}$$

Consequently, applying Lemma 3.17 (3), one obtains that $\lambda(t) > 0$ for all $t \in]0, T[$ which by (3.30) proves the result. The proofs of (2) and (3) use the same arguments. □

Lemma 3.23. *(1) If* (case I) *holds with* $x^*(0) \geq x_\sigma$ *and* $\lambda(0) < 0$*, then* $x_b(\overline{T}, T, x^*(0)) < x_\sigma$*.*

(2) If (case II) *holds with* $x^*(0) \leq x_\sigma$ *and* $\lambda(0) > 0$*, then* $x_a(\overline{T}, T, x^*(0)) > x_\sigma$*.*

(3) If (case III) *holds with* $x^*(0) \leq x_\sigma$ *and* $\lambda(0) < 0$*, then* $x_b(\overline{T}, 0, x^*(0)) \geq x_\sigma$*.*

Proof. (1): Suppose $x_b(\overline{T}, T, x^*(0)) \geq x_\sigma$. First, let us remark that as $x^*(0) < x_b^2$, one has $x_b(\cdot, T, x^*(0))$ increasing on $[\overline{T}, T]$, hence

$$x_\sigma \leq x_b(\overline{T}, T, x^*(0)). \tag{3.60}$$

This implies that

$$x_b(\overline{T}, T, x^*(0)) > x_b^1. \tag{3.61}$$

Indeed, assuming $x_b(\overline{T}, T, x^*(0)) = x_b^1$ implies by (3.60) that

$$x_b(0, T, x^*(0)) = x_b(\overline{T}, T, x^*(0)) < x^*(0), \tag{3.62}$$

giving by contrapositive of (2.9) that $x^*(0) < \tilde{x}_b(0)$, which is impossible. Therefore, one obtains by (3.61) that $x_b(\cdot, T, x^*(0))$ is decreasing on $[0, \overline{T}]$. Thus, by assumption, one has

$$x_b(t, T, x^*(0)) > x_\sigma, \quad \forall t \in [0, T] \setminus \{\overline{T}\}. \tag{3.63}$$

As $\lambda(0) = \lambda(T) < 0$, one has by (3.63) and Lemma 3.17 (4) that

$$\lambda(t) < 0, \quad \forall t \in [0, T]. \tag{3.64}$$

Thus, one has that $u_b : t \mapsto b$ is optimal, implying that $x^*(\cdot) = \tilde{x}_b(\cdot) = x_b(\cdot, T, x^*(0))$. In particular, one has

$$x_\sigma \leq x_b(\overline{T}, T, x^*(0)) = \min_{t \in [0,T]} x_b(t, T, x^*(0)) = \min_{t \in [0,T]} \tilde{x}_b(t), \tag{3.65}$$

which is a contradiction to (h2). (2): The arguments are the same that in the proof of (1). (3): Let us suppose that $x_b(\overline{T}, 0, x^*(0)) < x_\sigma$. Then as $\lambda(0) < 0$ one has directly from Lemma 3.17 that $\lambda(t) < 0$ for all $t \in [0, T]$. This implies by (3.30) that $u^* = u_b$ and so by uniqueness of the periodic solution for a given control, one has $x^*(\cdot) = \tilde{x}_b(\cdot) = x_b(\cdot, 0, x^*(0))$. Now, call $x_0 = x_b(0, \overline{T}, x_\sigma)$. As $x_0 > \tilde{x}_b(0)$, one has $x_b(T, 0, x_0) < x_0$ by Theorem 2.3. And as $x_b(\overline{T}, 0, x_0) = x_\sigma$ by definition of x_0, one has $x_b(\overline{T}, T, x_0) > x_\sigma$. Therefore there exists $\bar{t} \in]\overline{T}, T[$, such that $x_b(\bar{t}, T, x_0) = x_\sigma$. So considering

$$\overline{u}(t) := \begin{cases} b, & \text{if } t \in [0, \overline{T}[, \\ u_\sigma^2 & \text{if } t \in [\overline{T}, \bar{t}[, \\ b, & \text{if } t \in [\bar{t}, T], \end{cases} \tag{3.66}$$

with $f_2(x_\sigma, u_\sigma^2) = 0$, one has that $x_{\overline{u}}(\cdot, 0, x_0)$ is T-periodic. Moreover, it is not hard to see that by construction one has

$$x^*(t) = \tilde{x}_b(t) < x_{\overline{u}}(t, 0, x_0) \leq x_\sigma, \tag{3.67}$$

which contradicts by Hypothesis 6 the optimality of x^* and concludes the proof. □

Remark 3.24. It is worth noting that case (3) of Lemma 3.23 is different of the two other cases. Indeed, in case (3), assuming $x_b(\overline{T}, 0, x^*(0)) = x_\sigma$ do not lead to a contradiction of (h2) as in the two other cases. This will imply different statement for case (3) in the following Lemmas and in the following Theorem dealing with (case III).

Lemma 3.25. *(1) Suppose* (case I) *holds with* $x^*(0) \geq x_\sigma$*,* $\lambda(0) < 0$ *and* $x_b(\overline{T}, T, x^*(0)) < x_\sigma$*. Then one has* $\lambda(t) < 0$ *for all* $t \in]t_1, T]$ *with* t_1 *being the unique time* $t \in [0, \overline{T}[$ *such that* $x_b(t, T, x^*(0)) = x_\sigma$*.*

(2) Suppose (case II) *holds with* $x^*(0) \leq x_\sigma$*,* $\lambda(0) > 0$ *and* $x_a(\overline{T}, T, x^*(0)) > x_\sigma$*. Then one has* $\lambda(t) > 0$ *for all* $t \in]t_1, T]$ *with* t_1 *being the unique time* $t \in [0, \overline{T}[$ *such that* $x_a(t, T, x^*(0)) = x_\sigma$*.*

(3) Suppose (case III) *holds with* $x^*(0) \leq x_\sigma$*,* $\lambda(0) < 0$ *and* $x_b(\overline{T}, 0, x^*(0)) \geq x_\sigma$*. Then one has* $\lambda(t) < 0$ *for all* $t \in [0, t_0[$ *with* t_0 *being the unique time* $t \in [\overline{T}, T[$ *such that* $x_b(t, 0, x^*(0)) = x_\sigma$*.*

