Sequent Calculi with procedure calls

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Abstract

In this paper, we introduce two focussed sequent calculi, $\mathsf{LK}^p(\mathcal{T})$ and $\mathsf{LK}^+(\mathcal{T})$, that are based on Miller-Liang's LKF system [LM09] for polarised classical logic. The novelty is that those sequent calculi integrate the possibility to call a decision procedure for some background theory $\mathcal{T}$, and the possibility to polarise literals "on the fly" during proof-search.

These features are used in other works [FLM12, FGLM13] to simulate the $\mathsf{DPLL}(\mathcal{T})$ procedure [NOT06] as proof-search in the extension of $\mathsf{LK}^p(\mathcal{T})$ with a cut-rule.

In this report we therefore prove cut-elimination in $\mathsf{LK}^p(\mathcal{T})$.

Contrary to what happens in the empty theory, the polarity of literals affects the provability of formulae in presence of a theory $\mathcal{T}$. On the other hand, changing the polarities of connectives does not change the provability of formulae, only the shape of proofs.

In order to prove this, we introduce a second sequent calculus, $\mathsf{LK}^+(\mathcal{T})$ that extends $\mathsf{LK}^p(\mathcal{T})$ with a relaxed focussing discipline, but we then show an encoding of $\mathsf{LK}^+(\mathcal{T})$ back into the more restrictive system $\mathsf{LK}(\mathcal{T})$.

We then prove completeness of $\mathsf{LK}^p(\mathcal{T})$ (and therefore of $\mathsf{LK}^+(\mathcal{T})$) with respect to first-order reasoning modulo the ground propositional lemmas of the background theory $\mathcal{T}$.

Contents

1 $\mathsf{LK}^p(\mathcal{T})$: Definitions

The sequent calculus $\mathsf{LK}^p(\mathcal{T})$ manipulates the formulae of first-order logic, with the specificity that connectives are of one of two kinds: positive ones and negative ones, and each *boolean* connective comes in two versions, one of each kind. This section develops the preliminaries and the definition of the $\mathsf{LK}^p(\mathcal{T})$ system.

Definition 1 (Terms and literals) Consider an infinite set of elements called *variables*.

The set of *terms* over a first-order (function) signature F_Σ is defined by:

$$t, t_1, t_2, \ldots := x \mid f(t_1, \ldots, t_n)$$

with f/n (f of arity n) ranging over F_Σ and x ranging over variables.

Let P_Σ be a first-order predicate signature equipped with an involutive and arity-preserving function called *negation*. The negation of a predicate symbol P is denoted $P^\perp$.

Let $\mathcal{L}^\top$ be the set $\{P(t_1, \ldots, t_n) \mid P/n \in P_\Sigma, t_1, \ldots t_n \text{ terms}\}$, to which we extend the involutive function of negation with:

$$(P(t_1, \ldots, t_n))^\perp := P^\perp(t_1, \ldots, t_n)$$

The substitution, in a term t', of a term t for a variable x, denoted $\left\{{}^{t}\!/\!_{x}\right\} t'$, is defined as usual, and straightforwardly extended to elements of $\mathcal{L}^\top$.

In the rest of this chapter, we consider a subset $\mathcal{L} \subseteq \mathcal{L}^\top$, of elements called *literals* and denoted $l, l_1, l_2 \ldots$, that is closed under negation and under substitution.[1]

For a set $\mathcal{A}$ of literals, we write $\left\{{}^{t}\!/\!_{x}\right\}\mathcal{A}$ for the set $\{\left\{{}^{t}\!/\!_{x}\right\} l \mid l \in \mathcal{A}\}$. The closure of $\mathcal{A}$ under all possible substitutions is denoted $\mathcal{A}^\downarrow$. ※

Notation 2 We often write $\mathcal{V}, \mathcal{V}'$ for the set or multiset union of $\mathcal{V}$ and $\mathcal{V}'$.

Remark 1 Negation obviously commutes with substitution.

Definition 3 (Inconsistency predicates)

An *inconsistency predicate* is a predicate over sets of literals

1. satisfied by the set $\{l, l^\perp\}$ for every literal l;
2. that is upward closed (if a subset of a set satisfies the predicate, so does the set);
3. such that if the sets $\mathcal{A}, l$ and $\mathcal{A}, l^\perp$ satisfy it, then so does $\mathcal{A}$.
4. such that if a set $\mathcal{A}$ satisfies it, then so does $\left\{{}^{t}\!/\!_{x}\right\}\mathcal{A}$.

The smallest inconsistency predicate is called the *syntactical inconsistency* predicate[2]. If a set $\mathcal{A}$ of literals satisfies the syntactically inconsistency predicate, we say that $\mathcal{P}$ is *syntactically inconsistent*, denoted $\mathcal{P} \models$. Otherwise $\mathcal{A}$ is *syntactically consistent*.

In the rest of this chapter, we specify a "theory" $\mathcal{T}$ by considering another inconsistency predicate called the *semantical inconsistency* predicate. If a set $\mathcal{A}$ of literals satisfies it, we say that $\mathcal{A}$ is *semantically inconsistent*, denoted by $\mathcal{A} \models_\mathcal{T}$. Otherwise $\mathcal{A}$ is *semantically consistent*. ※

Remark 2

- In the conditions above, (1) corresponds to *basic inconsistency*, (2) corresponds to *weakening*, (3) corresponds to *cut-admissibility* and (4) corresponds to *stability under instantiation*. *Contraction* is built-in because inconsistency predicates are predicates over sets of literals (not multisets).
- If $\mathcal{A}$ is syntactically consistent, $\left\{{}^{t}\!/\!_{x}\right\}\mathcal{A}$ might not be syntactically consistent.

Definition 4 (Formulae)

The *formulae* of polarised classical logic are given by the following grammar:

$$\begin{array}{llll} \text{Formulae} & A, B, \ldots & ::= & l \\ & & \mid & A \wedge^+ B \mid A \vee^+ B \mid \exists x A \mid \top^+ \mid \bot^+ \\ & & \mid & A \wedge^- B \mid A \vee^- B \mid \forall x A \mid \top^- \mid \bot^- \end{array}$$

[1] Very often we will take $\mathcal{L} = \mathcal{L}^\top$, but it is not a necessity.

[2] It is the predicate that is true of a set $\mathcal{A}$ of literals iff $\mathcal{A}$ contains both l and $l^\perp$ for some $l \in \mathcal{L}$.

where l ranges over $\mathcal{L}$.

The set of *free variables* of a formula A, denoted $\mathsf{FV}(A)$, and α-conversion, are defined as usual so that both $\exists x A$ and $\forall x A$ bind x in A.

The size of a formula A, denoted $\sharp(A)$, is its size as a tree (number of nodes).

Negation is extended from literals to all formulae:

$(A\wedge^+ B)^\perp$	$:=$	$A^\perp \vee^- B^\perp$	$(A\wedge^- B)^\perp$	$:=$	$A^\perp \vee^+ B^\perp$
$(A\vee^+ B)^\perp$	$:=$	$A^\perp \wedge^- B^\perp$	$(A\vee^- B)^\perp$	$:=$	$A^\perp \wedge^+ B^\perp$
$(\exists x A)^\perp$	$:=$	$\forall x A^\perp$	$(\forall x A)^\perp$	$:=$	$\exists x A^\perp$
$(\top^+)^\perp$	$:=$	$\bot^-$	$(\top^-)^\perp$	$:=$	$\bot^+$
$(\bot^+)^\perp$	$:=$	$\top^-$	$(\bot^-)^\perp$	$:=$	$\top^+$

The *substitution* in a formula A of a term t for a variable x, denoted $\left\{{}^{t}\!/\!{}_{x}\right\}A$, is defined in the usual capture-avoiding way. ※

Notation 5 For a set (resp. multiset) $\mathcal{V}$ of literals / formulae, $\mathcal{V}^\perp$ denotes $\{A^\perp \mid A \in \mathcal{V}\}$ (resp. $\{\!\{A^\perp \mid A \in \mathcal{V}\}\!\}$). Similarly, we write $\left\{{}^{t}\!/\!{}_{x}\right\}\mathcal{V}$ for $\{\left\{{}^{t}\!/\!{}_{x}\right\}A \mid A \in \mathcal{V}\}$ (resp. $\{\!\{\left\{{}^{t}\!/\!{}_{x}\right\}A \mid A \in \mathcal{V}\}\!\}$), and $\mathsf{FV}(\mathcal{V})$ for the set $\bigcup_{A\in\mathcal{V}} \mathsf{FV}(A)$.

Definition 6 (Polarities)

A *polarisation set* $\mathcal{P}$ is a set of literals ($\mathcal{P} \subseteq \mathcal{L}$) that is syntactically consistent, and such that $\mathsf{FV}(\mathcal{P})$ is finite.

Given such a set, we define *$\mathcal{P}$-positive formulae* and *$\mathcal{P}$-negative formulae* as the formulae generated by the following grammars:

$\mathcal{P}$-positive formulae	$P, \ldots$	$::=$	$p \mid A\wedge^+ B \mid A\vee^+ B \mid \exists x A \mid \top^+ \mid \bot^+$
$\mathcal{P}$-negative formulae	$N, \ldots$	$::=$	$p^\perp \mid A\wedge^- B \mid A\vee^- B \mid \forall x A \mid \top^- \mid \bot^-$

where p ranges over $\mathcal{P}$.

In the rest of the chapter, p, p',... will denote a literal that is $\mathcal{P}$-positive, when the polarisation set $\mathcal{P}$ is clear from context.

