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Abstract

Motivated by microscopic traffic modeling, we analyze dynamical systems which have a piecewise linear concave dynamics not necessarily monotonic. We introduce a deterministic Petri net extension where edges may have negative weights. The dynamics of these Petri nets are uniquely defined and may be described by a generalized matrix with a submatrix in the standard algebra with possibly negative entries, and another submatrix in the minplus algebra. When the dynamics is additively homogeneous, a generalized additive eigenvalue is introduced, and the ergodic theory is used to define a growth rate. In the traffic example of two roads with one junction, we compute explicitly the eigenvalue and we show, by numerical simulations, that these two quantities (the additive eigenvalue and the average growth rate) are not equal, but are close to each other. With this result, we are able to extend the well-studied notion of fundamental traffic diagram (the average flow as a function of the car density on a road) to the case of roads with a junction and give a very simple analytic approximation of this diagram where four phases appear with clear traffic interpretations. Simulations show that the fundamental diagram shape obtained is also valid for systems with many junctions.

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I. Introduction

The main purpose of this paper is to explore some new points on dynamical systems that have emerged with the study of microscopic traffic modeling using minplus algebra and Petri nets. These points are the following :

- Standard Petri nets or linear maxplus algebra have not enough modeling power to describe the dynamics of a traffic system as simple as two roads with a crossing and the simplest vehicle move.
- A remedy to this difficulty is to introduce possibly negative weights on Petri net arcs (not to be confused with negative places or negative tokens as in [45]) or to use nonlinear minplus dynamics that are not monotonic.
- A consequence is that all known standard methodology based mainly on dynamic programming is ineffective.
- Nevertheless, it is sometimes possible to use the standard ergodic theory, and/or to compute a generalized eigenvalue for the subclass of additively homogeneous systems and get nice qualitative results on the system.
- When it is not possible to derive analytical results, we can do numerical simulations. In general, a microscopic modeling of a realistic system has a very large number of state variables difficult to manage. To build such models in a modular way, good notations are essential. The standard and the minplus matrix notations are combined to obtain a generalized matrix notation adapted to the piecewise affine concave dynamics used here.

All these points are illustrated on the traffic application for which we have obtained a new result: a good analytic approximation of the macroscopic law called the fundamental diagram in the case of roads with junctions.

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The traffic on a road has been studied from different points of view on a macroscopic level. For example:

- The Lighthill-Whitham-Richards (LWR) model [38], which is the most standard view, expresses the mass conservation of cars seen as a fluid:
$$\begin{cases} \partial_t \rho + \partial_x \varphi = 0 \ , \\ \varphi = f(\rho), \end{cases}$$
where $\varphi(x,t)$ denotes the flow at time t and position x on the road; $\rho(x,t)$ denotes the density; and f is a given function called the *fundamental traffic diagram*. For traffic, this diagram plays the role of the perfect gas law for the fluid dynamics. The diagram has been estimated using experimental data, and its behavior is quite different from standard gas at high density. We obtain an idea of the diagram's shape in the subsection titled "traffic on a circular road."
In [14], Daganzo, using the variable $N(t,x)$ counting the cumulated number of vehicles having reached the point x at time t, and remarking that $\varphi = \partial_t N$ and $\rho = -\partial_x N$, has interpreted the equation $\varphi = f(\rho)$ as a Hamilton-Jacobi equation when f is concave. The other equation $\partial_t \rho + \partial_x \varphi = 0$ telling that $\partial_{tx} N = \partial_{xt} N$. In [15], Daganzo and Geroliminis have generalized the variational formulation to a network by postulating the existence of a Hamiltonian in this case also and by trying to approximate it as an infimum of linear constraints that he obtains by physical considerations on the street involved. In [31], [32], Geroliminis and Daganzo have studied what happen experimentally on real towns.
- The Prigogine-Herman (PH) model [48] (kinetic model) gives the evolution of the density of particles $\rho(t,x,v)$ as a function of t, x and v, where v is the speed of particles. The model is given by:
$$\partial_t \rho + v \partial_x \rho = C(\rho, \rho) \ ,$$
where $C(\rho,\rho)$ is an interacting term that is, in general, quadratic in ρ and, as such, models the driver behavior in a simple way. From the distribution ρ, we can derive all the useful quantities, such as the average speed $\bar{v}(t,x) = \int v \rho(t,x,v) dv$.

The integration of the PH model is more time consuming and, therefore, not used in practice. The main interest of the PH equation is that we can derive macroscopic laws like the fundamental diagram from its solution.

The LWR model assumes the knowledge of the fundamental traffic diagram f. This function has been studied not only experimentally but also theoretically using simple microscopic models such as exclusion processes, [17], [8], cellular automata [3], [10], or simulation of individual car dynamics. Here, we recall a way to derive an approximation of this diagram. This derivation consists of computing the eigenvalue of a simple minplus linear system counting the number of vehicles entering in a road section.

The main result we give here is the generalization of the fundamental law to the two dimensional (2D) case where the roads cross each other. In statistical physics, a lot of numerical work ([44], [13], [43], [5], [9]) and good surveys [10], [33] have been done on idealized towns. These works analyze numerical experimentations based on various stochastic models with or without turning possibilities and show the existence of a threshold of the density at which the system blocks suddenly. The particular case of one junction is studied in [26], [27] where precise results are given for the stochastic case without turning possibilities. Let us mention also the attempt to derive a 2D fundamental diagram by Helbing [34] using queuing theory. In the saturated and unsaturated traffic case he derives a formula for an intersection that he extends to an area of a town that fits well to experimental data.

Here we present a deterministic model with turning possibilities, based on Petri nets and minplus algebra. The minplus linear model on a unique road can be described in terms of event graphs (a subclass of Petri nets). The presence of conflicts at junctions prevents the extension of this model to the 2D case. We propose a way to solve the difficulty by extending the class of weights used in Petri nets allowing negative weights. Due to such weights, the firing of a transition can consume tokens downstream, and the modeling power of the Petri nets is improved significantly. This possibility is used to model the authorization to enter into the junctions.

The dynamics of general Petri nets allowing negative weights can be written easily but is neither linear in minplus algebra nor monotone. Nevertheless, in the traffic applications given here, dynamics ($x^{k+1} = f(x^k)$) are always homogeneous of degree one ($f(\lambda \otimes x) = \lambda \otimes f(x)$, where $\otimes$ denotes the minplus multiplication that is the standard addition). For such systems, the eigenvalue problem (computation of λ such that $\lambda \otimes x = f(x)$) can be reduced to a fixed point problem. The existence of an (the) average growth rate $\chi = \lim_k x^k/k$ is due to the existence of a Birkhoff average. The quantities λ and χ coincide when f is also monotone, but this is not true in the general case. The monotone case has been studied carefully in [30], [42]. In traffic examples of roads with junctions, the dynamics is not monotone.

In all the traffic examplesgiven here, we study systems of roads on a torus without entries in such a way that the number of vehicles remains constant in the system. In this way, we represent an idealization of constant densities used in realistic system. Indeed, to maintain a constant density in an open system, a new vehicle has to enter, each time another one leaves the system. This is mathematically equivalent to consideration of circular roads.

The particular case of two circular roads with one junction is studied in detail. For this system, when the initial state is zero, a result on the existence of the average growth rate is obtained using the nondecreasing trajectory property. From this result, we show that the distances between the states stay bounded. The eigenvalue problem can be solved explicitly. By simulation, we see that the eigenvalue and the average growth rate generally do not coincide. However, these quantities are very close for any fixed density. Therefore, the simple formulas obtained by solving the eigenvalue problem give a good approximation of the 2D traffic fundamental diagram.

The definition of the fundamental diagram that we use here must not be confused with the one used by Daganzo and Geroliminis [14], [15] which is the Hamiltonian of the traffic dynamics interpreted as a dynamic programming equation in the simplest case of a single road, and generalized in heuristic way to the city case. The fundamental diagram given in [15] mainly coincides with our minplus dynamics, in the case of one street, but is observed experimentally and is not derived from a microscopic model as we do here. The dependence of the eigenvalue or the average growth rate with the density of vehicles in the system, which is a difficult result to obtain, even in the case of two streets with one intersection, is not done in the Daganzo work. Moreover, for the city case, we remark also, on the simple example studied here, that the dynamics is not a standard dynamic programming equation associated to a deterministic or a stochastic control problem (since the dynamics is not monotone), which could be a problem, at least at the conceptual level, for the point of view adopted by Daganzo.

The fundamental diagram of 2D-traffic given here shows four phases:

- *Free phase*: The density is low and the vehicles do not interact.
- *Saturation phase*: The junction is saturated, and when a vehicle wants to leave the junction, the locations downstream of the junction are free.
- *Recession phase*: The junction is saturated, but when a vehicle wants to leave the junction, the locations downstream of the junction are sometimes crowded.
- *Freeze phase*: The vehicles cannot move.

The four phases are derived from a unique simple model and are not postulated to obtain different models subsequently analyzed.

Preliminary results on the traffic dynamics have been presented in [19], [20], [21]. In [23], many developments, complementary results, and other completely solved examples can be found. The theorems given here on the eigenvalue problem and the growth rate complete some of the main results of [23] by relaxing some assumptions and clarifying the growth rate existence problem. In the companion paper [22], more traffic engineering oriented, the traffic interpretations are developed, generalized to more realistic systems and the influence of traffic lights is studied.

This paper is organized in three parts. First, we recall the basic results of minplus algebra, present matrix notations for polyhedral concave 1-homogeneous systems, and discuss their growth rate and eigenvalue problems. Second, we present a new class of Petri nets where we allow negative weights, and show its application in the modelling of junctions managed with a priority rule. Third, after giving a short review on the traffic on a circular road, we compute a good analytical approximation of the fundamental traffic diagram in the case of two circular roads with one junction managed using a priority rule.

II. Minplus Algebra and extensions

A. Review of minplus algebra

In this section, we revisit the main definitions and properties of the minplus algebra. An in-depth treatment of the subject is in [4].

The structure $\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \oplus, \otimes)$ is defined by the set $\mathbb{R} \cup \{+\infty\}$ endowed with the operations $\min$ (denoted by $\oplus$, called minplus *sum*) and $+$ (denoted by $\otimes$, called minplus *product*). The element $\varepsilon = +\infty$ is the *zero* element $\varepsilon \oplus x = x$, and is *absorbing* $\varepsilon \otimes x = \varepsilon$. The element $e = 0$ is the *unit* element $e \otimes x = x$. The main difference with respect to the conventional algebra is the idempotency of the addition $x \oplus x = x$ and the fact that the addition cannot be *simplified* $a \oplus b = c \oplus b \not\Rightarrow a = c$. This structure is called *minplus algebra*. We will call $\overline{\mathbb{R}}_{\min}$ the completion of $\mathbb{R}_{\min}$ by $-\infty$ with $-\infty \otimes \varepsilon = \varepsilon$.