Proof. (1): As, $x^*(0) \geq x_\sigma > x_b(\overline{T}, T, x^*(0))$, together with the strict monotonicity of $x_b(\cdot, T, x^*(0))$, one has the existence of a unique $t_2 \in]\overline{T}, T]$ such that $x_b(t_2, T, x^*(0)) = x_\sigma$. Also, as $x^*(0) \geq \tilde{x}_b(0)$, one has by the inequalities of Theorem 2.3 that $x_b(0, T, x^*(0)) \geq x^*(0)$. So by strict monotonicity of $x_b(\cdot, T, x^*(0))$ on $[0, \overline{T}]$, together with $x^*(0) \geq x_\sigma > x_b(\overline{T}, T, x^*(0))$, one has the existence of a unique $t_1 \in [0, \overline{T}[$ such that $x_b(t_1, T, x^*(0)) = x_\sigma$. One has to consider two cases: $t_2 = T$ (case a) and $t_2 < T$ (case b). Case a: Suppose $t_2 = T$. As $\lambda(0) = \lambda(T) \leq 0$, one has $\lambda(t_2) \leq 0$. Let us show that $\lambda(t) < 0$, for all $t \in]t_1, t_2[$. Suppose that there exists $\hat{t} \in]t_1, t_2[$ such that $\lambda(\hat{t}) \geq 0$. Without lost of generality, we can, by continuity of λ, assume that $\lambda(\hat{t}) = 0$. Also, by Lemma 2.5, one has that $x^*(t) \leq x_b(t, T, x^*(0))$, for all $t \in [0, T]$. Therefore, by definition of t_1 and t_2, one has

$$x^*(t) < x_\sigma, \quad \forall t \in]t_1, t_2[. \tag{3.68}$$

So using 3.68, together with Hypothesis 6, in (3.33), one has for all $t \in]t_1, \hat{t}[$, $\lambda(t) > 0$. Let $\bar{t} \in]t_1, \min\{\hat{t}, \overline{T}\}]$. Then one has $\lambda(\bar{t}) > 0$ with $x^*(\bar{t}) < x_\sigma \leq x_a^1$. So by strict monotonicity of $x_a(\cdot, \bar{t}, x^*(\bar{t})) < x_\sigma$ on $[0, \overline{T}]$, one has $x_a(t, \bar{t}, x^*(\bar{t})) < x_\sigma$, for all $t \in [0, \bar{t}]$. Therefore, applying Lemma 3.17 (3), one obtains that $\lambda(0) > 0$, contradicting $\lambda(0) \leq 0$. Case b: Suppose $t_2 < T$. Let us show that $\lambda(t) < 0$, for all $t \in [t_2, T[$. As $t_2 < T$, one has $x^*(0) > x_\sigma$. Therefore, by continuity of $x^*(\cdot)$, there exists $\eta > 0$ such that $x^*(t) > x_\sigma$, for all $t \in [T - \eta, T]$. So using $\lambda(0) = \lambda(T) \leq 0$ in (3.33) together with Hypothesis 6, one obtains that for all $t \in [T - \eta, T[$, $\lambda(t) < 0$. Let $\bar{t} \in]\max\{T - \eta, t_2\}, T[$. One has $\lambda(\bar{t}) < 0$, and by definition of t_2,

$$x_b(t, \bar{t}, x^*(\bar{t})) = x_b(t, T, x^*(0)) > x_\sigma, \quad \forall t \in]t_2, \bar{t}]. \tag{3.69}$$

So applying Lemma 3.17 (4), one obtains $\lambda(t) < 0$, for all $t \in [t_2, \bar{t}]$. Thus, $\lambda(t) < 0$, for all $t \in [t_2, T[$. We conclude the proof showing that $\lambda(t) < 0$, for all $t \in]t_1, t_2[$ as in case a. The proofs of (2) and (3) use the same arguments. □

Lemma 3.26. *(1) Under the hypotheses of Lemma 3.25 (1), one has $\lambda(t) < 0$, for all $t \in [0, t_0[$ with t_0 being the unique $t \in [0, t_1]$ such that $x_b(t, 0, x^*(0)) = x_\sigma$.*

(2) Under the hypotheses of Lemma 3.25 (2), one has $\lambda(t) > 0$, for all $t \in [0, t_0[$ with t_0 being the unique $t \in [0, t_1]$ such that $x_a(t, 0, x^*(0)) = x_\sigma$.

(3) Under the hypotheses of Lemma 3.25 (3), one has $\lambda(t) < 0$, for all $t \in]t_1, T]$ with t_1 being the unique $t \in [t_0, T]$ such that $x_b(t, T, x^*(0)) = x_\sigma$.

Proof. (1): First let us remark that by (2.9), one has $x_b(0, T, x^*(0)) \geq x^*(0)$, which implies that

$$x_b(t, 0, x^*(0)) \leq x_b(t, T, x^*(0)), \quad \forall t \in [0, T], \tag{3.70}$$

and so $x_b(\overline{T}, 0, x^*(0)) < x_\sigma$. Therefore, as $x^*(0) \geq x_\sigma$ and $x_b(\cdot, 0, x^*(0))$ is monotonous, there exists a unique $t_0 \in [0, \overline{T}[$ such that $x_b(t_0, 0, x^*(0)) = x_\sigma$. If $x^*(0) = x_\sigma$, then $t_0 = 0$ and the result is trivially true. Suppose $x^*(0) > x_\sigma$. We remark by definition of t_0 that one has

$$x_\sigma < x_b(t, 0, x^*(0)), \quad \forall t \in [0, t_0[, \tag{3.71}$$

hence by Lemma 2.5,

$$x_\sigma < x^*(t), \quad \forall t \in [0, t_0[. \tag{3.72}$$

Suppose also that there exists $\hat{t} \in [0, t_0[$ such that $\lambda(\hat{t}) = 0$. Combining (3.33) and Hypothesis 6, one gets that $\lambda(t) > 0$, for all $t \in]\hat{t}, t_0[$. Let $\bar{t} \in]\hat{t}, t_0[$. As x_σ fulfill (case I), one has that

$$x_a(t, \bar{t}, x^*(\bar{t})) > x_\sigma, \quad \forall t \in [\bar{t}, T]. \tag{3.73}$$

Therefore, by Lemma 3.17, one has $\lambda(t) > 0$, for all $t \in [\bar{t}, T]$, which contradicts $\lambda(T) = \lambda(0) < 0$ and concludes the proof. The proofs of (2) and (3) use the same arguments. □