Let $\mathsf{U}_\mathcal{P}$ be the set of all *$\mathcal{P}$-unpolarised literals*, i.e. literals that are neither $\mathcal{P}$-positive nor $\mathcal{P}$-negative. ※

Remark 3 Notice that the negation of a $\mathcal{P}$-positive formula is $\mathcal{P}$-negative and vice versa. On the contrary, nothing can be said of the polarity of the result of substitution on a literal w.r.t. the polarity of the literal: e.g. l could be in $\mathcal{P}$-positive, while $\left\{{}^{t}\!/\!{}_{x}\right\}l$ could be $\mathcal{P}$-negative or $\mathcal{P}$-unpolarised.

Definition 7 ($\mathsf{LK}^p(\mathcal{T})$) The sequent calculus $\mathsf{LK}^p(\mathcal{T})$ manipulates two kinds of sequents:

Focused sequents	$\Gamma \vdash^\mathcal{P} [A]$
Unfocused sequents	$\Gamma \vdash^\mathcal{P} \Delta$

where $\mathcal{P}$ is a polarisation set, Γ is a (finite) multiset of literals and $\mathcal{P}$-negative formulae, Δ is a (finite) multiset of formulae, and A is said to be in the *focus* of the (focused) sequent.

By $\mathsf{lit}_\mathcal{P}(\Gamma)$ we denote the sub-multiset of Γ consisting of its $\mathcal{P}$-positive literals (i.e. $\mathcal{P} \cap \Gamma$ as a set).

The rules of $\mathsf{LK}^p(\mathcal{T})$, given in Figure 1, are of three kinds: *synchronous rules*, *asynchronous rules*, and *structural rules*. These correspond to three alternating phases in the proof-search process that is described by the rules. ※

The gradual proof-tree construction defined by the inference rules of $\mathsf{LK}^p(\mathcal{T})$ is a goal-directed mechanism whose intuition can be given as follows:

Asynchronous rules are invertible: $(\wedge^-)$ and $(\vee^-)$ are applied eagerly when trying to construct the proof-tree of a given sequent; (Store) is applied when hitting a literal or a positive formula on the right-hand side of a sequent, storing its negation on the left.

When the right-hand side of a sequent becomes empty, a sanity check can be made with (Init_2) to check the semantical consistency of the stored (positive) literals (w.r.t. the theory), otherwise a choice must be made to place a formula in focus which is not $\mathcal{P}$-negative, before applying synchronous rules like $(\wedge^+)$ and $(\vee^+)$. Each such rule decomposes the formula in focus, keeping the revealed sub-formulae in the focus of the corresponding premises, until a

Synchronous rules

$$(\wedge^+)\frac{\Gamma\vdash^{\mathcal{P}}[A]\qquad\Gamma\vdash^{\mathcal{P}}[B]}{\Gamma\vdash^{\mathcal{P}}[A\wedge^+B]}\qquad(\vee^+)\frac{\Gamma\vdash^{\mathcal{P}}[A_i]}{\Gamma\vdash^{\mathcal{P}}[A_1\vee^+A_2]}\qquad(\exists)\frac{\Gamma\vdash^{\mathcal{P}}[\{{}^{t}\!/\!_{x}\}A]}{\Gamma\vdash^{\mathcal{P}}[\exists xA]}$$

$$(\top^+)\frac{}{\Gamma\vdash^{\mathcal{P}}[\top^+]}\qquad(\mathsf{Init}_1)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma),l^{\perp}\models_{\mathcal{T}}}{\Gamma\vdash^{\mathcal{P}}[l]}\;l\text{ is }\mathcal{P}\text{-positive}\qquad(\mathsf{Release})\frac{\Gamma\vdash^{\mathcal{P}}N}{\Gamma\vdash^{\mathcal{P}}[N]}\;N\text{ is not }\mathcal{P}\text{-positive}$$

Asynchronous rules

$$(\wedge^-)\frac{\Gamma\vdash^{\mathcal{P}}A,\Delta\qquad\Gamma\vdash^{\mathcal{P}}B,\Delta}{\Gamma\vdash^{\mathcal{P}}A\wedge^-B,\Delta}\qquad(\vee^-)\frac{\Gamma\vdash^{\mathcal{P}}A_1,A_2,\Delta}{\Gamma\vdash^{\mathcal{P}}A_1\vee^-A_2,\Delta}\qquad(\forall)\frac{\Gamma\vdash^{\mathcal{P}}A,\Delta}{\Gamma\vdash^{\mathcal{P}}(\forall xA),\Delta}\;x\notin\mathsf{FV}(\Gamma,\Delta,\mathcal{P})$$

$$(\perp^-)\frac{\Gamma\vdash^{\mathcal{P}}\Delta}{\Gamma\vdash^{\mathcal{P}}\Delta,\perp^-}\qquad(\top^-)\frac{}{\Gamma\vdash^{\mathcal{P}}\Delta,\top^-}\qquad(\mathsf{Store})\frac{\Gamma,A^{\perp}\vdash^{\mathcal{P};A^{\perp}}\Delta}{\Gamma\vdash^{\mathcal{P}}A,\Delta}\;\begin{array}{l}A\text{ is a literal}\\\text{or is }\mathcal{P}\text{-positive}\end{array}$$

Structural rules

$$(\mathsf{Select})\frac{\Gamma,P^{\perp}\vdash^{\mathcal{P}}[P]}{\Gamma,P^{\perp}\vdash^{\mathcal{P}}}\;P\text{ is not }\mathcal{P}\text{-negative}\qquad(\mathsf{Init}_2)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma)\models_{\mathcal{T}}}{\Gamma\vdash^{\mathcal{P}}}$$

where $\mathcal{P};A := \mathcal{P},A$ if $A\in\mathsf{U}_{\mathcal{P}}$
$\mathcal{P};A := \mathcal{P}$ if not

Figure 1: System $\mathsf{LK}^p(\mathcal{T})$

positive literal or a non-positive formula is obtained: the former case must be closed immediately with (Init_1) calling the decision procedure, and the latter case uses the ($\mathsf{Release}$) rule to drop the focus and start applying asynchronous rules again. The synchronous and the structural rules are in general not invertible, and each application of those yields a potential backtrack point in the proof-search.

Remark 4 The polarisation of literals (if not already polarised) happens in the (Store) rule, where the construction $\mathcal{P};A$ plays a crucial role. It will be useful to notice the commutation $\mathcal{P};A;B=\mathcal{P};B;A$ unless $A=B^{\perp}\in\mathsf{U}_{\mathcal{P}}$.

2 Admissibility of basic rules

In this section, we show the admissibility and invertibility of some rules, in order to prove the meta-theory of $\mathsf{LK}^p(\mathcal{T})$.

Lemma 5 (Weakening and contraction) The following rules are height-preserving admissible in $\mathsf{LK}^p(\mathcal{T})$:

$$(\mathsf{W}_l)\frac{\Gamma\vdash^{\mathcal{P}}\Delta}{\Gamma,A\vdash^{\mathcal{P}}\Delta}\qquad(\mathsf{W}_f)\frac{\Gamma\vdash^{\mathcal{P}}[B]}{\Gamma,A\vdash^{\mathcal{P}}[B]}$$

$$(\mathsf{C}_l)\frac{\Gamma,A,A\vdash^{\mathcal{P}}\Delta}{\Gamma,A\vdash^{\mathcal{P}}\Delta}\qquad(\mathsf{C}_f)\frac{\Gamma,A,A\vdash^{\mathcal{P}}[B]}{\Gamma,A\vdash^{\mathcal{P}}[B]}\qquad(\mathsf{C}_r)\frac{\Gamma\vdash^{\mathcal{P}}\Delta,A,A}{\Gamma\vdash^{\mathcal{P}}\Delta,A}$$

Proof: By induction on the derivation of the premiss. □

Lemma 6 (Identities) The identity rules are admissible in $\mathsf{LK}^p(\mathcal{T})$:

$$(\mathsf{Id}_1)\frac{}{\Gamma, l \vdash^{\mathcal{P}} [l]}\ l \text{ is } \mathcal{P}\text{-positive} \qquad (\mathsf{Id}_2)\frac{}{\Gamma, l, l^{\perp} \vdash^{\mathcal{P}}}$$

Proof: It is trivial to prove Id_1.

If l or $l^{\perp}$ is $\mathcal{P}$-positive, the Id_2 rule can be obtained by a derivation of the following form:

$$\frac{(\mathsf{Id}_1)\frac{}{\Gamma, l, l^{\perp} \vdash^{\mathcal{P}} [l]}}{\Gamma, l, l^{\perp} \vdash^{\mathcal{P}}}$$

where l is assumed to be the $\mathcal{P}$-positive literal.

If $l \in \mathsf{U}_{\mathcal{P}}$, we polarise it positively with

$$\frac{(\mathsf{Release})\frac{(\mathsf{Store})\frac{\Gamma, l, l^{\perp}, l \vdash^{\mathcal{P}, l}}{\Gamma, l, l^{\perp} \vdash^{\mathcal{P}} l^{\perp}}}{\Gamma, l, l^{\perp} \vdash^{\mathcal{P}} [l^{\perp}]}}{\Gamma, l, l^{\perp} \vdash^{\mathcal{P}}}$$

□

3 Invertibility of the asynchronous phase

We have mentioned that the asynchronous rules are invertible; now in this section, we prove it.

Lemma 7 (Invertibility of asynchronous rules) All asynchronous rules are invertible in $\mathsf{LK}(\mathcal{T})$.