This minplus structure on scalars induces an idempotent semiring structure on $m \times m$ square matrices with the element-wise minimum denoted by $\oplus$ and the matrix product defined by $(A \otimes B)_{ik} = \min_j (A_{ij} + B_{jk})$, where the zero and the unit matrices are still denoted by ε and e. We associate a precedence graph $\mathcal{G}(A)$ to a square matrix A where the nodes of the graph correspond to the columns of the matrix A and the edges of the graph correspond to the non-zero ($\neq \varepsilon$) entries of the matrix. The *weight* of the edge going from i to j is the non-zero entry A_{ji}. We define the weight of a path p in $\mathcal{G}(A)$, which we denote by $|p|_w$, as the minplus product of the weights of the edges composing the path (that is the standard sum of weights). The number of edges of a path p is denoted by $|p|_l$. We will use the following fundamental result.

Theorem 1 ([4]): If the graph $\mathcal{G}(A)$ associated with the $m \times m$ minplus matrix A is strongly connected, then the matrix A admits a unique *eigenvalue* $\lambda \in \mathbb{R}_{\min} \setminus \{\varepsilon\}$:

$$\exists\ X \in \mathbb{R}^m_{\min}, X \neq \varepsilon : A \otimes X = \lambda \otimes X \quad \text{with } \lambda = \min_{c \in \mathcal{C}} \frac{|c|_w}{|c|_l}\ , \tag{1}$$

where $\mathcal{C}$ is the set of circuits of $\mathcal{G}(A)$.

B. Generalized matrix notations

In a Petri net, two kinds of operations appear. One is the accumulation of resources in the places, and the other is the synchronization in the transitions. The first operation can be modeled by addition, and the second by a min or max (a task can start at the maximum of the arrival instants of the resources needed by the task). Matrix notations can be generalized to such situations.

We consider the set of $m \times m$ matrices where the rows [resp. columns] are partitioned in two sets: the standard and the minplus sets (here the m' first rows [resp. columns], and the last m" rows [resp. columns]) with entries in $\mathbb{R} \cup \{+\infty, -\infty\}$, equipped with the two operations $\boxplus$ and $\boxtimes$ defined by:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \boxplus \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} A + A' & B + B' \\ C \oplus C' & D \oplus D' \end{bmatrix}\ ,$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \boxtimes \begin{bmatrix} A' & B' \\ C' & D' \end{bmatrix} = \begin{bmatrix} AA' + BC' & AB' + BD' \\ C \otimes A' \oplus D \otimes C' & C \otimes B' \oplus D \otimes D' \end{bmatrix}\ .$$

Since the entries are in an extension of $\mathbb{R}$, we have to specify the scalar addition and multiplication table:

$$0 \times \pm\infty = \pm\infty \times 0 = 0, \quad +\infty \otimes (-\infty) = +\infty - \infty = +\infty\ .$$

These choices have been made to preserve the absorption properties for the multiplication of the null elements of the standard algebra (0) and the minplus algebra ($\varepsilon = +\infty$). This absorption property is useful to model the absence of the arc in the precedence graph $\mathcal{G}(A)$ associated to a square matrix A (defined in the same way as in the pure minplus case).

The addition $\boxplus$ is associative, commutative and has the null element $\begin{bmatrix} 0 & 0 \\ \varepsilon & \varepsilon \end{bmatrix}$ still denoted ε.

The multiplication $\boxtimes$ has no identity element. It is neither associative nor commutative nor distributive with respect to the addition. The main interest of this operation appears when the right operand is a vector $Y = A \boxtimes X$ where $X \in \overline{\mathbb{R}}_{\min}^m$ is a vector and A is a $m \times m$ matrix with entries in $\overline{\mathbb{R}}_{\min}$. Then Y, seen as a function of X, is a set of m' standard linear forms and of m" minplus linear forms on $\overline{\mathbb{R}}_{\min}^m$ with $m = m' + m$". The operation $Z = A \boxtimes (A \boxtimes X)$ corresponds to the compositions of these linear forms, but the compositions do not define anymore a set of standard and minplus linear forms and $Z \neq (A \boxtimes A) \boxtimes X$.

Moreover, when the right operand is a vector, the multiplication can be represented by the graph $\mathcal{G}(A)$ where we have two kinds of nodes (those corresponding to the standard $+$ operation, and those corresponding to $\oplus$ operation) and two kinds of edges (those which operate multiplicatively ($\times$) and those which operate additively ($\otimes$)).

Example 1: Let us consider the graph $\mathcal{G}(A)$ associated to the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (Figure-1) with one node associated to the standard algebra and one node to the minplus algebra. Then $y = A \boxtimes x$, where y and x are two vectors with two entries, means:

$$y_1 = ax_1 + bx_2, \quad y_2 = \min(c + x_1, d + x_2) \; .$$

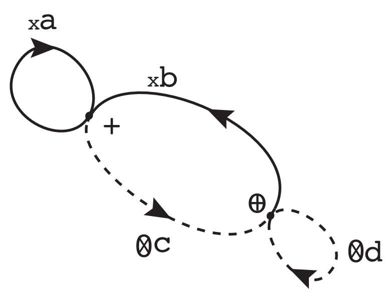

Fig. 1. The incidence graph associated to the matrix A with one $\oplus$node and one +node.

We adopt the following conventions to solve some ambiguities in the formula notations:

– As soon as a minplus symbol appears in a set of formulas, all the operations must be understood in the minplus sense with the exception of the exponent that must be understood in the standard sense. For example, $x^{a/b} \oplus 1$ or $\sqrt[b]{x^a} \oplus 1$ must be understood as $\min((a/b)x, 1)$ and not as $\min((a-b)x, 1)$. In minplus sense, $a/b = a - b$ since it is the solution[1] of $b \otimes x = a$. The equation $b \otimes x = a$ means $b + x = a$.

– In a minplus formula, the rational numbers (written with figures) are denoted in the standard algebra. For example, $\frac{1}{2}x \oplus 1$ (instead of $\sqrt{1}x \oplus 1$) means $\min(0.5 + x, 1)$ and not $\min(-1 + x, 1)$, but $(a/b)x \oplus 1$ means $\min(a - b + x, 1)$.

– A non-zero element in the minplus sense means a finite element in the usual sense. Positive element will be used always in the standard sense.

With these conventions the formulas are concise and not ambiguous.

[1]For the reader familiar with the residuation (see for example [4]), the minplus division used here means the standard minus operator with the convention previously given for infinite elements. This choice is incompatible with the residuation which chooses the smallest solution of $b \otimes x = a$.

C. Additively homogeneous autonomous dynamical systems

Let us now discuss one-homogeneous minplus dynamical systems, which form a large class of systems, used frequently when there is a conservation such as probability mass or number of tokens in Petri nets. However, because we accept negative entries, we obtain a generalization of two cases. One case is a measure not necessarily positive with a total mass equal to one. The other case is a conservation of tokens in a situation where negative entries may appear. Let us start with an academic example. The traffic modeling will give more concrete examples in the following sections.

Example 2: Let us go back to Example 1 and assume that $a+b=1$. Adding λ to each component of x (that we can write $\lambda \otimes x$) implies that the two components of y are augmented by λ. We have

$$A \boxtimes (\lambda \otimes x) = \lambda \otimes (A \boxtimes x) \ .$$

We say that the system is *additively homogeneous of degree 1* or, more simply, *homogeneous*. Indeed, using the minplus notation, $y = A \boxtimes x$ can be written:

$$y_1 = (x_1)^{\otimes a} \otimes (x_2)^{\otimes b}, \quad y_2 = c \otimes x_1 \oplus d \otimes x_2 \ ,$$

which is clearly homogeneous of degree 1 in the minplus algebra as long as $a+b=1$. Moreover, if a and b are nonnegative, then the transformation is *nondecreasing*.

To simplify the notations, we will write the transformation in the following way:

$$y_1 = (x_1)^a (x_2)^b, \quad y_2 = cx_1 \oplus dx_2 \ .$$

More generally, we say that the function $f : \overline{\mathbb{R}}^n_{\min} \mapsto \overline{\mathbb{R}}^n_{\min}$ is *homogeneous* if

$$f(\lambda \otimes x) = \lambda \otimes f(x) \ .$$

D. Eigenvalues of homogeneous systems

The *eigenvalue problem* of such a homogeneous function f can be formulated as finding non-zero x and λ such that:

$$\lambda \otimes x = f(x) \ .$$

When there is a risk of confusion with the standard eigenvalue problem of a standard linear operator we will call it additive eigenvalue problem.

In this section we do not specify intentionally if x and λ are finite or not, because, for some functions it is more natural to consider finite value (for example standard affine system), while for other functions, it is more natural to complete by adding infinite value (for example linear minplus system). Moreover, we will not give the precise hypotheses under which the existence is true. We survey only typical applications of this eigenvalue problem for which a complete general theory still does not exist.

Since f is homogeneous, by symmetry we may consider only those eigenvectors x such that $x_1 \neq \varepsilon$. The eigenvalue problem becomes:

$$\begin{cases} \lambda & = f_1(x/x_1) \ , \\ x_2/x_1 & = (f_2/f_1)(x/x_1) \ , \\ \cdots & = \cdots \\ x_n/x_1 & = (f_n/f_1)(x/x_1) \ , \end{cases}$$

where the division is in minplus sense, that is the subtraction. Denoting $y = (x_2/x_1, \cdots, x_n/x_1)$ and $g_{i-1}(y) = (f_i/f_1)(0,y)$, the problem is reduced to the computation of the fixed point problem $y = g(y)$. This fixed point gives the normalized eigenvector from which the eigenvalue is deduced by: $\lambda = f_1(0,y)$. We note that g is a non-homogeneous minplus function of y.