Proof of Theorem 3.10. We first show that $x^*(0) \geq x_\sigma$ by contradiction. Suppose that $x^*(0) < x_\sigma$. Then, as $u_a : t \mapsto a$ is not optimal by Lemma 3.18 (1), one has that $\lambda(0) < 0$ by contrapositive of Lemma 3.19 (1). Therefore by Lemma 3.20 (1), one obtains that $u_b : t \mapsto b$ is optimal which is a contradiction to Lemma 3.21 (1), hence $x_\sigma \leq x^*(0)$. Let us show now that $\lambda(0) < 0$. Suppose that $\lambda(0) \geq 0$. Then, by Lemma 3.22 (1), one obtains that $u_a : t \mapsto a$ is optimal, contradicting Lemma 3.18 (1), hence $\lambda(0) < 0$. Therefore, we are in position to apply Lemmas 3.23, 3.25 and 3.26, to obtain (a), (b) and (c) with

$$\lambda(t) < 0, \quad \forall t \in [0, t_0[\, \cup\,]t_1, T]. \tag{3.74}$$

Consequently, one deduces by (3.30), that

$$u^*(t) = b, \quad \text{for a.e. } t \in [0, t_0[\, \cup\,]t_1, T]. \tag{3.75}$$

Finally, as x_σ fulfills (case I), there exists $u^1_\sigma \in [a, b]$ such that

$$f_1(x_\sigma, u^1_\sigma) = 0. \tag{3.76}$$

Then, as $x^*(t_0) = x^*(t_1) = x_\sigma$ by (3.75) and the definitions of t_0 and t_1, and by Hypothesis 6, it is not hard to see that u^* defined as in (d) is optimal. □

As the proofs of the optimality in (case II) and (case III) follow the same steps that the proof of Theorem 3.10 we will omit these proofs.

3.4.2 Singular arc never controllable and bang-bang strategy

Lemma 3.27. *Suppose* (case IV). *Then* $x^*(0) \le x_\sigma$.

Proof. Suppose (case IV) with $x^*(0) > x_\sigma$. We first show that $u_a : t \mapsto a$ is optimal. Let us remark that $\lambda(0) \ge 0$. Indeed, if $\lambda(0) < 0$ then $\lambda(T) < 0$, and by (case IV) together with the inequalities of Theorem 2.3, one deduces that

$$x_b(t, T, x^*(0)) > x_\sigma, \quad \forall t \in [0, T]. \tag{3.77}$$

So by Lemma 3.17, one obtains that $\lambda(t) < 0$, for all $t \in [0, T]$, implying by (3.30) that $u_b : t \mapsto b$ is optimal, which gives, by uniqueness of the T-periodic positive solution, that $x^* = \tilde{x}_b$, contradicting (h2), hence $\lambda(0) \ge 0$. Now as $\lambda(0) \ge 0$ with $x^*(0) > x_\sigma$, one can copy the proof of Lemma 3.22 (1) to obtain that $u_a : t \mapsto a$ is optimal. However, as $x_\sigma < x^*(0) = \tilde{x}_a(0)$, one has by the inequalities of Theorem 2.3 that $x_a(T, 0, x_\sigma) > x_\sigma$. Likewise, one has by (h2) that

$$x_\sigma > \min_{t \in [0,T]} \tilde{x}_b(t) = \tilde{x}_b(0), \tag{3.78}$$

so by the inequalities of Theorem 2.3, one has $x_b(0, T, x_\sigma) > x_\sigma$. Therefore by continuity of $x_a(\cdot, 0, x_\sigma)$ and $x_b(\cdot, T, x_\sigma)$, there exists $t_0 \in]0, T[$ such that $x_a(t_0, 0, x_\sigma) = x_b(t_0, T, x_\sigma)$. Thus, setting

$$u_{a,b}(t) := \begin{cases} a, & \text{if } t \in [0, t_0[, \\ b, & \text{if } t \in [t_0, T], \end{cases} \tag{3.79}$$

one has $x_{u_{a,b}}(\cdot, 0, x_\sigma)$ T-periodic and positive. Moreover, one has by (case IV) that

$$x_\sigma \le x_{u_{a,b}}(t, 0, x_\sigma) < \tilde{x}_a(t), \quad \forall t \in [0, T], \tag{3.80}$$

which implies by Hypothesis 6 that $u_a : t \mapsto a$ is not optimal which is a contradiction. □

Lemma 3.28. *Suppose* (case IV) *with* $x^*(0) = x_\sigma$. *Then* $\lambda(0) > 0$ *and* $u_a : t \mapsto a$ *is optimal.*

Proof. : First, let us suppose that $\lambda(0) \ge 0$. As $x^*(0) = x_\sigma < x^1_b$, one has by Lemma 2.5 and Corollary 3.4 that

$$x^*(t) \ge x_b(t, 0, x^*(0)) > x_\sigma, \quad \forall t \in]0, \overline{T}].$$

We dedue by Hypothesis 6 in (3.33) that $\lambda(t) > 0$, for all $t \in]0, \overline{T}]$, hence $x^*(t) = x_a(t, 0, x_\sigma)$, for all $t \in [0, \overline{T}]$. On the other hand, one has by Theorem 2.3 that $x_a(T, 0, x_\sigma) > x_\sigma$, so we deduce from (case IV) that $x_a(t, 0, x_\sigma) > x_\sigma$, for all $t \in]0, T]$. Consequently, applying Lemma 3.17, one obtains that $\lambda(t) > 0$, for all $t \in]0, T]$. This implies in particular that $\lambda(0) = 0$ is not possible, otherwise one would obtain $0 = \lambda(0) \neq \lambda(T) > 0$ contradicting the periodicity of $\lambda(\cdot)$. It remains to show that $\lambda(0) < 0$ leads to a contradiction to conclude. Suppose $\lambda(0) < 0$, then $\lambda(T) < 0$. By the same arguments in backward time one has $\lambda(t) < 0$, for all $t \in [0, T]$, implying by (3.30) that $x^*(t) = \tilde{x}_b(t)$, for all $t \in [0, T]$. However, (case IV) gives

$$m_1 = \tilde{x}_b(0) = x^*(0) = x_\sigma,$$

which contradicts (h2) and concludes the proof. □

Lemma 3.29. *Suppose* (case IV) *with* $x^*(0) < x_\sigma$ *and* $\lambda(0) = 0$. *Then, defining* $t_0 \in [0, T]$ *as the unique time such that* $x_b(\cdot, 0, x^*(0)) = x_a(\cdot, T, x^*(0))$, *the control*

$$u_{b,a} : t \mapsto \begin{cases} b, & \text{if } t \in [0, t_0[, \\ a, & \text{if } t \in [t_0, T], \end{cases}$$

is optimal.