Proof: By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of $A \wedge^- B$: by case analysis on the last rule actually used
  - $(\wedge^-)$

$$\frac{\Gamma \vdash^{\mathcal{P}} A \wedge^- B, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} A \wedge^- B, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \wedge^- B, C \wedge^- D, \Delta'}$$

    By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, \Delta' \qquad \Gamma \vdash^{\mathcal{P}} A, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C \wedge^- D, \Delta'}$$

    and

$$\frac{\Gamma \vdash^{\mathcal{P}} B, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} B, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} B, C \wedge^- D, \Delta'}$$

  - $(\vee^-)$

$$\frac{\Gamma \vdash^{\mathcal{P}} A \wedge^- B, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \wedge^- B, C \vee^- D, \Delta'}$$

    By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C \vee^- D, \Delta'}$$

    and

$$\frac{\Gamma \vdash^{\mathcal{P}} B, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} B, C \vee^- D, \Delta'}$$

– (∀)

$$\frac{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, C, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta', A \wedge^{-} B)$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta', A)$$

and

$$\frac{\Gamma \vdash^{\mathcal{P}} B, C, \Delta'}{\Gamma \vdash^{\mathcal{P}} B, (\forall x C), \Delta'}\, x \notin \mathsf{FV}(\Gamma, \Delta', B)$$

– (Store)

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P}; C^{\perp}} A \wedge^{-} B, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, C, \Delta'}\, C \text{ literal or } \mathcal{P}\text{-positive formula}$$

By induction hypothesis we get

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P}; C^{\perp}} A, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C, \Delta'}\, C \text{ literal or } \mathcal{P}\text{-positive formula}$$

and

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P}; C^{\perp}} B, \Delta'}{\Gamma \vdash^{\mathcal{P}} B, C, \Delta'}\, C \text{ literal or } \mathcal{P}\text{-positive formula}$$

– ($\perp^{-}$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, \perp^{-}, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, \perp^{-}, \Delta'}$$

and

$$\frac{\Gamma \vdash^{\mathcal{P}} B, \Delta'}{\Gamma \vdash^{\mathcal{P}} B, \perp^{-}, \Delta'}$$

– ($\top^{-}$)

$$\frac{}{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, \top^{-}, \Delta'}$$

We get

$\frac{}{\Gamma \vdash^{\mathcal{P}} A, \top^{-}, \Delta'}$ and $\frac{}{\Gamma \vdash^{\mathcal{P}} B, \top^{-}, \Delta'}$

- Inversion of $A \vee^{-} B$: by case analysis on the last rule
  – ($\wedge^{-}$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} A \vee^{-} B, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, C \wedge^{-} D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, B, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} A, B, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, B, C \wedge^{-} D, \Delta'}$$

  – ($\vee^{-}$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, C \vee^{-} D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, B, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, B, C \vee^{-} D, \Delta'}$$

– (∀)

$$\frac{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, C, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, (\forall x C), \Delta'} \; x \notin \mathsf{FV}(\Gamma, \Delta')$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, B, C, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, B, (\forall x C), \Delta'} \; x \notin \mathsf{FV}(\Gamma, \Delta')$$

– (Store)

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P}; C^{\perp}} A \vee^{-} B, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, C, \Delta'} \; C \text{ literal or } \mathcal{P}\text{-postive formula}$$

By induction hypothesis we get

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P}; C^{\perp}} A, B, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, B, C, \Delta'} \; C \text{ literal or } \mathcal{P}\text{-positive formula}$$

– ($\perp^{-}$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, \Delta'}{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, \perp^{-}, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, B, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, B, \perp^{-}, \Delta'}$$

– ($\top^{-}$)

$$\frac{}{\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, \top^{-}, \Delta'}$$

We get

$$\frac{}{\Gamma \vdash^{\mathcal{P}} A, B, \top^{-}, \Delta'}$$

- Inversion of $\forall x A$: by case analysis on the last rule
  – ($\wedge^{-}$)

$$\frac{\Gamma \vdash^{\mathcal{P}} (\forall x A), C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} (\forall x A), D, \Delta'}{\Gamma \vdash^{\mathcal{P}} (\forall x A), C \wedge^{-} D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} A, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C \wedge^{-} D, \Delta'} \; x \notin \mathsf{FV}(\Gamma, \Delta')$$

– ($\vee^{-}$)

$$\frac{\Gamma \vdash^{\mathcal{P}} (\forall x A), C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} (\forall x A), C \vee^{-} D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C \vee^{-} D, \Delta'}$$

– (∀)

$$\frac{\Gamma \vdash^{\mathcal{P}} (\forall x A), D, \Delta'}{\Gamma \vdash^{\mathcal{P}} (\forall x A), (\forall x D), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta')$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, (\forall x D), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta')$$

– (Store)

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} (\forall x A), \Delta'}{\Gamma \vdash^{\mathcal{P}} (\forall x A), C, \Delta'}\ C \text{ literal or } \mathcal{P}\text{-positive formula}$$

By induction hypothesis we get

$$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} A, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C, \Delta'}\ C \text{ literal or } \mathcal{P}\text{-positive formula}$$

– ($\perp^-$)

$$\frac{\Gamma \vdash^{\mathcal{P}} (\forall x A), \Delta'}{\Gamma \vdash^{\mathcal{P}} (\forall x A), \perp^-, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, \perp^-, \Delta'}$$

– ($\top^-$)

$$\frac{}{\Gamma \vdash^{\mathcal{P}} (\forall x A), \top^-, \Delta'}$$

We get

$$\frac{}{\Gamma \vdash^{\mathcal{P}} A, \top^-, \Delta'}$$

- Inversion of (Store): where A is a literal or $\mathcal{P}$-positive formula.
  By case analysis on the last rule
  – ($\wedge^-$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} A, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C \wedge^- D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} C, \Delta' \quad \Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} D, \Delta'}{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} C \wedge^- D, \Delta'}$$

– ($\vee^-$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, C \vee^- D, \Delta'}$$

By induction hypothesis

$$\frac{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} C, D, \Delta'}{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} C \vee^- D, \Delta'}$$

– (∀)

$$\frac{\Gamma \vdash^{\mathcal{P}} A, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, (\forall x D), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta')$$

By induction hypothesis we get

$$\frac{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} D, \Delta'}{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} (\forall x D), \Delta'}\ x \notin \mathsf{FV}(\Gamma, \Delta')$$

– (Store)

$$\frac{\Gamma, B^\perp \vdash^{\mathcal{P};B^\perp} A, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, B, \Delta'}\; B \text{ literal or } \mathcal{P}\text{-positive formula}$$

By induction hypothesis we can construct:

$$\frac{\Gamma, A^\perp, B^\perp \vdash^{\mathcal{P};B^\perp;A^\perp} \Delta'}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} B, \Delta'}$$

provided $\mathcal{P};B^\perp;A^\perp = \mathcal{P};A^\perp;B^\perp$, which is always the case unless $A = B^\perp$ and $A \in \mathsf{U}_{\mathcal{P}}$, in which case we build:

$$\frac{\overline{\Gamma, A^\perp, B^\perp \vdash^{\mathcal{P};A^\perp;B^\perp} \Delta'}\,(\mathsf{Id}_2)}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} B, \Delta}$$

– ($\bot^-$)

$$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta'}{\Gamma \vdash^{\mathcal{P}} A, \bot^-, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} \Delta'}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} \bot^-, \Delta'}$$

– ($\top^-$)

$$\frac{}{\Gamma \vdash^{\mathcal{P}} A, \top^-, \Delta'}$$

We get

$$\frac{}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} \top^-, \Delta'}$$

- Inversion of ($\bot^-$): by case analysis on the last rule
  – ($\wedge^-$)

$$\frac{\Gamma \vdash^{\mathcal{P}} \bot^-, C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} \bot^-, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} \bot^-, C \wedge^- D, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} C, \Delta' \quad \Gamma \vdash^{\mathcal{P}} D, \Delta'}{\Gamma \vdash^{\mathcal{P}} C \wedge^- D, \Delta'}$$

– ($\vee^-$)

$$\frac{\Gamma \vdash^{\mathcal{P}} \bot^-, C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} \bot^-, C \vee^- D, \Delta'}$$

By induction hypothesis

$$\frac{\Gamma \vdash^{\mathcal{P}} C, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} C \vee^- D, \Delta'}$$

– ($\forall$)

$$\frac{\Gamma \vdash^{\mathcal{P}} \bot^-, D, \Delta'}{\Gamma \vdash^{\mathcal{P}} \bot^-, (\forall x D), \Delta'}\; x \notin \mathsf{FV}(\Gamma, \Delta')$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} D, \Delta'}{\Gamma \vdash^{\mathcal{P}} (\forall x D), \Delta'}\; x \notin \mathsf{FV}(\Gamma, \Delta')$$

– (Store)

$$\frac{\Gamma, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} \bot^{-}, \Delta'}{\Gamma \vdash^{\mathcal{P}} \bot^{-}, B, \Delta'}\ B \text{ literal or } \mathcal{P}\text{-positive formula}$$

By induction hypothesis we get

$$\frac{\Gamma, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} \Delta'}{\Gamma \vdash^{\mathcal{P}} B, \Delta'}\ B \text{ literal or } \mathcal{P}\text{-positive formula}$$

– $(\bot^{-})$

$$\frac{\Gamma \vdash^{\mathcal{P}} \bot^{-}, \Delta'}{\Gamma \vdash^{\mathcal{P}} \bot^{-}, \bot^{-}, \Delta'}$$

By induction hypothesis we get

$$\frac{\Gamma \vdash^{\mathcal{P}} \Delta'}{\Gamma \vdash^{\mathcal{P}} \bot^{-}, \Delta'}$$

– $(\top^{-})$

$$\frac{}{\Gamma \vdash^{\mathcal{P}} \top^{-}, \bot^{-}, \Delta'}$$

We get

$$\frac{}{\Gamma \vdash^{\mathcal{P}} \top^{-}, \Delta'}$$

- Inversion of $(\top^{-})$: Nothing to do.