The fixed point problem does not always have a solution, but nevertheless, there are cases where we are able to find one:

- f *is affine in standard algebra*. In this case $f(x) = Ax + b$. The homogeneity[2] implies that the kernel of $A - I_d$ is not empty. When the standard eigenvalue 0 of $A - I_d$ is simple, the additive eigenvalue of the affine system is equal to $\lambda = pb$, where p is the normalized ($p\bar{1} = 1$) left standard eigenvector of A associated to the (standard) eigenvalue 1. But even in this case, all the standard eigenvalues do not have a module necessarily smaller than one, and the dynamical system may be unstable. We note that when all the entries of the matrix A are nonnegative, f is monotone nondecreasing, but when there are positive entries and negative entries, the system is not monotone.
- f *is minplus linear*: $f(x) = A \otimes x$. In this case the system is monotone.
- f *corresponds to stochastic control*. In this case $f(x) = D \otimes (Hx)$ where H is a standard matrix with rows that define discrete probability laws. Such a matrix is called a stochastic matrix. Then the dynamics $x^{k+1} = f(x^k)$ has the interpretation of a dynamic programming equation associated to a stochastic control problem. When the system is communicating (see [6] for a precise definition), there exists a unique additive eigenvalue (with a finite eigenvector) which is the optimal average cost of the corresponding stochastic control problem. Note that in this case, x^k are components of the dynamics which can be written
$$x^{k+1} = A \boxtimes x^k, \text{ with } A = \begin{bmatrix} 0 & H \\ D & \varepsilon \end{bmatrix} .$$
- f *corresponds to stochastic games*. In this case $f = D_1 \odot (D_2 \otimes (Hx))$ where $\odot$ denotes the maxplus product (obtained by replacing min by max in the minplus matrix product $\otimes$), and H is a stochastic matrix. This case corresponds to dynamic programming equations associated to stochastic games see [35]. In this case, f is monotone.
- f *has a particular triangular structure* for example:
$$\begin{cases} x^{k+1} = A \otimes x^k \ , \\ y^{k+1} = B(x^k) \otimes y^k \ , \end{cases}$$
with $B(x)$ additively homogeneous of degree 0 but not necessarily monotone. For such systems, it is easy to find the eigenvalue and eigenvector by applying the minplus algebra results. In this case, f is not always monotone. See [23], [25] for discussions and generalizations.

In general, it is possible to compute the fixed point using the Newton's method[3]. This method corresponds to the policy iteration when the dynamic programming interpretation holds true. For stochastic control problems, the policy iteration is globally stable. In the game case, the policy iteration is only locally stable but a global policy iteration algorithm can be designed, see [12], where at each iteration one player solve completely its dynamic programming equation, given the strategy of the other player.

We may have unstable fixed points which are not accessible by integrating the dynamics. In this case, the eigenvalue, computable by the Newton's method, gives no information on the time asymptotic of the system. When all the fixed points are unstable, we may have a linear growth of the state trajectories. This point is illustrated by the chaotic tent dynamics example given in the next section.

E. Growth rate of homogeneous systems

We define the *average growth rate*[4], $\chi(f)$, of a dynamic system $x^{k+1} = f(x^k)$, where $x \in \overline{\mathbb{R}}^n_{\min}$, by the common limit $\lim_k x_i^k/k$ of all the components i when this limit exists. In [30] it has been proved, with a special definition of connexity (satisfied for a system defined by $f(x) = A \boxtimes x$ as long as the graph $\mathcal{G}(A)$ is strongly connected[5]), that the growth rate and the eigenvalue of a homogeneous and monotone

[2] $A\bar{1} = \bar{1}$ with $\bar{1}$ the vector with all its entries equal to 1.

[3] we have to solve a piecewise linear system of equations

[4] It corresponds to what is often called the cycle time vector when all the components of this vector are equal.

[5] Here there is an edge from i towards j in $\mathcal{G}(A)$ if $A_{ji} \neq \varepsilon$) if it is a minplus edge or $A_{ji} \neq 0$ if it is a standard one.

system exist and are equal. Let us show that chaos may appear and that the eigenvalue and growth rate may differ on a system which is only homogeneous.

Let us consider the homogeneous dynamic system where k is the time index:

$$\begin{cases} x_1^{k+1} = x_2^k \ , \\ x_2^{k+1} = (x_2^k)^3/(x_1^k)^2 \oplus 2(x_1^k)^2/x_2^k \ . \end{cases}$$

The corresponding eigenvalue problem is

$$\begin{cases} \lambda x_1 = x_2 \ , \\ \lambda x_2 = x_2^3/x_1^2 \oplus 2x_1^2/x_2 \ , \end{cases}$$

where the minplus power exponent must not be confused with a time index.

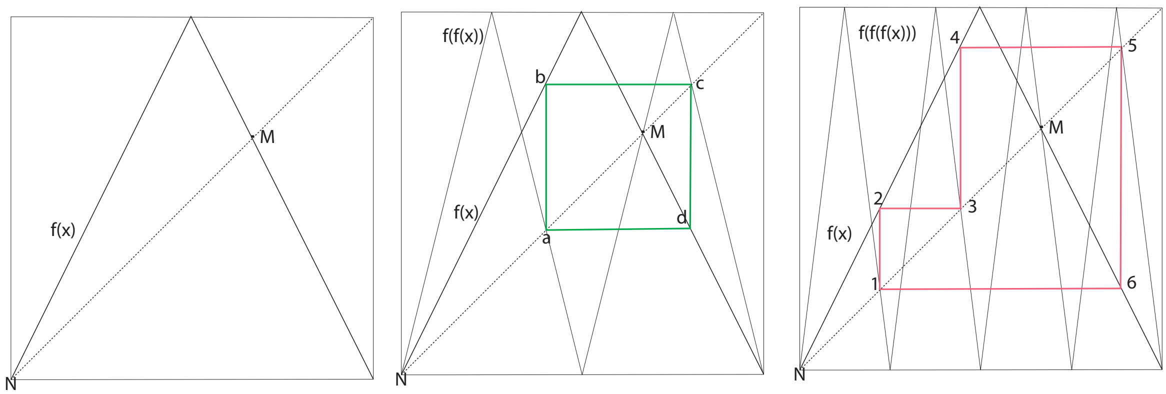

Fig. 2. Cycles of the tent transformation. The abscissa x_M of M is a fixed point of f: $x_M = f(x_M)$. The pair (x_a, x_c) is a cycle of f composed of two fixed points of $f \circ f$: $x_c = f(x_a)$, $x_a = f(x_c) = f(f(x_a))$. The triplet (x_1, x_3, x_5) is a circuit of f is composed of three fixed points of $f \circ f \circ f$: $x_3 = f(x_1)$, $x_5 = f(x_3) = f(f(x_1))$, $x_1 = f(x_5) = f(f(f(x_1)))$.

The solution is $\lambda = y$ with $y = x_2/x_1$ satisfying the equation

$$y = y^2 \oplus 2/y^2,$$

which has the solutions $y = 0$ and $y = \frac{2}{3} = 0.66....$ These two solutions are unstable fixed points of the transformation $f(y) = y^2 \oplus 2/y^2$. However, the system $y^{k+1} = f(y^k)$ is chaotic since f is the tent transform (see [7] for a clear discussion of this dynamics). In Figure-2, we show the graph of $x \mapsto f(x)$, $x \mapsto f(f(x))$, $x \mapsto f(f(f(x)))$, their fixed points, and periodic trajectories.

It has been proved that the tent iteration has a unique invariant measure absolutely continuous with respect to the Lebesgue measure (the uniform law on $[0, 1]$). Therefore, the system is ergodic. The growth rate

$$(x_1^N - x_1^0)/N = \frac{1}{N} \sum_{k=1}^{N} (x_1^k - x_1^{k-1}) \ ,$$

can be computed by averaging, with respect to the uniform law, an increase in one step: $f_1(x) - x_1$ with the standard notations (that is $f_1(x)/x_1 = y$ with the minplus notations). Therefore $\chi(f) = \int_0^1 y dy = 0.5$, for almost all initial conditions, which is different from the eigenvalues (0 and $\frac{2}{3}$).

More generally, for a homogeneous system, we can write the system dynamics:

$$x_1^{k+1}/x_1^k = f_1(x^k)/x_1^k = h(y^k), \quad y^{k+1} = g(y^k),$$

with $y_{i-1}^k = x_i^k/x_1^k$ and $g_{i-1} = f_i/f_1$ for $i = 2, \cdots, n$. As long as y^k belong to a bounded closed (compact) set for all k, we remark (after Kryloff and Bogoliuboff [36]) that the set of measures:

$$\left\{ P_{y^0}^N = \frac{1}{N} \left(\delta_{y^0} + \delta_{g(y_0)} + \cdots + \delta_{g^{N-1}(y^0)} \right) , \ n \in \mathbb{N} \right\} \ ,$$

(where δ_a denotes the Dirac mass on a) is tight. Therefore, we can extract convergent subsequences which converge toward invariant measures Q_{y^0} that we will call Kryloff-Bogoliuboff invariant measure. Then we can apply the ergodic theorem at the sequence $(y^k)_{k\in\mathbb{N}}$. Application of this theorem shows that, for almost all new initial conditions chosen randomly according to Q_{y^0}, we have :

$$\chi(f) = \lim_N \frac{1}{N}(x_1^N - x_1^0) = \lim_N \frac{1}{N}\left(\sum_{k=0}^{N-1} h(y^k)\right) = \int h(y)dQ_{y^0}(y) \ . \tag{2}$$

It may happen that the initial condition y^0 is transient; therefore, it is in the attractive basin of Q_{y^0} not in the support of Q_{y^0}. It would be very useful to prove that y^0 is generic (in the sense of Furstenberg[29]), that is the limit exists for y^0. A priori homogeneous systems do not have the uniform continuity property required to prove the convergence of the Cesaro means for all initial condition y^0. The following classic example shows the case of an ergodic system with non generic points $f : x \in \mathbb{T}^1 \to 2x \in \mathbb{T}^1$ with: $x^0 = 0.1\overset{2}{00}\overset{4}{1111}\overset{8}{00000000}\cdots$ where $\mathbb{T}^1$ denotes the torus of dimension 1. In Figure-3 we see that the Cesaro does not converge.

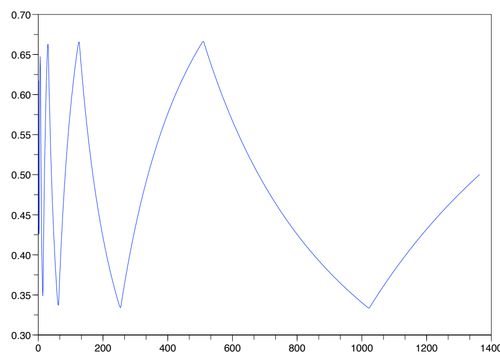

Fig. 3. Plot of $\frac{1}{n}\sum_{k=0}^{n-1} x^k$ function of n.