Proof. Note first that by the inequations of Theorem 2.3 such a t_0 is well-defined. Also, there exists $t_1 \in]0, \overline{T}]$ such that $x_b(t_1, =, x^*(0)) = x_\sigma$, otherwise one would have $x^*(\cdot) = \tilde{x}_b(\cdot)$ and one could construct a better periodic solution for the criterion J (cf. proof of Lemma 3.30). Likewise, there exists $t_2 \in [\overline{T}, T[$ such that $x_a(t_2, T, x^*(0)) = x_\sigma$ (cf. proof of Lemma 3.30). Moreover, one have by Lemma 3.17 that

$$\lambda(t) < 0, \ \forall t \in [0, t_1], \quad \text{and} \quad , \lambda(t) > 0, \ \forall t \in [t_2, T]. \tag{3.81}$$

On the other hand, one has by Lemma 2.5 that

$$x^*(t) \geq x_b(t, 0, x^*(0)), \ \forall t \in [0, t_0[, \quad \text{and} \quad , x^*(t) \geq x_a(t, T, x^*(0)), \ \forall t \in [0, t_0[. \tag{3.82}$$

This implies in particular that $t_0 \in [t_1, t_2]$. Therefore one has by Equations (3.81) and (3.82) that

$$\begin{cases} x^*(t) = x_b(t, 0, x^*(0)), \ \forall t \in [0, t_1[, \\ x^*(t) \geq x_b(t, 0, x^*(0)) > x_\sigma, \ \forall t \in [t_1, t_0[, \\ x^*(t) \geq x_a(t, T, x^*(0)) > x_\sigma, \ \forall t \in [t_0, t_2[, \\ x^*(t) = x_a(t, T, x^*(0)), \ \forall t \in [t_2, T]. \end{cases}$$

One deduces from Hypothesis 6 that

$$\int_{t_1}^{t_0} \frac{\partial \ell}{\partial s}(s, x_b(s, 0, x^*(0)))ds + \int_{t_0}^{t_2} \frac{\partial \ell}{\partial s}(s, x_a(s, T, x^*(0)))ds \leq \int_{t_1}^{t_2} \frac{\partial \ell}{\partial s}(s, x^*(s))ds,$$

which implies that $J(u_{a,b}) \leq J(u^*)$ and proves the result. □

Lemma 3.30. *(1) Suppose* (case IV) *with* $x^*(0) < x_\sigma$ *and* $\lambda(0) > 0$. *Then* $x_a(\overline{T}, T, x^*(0)) > x_\sigma$.

(2) Suppose (case IV) *with* $x^*(0) < x_\sigma$ *and* $\lambda(0) < 0$. *Then* $x_b(\overline{T}, 0, x^*(0)) \geq x_\sigma$.

Proof. (1): Suppose $x_a(\overline{T}, T, x(0)) \leq x_\sigma$. Then by Lemma 3.17 this implies that $u_a : t \mapsto a$ is optimal which gives by (case IV) that

$$m_2 = x_a(\overline{T}, T, x^*(0)) \leq x_\sigma,$$

contradicting (h2). (2): Suppose $x_b(\overline{T}, T, x^*(0)) < x_\sigma$. Then by Lemma 3.17 $u^* = u_b$. Define $x_0 := x_b(\overline{T}, T, x^*(0))$. By Theorem 2.3, there exists a unique $t_0 \in]0, T[$ such that $x_a(\cdot, 0, x_0) = x_b(\cdot, T, x_0)$ and it is not difficult to see that

$$\tilde{x}_b(t) < \overline{x}(t) \leq x_\sigma, \quad \forall t \in [0, T],$$

where $\overline{x}$ is the solution associated to the control

$$\overline{u}_{a,b}(t) := \begin{cases} a, & \text{if } t \in [0, t_0[, \\ b, & \text{if } t \in [t_0, T]. \end{cases}$$

Therefore, one obtains $J(u_{a,b}) < J(u_b) = J(u^*)$ which is a contradiction to the optimality of u^*. □

Lemma 3.31. *Suppose* (case IV).

(1) Assume that $x^*(0) < x_\sigma$ *and* $\lambda(0) > 0$. *Then there exists* $\bar{t}_0, \bar{t}_1 \in]0, \overline{T}[$ *and* $\bar{t}_2 \in]\overline{T}, T[$ *such that*

$$x_a(\bar{t}_0, 0, x^*(0)) = x_a(\bar{t}_1, T, x^*(0)) = x_a(\bar{t}_2, T, x^*(0)) = x_\sigma,$$

with $\bar{t}_0 \leq \bar{t}_1 < \bar{t}_2$.

- *If* $\bar{t}_0 = \bar{t}_1$, *then* $\lambda(t) > 0$, *for almost every* $t \in [0, T]$.
- *If* $\bar{t}_0 < \bar{t}_1$, *then there exists* $t_0 \in]0, \bar{t}_0[$ *and* $t_1 \in]\bar{t}_1, \bar{t}_2[$ *such that*

$$\begin{cases} \lambda(t) > 0, & \text{for a.e. } t \in [0, t_0[\cup]t_1, T], \\ \lambda(t) < 0, & \text{for a.e. } t \in]t_0, t_1[. \end{cases}$$

(2) Assume that $x^*(0) < x_\sigma$ *and* $\lambda(0) < 0$. *Then there exists* $\bar{t}_0 \in]0, \overline{T}]$ *and* $\bar{t}_1, \bar{t}_2 \in [\overline{T}, T[$ *such that*

$$x_b(\bar{t}_0, 0, x^*(0)) = x_b(\bar{t}_1, 0, x^*(0)) = x_b(\bar{t}_2, T, x^*(0)) = x_\sigma,$$

with $\bar{t}_0 \leq \bar{t}_1 \leq \bar{t}_2$.