□

4 On-the-fly polarisation

The side-conditions of the $\mathsf{LK}^p(\mathcal{T})$ rules make it quite clear that the polarisation of literals plays a crucial role in the shape of proofs. The less flexible the polarisation of literals is, the more structure is imposed on proofs. We therefore concentrated the polarisation of literals in just one rule: (Store). In this section, we describe more flexible ways of changing the polarity of literals without modifying the provability of sequents. We do this by showing the admissibility and invertibility of some "on-the-fly" polarisation rules.

Lemma 8 (Invertibility) The following rules are invertible in $\mathsf{LK}^p(\mathcal{T})$:

$$(\mathsf{Pol})\frac{\Gamma \vdash^{\mathcal{P},l} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{lit}_{\mathcal{P},l}(\Gamma, \Delta^{\perp}), l^{\perp} \models_{\mathcal{T}} \qquad (\mathsf{Pol}_i)\frac{\Gamma \vdash^{\mathcal{P},l} [A]}{\Gamma \vdash^{\mathcal{P}} [A]}\ \mathsf{lit}_{\mathcal{P},l}(\Gamma), l^{\perp} \models_{\mathcal{T}}$$

where $l \in \mathsf{U}_{\mathcal{P}}$.

Proof: By simultaneous induction on the derivation of the conclusion (by case analysis on the last rule used in that derivation):

- $(\wedge^{-})$,$(\vee^{-})$,$(\forall\ )$,$(\bot^{-})$,$(\top^{-})$
  For these rules, whatever is done with the polarisation set $\mathcal{P}$ can be done with the polarisation set $\mathcal{P}, l$:

$$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta \quad \Gamma \vdash^{\mathcal{P}} B, \Delta}{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, \Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} A, \Delta \quad \Gamma \vdash^{\mathcal{P},l} B, \Delta}{\Gamma \vdash^{\mathcal{P},l} A \wedge^{-} B, \Delta}$$

$$\frac{\Gamma \vdash^{\mathcal{P}} A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} A \wedge^{-} B, \Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} A, B, \Delta}{\Gamma \vdash^{\mathcal{P},l} A \vee^{-} B, \Delta}$$

$$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta}{\Gamma \vdash^{\mathcal{P}} \forall x A, \Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} A, \Delta}{\Gamma \vdash^{\mathcal{P},l} \forall x A, \Delta}$$

$$\frac{\Gamma \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \bot^{-}, \Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} \Delta}{\Gamma \vdash^{\mathcal{P},l} \bot^{-}, \Delta}$$

$$\overline{\Gamma \vdash^{\mathcal{P}} \top^-, \Delta} \qquad \overline{\Gamma \vdash^{\mathcal{P},l} \top^-, \Delta}$$

- (Store): We assume

$$\frac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} \Delta}{\Gamma \vdash^{\mathcal{P}} A, \Delta} A \text{ is a literal or is } \mathcal{P}\text{-positive}$$

Notice that A is either a literal or a $\mathcal{P}, l$-positive formula, so can prove

$$\frac{\Gamma, A^\perp \vdash^{\mathcal{P},l;A^\perp} \Delta}{\Gamma \vdash^{\mathcal{P},l} A, \Delta}$$

provided we can prove the premiss.
  - If $A \neq l$, then $\mathcal{P}, l; A^\perp = \mathcal{P}; A^\perp, l$ and applying the induction hypothesis finishes the proof (unless $A = l^\perp$ in which case the derivable sequent $\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} \Delta$ is the same as the premiss to be proved);
  - If $A = l$, we build

$$(\mathsf{Store})\dfrac{(\mathsf{Store})\dfrac{\dfrac{(\mathsf{Init}_1)\dfrac{\mathsf{lit}_{\mathcal{P}',l}(\Gamma, l^\perp, \Gamma'), l^\perp \models_{\mathcal{T}}}{\Gamma, l^\perp, \Gamma' \vdash^{\mathcal{P}',l} [l]}}{\Gamma, l^\perp, \Gamma' \vdash^{\mathcal{P}',l}}}{\Gamma, \Gamma' \vdash^{\mathcal{P}',l} l}}{\Gamma \vdash^{\mathcal{P},l} l, \Delta}$$

for some $\mathcal{P}' \supseteq \mathcal{P}$ and some $\Gamma' \supseteq \mathsf{lit}_{\mathcal{L}}(\Delta^\perp)$. The closing condition $\mathsf{lit}_{\mathcal{P}',l}(\Gamma, l^\perp, \Gamma'), l^\perp \models_{\mathcal{T}}$ holds, since $\mathsf{lit}_{\mathcal{P},l}(\Gamma, l^\perp, \Delta^\perp), l^\perp \subseteq \mathsf{lit}_{\mathcal{P}',l}(\Gamma, l^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp)), l^\perp$ is assumed inconsistent.

- (Select): We assume

$$\frac{\Gamma \vdash^{\mathcal{P}} [A]}{\Gamma \vdash^{\mathcal{P}}} \begin{array}{l} A \text{ is not } \mathcal{P}\text{-negative} \\ A^\perp \in \Gamma \end{array}$$

  - If $A \neq l^\perp$, then A is not $\mathcal{P}, l$-negative and we can use the induction hypothesis (invertibility of Pol_i) to construct:

$$\frac{\Gamma \vdash^{\mathcal{P},l} [A]}{\Gamma \vdash^{\mathcal{P},l}}$$

  - If $A = l^\perp$, then $l \in \Gamma$ and the hypothesis can only be derived by

$$\dfrac{\dfrac{\dfrac{\Gamma, l \vdash^{\mathcal{P},l}}{\Gamma \vdash^{\mathcal{P}} l^\perp}}{\Gamma \vdash^{\mathcal{P}} [l^\perp]}}{\Gamma \vdash^{\mathcal{P}}}$$

as $\mathcal{P}; l = \mathcal{P}, l$; then we can construct:

$$(\mathsf{C}_l)\frac{\Gamma, l \vdash^{\mathcal{P},l}}{\Gamma \vdash^{\mathcal{P},l}}$$

- (Init_2): We assume

$$\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}}}$$

We build

$$\frac{\mathsf{lit}_{\mathcal{P},l}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P},l}}$$

- $(\wedge^+)$,$(\vee^+)$,$(\exists\ )$,$(\top^+)$

Again, for these rules, whatever is done with the polarisation set $\mathcal{P}$ can be done with the polarisation set $\mathcal{P}, l$:

$$\frac{\Gamma \vdash^{\mathcal{P}} [A] \quad \Gamma \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [A \wedge^+ B]} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} [A] \quad \Gamma \vdash^{\mathcal{P},l} [B]}{\Gamma \vdash^{\mathcal{P},l} [A \wedge^+ B]}$$

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} [A_i]}{\Gamma \vdash^{\mathcal{P},l} [A_1 \vee^+ A_2]}$$

$$\frac{\Gamma \vdash^{\mathcal{P}} [\{{}^{t}\!/\!_{x}\} A]}{\Gamma \vdash^{\mathcal{P}} [\exists x A]} \qquad \frac{\Gamma \vdash^{\mathcal{P},l} [\{{}^{t}\!/\!_{x}\} A]}{\Gamma \vdash^{\mathcal{P},l} [\exists x A]}$$

$$\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]} \qquad \frac{}{\Gamma \vdash^{\mathcal{P},l} [\top^+]}$$

- $(\mathsf{Release})$: We assume

  $$\frac{\Gamma \vdash^{\mathcal{P}} A}{\Gamma \vdash^{\mathcal{P}} [A]}$$

  where A is not $\mathcal{P}$-positive.
  - If $A \neq l$, then we build:

    $$\frac{\Gamma \vdash^{\mathcal{P},l} A}{\Gamma \vdash^{\mathcal{P},l} [A]}$$

    since A is not $\mathcal{P}, l$-positive, and we close the branch by applying the induction hypothesis (invertibility of Pol), whose side-condition $\mathsf{lit}_{\mathcal{P},l}(\Gamma, A^\perp), l^\perp \models_{\mathcal{T}}$ is implied by $\mathsf{lit}_{\mathcal{P},l}(\Gamma), l^\perp \models_{\mathcal{T}}$.
  - if $A = l$ then we build

    $$\frac{\mathsf{lit}_{\mathcal{P},l}(\Gamma), l^\perp \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P},l} [l]}$$

    where $\mathsf{lit}_{\mathcal{P},l}(\Gamma), l^\perp \models_{\mathcal{T}}$ is the side-condition of (Pol_i) that we have assumed.
- (Init_1) We assume

  $$\frac{}{\Gamma \vdash^{\mathcal{P}} [l']}$$

  with $\mathsf{lit}_{\mathcal{P}}(\Gamma), {l'}^\perp \models_{\mathcal{T}}$ and l' is $\mathcal{P}$-positive.
  We build:

  $$\frac{}{\Gamma \vdash^{\mathcal{P},l} [l']}$$

  since l' is $\mathcal{P}, l$-positive and $\mathsf{lit}_{\mathcal{P},l}(\Gamma), {l'}^\perp \models_{\mathcal{T}}$.

□

Corollary 9 The following rules are admissible in $\mathsf{LK}^p(\mathcal{T})$:

$$(\mathsf{Store}^=)\frac{\Gamma, A^\perp \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} A, \Delta} \qquad (\mathsf{W}_r)\frac{\Gamma \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta, \Delta'}$$

where A is a literal or a $\mathcal{P}$-positive formula.

Proof: For the first rule: if A is polarised, we use (Store) and it does not change $\mathcal{P}$; otherwise A is an unpolarised literal l and we build

$$(\mathsf{Store})\frac{\dfrac{\Gamma, l^\perp \vdash^{\mathcal{P}} \Delta}{\Gamma, l^\perp \vdash^{\mathcal{P},l^\perp} \Delta}}{\Gamma \vdash^{\mathcal{P}} l, \Delta}$$

The topmost inference is the invertibility of (Pol), given that $\mathsf{lit}_{\mathcal{P},l^\perp}(\Gamma, l^\perp), l \models_{\mathcal{T}}$.