In the case where the compact set is finite, we can apply the ergodicity results on Markov chains with a finite state number to show the convergence of $P_{y^0}^N$ towards Q_{y^0}. Instead of the subsequence convergence, this convergence proves the genericity of y^0.

The discussion above is summarized by the following result.

Theorem 2: For any additively 1-homogeneous dynamical system $x^{k+1} = f(x^k)$, and for any closed set of initial conditions x^0 such that x_j^k/x_1^k stays bounded for all j and k, there exists a measure on the set of normalized initial conditions x_j^0/x_1^0 such that the average growth rate exists for almost all normalized initial conditions □

We can also see [2] for construction of invariant measures of stochastic recursions.

Coming back to the tent example, according to the initial value y^0, the tent iterations y^k stay in circuits or follow trajectories without circuit (possibly dense in $[0, 1]$). For example, assuming that the initial condition is such that $y = \frac{2}{5}$, the trajectory is periodic of period 2. The invariant measure is $Q_{y^0} = \frac{1}{2}(\delta_{\frac{2}{5}} + \delta_{\frac{6}{5}})$. The growth rate is $\frac{4}{5}$, which is again different from the eigenvalues 0 and $\frac{2}{3}$. Moreover, it can be shown that for all initial conditions with a finite binary development (this set contains all the float numbers of computers), the trajectory stays in the unstable fixed point 0 after a finite number of steps. That is, for a dense set of initial conditions the invariant measure is δ_0 and the growth rate is 0.

III. Petri Net Dynamics

A. Autonomous Petri nets

Let us give, in min-plus-times algebra, a presentation of timed continuous Petri nets with weights. The weight can be negative and the numbers of tokens are not necessary integer (in continuous Petri nets, what we call tokens are in fact fluid amounts).

A Petri net $\mathcal{N}$ is a graph with two sets of nodes (the *transitions* $\mathcal{Q}$ (with $|\mathcal{Q}|$ elements) and the *places* $\mathcal{P}$ (with $|\mathcal{P}|$ elements)) and two sorts of edges, (the *synchronization edges* (from a place to a transition) and the *production edges* (from a transition to a place)).

A minplus $|\mathcal{Q}| \times |\mathcal{P}|$ matrix D, called *synchronization*[6] *matrix* is associated to the synchronization edges. $D_{qp} = a_p$ if there exists an edge from the place $p \in \mathcal{P}$ to the transition $q \in \mathcal{Q}$, and $D_{qp} = \varepsilon$ elsewhere, where a_p is the *initial marking* of the place p, which is, graphically, the number of tokens in p. We suppose here that the sojourn time in all the places is one unit of time[7].

A standard algebra $|\mathcal{P}| \times |\mathcal{Q}|$ matrix H, called *production*[8] *matrix* is associated with the production edges. It is defined by $H_{pq} = m_{pq}$ if there exists an edge from q to p, and 0 elsewhere, where m_{pq} is the *multiplicity* of the edge [9].

Therefore, a Petri net is characterized by the quadruple:

$$(\mathcal{P}, \mathcal{Q}, H, D) \ .$$

A Petri net is a dynamical system in which the token (fluid) evolution is partially defined by the transition firings, saying that a transition can fire as soon as all its upstream places contain a positive quantity of tokens (fluid) having stayed at least one unit of time. When a transition fires, it consumes a quantity of tokens (fluid) equal to the minimum of all the available quantities being in the upstream places. Cumulating the firings done up to present time defines the *cumulated transition firing* of the transition. The firing produces a quantity of tokens (fluid) in each downstream place equal to the firing of the transition multiplied by the multiplicity of the corresponding production edge. If the multiplicity of a production edge, going from q to p, is negative, the firing of q consumes tokens (fluid) of p.

A general Petri Net defines constraints on the transition firing. Denoting by q^k the cumulated firings of transitions $q \in \mathcal{Q}$ up to instant k, they satisfy the constraints:

$$\min_{p \in q^{in}} \left[a_p + \sum_{q \in p^{in}} m_{pq} q^{k-1} - \sum_{q \in p^{out}} q^k \right] = 0, \ \forall q \in \mathcal{Q}, \ \forall k, \tag{3}$$

where $p \in \mathcal{P}$ is a place of the Petri net; $q^{in} \subset \mathcal{P}$ [resp. $q^{out} \subset \mathcal{P}$] denotes set of places upwards [resp. downwards] the transition q; and $p^{in} \subset \mathcal{Q}$ [resp. $p^{out} \subset \mathcal{Q}$] denotes the set of transitions upwards [resp. downwards] the place p.

Indeed, being at time k, we know (from the firing definition of transitions) that after the firing (which is instantaneous), there is at least one place upstream of any transition in which no token entered before time $k-1$. For each transition q, the equation (3) computes the number of tokens which have stayed at least one unit of time in each place $p \in q^{in}$. The equation expresses that at least one place is empty and the others have nonnegative numbers of tokens.

As long as there is more than one edge leaving a place, the trajectory of the system is not uniquely defined because we do not know the path of a token leaving this place.

[6]Decision matrix in stochastic control.

[7]When different integer sojourn times are considered, an equivalent Petri net with a unique sojourn time can be obtained by adding places and transitions and solving the implicit relations.

[8]Hazard matrix in stochastic control

[9]Here the multiplicity appears only with the output transition edges. The multiplicity of input transition edges is supposed to be always equal to one. Looking at the more general case [11], we see that we do not lose generality by doing so (the dynamics class obtained is the same)

In the case of a *deterministic* Petri net (generally called conflict free Petri net) where all the places have only one downstream edge, the dynamics are uniquely defined, meaning there is no token consumption conflicts between the transitions downstream of each place[10]. Then, denoting by $Q = (q^k)_{q\in\mathcal{Q},k\in\mathbb{N}}$ the vector of sequences of cumulated firing quantities of transitions, and by $P = (p^k)_{p\in\mathcal{P},k\in\mathbb{N}}$ the vector of sequences of cumulated token quantities arrived in the places at time k, we have:

$$\begin{bmatrix}P^{k+1}\\Q^{k+1}\end{bmatrix} = \begin{bmatrix}0 & H\\D & \varepsilon\end{bmatrix} \boxtimes \begin{bmatrix}P^{k+1}\\Q^{k}\end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix}HQ^k\\D\otimes P^{k+1}\end{bmatrix} . \tag{4}$$

In Equation (4), P^{k+1} counts the number of tokens available (for firing in the downstream transitions) at time $k+1$ coming from the upstream transition firings. It is obtained by summing the weighted firing numbers of the transitions upstream the places up to time k since the tokens are supposed to stay at least one unit of time in the places.

The part about Q^{k+1} in Equation (4) tells that the transition firing numbers at time $k+1$ are equal to the minimal token numbers in the places upstream the transitions. To obtain these quantities we have only to add the token numbers in the place at initial time to the numbers entered by the firings that are the entries of P^{k+1}.

From these dynamics, we deduce the dynamics of the cumulated firing quantities by eliminating the place variables. We deduce the dynamics of the cumulated token quantities by eliminating the transition variables.

$$Q^{k+1} = D\otimes (HQ^k), \quad P^{k+1} = H(D\otimes P^k) .$$

In the case of *event graphs* (particular deterministic Petri nets where all the multiplicities m_{pq} are equal to 1 and all the places have exactly one edge upstream), the dynamics are linear in the minplus sense given by:

$$Q^{k+1} = A\otimes Q^k,$$

where $A_{q'q} = a_p$ with p the unique place between q and q'.

B. Deterministic Petri nets

By using the negative weights and/or fixing a routing policy, it is possible to transform a Petri net with conflicts in a deterministic Petri net. Let us discuss these points more precisely on the simple system given in the first picture of Figure-4.

The incomplete dynamics of this system can be written in minplus algebra[11]:

$$q_4^k q_3^k = a q_1^{k-1} q_2^{k-1} . \tag{5}$$

Clearly q_3 and q_4 are not uniquely defined. In the following two examples, we complete the dynamics in two different ways. These two ways are equally useful for traffic applications as we will see later.

- By specifying the routing policy (for example we choose arbitrarily that half of the total tokens available are given to q_3 and half to q_4, see [11] for results on the general routing case) [12]:

  $$q_4^k = \sqrt{q_1^{k-1}q_2^{k-1}}, \quad q_3^k = a q_4^k .$$

  The minplus product of the two equations gives the constraint (5).
- By choosing a priority rule (top priority to q_3 against q_4)[13]:

  $$q_3^k = a q_1^{k-1} q_2^{k-1}/q_4^{k-1}, \quad q_4^k = a q_1^{k-1} q_2^{k-1}/q_3^k .$$

[10] In the non-deterministic case, we have to specify the rules which resolve the conflicts by, for example, giving priorities to the consuming transitions or by imposing ratios to be followed. As long as these rules are added, the initial non-deterministic Petri net becomes a deterministic one.

[11] Which means in standard algebra: $q_4^k + q_3^k = a + q_1^{k-1} + q_2^{k-1}$.

[12] Which means in standard algebra: $q_4^k = \frac{1}{2}\left(q_1^{k-1} + q_2^{k-1}\right), \quad q_3^k = a + q_4^k$.

[13] Which means in standard algebra: $q_3^k = a + q_1^{k-1} + q_2^{k-1} - q_4^{k-1}, \quad q_4^k = a + q_1^{k-1} + q_2^{k-1} - q_3^k$.

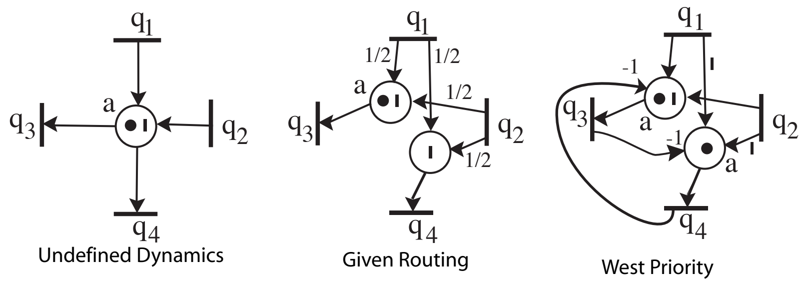

Fig. 4. The nondeterministic Petri net, given in the left figure, is made deterministic by: – choosing a routing policy: 1/2 (to read in standard algebra) towards q_3, 1/2 towards q_4; in the central figure, – or giving top priority to q_3 over q_4; in the right figure. In the left and central figures, the minimal time spent by a token in a place is one time unit, and is not represented in those figures. In the right figure, the minimal time spent in the top-left place by a token is one time unit, represented by a stick in that place, while the minimal time spent in the down-right place by a token is one time unit if the token is produced by q_1 or q_2, represented by sticks on the production edges, and is zero time unit if the token is consumed (produced with a multiplicity (-1)) by q_3.