- *If* $\bar{t}_0 = \bar{t}_1$ *or* $\bar{t}_1 = \bar{t}_2$, *then* $\lambda(t) < 0$, *for almost every* $t \in [0, T]$.
- *If* $\bar{t}_0 < \bar{t}_1 < \bar{t}_2$, *then there exists* $t_0 \in]\bar{t}_0, \bar{t}_1[$ *and* $t_1 \in]\bar{t}_2, T[$ *such that*

$$\begin{cases} \lambda(t) < 0, & \text{for a.e. } t \in [0, t_0[\cup]t_1, T], \\ \lambda(t) > 0, & \text{for a.e. } t \in]t_0, t_1[. \end{cases}$$

Proof. (1): From Lemma 3.30 one has $x_a(\overline{T}, T, x^*(0)) > x_\sigma$ with $x^*(0) < x_\sigma$. Therefore, there exists $\bar{t}_2 \in]\overline{T}, T[$ such that $x_a(\bar{t}_2, T, x^*(0)) = x_\sigma$. Moreover, as $x^*(0) = \tilde{x}_a(0)$, one has by Theorem 2.3

$$x_a(0, T, x^*(0)) \leq x^*(0) < x_\sigma,$$

so there exists $\bar{t}_1 \in]0, \overline{T}[$ such that $x_a(\bar{t}_1, T, x^*(0)) = x_\sigma$. Also, because $x_a(0, T, x^*(0)) \leq x^*(0)$, one has

$$x_a(t, 0, x^*(0)) \geq x_a(t, T, x^*(0)), \quad \forall t \in [0, T],$$

hence the existence of $\bar{t}_0 \in]0, \bar{t}_1]$ such that $x_a(\bar{t}_0, 0, x^*(0)) = x_\sigma$. We claim that if $\bar{t}_0 = \bar{t}_1$, then $\lambda(t) > 0$, for almost every $t \in [0, T]$. By Lemma 3.17, one has $\lambda(t) > 0$, for all $t \in [\bar{t}_2, T]$. If $\lambda(t) > 0$, for all $t \in [\bar{t}_1, \bar{t}_2[$, then one can apply Lemma 3.17 once again to obtain that $\lambda(t) > 0$, for all $t \in [0, \bar{t}_1]$ and the claim is proved. Suppose now there exists $\bar{t} \in [\bar{t}_1, \bar{t}_2[$, such that $\lambda(t) = 0$, and assume first that $\bar{t} \in]\bar{t}_1, \bar{t}_2[$. Then, as by Lemma 2.5, $x^*(t) \geq x_a(t, T, x^*(0))$, for all $t \in [0, T]$, and as $x_a(t, T, x^*(0)) > x_\sigma$, for all $t \in]\bar{t}_1, \bar{t}_2[$, one deduces from Hypothesis 6 in (3.33) that $\lambda(t) < 0$, for all $t \in [\bar{t}_1, \bar{t}[$. Consequently, $x^*(t) > x_a(t, T, x^*(0))$, for all $t \in [0, \bar{t}[$. On the other hand, as $\lambda(0) > 0$, one has by (3.30) that $x^*(t) = x_a(t, 0, x^*(0))$, for all $t \in [0, \eta]$, for some $\eta > 0$. Therefore, $x_a(t, 0, x^*(0)) > x_a(t, T, x^*(0))$, for all $t \in [0, \eta]$. However, this is a contradiction to $\bar{t}_0 = \bar{t}_1$, as $\bar{t}_0 = \bar{t}_1$ implies $x_a(\cdot, 0, x^*(0)) > x_a(\cdot, T, x^*(0))$. Suppose finally that $\bar{t} = \bar{t}_1$. Then $x^*(\bar{t}) = x_\sigma$ and, as $\bar{t} \in]0, \overline{T}[$, one has by (case IV) together with Lemma 2.5 that

$$x^*(t) \leq x_b(t, \bar{t}, x_\sigma) < x_\sigma, \quad \forall t \in [0, \bar{t}[.$$

Therefore by Hypothesis 6 and (3.33), one obtains that $\lambda(t) > 0$, for all $t \in [0, \bar{t}[$, which proves the claim. Suppose now that $\bar{t}_0 < \bar{t}_1$. This implies that $\lambda(\cdot)$ is not positive almost everywhere on $[0, T]$. Indeed, if $\lambda(t) > 0$, for almost every $t \in [0, T]$, then by (3.30), one has that $x^*(\cdot) = \tilde{x}_a(\cdot)$. This gives that $x_a(\cdot, 0, x^*(0)) = x_a(\cdot, T, x^*(0))$ imlying $\bar{t}_0 = \bar{t}_1$ which contradicts $\bar{t}_0 < \bar{t}_1$. Therefore, one can define

$$t_0 := \inf\{t \in [0, T] : \lambda(t) \leq 0\} \quad \text{and} \quad t_1 := \sup\{t \in [0, T] : \lambda(t) \leq 0\}.$$

By definition, one has

$$\lambda(t) > 0, \quad \forall t \in [0, t_0[\cup]t_1, T], \tag{3.83}$$

and by continuity of $\lambda(\cdot)$, $\lambda(t_0) = \lambda(t_1) = 0$. We show that $t_1 \in]\bar{t}_1, \bar{t}_2[$. As $\lambda(0) = \lambda(T) > 0$ and $x^*(0) < x_\sigma$, we get from Lemma 3.17 that $\lambda(t) > 0$, for all $t \in [\bar{t}_2, T]$, hence $t_1 < \bar{t}_2$. Suppose now that $t_1 \leq \bar{t}_1$. By (3.83) and (3.30), one has $x^*(\cdot) = x_a(\cdot, T, x^*(0))$ on $[t_1, T]$, so $t_1 \leq \bar{t}_1$ implies that

$$x^*(\bar{t}_1) = x_a(\bar{t}_1, T, x^*(0)) = x_\sigma. \tag{3.84}$$

So by (case IV) one has $x^*(t) < x_\sigma$, for all $t \in [0, \bar{t}_1[$, hence if $t_1 \leq \bar{t}_1$, $x^*(t) < x_\sigma$, for all $t \in [0, t_1[$ with $\lambda(t_1) = 0$. We deduce by Hypothesis 6 in (3.33) that $\lambda(t) > 0$, for all $t \in [0, t_1[$ implying that $\lambda(\cdot)$ is positive almost everywhere on $[0,T]$ and contradicting $\bar{t}_0 < \bar{t}_1$. W show now that $t_0 < \bar{t}_0$. Suppose $\bar{t}_0 \leq t_0$. First we remark that $t_0 < t_1$, as $t_0 = t_1$ implies that $\lambda(\cdot)$ is positive almost everywhere which contradicts $\bar{t}_0 < \bar{t}_1$. On the other hand, as $\bar{t}_0 < \bar{t}_1$ implies $x^*(\cdot) \neq \tilde{x}_a(\cdot)$, one has $x^*(0) < \tilde{x}_a(0)$ implying by Theorem 2.3 that $x_a(T,0,x^*(0)) > x^*(0)$, hence