For the second case, we simply do a multiset induction on Δ', using rule $(\mathsf{Store}^=)$ for the base case, followed by a left weakening. □

Now we can show that removing polarities is admissible:

Lemma 10 (Admissibility) The following rules are admissible in $\mathsf{LK}^p(\mathcal{T})$:

$$(\mathsf{Pol})\frac{\Gamma \vdash^{\mathcal{P},l} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta} \qquad (\mathsf{Pol}_a)\frac{\Gamma \vdash^{\mathcal{P},l} [A]}{\Gamma \vdash^{\mathcal{P}} [A]}\ l \notin \Gamma \text{ or } \mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}}$$

where $l \in \mathsf{U}_{\mathcal{P}}$.

Proof: By a simultaneous induction on the derivation of the premiss, again by case analysis on the last rule used in the assumed derivation.

- $(\wedge^-)$,$(\vee^-)$,$(\forall\ )$,$(\bot^-)$,$(\top^-)$
  For these rules, whatever is done with the polarisation set $\mathcal{P}, l$ can be done with the polarisation set $\mathcal{P}$:

$$\frac{\Gamma \vdash^{\mathcal{P},l} A,\Delta \quad \Gamma \vdash^{\mathcal{P},l} B,\Delta}{\Gamma \vdash^{\mathcal{P},l} A\wedge^- B,\Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P}} A,\Delta \quad \Gamma \vdash^{\mathcal{P}} B,\Delta}{\Gamma \vdash^{\mathcal{P}} A\wedge^- B,\Delta}$$

$$\frac{\Gamma \vdash^{\mathcal{P},l} A,B,\Delta}{\Gamma \vdash^{\mathcal{P},l} A\vee^- B,\Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P}} A,B,\Delta}{\Gamma \vdash^{\mathcal{P}} A\wedge^- B,\Delta}$$

$$\frac{\Gamma \vdash^{\mathcal{P},l} A,\Delta}{\Gamma \vdash^{\mathcal{P},l} \forall x A,\Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P}} A,\Delta}{\Gamma \vdash^{\mathcal{P}} \forall x A,\Delta}$$

$$\frac{\Gamma \vdash^{\mathcal{P},l} \Delta}{\Gamma \vdash^{\mathcal{P},l} \bot^-,\Delta} \qquad \frac{\Gamma \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \bot^-,\Delta}$$

$$\frac{}{\Gamma \vdash^{\mathcal{P},l} \top^-,\Delta} \qquad \frac{}{\Gamma \vdash^{\mathcal{P}} \top^-,\Delta}$$

- (Store): We assume

$$\frac{\Gamma, A^\bot \vdash^{\mathcal{P},l;A^\bot} \Delta}{\Gamma \vdash^{\mathcal{P},l} A,\Delta}\ A \text{ is a literal or } \mathcal{P},l\text{-positive}$$

  Notice that A is either a literal or a $\mathcal{P}$-positive formula.
  - If $A = l^\bot$, we build

$$(\mathsf{Store})\frac{\Gamma, A^\bot \vdash^{\mathcal{P},A^\bot} \Delta}{\Gamma \vdash^{\mathcal{P}} A,\Delta}$$

    whose premiss is the derivable sequent $\Gamma, A^\bot \vdash^{\mathcal{P},l;A^\bot} \Delta$.
  - If $A = l$, we build

$$(\mathsf{Store}^=)\frac{\Gamma, A^\bot \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} A,\Delta}$$

    using the admissibility of $\mathsf{Store}^=$, and we can prove the premiss from the induction hypothesis, as we have $\mathcal{P},l;A^\bot = \mathcal{P},l$.
  - In all other cases, we build

$$(\mathsf{Store})\frac{\Gamma, A^\bot \vdash^{\mathcal{P};A^\bot} \Delta}{\Gamma \vdash^{\mathcal{P}} A,\Delta}$$

    whose premiss is provable from the induction hypothesis, as we have $\mathcal{P},l;A^\bot = \mathcal{P};A^\bot,l$.
- (Select): We assume

$$\frac{\Gamma \vdash^{\mathcal{P},l} [A]}{\Gamma \vdash^{\mathcal{P},l}}\ \begin{array}{l} A^\bot \in \Gamma \\ \text{and } A \text{ not } \mathcal{P},l\text{-negative}\end{array}$$

  - If $l \in \Gamma$ then we can build:

$$\frac{\dfrac{\dfrac{\dfrac{(\mathsf{W}_l)\dfrac{\Gamma \vdash^{\mathcal{P};l}}{\Gamma, l \vdash^{\mathcal{P};l}}}{\Gamma \vdash^{\mathcal{P}} l^\bot}}{\Gamma \vdash^{\mathcal{P}} [l^\bot]}}{}}{\Gamma \vdash^{\mathcal{P}}}$$

    and we close with the assumption since $\mathcal{P};l = \mathcal{P},l$.
  - If $l \notin \Gamma$ then $\mathsf{lit}_{\mathcal{P},l}(\Gamma) = \mathsf{lit}_{\mathcal{P}}(\Gamma)$

Using the induction hypothesis (admissibility of Pol_a) we construct :

$$\frac{\Gamma \vdash^{\mathcal{P}} [A]}{\Gamma \vdash^{\mathcal{P}}}$$

since A is not $\mathcal{P}$-negative.

- (Init_2): We assume

$$\frac{\mathsf{lit}_{\mathcal{P},l}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P},l}}$$

  - If $l \in \Gamma$ then again we can build:

$$\dfrac{\dfrac{\dfrac{(\mathsf{W}_l)\dfrac{\Gamma \vdash^{\mathcal{P};l}}{\Gamma, l \vdash^{\mathcal{P};l}}}{\Gamma \vdash^{\mathcal{P}} l^{\perp}}}{\Gamma \vdash^{\mathcal{P}} [l^{\perp}]}}{\Gamma \vdash^{\mathcal{P}}}$$

    and we close with the assumption since $\mathcal{P}; l = \mathcal{P}, l$.
  - If $l \notin \Gamma$, $\mathsf{lit}_{\mathcal{P},l}(\Gamma) = \mathsf{lit}_{\mathcal{P}}(\Gamma)$, then we can build:

$$\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}}}$$

- $(\wedge^+)$,$(\vee^+)$,$(\exists\ )$,$(\top^+)$
  Again, for these rules, whatever is done with the polarisation set $\mathcal{P}, l$ can be done with the polarisation set $\mathcal{P}$:

$$\frac{\Gamma \vdash^{\mathcal{P},l} [A] \quad \Gamma \vdash^{\mathcal{P},l} [B]}{\Gamma \vdash^{\mathcal{P},l} [A \wedge^+ B]} \qquad \frac{\Gamma \vdash^{\mathcal{P}} [A] \quad \Gamma \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [A \wedge^+ B]}$$

$$\frac{\Gamma \vdash^{\mathcal{P},l} [A_i]}{\Gamma \vdash^{\mathcal{P},l} [A_1 \vee^+ A_2]} \qquad \frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]}$$

$$\frac{\Gamma \vdash^{\mathcal{P},l} [\left\{ {}^{t}\!/\!_{x} \right\} A]}{\Gamma \vdash^{\mathcal{P},l} [\exists x A]} \qquad \frac{\Gamma \vdash^{\mathcal{P}} [\left\{ {}^{t}\!/\!_{x} \right\} A]}{\Gamma \vdash^{\mathcal{P}} [\exists x A]}$$

$$\frac{}{\Gamma \vdash^{\mathcal{P},l} [\top^+]} \qquad \frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]}$$

- $(\mathsf{Release})$: We assume

$$\frac{\Gamma \vdash^{\mathcal{P},l} A}{\Gamma \vdash^{\mathcal{P},l} [A]}$$

  where A is not $\mathcal{P}, l$-positive.
  By induction hypothesis (admissibility of Pol) we can build:

$$\frac{\Gamma \vdash^{\mathcal{P}} A}{\Gamma \vdash^{\mathcal{P}} [A]}$$

- (Init_1): We assume

$$\frac{}{\Gamma \vdash^{\mathcal{P},l} [l']}$$

  where l' is $\mathcal{P}, l$-positive and $\mathsf{lit}_{\mathcal{P},l}(\Gamma), {l'}^{\perp} \models_{\mathcal{T}}$.
  - If $l' \neq l$, then l' is $\mathcal{P}$-positive and we can build

$$\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma), {l'}^{\perp} \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [l']}$$

    The condition $\mathsf{lit}_{\mathcal{P}}(\Gamma), {l'}^{\perp} \models_{\mathcal{T}}$ holds for the following reasons:
    If $l \notin \Gamma$, then $\mathsf{lit}_{\mathcal{P}}(\Gamma) = \mathsf{lit}_{\mathcal{P},l}(\Gamma)$ and the condition is that of the hypothesis.

If $l \in \Gamma$, then the side-condition of (Pol_a) implies $\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}}$; moreover, the condition of the hypothesis can be rewritten as $\mathsf{lit}_{\mathcal{P}}(\Gamma), l, l'^{\perp} \models_{\mathcal{T}}$; the fact that semantical inconsistency admits cuts then proves the desired condition.

– If $l' = l$ then we build

$$\dfrac{\dfrac{\Gamma, l^{\perp} \vdash^{\mathcal{P}, l^{\perp}}}{\Gamma \vdash^{\mathcal{P}} l}}{\Gamma \vdash^{\mathcal{P}} [l]}$$

which we close as follows: If $l \in \Gamma$ then we can apply Id_2, otherwise we apply Init_2: the condition $\mathsf{lit}_{\mathcal{P}, l^{\perp}}(\Gamma, l^{\perp}) \models_{\mathcal{T}}$ holds because $\mathsf{lit}_{\mathcal{P}, l^{\perp}}(\Gamma, l^{\perp}) = \mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} = \mathsf{lit}_{\mathcal{P}, l}(\Gamma), l^{\perp}$ and the condition of the hypothesis is $\mathsf{lit}_{\mathcal{P}, l}(\Gamma), l^{\perp} \models_{\mathcal{T}}$.