The last equation implies that the initial constraint (5) is satisfied. We see that the negative weights on q_4^{k-1} and on q_3^k are essential to express this priority. In this very simple situation by substituting q_3^k by its value in the left equation we see that $q_4^k = q_4^{k-1}$ meaning that transition q_4 never fires, consistlently with the priority of q_3 over q_4. We shall see in the next section that more interesting behaviors occur when the firings of transition having priority (here q_3) can be blocked, allowing the other transition (here q_4) to ?re.

In the two cases, we obtain a *degree one homogeneous minplus* system.

IV. Traffic Application

In this section, for the convenience of the reader, we first recall some known results on the traffic on a circular road, before developping the original part on the traffic on two circular words with a junction, which is much more difficult.

A. *Traffic on a circular road*

Let us recall the simplest model to derive the fundamental traffic diagram on a single road. The best way to obtain the diagram is to study the stationary regime on a circular road with a given number of vehicles[14]. We present two ways to obtain the fundamental diagram: one is by logical deduction from an exclusion process point of view [8] (it shows clearly the presence of two distinct phases), and the other by computing the eigenvalue of a minplus system derived from a simple Petri net modeling of the road [19], [23], [39] (this way will be extended to the case of roads with junctions). In the following, the road is cut in m sections, each of them can contain at most one vehicle.

B. *Exclusion process modeling*

Following [8], we can consider the dynamical system defined by the rule $10 \to 01$ applied to a binary word w. The word w_k describes the vehicle positions at instant k on a road cut in sections (each bit representing a section, 1 meaning occupied and 0 meaning free, see II in Figure-5). Let us take an

[14]We consider that the stationary regime on the circular road is reached locally on a standard road when its density is constant at the considered zone.

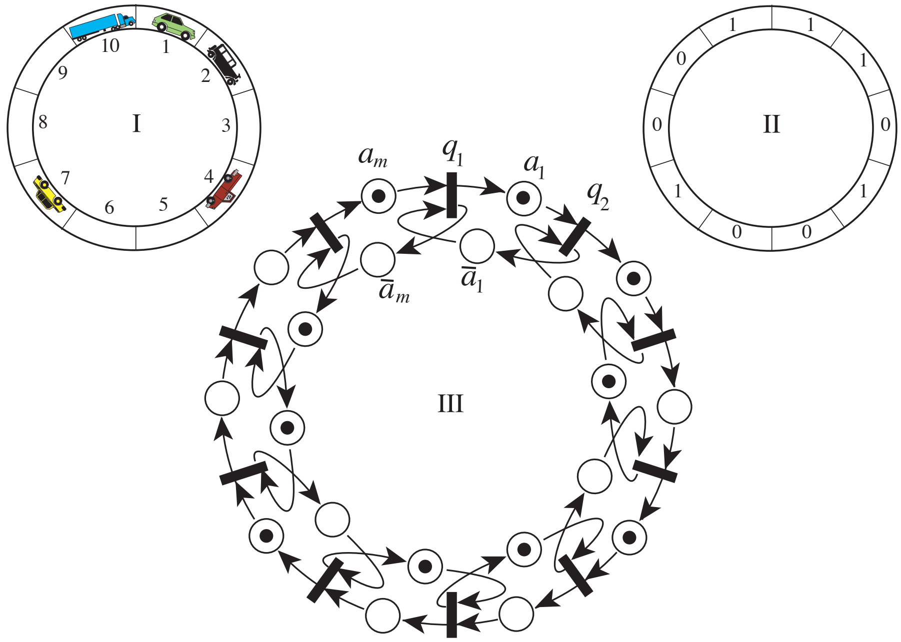

Fig. 5. On the top-left side, a circular road cut in sections. On the top-right side, its exclusion process, where 1 means that the section is occupied. On the middle, its Petri net representation where the ticks (1 time delay) in each place are not represented.

example:

$$\begin{aligned} w_1 &= 1101001001, \quad & w_2 &= 1010100101, \\ w_3 &= 0101010011, \quad & w_4 &= 1010101010, \\ w_5 &= 0101010101. \end{aligned}$$

Let us define: – the *density* ρ by the number of vehicles n divided by the number of sections m: $\rho = n/m$, – the flow $\varphi(t)$ at time t by the number of vehicles going one step forward at time t divided by the number of sections. Then the *fundamental traffic diagram* gives the relation between $\varphi(t)$ and ρ.

If $\rho \leq 1/2$, after a transient period, all the vehicle groups split off, and then all the vehicles can move forward without other vehicles in the way, and we have:

$$\varphi(t) = \varphi = n/m = \rho \; .$$

If $\rho \geq 1/2$, the free place groups split off after a finite time and move backward without other free places in the way. Then $m - n$ vehicles move forward and we have:

$$\varphi(t) = \varphi = (m - n)/m = 1 - \rho \; .$$

Theorem 3 ([8]):

$$\exists T : \; \forall t \geq T \quad \varphi(t) = \varphi = \begin{cases} \rho & \text{if } \; \rho \leq 1/2 \; , \\ 1 - \rho & \text{if } \; \rho \geq 1/2 \; . \end{cases}$$

C. Event Graph modeling

Let us recall some results previously given in[19], [23] (see also [39] where similar results have been obtained). The Petri net given in III of Figure 5 describes, in a different way, the same dynamics described above by an exclusion process model. In fact, this Petri net is an event graph. Therefore, the dynamics are linear in minplus algebra. The vehicle number entered in the section s before time k is denoted q_s^k.

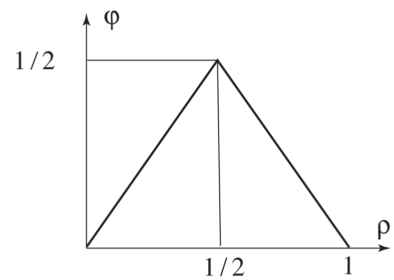

Fig. 6. The fundamental traffic diagram showing the dependence on the average flow with respect of the vehicle density.

The initial vehicle position is given by the booleans a_s which take the value 1 when the cell contains a vehicle and take the value 0 otherwise. The booleans $\bar{a}_s$ are equal to $1 - a_s$. Therefore the presence of a token in the place $\bar{a}_s$ indicates that the section s is free.

Then, the dynamics are given by:

$$q_s^{k+1} = \min\{a_{s-1} + q_{s-1}^k, \bar{a}_s + q_{s+1}^k\} ,$$

which can be written linearly in minplus algebra:

$$q_s^{k+1} = a_{s-1} q_{s-1}^k \oplus \bar{a}_s q_{s+1}^k ,$$

where the index addition is done modulo m.

Theorem 4 ([19], [23], [39]): The average car flow φ depends on the car density ρ according to the law:

$$\varphi = \min(\rho, 1 - \rho) \; \square$$

Proof: The event graph of Figure 5 has three kinds of elementary circuits: the outside circuit with average weight n/m; the inside circuit with average weight $(m - n)/m$; and the circuits on which some steps are made forward and then back with the average mean 1/2. Therefore, using Theorem 1, its eigenvalue is

$$\varphi = \min(n/m, (m - n)/m, 1/2) = \min(\rho, 1 - \rho) ,$$

which gives the average speed as a function of the car density since the minplus eigenvalue is equal to $\lim_k q_i^k / k = \varphi$ for all i.

D. Traffic on two roads with one junction

We now study in detail the case of two circular roads with a junction (see the top-right side of Figure-7). The description of the dynamics is given here in term of Petri nets in order to illustrate the modeling power of the negative weights. To our knowledge, this modelling is new. In the companion paper [22] we have described the same system avoiding the help of Petri net for people not familiar with these nets.

A first trial to model a junction is to consider the Petri net given in the middle of Figure-7. This Petri net is not an event graph. The junction is modeled by the places a_n and $\bar{a}_n$. A token in a_n indicates the presence of a car in the junction. A token in $\bar{a}_n$ indicates that the junction is free (it represents an authorization to enter into the junction). The Petri net given in the middle of Figure 7 is a general non deterministic Petri Net.

We can write the dynamics of the Petri net using Equation (3), but these equations do not uniquely determine the trajectories of the system. We have two places a_n and $\bar{a}_n$ with two outgoing edges. At place a_n, we have to specify the *routing policy* giving the proportion of cars going West and the proportion of cars going South. At place $\bar{a}_n$, we follow the *first arrived first served* rule with the *right priority rule* when there is a conflict, to deliver an authorization to enter into the junction. Allowing negative weights, we obtain the Petri Net of Figure-7 where the junction is described precisely in the top-left part of the

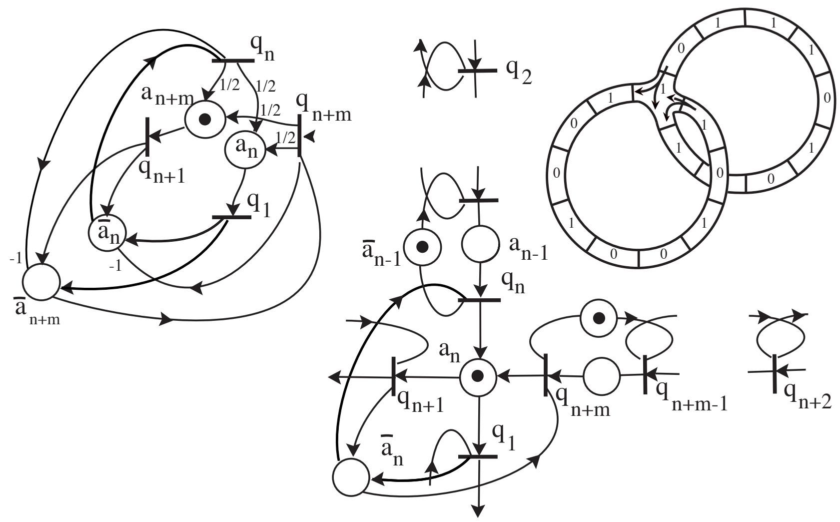

Fig. 7. A junction with two circular roads cut in sections (top-right), its Petri net simplified modeling (middle) and the precise modeling of the junction (top left). The ticks representing the time delays present in each place are not represented.

figure. In the place a_n [resp a_{n+m}] are accumulated the cars going towards West [resp. towards South]. In the place $\bar{a}_n$ [resp. $\bar{a}_{n+m}$] are accumulated the authorizations to enter into the junction from North [resp. from East]. The authorizations are obtained by subtracting the authorizations to enter from East [resp. from North] from the total number of authorizations to enter into the junction; see Section III-B to have a precise description of the management of the right priority. We remark that we have obtained a deterministic (conflict free) Petri net with some negative weights (not to be confused with an event graph with negative tokens).