$$x_a(t,0,x^*(0)) > x_a(t,T,x^*(0)), \quad \forall t \in [0,T].$$

Therefore, by definition of $\bar{t}_0$ and $\bar{t}_2$, one has $x_a(t,0,x^*(0)) > x_\sigma$, for all $t \in]\bar{t}_0, \bar{t}_2[$. Moreover, as $\lambda(t_0) = 0$ with $\bar{t}_0 \leq t_0$, we deduce from Hypothesis 6 in (3.33) that $\lambda(t) > 0$, for all $t \in]t_0, \bar{t}_2]$. This gives that $\lambda(t) > 0$, for all $t \in]t_0, T]$, implying that $t_1 \leq t_0$, which is a contradiction. Also, there exists $\bar{t}_3 \in]0, t_1[$ such that $x_b(\bar{t}_3, t_1, x^*(t_1)) = x_\sigma$ otherwise we would have by Lemma 3.17 that $\lambda(t) < 0$, for all $t \in [0, t_1[$, contradicting $\lambda(0) > 0$. Moreover, by definition of $\bar{t}_3$ and (case IV) one has $x_b(t, t_1, x^*(t_1)) > x_\sigma$, for all $t \in]\bar{t}_3, t_1]$, with $\lambda(t_1) = 0$, so by Hypothesis 6 and (3.33) one has $\lambda(t) < 0$, for all $t \in [\bar{t}_3, t_1[$. Likewise, one as $\lambda(t) < 0$, for all $t \in]t_0, \bar{t}_3]$. This implies that $\bar{t}_3 \in]t_0, t_1[$ and that $\lambda(t) < 0$, for all $t \in]t_0, t_1[$, which concludes the proof. $\square$

4 Application and Simulations

In this section we illustrate Theorem 3.12 and Theorem 3.14 through two examples together with numerical simulations. The first example is inspired by periodic bioprocess optimization problem and shows the links with some applications in ecology, whereas the second example is purely academical.

4.1 Simulations of Theorem 3.12

As an example of periodicity in the field of bioprocesses, we cite [2] in which the authors are interested by maximizing a production of microalgae limited by light in a photobioreactor under day/night cycles. To simplify the study, they approximate the light source by a step function equal to some positive real during the day and equal to zero during the night. Many other examples of periodicity problems in reactors can be found in [11]. Following the approach of [2], we consider a chemostat with a two phases growth function. The main difference is that we consider a classical model of biomass growth limited by a nutrient, with a (natural or forced) periodic variation of the environment (e.g. light, pH, or aeration) which affects the maximum growth rate. Therefore, we consider the chemostat of biomass x and substrate s given by

$$\begin{cases} \dot{x} = \mu(t,s)x - ux, \\ \dot{s} = -\mu(t,s)x + u(s_{\text{in}} - s), \end{cases} \tag{4.85}$$

where the growth rate function is defined by

$$\mu(t,s) = \begin{cases} \bar{\mu}\frac{s}{s+c}, & \text{if } t \in [0, \frac{T}{2}[, \\ \varepsilon\frac{s}{s+c}, & \text{if } t \in [\frac{T}{2}, T[, \end{cases}$$

with $0 < \varepsilon < \bar{\mu}$ and $0 < T$ and $c > 0$. The dilution rate u is the control variable and takes it values in $[0,1]$ and $s_{\text{in}} > 0$ represents the substrate concentration in the input flow rate. We are interested by maximizing the biomass production over a period T under a T-periodic constraint on the states variables, that is,

$$\max_{u(\cdot)\in[0,1]} \int_0^T u(t)x(t)dt, \quad \text{with } x(0) = x(T) \text{ and } s(0) = s(T). \tag{4.86}$$

Setting $M = x + s$, one gets $\dot{M} + uM = us_{\text{in}}$ which implies by periodicity of M that $M = s_{\text{in}}$, i.e., $s = s_{\text{in}} - x$. Thus

$$\dot{x} = (\mu(t, s_{\text{in}} - x) - u)x,$$

which implies by periodicity of x that $ux = \mu(t, \sin - x$, implying also the periodicity of u. Therefore, setting

$$\mathcal{U}_T := \{u : \mathbb{R} \to [0,1] : u \text{ measurable and T-periodic}\},$$

the problem (4.86) can be restated as

$$\min_{u\in\mathcal{U}_T} \int_0^T -\mu(t, s_{\text{in}} - x)x dt, \quad \text{with } \dot{x} = (\mu(t, s_{\text{in}} - x) - u)x, \text{ and } x(0) = x(T). \tag{4.87}$$

At this point, the problem is of the form of subsection 3.1 but the Hypotheses are not fulfilled since for each t, the function $x \mapsto \mu(t, s_{\text{in}} - x)$ is not continuous at $x = s_{\text{in}} + c$. However, since $x + s = s_{\text{in}}$, one has $x(t) \in [0, s_{\text{in}}]$ for all t. Therefore, one can restrain our study to $[0, s_{\text{in}}]$. Nevertheless, we need $x \mapsto \mu(t, s_{\text{in}} - x)$ to be negative at some point to fulfill Hypothesis 4. Thus, it suffices to choose $f_1 : [0, +\infty[\times [0, 1] \to \mathbb{R}$, resp. $f_2 : [0, +\infty[\times [0, 1] \to \mathbb{R}$, continuous such that $f_1(x, u) = \overline{\mu} \frac{s_{\text{in}} - x}{s_{\text{in}} - x + c} - u$, resp. $f_2(x, u) = \varepsilon \frac{s_{\text{in}} - x}{s_{\text{in}} - x + c} - u$, for $(x, u) \in [0, s_{\text{in}}] \times [0, 1]$ with f_1, resp. f_2, negative on $]s_{\text{in}}, +\infty[\times [0, 1]$ and satisfying Hypotheses 1 and 2. Likewise, by the restriction $x(t) \in [0, s_{\text{in}}]$ for all t, one can choose a lagrangian ℓ equal to $-\mu(t, s_{\text{in}} - x)x$ on $[0, T] \times [0, s_{\text{in}}]$ and satisfying Hypothesis 5, strictly increasing with respect to x on $]s_{\text{in}}, +\infty[$ in order to have Hypothesis 6. In this case, one obtains that

$$x_\sigma = c + s_{\text{in}} - \sqrt{c(c + \sin)}, \tag{4.88}$$

with $0 < x_\sigma < s_{\text{in}}$. Then we assume that

$$(\overline{\mu} + \varepsilon) \left(\frac{s_{\text{in}}}{s_{\text{in}} + c} \right) > 2 \tag{4.89}$$