□

Corollary 11 The $(\mathsf{Store}^{=})$ rule is invertible, and the (Select^{-}) rule is admissible:

$$(\mathsf{Store}^{=})\dfrac{\Gamma, A^{\perp} \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} A, \Delta}\ \begin{array}{l} A \text{ is literal} \\ \text{or is } \mathcal{P}\text{-positive} \end{array} \qquad (\mathsf{Select}^{-})\dfrac{\Gamma, l^{\perp} \vdash^{\mathcal{P}, l^{\perp}} [l]}{\Gamma, l^{\perp} \vdash^{\mathcal{P}, l^{\perp}}}$$

Proof:

$(\mathsf{Store}^{=})$ Using the invertibility of (Store), we get a proof of $\Gamma, A^{\perp} \vdash^{\mathcal{P}; A^{\perp}} \Delta$. If A is polarised, then $\mathcal{P}; A^{\perp} = \mathcal{P}$ and we are done. Otherwise we have a proof of $\Gamma, A^{\perp} \vdash^{\mathcal{P}, A^{\perp}} \Delta$ and we apply the admissibility of (Pol) to conclude.

(Select^{-}) We first apply the admissibility of (Pol_a) to prove $\Gamma, l^{\perp} \vdash^{\mathcal{P}} [l]$, then the standard (Select) rule, then the invertibility of (Pol_i) to get $\Gamma, l^{\perp} \vdash^{\mathcal{P}, l^{\perp}}$.

□

5 Cut-elimination

Cut-elimination is an important feature of all sequent calculi. In this section we present some admissible cut-rules in $\mathsf{LK}^p(\mathcal{T})$ and show how to eliminate them.

5.1 Cuts with the theory

Theorem 12 (cut_1 and cut_2)

The following rules are admissible in $\mathsf{LK}^p(\mathcal{T})$, assuming $l \notin \mathsf{U}_{\mathcal{P}}$:

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_1 \qquad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [B]}\,\mathsf{cut}_2$$

Proof:

By simultaneous induction on the derivation of the right premiss.

We reduce cut_1 by case analysis on the last rule used to prove the right premiss.

- $(\wedge^{-})$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \dfrac{\Gamma, l \vdash^{\mathcal{P}} B, \Delta \quad \Gamma, l \vdash^{\mathcal{P}} C, \Delta}{\Gamma, l \vdash^{\mathcal{P}} B \wedge^{-} C, \Delta}}{\Gamma \vdash^{\mathcal{P}} B \wedge^{-} C, \Delta}\,\mathsf{cut}_1$$

reduces to

$$\dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} B, \Delta}{\Gamma \vdash^{\mathcal{P}} B, \Delta}\,\mathsf{cut}_1 \quad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} C, \Delta}{\Gamma \vdash^{\mathcal{P}} C, \Delta}\,\mathsf{cut}_1}{\Gamma \vdash^{\mathcal{P}} B \wedge^{-} C, \Delta}$$

- $(\vee^{-})$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\Gamma, l \vdash^{\mathcal{P}} B_1, B_2, \Delta}{\Gamma, l \vdash^{\mathcal{P}} B_1 \vee^{-} B_2, \Delta}}{\Gamma \vdash^{\mathcal{P}} B_1 \vee^{-} B_2, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} B_1, B_2, \Delta}{\Gamma \vdash^{\mathcal{P}} B_1, B_2, \Delta}\,\mathsf{cut}_1}{\Gamma \vdash^{\mathcal{P}} B_1 \vee^{-} B_2, \Delta}$$

- $(\forall\ )$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\Gamma, l \vdash^{\mathcal{P}} B, \Delta}{\Gamma, l \vdash^{\mathcal{P}} \forall x B, \Delta}}{\Gamma \vdash^{\mathcal{P}} \forall x B, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} B, \Delta}{\Gamma \vdash^{\mathcal{P}} B, \Delta}\,\mathsf{cut}_1}{\Gamma \vdash^{\mathcal{P}} \forall x B, \Delta}$$

- (Store) where B is a literal or $\mathcal{P}$-positive formula.

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\Gamma, l, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} \Delta}{\Gamma, l \vdash^{\mathcal{P}} B, \Delta}}{\Gamma \vdash^{\mathcal{P}} B, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P};B^{\perp}}(\Gamma, B^{\perp}), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} \Delta}{\Gamma, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} \Delta}\,\mathsf{cut}_1}{\Gamma \vdash^{\mathcal{P}} B, \Delta}$$

  We have $\mathsf{lit}_{\mathcal{P};B^{\perp}}(\Gamma, B^{\perp}), l^{\perp} \models_{\mathcal{T}}$ since $\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \subseteq \mathsf{lit}_{\mathcal{P};B^{\perp}}(\Gamma, B^{\perp}), l^{\perp}$ and we assume semantical inconsistency to satisfy weakening.

- $(\perp^{-})$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\Gamma, l \vdash^{\mathcal{P}} \Delta}{\Gamma, l \vdash^{\mathcal{P}} \perp^{-}, \Delta}}{\Gamma \vdash^{\mathcal{P}} \perp^{-}, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_1}{\Gamma \vdash^{\mathcal{P}} \perp^{-}, \Delta}$$

- $(\top^{-})$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{}{\Gamma, l \vdash^{\mathcal{P}} \top^{-}, \Delta}}{\Gamma \vdash^{\mathcal{P}} \top^{-}, \Delta}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{}{\Gamma \vdash^{\mathcal{P}} \top^{-}, \Delta}$$

- (Select) where $P^{\perp} \in \Gamma, l$ and P is not $\mathcal{P}$-negative.

  If $P^{\perp} \in \Gamma$,

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\Gamma, l \vdash^{\mathcal{P}} [P]}{\Gamma, l \vdash^{\mathcal{P}}}}{\Gamma \vdash^{\mathcal{P}}}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} [P]}{\Gamma \vdash^{\mathcal{P}} [P]}\,\mathsf{cut}_2}{\Gamma \vdash^{\mathcal{P}}}$$

  If $P^{\perp} = l$, then as P is not $\mathcal{P}$-negative and $l \notin \mathsf{U}_{\mathcal{P}}$ we get that $l^{\perp}$ is $\mathcal{P}$-positive, so

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l \models_{\mathcal{T}}}{\Gamma, l \vdash^{\mathcal{P}} [l^{\perp}]}}{\Gamma, l \vdash^{\mathcal{P}}}}{\Gamma \vdash^{\mathcal{P}}}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}}}\,\mathsf{Init}_2$$

  since semantical inconsistency admits cuts.

- (Init_2)

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^{\perp} \models_{\mathcal{T}} \qquad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l \models_{\mathcal{T}}}{\Gamma, l \vdash^{\mathcal{P}}}}{\Gamma \vdash^{\mathcal{P}}}\,\mathsf{cut}_1 \quad \text{reduces to} \quad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}}}$$

  since semantical inconsistency admits cuts.

We reduce cut_2 again by case analysis on the last rule used to prove the right premiss.

- $(\wedge^{+})$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \dfrac{\Gamma, l \vdash^{\mathcal{P}} [B] \quad \Gamma, l \vdash^{\mathcal{P}} [C]}{\Gamma, l \vdash^{\mathcal{P}} [B \wedge^+ C]}}{\Gamma \vdash^{\mathcal{P}} [B \wedge^+ C]}\ \mathsf{cut}_2$$

reduces to

$$\dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [B]}\ \mathsf{cut}_2 \qquad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} [C]}{\Gamma \vdash^{\mathcal{P}} [C]}\ \mathsf{cut}_2}{\Gamma \vdash^{\mathcal{P}} [B \wedge^+ C]}$$

- $(\vee^+)$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \dfrac{\Gamma, l \vdash^{\mathcal{P}} [B_i]}{\Gamma, l \vdash^{\mathcal{P}} [B_1 \vee^+ B_2]}}{\Gamma \vdash^{\mathcal{P}} [B_1 \vee^+ B_2]}\ \mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} [B_i]}{\Gamma \vdash^{\mathcal{P}} [B_i]}\ \mathsf{cut}_2}{\Gamma \vdash^{\mathcal{P}} [B_1 \vee^+ B_2]}$$

- $(\exists\ )$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \dfrac{\Gamma, l \vdash^{\mathcal{P}} [\{^t/_x\} B]}{\Gamma, l \vdash^{\mathcal{P}} [\exists x B]}}{\Gamma \vdash^{\mathcal{P}} [\exists x B]}\ \mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} [\{^t/_x\} B]}{\Gamma \vdash^{\mathcal{P}} [\{^t/_x\} B]}\ \mathsf{cut}_2}{\Gamma \vdash^{\mathcal{P}} [\exists x B]}$$

- $(\top^+)$

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \dfrac{}{\Gamma, l \vdash^{\mathcal{P}} [\top^+]}}{\Gamma \vdash^{\mathcal{P}} [\top^+]}\ \mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]}$$

- (Release)

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \dfrac{\Gamma, l \vdash^{\mathcal{P}} N}{\Gamma, l \vdash^{\mathcal{P}} [N]}}{\Gamma \vdash^{\mathcal{P}} [N]}\ \mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \Gamma, l \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} N}\ \mathsf{cut}_1}{\Gamma \vdash^{\mathcal{P}} [N]}$$

- (Init_1)

$$\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \quad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l, p^\perp \models_{\mathcal{T}}}{\Gamma, l \vdash^{\mathcal{P}} [p]}}{\Gamma \vdash^{\mathcal{P}} [p]}\ \mathsf{cut}_2 \qquad \text{reduces to} \qquad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), p^\perp \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [p]}$$

since weakening gives $\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp, p^\perp \models_{\mathcal{T}}$ and semantical inconsistency admits cuts.