The dynamics describing the evolution of the vehicle number entered in section s before time k denoted q_s^k can be obtained immediately from the Petri net in top left corner of Figure-7. Using the minplus notations discussed in Section-II-B, we have :

$$\begin{cases} q_i^{k+1} = a_{i-1}q_{i-1}^k \oplus \bar{a}_i q_{i+1}^k, \ i \neq 1, n, n+1, n+m, \\ q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k \oplus a_{n-1} q_{n-1}^k \ , \\ q_{n+m}^{k+1} = \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1} \oplus a_{n+m-1} q_{n+m-1}^k \ , \\ q_1^{k+1} = a_n \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_1 q_2^k \ , \\ q_{n+1}^{k+1} = a_{n+m} \sqrt{q_n^k q_{n+m}^k} \oplus \bar{a}_{n+1} q_{n+2}^k \ , \end{cases} \tag{6}$$

where the entries satisfy the following constraints (written with the standard notations):

- $0 \leq a_i \leq 1$ for $i = 1, \cdots, n+m$. These initial markings give the presence, 1, or absence, 0, of a vehicle in the road sections. However, here we see vehicles as fluid and can relax this integer constraint. Moreover, the dynamics equation apply to integer entries return rational numbers which are not integer. Therefore, it is better to accept real numbers of tokens belonging to the $[0, 1]$ interval;
- $\bar{a}_i = 1 - a_i$ for $i \neq n, n+m$ they give the initial free spaces in the places;
- $a_n + a_{n+m} \leq 1$ the maximum number of cars in the junction is 1;
- $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$ give the free place in the junction.

The first equation of (6) counts the number of cars entered in a section that is not a junction. It is the minimum between the number of cars available to enter into the section and the number of authorization to enter into the section. The second [resp. third] equation counts the numbers entered into the junction from the North [resp. East]. It is the minimum between the number of authorization to enter from the North [resp. East] into the junction according to the right priority rule described in Section-III-B, and the number of cars arrived from the North [resp. East]. The fourth [resp. fifth] equation counts the number of cars leaving the junction to the South [resp. West]. It is the minimum between the half of the total number of cars entered into the junction and the number of authorization to enter into the section just after the junction in the South [rep. West] direction.

We remark that the system is homogeneous of degree 1 and that it is easy to write the dynamics using the generalized matrix product. Indeed, all the "monomials" appearing in the right hand side of System (6) (for example $\sqrt{q_n^k q_{n+m}^k}$) are linear in the standard algebra and can be computed by a standard matrix product from the q vector. Then it is easy to obtain the complete right hand side from all the appearing "monomials" by a minplus matrix product. This matrix form is very useful to simulate the system in ScicosLab software [50] since the minplus matrix product is implemented in this software. We remark also that the "monomials" like $q_1^k q_{n+1}^k / q_{n+m}^k$ introduce negative entries in the standard matrices. Therefore the dynamics is not monotonic.

We prove the existence of a growth rate thanks to the following two results.

Theorem 5: Starting from 0, the state trajectory $(q_i^k)_{k\in\mathbb{N}}$ of the dynamics (6), is nonnegative and nondecreasing :

$$q^0 = 0 \Rightarrow q^{k+1} \geq q^k, \ \forall k \in \mathbb{N}.$$

Proof: Computing q^1 using the fact that all the a_i and $\bar{a}_i$ are nonnegative, it is clear that $q^1 \geq 0$. Let us prove by induction that $q^{k+1} \geq q^k, \ \forall k \in \mathbb{N}$.

We rewrite (6) as follows $q_i^{k+1} = f_i(q^k)$ for $i = 1, \cdots, n+m$. The functions f_i for $i \neq n, n+m$ are nondecreasing. Therefore, for such an i, we have:

$$q_i^{k+1} = f_i(q^k) \geq f_i(q^{k-1}) \geq q_i^k \ ,$$

using first the induction hypothesis and then the dynamics definition.

Let us prove that $q_n^{k+1} \geq q_n^k$.

- If $q_n^{k+1} = a_{n-1} q_{n-1}^k$ we have

$$q_n^{k+1} = a_{n-1} q_{n-1}^k \geq a_{n-1} q_{n-1}^{k-1} \geq f_n(q^{k-1}) = q_n^k \ .$$

- If $q_n^{k+1} = \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$, using the dynamics, we have $q_{n+m}^k \leq \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1} / q_n^k$. Therefore,

$$q_n^{k+1} \geq \bar{a}_n q_n^k q_1^k q_{n+1}^k / \bar{a}_{n+m} q_1^{k-1} q_{n+1}^{k-1}$$

  which gives $q_n^{k+1} \geq q_n^k$ using the induction hypothesis and the assumption $\bar{a}_n = \bar{a}_{n+m}$.

The nondecreasing property of q_{n+m} is proved in the same way. If $q_{n+m}^{k+1} = a_{n+m-1} q_{n+m-1}^k$, we have $q_{n+m}^{k+1} = a_{n+m-1} q_{n+m-1}^k \geq a_{n+m-1} q_{n+m-1}^{k-1} \geq f_{n+m}(q^{k-1}) = q_{n+m}^k$. If $q_{n+m}^{k+1} = \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1}$, we have $q_n^{k+1} \leq \bar{a}_n q_1^k q_{n+1}^k / q_{n+m}^k$ using the dynamics. Therefore, $q_{n+m}^{k+1} \geq \bar{a}_{n+m} q_{n+m}^k / \bar{a}_n$, which gives the result using $\bar{a}_n = \bar{a}_{n+m}$. ■

Let us consider the set $\mathcal{J} = \{q^0 \mid q^{k+1} \geq q^k, \ \forall k \in \mathbb{N}\}$, which is closed and not empty since $q_i^0 = 0, \forall i$ belongs to it.

Theorem 6: If $q^0 \in \mathcal{J}$, then the distances between any pair of states stay bounded:

$$\exists c_1 : \sup_k |q_i^k - q_j^k| \leq c_1, \forall i, j.$$

Moreover

$$\exists c_2 : \sup_k |q_i^{k+n+m} - q_i^k| \leq c_2, \forall i \ \square$$

Proof: The result is obtained thanks to the nondecreasing property of the trajectory and the fact that the dynamics is upper bounded by a dynamics of a connected stochastic control problem. More precisely we find a circuit of size $n+m$ on the state indices which covers all the indices and a timing on this circuit along which we have estimates. For that let us prove some inequalities.

Using the first equation of the dynamics (6) we have :

$$q_i^{k+1} \leq a_{i-1} q_{i-1}^k, \ i \neq 1, n+1 \ . \tag{7}$$

Moreover we have :

$$
\begin{aligned}
q_1^{k+1} &\le a_n \sqrt{q_n^k q_{n+m}^k}\ , && \text{by the fourth equation of (6),} && (8)\\
&\le a_n \sqrt{q_n^{k+n} q_{n+m}^k}\ , && \text{by non decreasing property of } q^n\ , && (9)\\
&\le a_n \sqrt{b_1^{n-1} q_1^{k+1} q_{n+m}^k}\ , && \text{using } n \text{ times (7)}\ , && (10)\\
q_1^{k+1} &\le a_n^2 b_1^{n-1} q_{n+m}^k = a_n b_1^n q_{n+m}^k\ , && \text{by simplifying the last inequality}\ , && (11)
\end{aligned}
$$

where the notation $b_j^l = \bigotimes_{i=j}^l a_i$ is used.

By the same kind of arguments we prove that :

$$q_{n+1}^{k+1} \le a_{n+m}\sqrt{q_n^k q_{n+m}^k} \le a_{n+m}\sqrt{q_n^k q_{n+m}^{k+m}} \le a_{n+m}\sqrt{q_n^k b_{n+1}^{n+m-1} q_{n+1}^{k+1}}$$

and by simplifying the last inequality we obtain :

$$q_{n+1}^{k+1} \le a_{n+m}^2 b_{n+1}^{n+m-1} q_n^k = a_{n+m} b_{n+1}^{n+m} q_n^k\ . \tag{12}$$

Now we are able to build the covering circuit of indices along which we have the estimates :

$$
\begin{aligned}
q_n^k &\le a_{n-1} q_{n-1}^{k-1} \le \cdots \le b_1^{n-1} q_1^{k+1-n}\ , && \text{by (7)}\ ,\\
&\le (b_1^n)^2 q_{n+m}^{k-n}\ , && \text{by (11)}\ ,\\
&\le (b_1^n)^2 a_{n+m-1} q_{n+m-1}^{k-n-1} \le \cdots \le (b_1^n)^2 b_{n+1}^{n+m-1} q_{n+1}^{k+1-n-m}\ , && \text{by (7)}\ ,\\
&\le (b_1^{n+m})^2 q_n^{k-n-m}\ , && \text{by (12)}\ .
\end{aligned}
$$

Using the existence of this covering circuit on indices along which we have estimates, and the increasing property of each state component it is clear that $|q_i^k - q_{i'}^k|$ is smaller than 3 times $(b_1^{n+m})^2$ for all i, i' and k (which is the wanted result). The factor 3 comes from the fact that q_i^k or $q_{i'}^k$ do not necessarily belongs to this circuit. In the worst case we have to use successively 3 times the circuit and use the nondecreasing property of the components to conclude. ■

Using Theorem 6 and Theorem 2, we obtain a result (Theorem 7) on the existence of the average growth rate of the dynamics (6). The average growth rate has the interpretation of the average traffic flow.

Theorem 7: There exists an initial distribution on the set $\{(q_j^0/q_1^0)_{j=2,n+m} \mid q^0 \in \mathcal{J}\}$, called the Kryloff-Bogoljuboff invariant measure, such that the growth rate $\chi = \lim_k q_i^k/k,\ \ \forall i$, of the dynamical system (6) exists almost everywhere □

Theorem 7 is not completely satisfactory. We would like to have the existence of the growth rate for the initial condition $q_i^0 = 0$ for all i. In Theorem 7, the "almost everywhere" part is worrying since $q^0 = 0$ could correspond to a transient point not charged by the invariant measure.