to obtain Hypothesis 3 and it is not hard to see that Hypotheses of subsection 3.1 are all fulfilled. In order to apply Theorem 3.12, we make the following assumptions:

$$\overline{\mu} > 1 + \max \left\{ \frac{c}{s_{\text{in}}}, \frac{1}{\sqrt{1 + \frac{s_{\text{in}}}{c}} - 1} \right\}, \tag{4.90}$$

and

$$\varepsilon < 1. \tag{4.91}$$

Indeed, as $\varepsilon \frac{s_{\text{in}} - x}{s_{\text{in}} - x + c} - 1 = 0$ if and only if $x = s_{\text{in}} - \frac{c}{\varepsilon - 1}$, Equation (4.91) implies that $f_2(\cdot, 1) < 0$ on $[0, +\infty[$, hence setting $x_b^2 = 0$, one has that x_b^2 satisfies Definition 3.2. Moreover, as $\overline{\mu} \frac{s_{\text{in}} - x}{s_{\text{in}} - x + c} - 1 = 0$ if and only if $x = s_{\text{in}} - \frac{c}{\overline{\mu} - 1}$, Equation (4.90) implies that $\overline{\mu} > 1 + \frac{c}{s_{\text{in}}}$ which gives that $s_{\text{in}} - \frac{c}{\overline{\mu} - 1} \in]0, s_{\text{in}}[$. Therefore, setting $x_b^1 = s_{\text{in}} - \frac{c}{\overline{\mu} - 1}$, one has by definition of $f_1(\cdot, 1)$ that x_b^1 fulfills Definition 3.2. Also, the fact that $\overline{\mu} > 1 + \frac{1}{\sqrt{1 + \frac{s_{\text{in}}}{c}} - 1}$ ensures that $x_\sigma < x_b^1$. Finally, by the definitions of $f_1(\cdot, 0)$ and $f_2(\cdot, 0)$, setting $x_a^1 = x_a^2 = s_{\text{in}}$ gives that Definition 3.2 is fulfilled, and we obtain (case III). We are in position to apply Theorem 3.12 which gives us the structure of the optimal control u^*. Therefore, given an initial condition $x_0 < x_\sigma$, one can graph $x_{u^*}(\cdot, 0, x_0)$ the solution associated to u^* starting at x_0 as done in Figure 1 for the values of Table 2. Thus, for each suitable initial conditions $x_0 < x_\sigma$, one can calculate the cost

$$J(x_0) = \int_0^T -\mu(t, s_{\text{in}} - x_{u^*}(t, 0, x_0)) x_{u^*}(t, 0, x_0) dt,$$

and graph the function J, cf. Figure 2, to do numerical optimization.

Table 2: Parameters

Parameter	Value
$\overline{\mu}$	3
ε	0.1
s_{in}	8
c	1
T	10

4.2 Simulations of Theorem 3.14

In this subsection we consider the quasi affine case as an academical example in order to illustrate Theorem 3.14. Given $0 < \overline{T} < T$, we consider the control problem of section 3.1:

$$\dot{x} = xF(t, x, u), \quad \text{where} \quad F(t, x, u) := \begin{cases} f_1(x, u) := \alpha_1 - \beta_1 x - \gamma_1 u, & \text{if } t \in [0, \overline{T}[, \\ f_2(x, u) := \alpha_2 - \beta_2 x - \gamma_2 u, & \text{if } t \in [\overline{T}, T[, \end{cases}$$

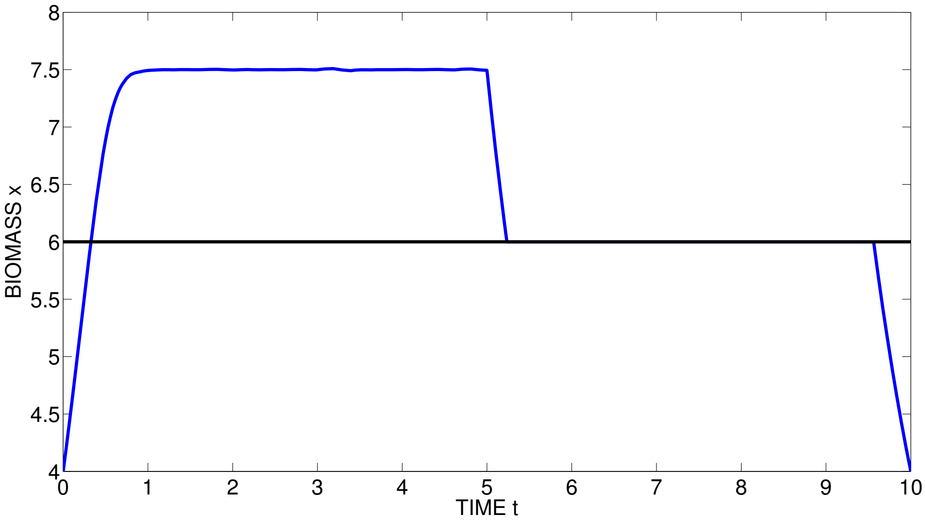

Figure 1: Graph of $x_{u^*}(\cdot, 0, x_0)$ for $x_0 = 4$. The black line represents the singular arc x_σ.

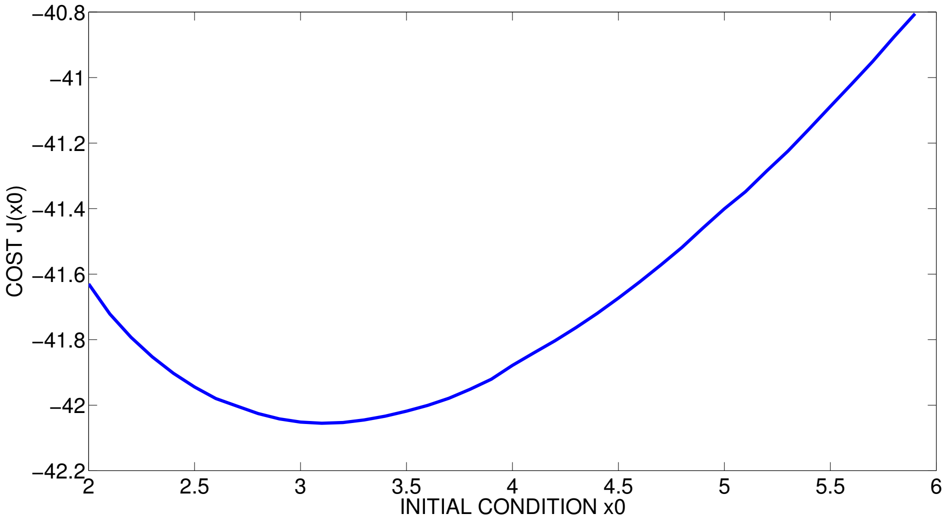

Figure 2: Graph of the cost function J in function of the initial condition x_0.