□

5.2 Safety and instantiation

Now we would like to prove the admissibility of other cuts, where both premisses are derived as a judgement of $\mathsf{LK}^p(\mathcal{T})$. Unfortunately, the expected cut-rules are not necessarily admissible unless we consider the following notion of safety.

Definition 8 (Safety)

- A pair $(\Gamma, \mathcal{P})$ (of a context and a polarisation set) is said to be *safe* if:
  for all $\Gamma' \supseteq \Gamma$, for all semantically consistent sets of literals $\mathcal{R}$ with $\mathsf{lit}_{\mathcal{P}}(\Gamma') \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P}}(\Gamma') \cup \mathsf{U}^{\downarrow}_{\mathcal{P}}$, and for all $\mathcal{P}$-positive literal l, if $\mathcal{R}, l^\perp \models_{\mathcal{T}}$ then $\mathsf{lit}_{\mathcal{P}}(\Gamma'), l^\perp \models_{\mathcal{T}}$.
- A sequent $\Gamma \vdash^{\mathcal{P}} [A]$ (resp. $\Gamma \vdash^{\mathcal{P}} \Delta$) is said to be *safe*
  if the pair $(\Gamma, \mathcal{P})$ (resp. $((\Gamma, \Delta^\perp), \mathcal{P})$) is safe.

※

Remark 13 Safety is a property that is monotonic in its first argument: if $(\Gamma, \mathcal{P})$ is safe and $\Gamma \subseteq \Gamma'$ then $(\Gamma', \mathcal{P})$ is safe (this property is built into the definition by the quantification over Γ').

When restricted to safe sequents, the expected cuts are indeed admissible. In order to show that the safety condition is not very restrictive, we show the following lemma:

Lemma 14 (Cases of safety)

1. Empty theory:
   When the theory is empty (semantical inconsistency coincides with syntactical inconsistency), the safety of $(\Gamma, \mathcal{P})$ means that either $\mathsf{lit}_{\mathcal{P}}(\Gamma)$ is syntactically inconsistent, or every $\mathcal{P}$-positive literal that is an instance of a $\mathcal{P}$-unpolarised literal must be in Γ (i.e. $\mathcal{P} \cap \mathsf{U}^{\downarrow}_{\mathcal{P}} \subseteq \Gamma$). In the particular case of propositional logic ($\left\{{}^{t}\!/\!_{x}\right\} l = l$ for every $l \in \mathcal{L}$), every sequent is safe.
2. Full polarisation:
   When every literal is polarised ($\mathsf{U}_{\mathcal{P}} = \emptyset$), every sequent (with polarisation set $\mathcal{P}$) is safe.
3. No polarisation:
   When every literal is unpolarised ($\mathsf{U}_{\mathcal{P}} = \mathcal{L}$), every sequent (with polarisation set $\mathcal{P}$) is safe.
4. Safety is an invariant of proof-search:
   for every rule of $\mathsf{LK}^p(\mathcal{T})$, if its conclusion is safe then each of its premisses is safe.

Proof:

1. In the case of the empty theory, if $\mathcal{R}$ is consistent then $\mathcal{R}, l^{\perp} \models_{\mathcal{T}}$ means that $l \in \mathcal{R}$, so either $l \in \mathsf{lit}_{\mathcal{P}}(\Gamma')$ or $l \in \mathsf{U}^{\downarrow}_{\mathcal{P}}$; that this should imply $\mathsf{lit}_{\mathcal{P}}(\Gamma'), l^{\perp} \models_{\mathcal{T}}$ means that $l \in \mathsf{lit}_{\mathcal{P}}(\Gamma')$ anyway, unless $\mathsf{lit}_{\mathcal{P}}(\Gamma')$ is syntactically inconsistent. In particular for $\Gamma' = \Gamma$. In the case of propositional logic, there are no $\mathcal{P}$-positive literals that are in $\mathsf{U}^{\downarrow}_{\mathcal{P}} = \mathsf{U}_{\mathcal{P}}$, so every sequent is safe.
2. When every literal is polarised ($\mathsf{U}_{\mathcal{P}} = \emptyset$), then $\mathcal{R} = \mathsf{lit}_{\mathcal{P}}(\Gamma')$ and the result is trivial.
3. When every literal is unpolarised ($\mathsf{U}_{\mathcal{P}} = \mathcal{L}$), the property holds trivially.
4. For every rule of $\mathsf{LK}^p(\mathcal{T})$, if its conclusion is safe then each of its premisses is safe.
   Every rule is trivial (considering monotonicity) except (Store), for which it suffices to show:
   Assume $(\Gamma, \mathcal{P})$ is safe and $A \in \Gamma$; then $(\Gamma, (\mathcal{P}; A))$ is safe.
   Consider $\Gamma' \supseteq \Gamma$ and $\mathcal{R}$ such that $\mathsf{lit}_{\mathcal{P};A}(\Gamma') \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P};A}(\Gamma') \cup \mathsf{U}^{\downarrow}_{\mathcal{P};A}$.
   - If $A \in \mathsf{U}_{\mathcal{P}}$, then $\mathcal{P}; A = \mathcal{P}, A$ and the inclusions can be rewritten as
     $$\mathsf{lit}_{\mathcal{P}}(\Gamma'), A \ \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P}}(\Gamma'), A \ \cup \mathsf{U}^{\downarrow}_{\mathcal{P},A}$$
     Since $\mathsf{U}_{\mathcal{P},A} \subseteq \mathsf{U}_{\mathcal{P}}$ we have $\mathsf{U}^{\downarrow}_{\mathcal{P},A} \subseteq \mathsf{U}^{\downarrow}_{\mathcal{P}}$ and therefore
     $$\mathsf{lit}_{\mathcal{P}}(\Gamma') \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P}}(\Gamma') \cup \mathsf{U}^{\downarrow}_{\mathcal{P}}$$
     Hence, $\mathcal{R}$ is a set for which safety of $(\Gamma, \mathcal{P})$ implies $\mathsf{lit}_{\mathcal{P}}(\Gamma'), l^{\perp} \models_{\mathcal{T}}$ for every $l \in \mathcal{P}$ such that $\mathcal{R}, l^{\perp} \models_{\mathcal{T}}$.
     For $l = A$, then trivially $\mathsf{lit}_{\mathcal{P},A}(\Gamma'), l^{\perp} \models_{\mathcal{T}}$ as $A \in \Gamma'$.
   - If $A \notin \mathsf{U}_{\mathcal{P}}$, then $\mathcal{P}; A = \mathcal{P}$ and the result is trivial.

□

Now cut-elimination in presence of quantifiers relies heavily on the fact that, if a proof can be constructed with a free variables x, then it can be replayed when x is instantiated by a particular term throughout the proof. In a polarised world, this is made difficult by the fact that a polarisation set $\mathcal{P}$ (i.e. a set that is syntactically consistent) might not remain a polarisation set after instantiation (i.e. $\left\{{}^{t}\!/\!_{x}\right\}\mathcal{P}$ might not be syntactically consistent: imagine $p(x, 3)$ is $\mathcal{P}$-positive and $p(3, x)$ is $\mathcal{P}$-negative, then after substituting 3 for x, what is the polarity of $p(3, 3)$?). Hence, polarities will have to be changed and therefore the exact same proof may not be replayed, but under the hypothesis that the substituted sequent is safe, we manage to reconstruct *some* proof. The first step to prove this is the following lemma:

Lemma 15 (Admissibility of instantiation with the theory) Let $\mathcal{P}$ be a polarisation set such that $x \notin \mathsf{FV}(\mathcal{P})$, let $l_1, \ldots, l_n$ be n literals, $\mathcal{A}$ be a set of literals, x be a variable and t be a term with $x \notin \mathsf{FV}(t)$.

Let $\mathcal{P}_i := \mathcal{P}; l_1; \ldots; l_i$ with $\mathcal{P}_0 := \mathcal{P}$, and similarly let $\mathcal{P}'_i := \mathcal{P}; \left\{{}^t\!/\!_x\right\} l_1; \ldots; \left\{{}^t\!/\!_x\right\} l_i$ with $\mathcal{P}'_0 := \mathcal{P}$.

Assume

- for all i such that $1 \leq i \leq n$, we have $l_i \in \Gamma$;
- $(\left\{{}^t\!/\!_x\right\}\Gamma, \mathcal{P}'_n)$ is safe;
- $\mathsf{lit}_{\mathcal{P}_n}(\Gamma), \mathcal{A} \models_{\mathcal{T}}$.

Then either $\mathsf{lit}_{\mathcal{P}'_n}(\Gamma), \left\{{}^t\!/\!_x\right\}\mathcal{A} \models_{\mathcal{T}}$ or $\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}$ is derivable in $\mathsf{LK}^p(\mathcal{T})$.

Proof: Let $\{l'_1, \ldots, l'_m\}$ be the set of literals $\{l \in \mathsf{lit}_{\mathcal{P}_n}(\Gamma) \mid \left\{{}^t\!/\!_x\right\} l \text{ is not } \mathcal{P}'_n\text{-positive}\}$. We have

$$\left\{{}^t\!/\!_x\right\}\mathsf{lit}_{\mathcal{P}_n}(\Gamma) \subseteq \mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} l'_1, \ldots, \left\{{}^t\!/\!_x\right\} l'_m$$

Since $\mathsf{lit}_{\mathcal{P}_n}(\Gamma), \mathcal{A} \models_{\mathcal{T}}$ and semantical inconsistency is stable under instantiation and weakening, we have $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} l'_1, \ldots, \left\{{}^t\!/\!_x\right\} l'_m, \left\{{}^t\!/\!_x\right\}\mathcal{A} \models_{\mathcal{T}}$.