Theorem 8: If $q^0 \in \mathcal{J}$ and if an average growth rate χ of the dynamics (6) exists, then $\chi \le 1/4$ □

Proof: From the dynamics (6) we have :

$$
\begin{aligned}
q_{n+m}^{k+1} &\le \bar{a}_{n+m} q_1^k q_{n+1}^k / q_n^{k+1}\ ,\\
q_1^{k+1} &\le a_n \sqrt{q_n^k q_{n+m}^k}\ ,\\
q_{n+1}^{k+1} &\le a_{n+m}\sqrt{q_n^k q_{n+m}^k}\ .
\end{aligned}
$$

then by summing (standard sum) these inequalities we get :

$$(q_1^{k+1} - q_1^k) + (q_n^{k+1} - q_n^k) + (q_{n+1}^{k+1} - q_{n+1}^k) + (q_{n+m}^{k+1} - q_{n+m}^k) \le 1.$$

Hence, when $k \to +\infty$, and taking into account Theorem 6, we obtain $4\chi \le 1$ ■

E. Eigenvalue Existence of the Junction Dynamics

Let us consider the eigenvalue problem associated to the dynamics (6). It is defined as finding λ and q such that:

$$\begin{cases} \lambda q_i = a_{i-1} q_{i-1} \oplus \bar{a}_i q_{i+1} \ , \ i \neq 1, n, n+1, n+m \ , \\ \lambda q_n = \bar{a}_n q_1 q_{n+1} / q_{n+m} \oplus a_{n-1} q_{n-1} \ , \\ \lambda q_{n+m} = \bar{a}_{n+m} q_1 q_{n+1} / (\lambda q_n) \oplus a_{n+m-1} q_{n+m-1} \ , \\ \lambda q_1 = a_n \sqrt{q_n q_{n+m}} \oplus \bar{a}_1 q_2 \ , \\ \lambda q_{n+1} = a_{n+m} \sqrt{q_n q_{n+m}} \oplus \bar{a}_{n+1} q_{n+2} \ , \end{cases} \tag{13}$$

with: $0 \leq a_i \leq 1$ for $i = 1, \cdots, n+m$, $\bar{a}_i = 1 - a_i$ for $i \neq n, n+m$, $a_n + a_{n+m} \leq 1$ and $\bar{a}_n = \bar{a}_{n+m} = 1 - a_n - a_{n+m}$.

The eigenvalue problem can be solved explicitly.

Theorem 9: The nonnegative eigenvalues λ as a function of the density d, written in the standard algebra, are given by:

d	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta,\gamma) < d < \max(\beta,\gamma)$	$\gamma \leq d \leq 1$
λ	$(1-\rho)d$	$1/4$	$\frac{r-(1-\rho)d}{2r-1+2\rho}$	0

with $N = n + m$, $\rho = 1/N$, $r = m/N$ and $d = \left(\sum_{i=1}^{n+m} a_i\right)/(N-1)$ the density of vehicles, $\alpha = \frac{1}{4(1-\rho)}$, $\beta = \frac{r+1/2-\rho}{2(1-\rho)}$ and $\gamma = \frac{r}{1-\rho}$. When $m \geq n$ the intervals given in the first line define a partition of $[0,1]$ and the eigenvalue is unique. In the other case the intervals overlap and there are up to three eigenvalues for some densities.

Moreover, when N is large with $r > 1/2$, the positive eigenvalue λ is unique and has the simple limit:

$$\lim_{N\to\infty,\, r>1/2} \lambda = \max\left\{0,\ \min\left\{d, \frac{1}{4}, \frac{r-d}{2r-1}\right\}\right\}.$$

Proof: The whole proof is given in the appendix. A less compact version of the proof is also available in [24], where the role of m and n are inverted. We give here a sketch of the proof. The proof has two parts. The first part consists of reducing the problem to a generalized eigenvalue problem in a four dimensional space. This part is Lemma 1 of the Appendix. The second part consists of a verification of the generalized minplus eigenvalue system of equations since we give explicit formulas for all the eigenelements. This part is given in the Appendix, and is also available with more explanations in [24]).

The eigenelements have been obtained by solving explicitly the homogeneous affine systems with five unknowns achieving the minimum in the reduced system.

In Lemma 1 of the Appendix, we show that $\lambda \leq 1/4$, and then by elimination of $q_i,\ i \neq 1, n, n+1, n+m,$, and thanks to the minplus linearity of the first equation of (13), we obtain the closed set of equations defining q_i, $i = 1, n, n+1, n+m$.

$$\begin{cases} q_n = (\bar{a}_n/\lambda) q_1 q_{n+1} / q_{n+m} \oplus (b_n/\lambda^{n-1}) q_1 \ , \\ q_{n+m} = (\bar{a}_{n+m}/\lambda^2) q_1 q_{n+1} / q_n \oplus (b_m/\lambda^{m-1}) q_{n+1} \ , \\ q_1 = (a_n/\lambda) \sqrt{q_n q_{n+m}} \oplus (\bar{b}_n/\lambda^{n-1}) q_n \ , \\ q_{n+1} = (a_{n+m}/\lambda) \sqrt{q_n q_{n+m}} \oplus (\bar{b}_m/\lambda^{m-1}) q_{n+m} \ , \end{cases} \tag{14}$$

where $b_n = \bigotimes_{i=1}^{n-1} a_i$ is the number of cars in the priority road; $\bar{b}_n = \bigotimes_{i=1}^{n-1} \bar{a}_i$ is the number of free places in the priority road; $b_m = \bigotimes_{i=n+1}^{n+m-1} a_i$ is the number of cars in the non priority road; and $\bar{b}_m = \bigotimes_{i=n+1}^{n+m-1} \bar{a}_i$ is the number of free places in the non priority raod.

In the second part of the proof, we show that Table I (written in standard algebra) gives the eigenvalues and the eigenvector formulas of the system (14). The eigenvqlues and the eigenvectors are given as functions of the density d.

	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta,\gamma) < d < \max(\beta,\gamma)$	$\gamma \leq d \leq 1$
λ	$(1-\rho)d$	$\frac{1}{4}$	$\frac{r-(1-\rho)d}{2r-1+2\rho}$	0
q_n	$b_n-(n-1)\lambda$	$b_n-(n-1)\lambda$	$b_n-(n-1)\lambda$	$\bar{a}_{n+m}+\bar{b}_m$
q_{n+m}	$(n+1)\lambda-2a_n-b_n$	$(n+1)\lambda-2a_n-b_n$	$(n+1)\lambda-2a_n-b_n$	$-2a_n-\bar{a}_{n+m}-\bar{b}_m$
q_1	0	0	0	0
q_{n+1}	$a_{n+m}-a_n$	$a_{n+m}-a_n$	$4\lambda-1+a_{n+m}-a_n$	$-2a_n-\bar{a}_{n+m}$

TABLE I

EIGENVALUES AND EIGENVECTORS OF THE SYSTEM (14), AS FUNCTIONS OF THE CAR DENSITY.

The limit given in Theorem 13 is obtained from the Table given in the same theorem, in the case where N is large with $m > n-2$, which case corresponds to $r > 1/2$. Note that, in this case we have $\beta < \gamma$. ■

To prove Theorem 9, numerical simulations of the car dynamics have suggested the affine system achieving the minimum in (14). Knowing it, we had only to verify the inequalities proving that this particular affine system achieves actually the minimum.

Since we want the eigenvalue as a function of the density, we cannot use a numerical approach. Instead, we have to find explicit formulas.

Knowing the existence of the four phases, it is possible to determine analytically their domains, and understand their traffic meaning by observing the corresponding asymptotic regimes. When $m > n-2$ ($r > 1/2$ for large N) we have $\beta < \gamma$ then the four following traffic phases appears:

- *Free moving*. When the density is small, $0 \leq d \leq \alpha$, after a finite time, all the vehicles move freely.
- *Saturation*. When $\alpha \leq d \leq \beta$, the junction is used at its maximal capacity without being bothered by downstream vehicles.
- *Recession*. When $\beta < d < \gamma$, the crossing is fully occupied, but vehicles sometimes cannot leave the crossing, because the road they want to enter into is crowded.
- *Freeze*. When $\gamma \leq d \leq 1$, the road without priority is full of vehicles. No vehicle can leave this road while the vehicle being in the junction wants to enter into.

Note that in the case where $\gamma < \beta$, three eigenvalues exist on the interval $[\gamma, \beta]$. In this case the growth rate of the car dynamics is zero and only three phases appear since the interval $[\gamma, \beta]$ is included in the freeze phase.

In Figure-8, we show the fundamental diagram obtained by simulation (using the maxplus arithmetic of the ScicosLab software [50]) for a particular relative size r of the two roads, and the eigenvalue λ given in Table I. We see clearly the four phases described above. On this figure, we see also that the average growth rate and the eigenvalue are very close to each other at least for three out of four phases.

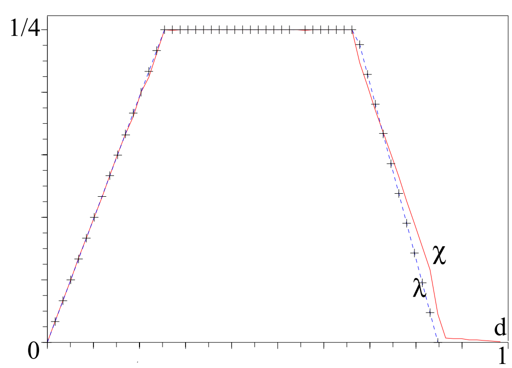

Fig. 8. The traffic fundamental diagram $\chi(d)$ when $r = 5/6$ (continuous line) obtained by simulation and its comparison with the eigenvalue $\lambda(d)$ given in Table I.