for some $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$. We are interested in finding

$$\min_{u(\cdot)\in[0,1]} \int_0^T \ell(x(t))dt, \tag{4.92}$$

where $\ell(x) := \frac{1}{2}(x - x_\sigma)^2$, for some $x_\sigma \in \mathbb{R}$. In order to fulfill the hypotheses of Theorem 3.14, we make the following assumptions:

(i) for all i, $\beta_i, \gamma_i \in]0, +\infty[$;

(ii) for all i, $\alpha_i > \gamma_i$;

(iii) $\frac{\alpha_2}{\beta_2} < x_\sigma < \frac{\alpha_1-\gamma_1}{\beta_1}$.

Indeed, under (i), (ii) and (iii), it is not hard to see that Hypotheses 1 to 6 are fulfilled. By calculation, we obtain that the equilibria of Definition 3.2 are given by

$$x_a^i = \frac{\alpha_i}{\beta_i}, \quad \text{and} \quad x_b^i = \frac{\alpha_i - \gamma_i}{\beta_i},$$

which are positive by (ii). Then, (iii) ensures that we are in (case IV). In the following, we only treat the case $x^*(0) < x_\sigma$ and $\lambda(0) > 0$ of Theorem 3.14. Applying Theorem 3.14 with $x^*(0) < x_\sigma$ and $\lambda(0) > 0$ gives that the optimal control is of the form $a - b - a$. Therefore, given $x^*(0) < x_\sigma$ and $x_0 \in]x^*(0), x_\sigma[$, one can graph the solution of the bang-bang type $a - b - a$ associated to $(x^*(0), x_0)$ as in Figure 3. The simulation has been made with the parameters given in Table 3. Therefore, given $(x^*(0), x_0)$, one can calculate and graph the cost

$$J(x^*(0), x_0) := \int_0^T \ell(x(t))dt,$$

where $x(\cdot)$ is the solution associated to $u_{a,b,a}$ defined in (3) of Theorem 3.14. This has been done in Figure 4. Finally, it is worth noting that to solve entirely the problem (4.92), one would have to graph J for the other three cases, i.e., (1), (2) and (4), of Theorem 3.14.

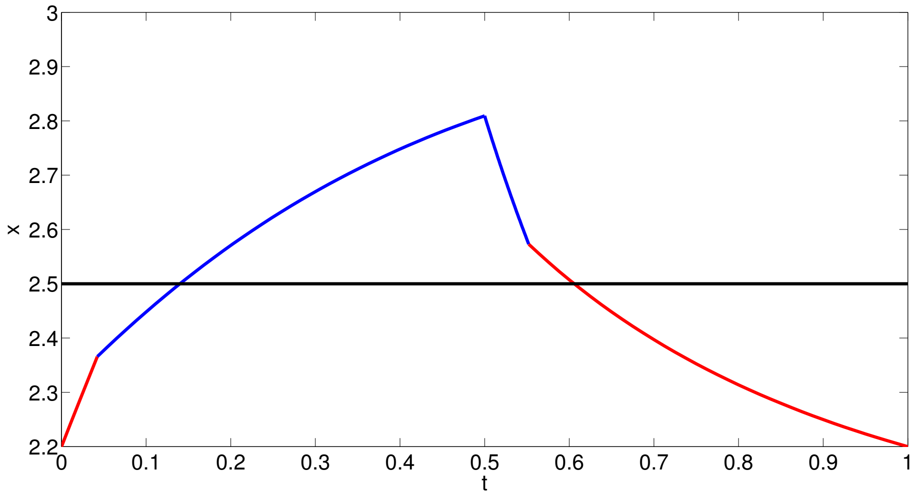

Figure 3: Graph of the trajectory associated to the control $u_{a,b,a}$ defined in (3) of Theorem 3.14 for $x^*(0) = 2.2$, and $x_0 = 2.3$. The color red, resp. blue, is used for the control a, resp. b. The black line represents the singular arc x_σ which is not controllable.

Table 3: Parameters

Parameter	Value
T	1
$\overline{T}$	0.5
x_σ	2.5
α_1	4
β_1	1
γ_1	1
α_2	2
β_2	1
γ_2	1

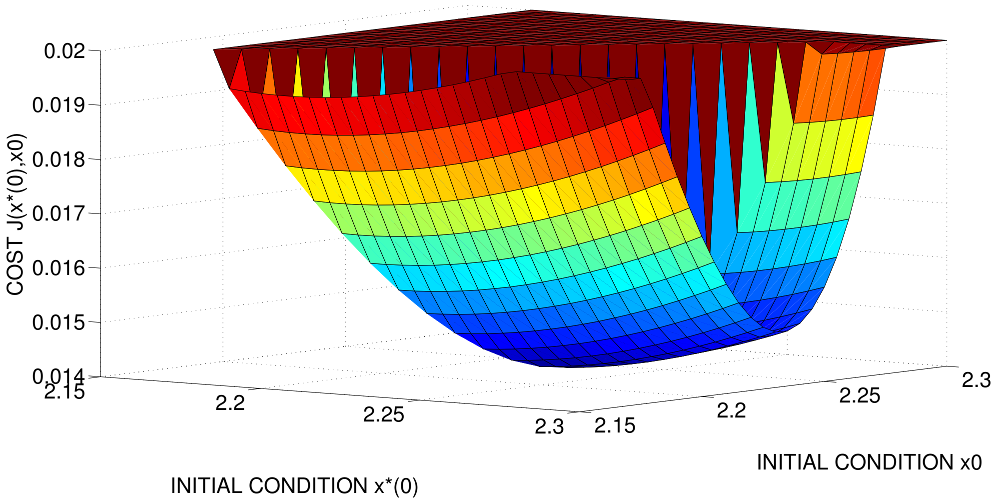

Figure 4: Graph of the cost function J in function of the initial conditions $x^*(0)$ and x_0. We have truncated the values over 0.02 to have a better view of the minimum of the graph.

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