A more detailed discussion of the traffic phases, their extension to more general road networks, and

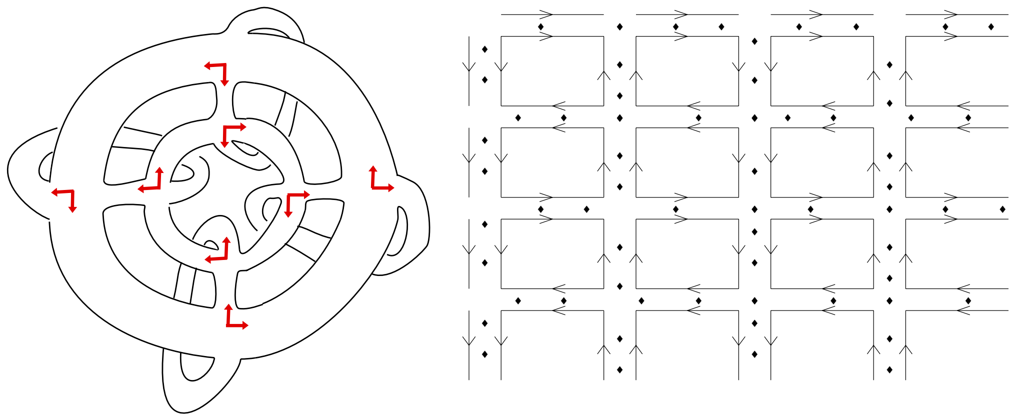

Fig. 9. Roads on a torus of 4×2 streets with its authorized turn at junctions (left) and the asymptotic car distribution in the streets on a torus of 4×4 streets obtained by simulation.

their control using traffic lights are given in [22].

F. Regular City Modeling.

We can generalize the modeling approach used in the case of one junction to derive the fundamental diagram of traffic on a regular city on a torus, described on the left side of Figure 9. The asymptotic vehicle distribution for a small city composed of two North-South, South-North, East-West and West-East avenues is given on the right side of Figure-9. The fundamental diagram presents a four-phase shape analogous to the case of two roads with one junction. In this more general case, the role of the non-priority road is played by a circuit of non-priority roads which blocks the whole system when the circuit is full (see [22]).

V. Conclusion

The Petri net modeling of traffic in a junction has be done thanks to the introduction of negative weights on output transition edges. The dynamics have a nice degree one homogeneous minplus property but are not monotone. This loss of monotonicity implies that the eigenvalue and the growth rate are no longer equal. Experimental results show that they are close in the case of two roads with one crossing where we are able to solve explicitly the eigenvalue problem and compute numerically the average growth rate. The fundamental traffic diagram, which gives the dependence of the average car-flow (given by the average growth rate) on the car-density, presents four phases with traffic interpretations. This set of 1-homogeneous minplus systems seems to be a good class of systems that we can describe by two matrices, one in the standard algebra and one in the minplus algebra.

In a companion paper [22], more traffic-oriented, we discussed further the traffic interpretation of the four phases, valid also for more general systems like regular cities. The influence on the fundamental diagram of the traffic control, using signal lights, is also studied in [22].

VI. Appendix

Lemma 1: The eigenvalue problem (13) of size N can be reduced to the eigenvalue problem (14) of size 4.

Proof: First let us verify that any solution of (13) satisfies $\lambda \leq 1/4$. Indeed (13) imply that $\lambda q_{n+m} \leq \bar{a}_{n+m} q_1 q_{n+1}/(\lambda q_n)$, $\lambda q_1 \leq a_n \sqrt{q_n q_{n+m}}$ and $\lambda q_{n+1} \leq a_{n+m} \sqrt{q_n q_{n+m}}$. Multiplying these three inequalities we obtain $\lambda^4 \leq \bar{a}_{n+m} a_n a_{n+m} = 1$.

To obtain the result we have to eliminate $q_i, i \neq 1, n, n{+}1, n{+}m$, that is to solve a linear minplus system which has a unique solution as soon as $\lambda < 1/2$. Indeed the loops of the precedence graph associated to the linear system has all its loops positive when $\lambda < 1/2$.

Moreover we can compute explicitly its solution. For $i = 2, n-1$ we have :

$$q_i = \left[\bigotimes_{j=1}^{i-1}(a_j/\lambda)\right] q_1 \oplus \left[\bigotimes_{j=i}^{n-1}(\bar{a}_j/\lambda)\right] q_n \ ,$$

for $i = n+1, n+m-1$:

$$q_i = \left[\bigotimes_{j=n+1}^{i-1}(a_j/\lambda)\right] q_{n+1} \oplus \left[\bigotimes_{j=i}^{n+m-1}(\bar{a}_j/\lambda)\right] q_{n+m} \ .$$

Using this explicit solution in the four last equations of (13) we obtain the reduced system (14). ■

To verify the results given Table I, let us rewrite the system (14) with simplified notations : $U = q_n$, $V = q_{n+m}$, $X = q_1$, $Y = q_{n+1}$, $g = b_n$, $h = b_m$, $k = a_n$, $l = a_{n+m}$, w$n' = n-1$, and $m' = m-1$.

$$\begin{cases} U = \bar{k}XY/\lambda V \oplus gX/\lambda^{n'} \ , \\ V = \bar{k}XY/\lambda^2 U \oplus hY/\lambda^{m'} \ , \\ X = k\sqrt{UV}/\lambda \oplus \bar{g}U/\lambda^{n'} \ , \\ Y = l\sqrt{UV}/\lambda \oplus \bar{h}V/\lambda^{m'} \ , \end{cases} \tag{15}$$

with $\bar{k} = 1 - k - l \geq 0$, $\bar{g} = n' - g \geq 0$ and $\bar{h} = m' - h \geq 0$ are the free places in the crossing and the two roads.

Lemma 2: The eigenvector (U, V, X, Y) of the minplus nonlinear system (15) is given (using minplus notations) by :

	$0 \leq d \leq \alpha$	$\alpha \leq d \leq \beta$	$\min(\beta,\gamma) < d < \max(\beta,\gamma)$	$\gamma \leq d \leq 1$
U	$g/\lambda^{n'}$	$g/\lambda^{n'}$	$g/\lambda^{n'}$	$\bar{k}\bar{h}$
V	$\lambda^{n'+2}/k^2 g = hl/k\lambda^{m'}$	$1\lambda^{n'-2}/k^2 g = \lambda^{n'+2}/k^2 g$	$\lambda^{n'+2}/k^2 g$	$e/k^2\bar{k}\bar{h}$
X	e	e	e	e
Y	l/k	l/k	$\lambda^{2+n'-m'}\bar{h}/k^2 g = \lambda^4 l/1k$	$e/k^2\bar{k}$

(16)

where the eigenvalue λ of the minplus nonlinear system (15) has been given in Theorem 9 □

Proof: Let us consider the four density regions corresponding to the four columns of this table :

1) $0 \leq d \leq \alpha$: the first column of Table I is solution of the standard linear system :

$$U = gX/\lambda^{n'}, \quad V = hY/\lambda^{m'}, \quad X = k\sqrt{UV}/\lambda \ , \quad Y = l\sqrt{UV}/\lambda \ , \tag{17}$$

which is itself a solution of System (15) since :

a) $\bar{k}XY/\lambda VU = \bar{k}kl/\lambda^3 = 1/\lambda^3 \geq 0$ (since $\lambda \leq 1/4$ indeed, once more, using (15) we have $VXY \leq \bar{k}klVXY/\lambda^4$),

b) $\bar{k}XY/\lambda^2 VU = 1/\lambda^4 \geq 0$,

c) $\bar{g}U/X\lambda^{n'} = (1/\lambda^2)^{n'} \geq 0$,

d) $\bar{h}V/Y\lambda^{m'} = \bar{h}h/\lambda^{m'} = (1/\lambda^2)^{m'} \geq 0$.

Moreover, multiplying the 4 equalities of (17) we obtain $\lambda^{n'+m'+2} = ghlk$ which gives the value of λ given in Table I.

2) $\alpha \leq d \leq \beta$: the second column of Table I is solution of the standard linear system :

$$U = gX/\lambda^{n'}, \quad V = \bar{k}XY/\lambda^2 U \ , \quad X = k\sqrt{UV}/\lambda \ , \quad Y = l\sqrt{UV}/\lambda \ , \tag{18}$$

which is itself solution of System (15) since :

a) $\bar{k}XY/\lambda VU = 1/\lambda^3 \geq 0$,

b) $hY/V\lambda^{m'} = hglk/\lambda^{n'+m'+2} = d^{m'+n'+1}/(mn)^{1/4} \geq \alpha^{m'+n'+1}/(mn)^{1/4} = 0$ since $d \geq \alpha$,
c) $\bar{g}U/X\lambda^{n'} = \bar{g}g/\lambda^{2n'} = (1/\lambda^2)^{n'} \geq 0$,
d) $\bar{h}V/Y\lambda^{m'} = m'\lambda^{n'+2-m'}/hkgl = m'(2n'/m')^{1/4}/d^{m'+n'+1} \geq m'(2n'/m')^{1/4}/\beta^{m'+n'+1} = 0$ since $d \leq \beta$.

Moreover using the equality giving two expressions for the value of V in (16) we obtain $\lambda = 1/4$.

3) $\min(\beta,\gamma) \leq d \leq \max(\beta,\gamma)$: the third column of Table I is solution of the standard linear system :

$$U = gX/\lambda^{n'}, \quad V = \bar{k}XY/\lambda^2 U\ , \quad X = k\sqrt{UV}/\lambda\ , \quad Y = \bar{h}V/\lambda^{m'}\ , \tag{19}$$

which is itself solution of System (15) since :
a) $\bar{k}XY/\lambda VU = \lambda \geq 0$,
b) $hY/V\lambda^{m'} = h\bar{h}/\lambda^{2m'} = (1/\lambda^2)^{m'} \geq 0$ since $\lambda \leq 1/4$,
c) $\bar{g}U/X\lambda^{n'} = \bar{g}g/\lambda^{2n'} = (1/\lambda^2)^{n'} \geq 0$,
d) $l\sqrt{UV}/\lambda Y = 1/\lambda^4 \geq 0$.

Moreover using the equality giving two expressions for the value of Y in (16) we obtain $\lambda^{-2+n'-m'}m' = klgh/1$ (equal $(m'+n'+1)d-1$ in standard algebra) which gives the value of λ given in Table I.

4) $\gamma \leq d \leq 1$: the fourth column of Table I is solution of the standard linear system :

$$U = \bar{k}XY/\lambda V\ , \quad V = \bar{k}XY/\lambda^2 U\ , \quad X = k\sqrt{UV}/\lambda\ , \quad Y = \bar{h}V/\lambda^{m'}\ , \tag{20}$$

which is itself solution of System (15) since :
a) $gX/\lambda^{n'}U = g/\bar{k}\bar{h} = ghkl/m = d^{n+m-1}/m \geq 0$ (since $d \geq \gamma$) ,
b) $hY/\lambda^{m'}V = h\bar{h} = m' \geq 0$,
c) $\bar{g}U/X\lambda^{n'} = \bar{g}\bar{k}\bar{h} \geq 0$,
d) $l\sqrt{UV}/\lambda Y = lk\bar{k} = 1$.

Moreover the compatibility of first two equalities of (20) implies that $\lambda = 0$.

■

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