- If all of the sets $(\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} {l'_j}^{\perp})_{1 \leq j \leq n}$ are semantically inconsistent, then from

  $$\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} l'_1, \ldots, \left\{{}^t\!/\!_x\right\} l'_m, \left\{{}^t\!/\!_x\right\}\mathcal{A} \models_{\mathcal{T}}$$

  we get $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}\mathcal{A} \models_{\mathcal{T}}$, since semantically inconsistency admits cuts.
- Otherwise, there is some $l'_j \in \mathsf{lit}_{\mathcal{P}_n}(\Gamma)$ such that $\left\{{}^t\!/\!_x\right\} l'_j$ is not $\mathcal{P}'_n$-positive and such that $\mathcal{R} := \mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} {l'_j}^{\perp}$ is semantically consistent.

  Notice that l'_j is not $\mathcal{P}$-positive, otherwise $\left\{{}^t\!/\!_x\right\} l'_j$ would also be $\mathcal{P}$-positive (since $x \notin \mathsf{FV}(\mathcal{P})$), so $l'_j = l_i$ for some i such that $1 \leq i \leq n$, with $l_i \in \mathsf{U}_{\mathcal{P}_{i-1}}$.

  Now, if $\left\{{}^t\!/\!_x\right\}\Gamma$ is syntactically inconsistent, we build

  $$\mathsf{Id}_2\,\frac{}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}}$$

  If on the contrary $\left\{{}^t\!/\!_x\right\}\Gamma$ is syntactically consistent, then $\{\left\{{}^t\!/\!_x\right\} l_1, \ldots, \left\{{}^t\!/\!_x\right\} l_n\}$ is also syntactically consistent (as every element is assumed to be in $\left\{{}^t\!/\!_x\right\}\Gamma$).

  Therefore, $\left\{{}^t\!/\!_x\right\} l_i$ must be $\mathcal{P}$-negative, otherwise it would ultimately be $\mathcal{P}'_n$-positive.

  So $\left\{{}^t\!/\!_x\right\} l_i^{\perp}$ is $\mathcal{P}$-positive, and ultimately $\mathcal{P}'_n$-positive.

  Now $(\left\{{}^t\!/\!_x\right\}\Gamma, \mathcal{P}'_n)$ is assumed to be safe, so we want to apply this property to $\Gamma' := \Gamma$, to the semantically consistent set $\mathcal{R}$, and to the $\mathcal{P}'_n$-positive literal $\left\{{}^t\!/\!_x\right\} l_i^{\perp}$, so as to conclude

  $$\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} l_i \models_{\mathcal{T}}$$

  To apply the safety property, we note that that $\mathcal{R}, \left\{{}^t\!/\!_x\right\} l_i \models_{\mathcal{T}}$ and that

  $$\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma) \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma) \cup \mathsf{U}^{\downarrow}_{\mathcal{P}'_n}$$

  provided we have $l_i \in \mathsf{U}_{\mathcal{P}'_n}$.

  In order to prove that proviso, first notice that $l_i \in \mathsf{U}_{\mathcal{P}}$, since $l_i \in \mathsf{U}_{\mathcal{P}_i}$. Now we must have $x \in \mathsf{FV}(l_i)$, otherwise $l_i = \left\{{}^t\!/\!_x\right\} l_i$ and we know that $\left\{{}^t\!/\!_x\right\} l_i$ is $\mathcal{P}$-negative. Since none of the literals $(\left\{{}^t\!/\!_x\right\} l_k)_{1 \leq k \leq n}$ have x as a free variable, we conclude the proviso $l_i \in \mathsf{U}_{\mathcal{P}'_n}$.

  Therefore safety ensures $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\} l_i \models_{\mathcal{T}}$ and we can finally build

  $$\mathsf{Select}\,\frac{\mathsf{Init}_1\,\dfrac{}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\} l_i^{\perp}]}}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}}$$

  as $\left\{{}^t\!/\!_x\right\} l_i^{\perp}$ is $\mathcal{P}'_n$-positive.

□

We can finally state and prove the admissibility of instantiation:

Lemma 16 (Admissibility of instantiation) Let $\mathcal{P}$ be a polarisation set such that $x \notin \mathsf{FV}(\mathcal{P})$, let $l_1, \ldots, l_n$ be n literals, x be a variable and t be a term with $x \notin \mathsf{FV}(t)$.

Let $\mathcal{P}_i := \mathcal{P}; l_1; \ldots; l_i$ with $\mathcal{P}_0 := \mathcal{P}$, and similarly let $\mathcal{P}'_i := \mathcal{P}; \left\{{}^t\!/\!_x\right\} l_1; \ldots; \left\{{}^t\!/\!_x\right\} l_i$ with $\mathcal{P}'_0 := \mathcal{P}$.

The following rules are admissible in $\mathsf{LK}^p(\mathcal{T})$:[3]

$$(\mathsf{Inst})\frac{\Gamma \vdash^{\mathcal{P}_n} \Delta}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} \left\{{}^t\!/\!_x\right\}\Delta} \qquad (\mathsf{Inst}_f)\frac{\Gamma \vdash^{\mathcal{P}_n} [B]}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\}B] \text{ or } \left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}}$$

where we assume

- for all i such that $1 \leq i \leq n$, we have $l_i \in \Gamma$;
- $\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} \left\{{}^t\!/\!_x\right\}\Delta$ is safe in (Inst);
- $(\left\{{}^t\!/\!_x\right\}\Gamma, \mathcal{P}'_n)$ is safe in (Inst_f).

Proof: By induction on the derivation of the premiss.

- $(\wedge^-)$,$(\vee^-)$,$(\forall\ )$,$(\bot^-)$,$(\top^-)$,$(\wedge^+)$,$(\vee^+)$,$(\exists\ )$,$(\top^+)$
  These rules are straightforward as the polarisation set is not involved.
- (Store) We assume
  $$\frac{\Gamma, A^\bot \vdash^{\mathcal{P}_n; A^\bot} \Delta}{\Gamma \vdash^{\mathcal{P}_n} A, \Delta}$$
  where A is a literal or is $\mathcal{P}_n$-positive.
  Using the induction hypothesis on the premiss we can build
  $$\frac{\left\{{}^t\!/\!_x\right\}\Gamma, \left\{{}^t\!/\!_x\right\}A^\bot \vdash^{\mathcal{P}'_n; \left\{{}^t\!/\!_x\right\}A^\bot} \left\{{}^t\!/\!_x\right\}\Delta}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} \left\{{}^t\!/\!_x\right\}A, \left\{{}^t\!/\!_x\right\}\Delta}$$
  since $\left\{{}^t\!/\!_x\right\}A$ is a literal or is $\mathcal{P}'_n$-positive.
- (Select) We assume
  $$\frac{\Gamma, P^\bot \vdash^{\mathcal{P}_n} [P]}{\Gamma, P^\bot \vdash^{\mathcal{P}_n}}$$
  where P is not $\mathcal{P}_n$-negative.
  If $\left\{{}^t\!/\!_x\right\}P$ is not $\mathcal{P}'_n$-negative, then we can apply the induction hypothesis and build
  $$\frac{\left\{{}^t\!/\!_x\right\}\Gamma, \left\{{}^t\!/\!_x\right\}P^\bot \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\}P]}{\left\{{}^t\!/\!_x\right\}\Gamma, \left\{{}^t\!/\!_x\right\}P^\bot \vdash^{\mathcal{P}'_n}}$$
  Otherwise, $\left\{{}^t\!/\!_x\right\}P$ is a $\mathcal{P}'_n$-negative literal and we can do the same as above with the (Select^-) rule instead of (Select).
- (Init_2) We assume
  $$\frac{\mathsf{lit}_{\mathcal{P}_n}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}_n}}$$
  We use Lemma 15 with $\mathcal{A} := \emptyset$, since we know $\mathsf{lit}_{\mathcal{P}_n}(\Gamma) \models_{\mathcal{T}}$.
  If we get $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma) \models_{\mathcal{T}}$, we build a proof with the same rule (Init_2):
  $$\frac{\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma) \models_{\mathcal{T}}}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}}$$
  If not, we directly get a proof of $\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}$.
- (Init_1) We assume
  $$\frac{\mathsf{lit}_{\mathcal{P}_n}(\Gamma), p^\bot \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}_n} [p]}$$
  where p is $\mathcal{P}_n$-positive.
  We use Lemma 15 with $\mathcal{A} := \{p\}$, since we know $\mathsf{lit}_{\mathcal{P}_n}(\Gamma), p^\bot \models_{\mathcal{T}}$.

[3] The admissibility of (Inst_f) means that if $\Gamma \vdash^{\mathcal{P}_n} [B]$ is derivable in $\mathsf{LK}^p(\mathcal{T})$ then either $\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\}B]$ or $\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}$ is derivable in $\mathsf{LK}^p(\mathcal{T})$.

If we get $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}p^\perp \models_{\mathcal{T}}$, we build a proof with the same rule (Init_1):

$$\frac{\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}p^\perp \models_{\mathcal{T}}}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\}p]}$$

If not, we directly get a proof of $\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}$.

- ($\mathsf{Release}$) We assume

$$\frac{\Gamma \vdash^{\mathcal{P}_n} N}{\Gamma \vdash^{\mathcal{P}_n} [N]}$$

where N is not $\mathcal{P}_n$-positive.
If $\left\{{}^t\!/\!_x\right\}N$ is not $\mathcal{P}'_n$-positive, then we can apply the induction hypothesis and build

$$\frac{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} \left\{{}^t\!/\!_x\right\}N}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\}N]}$$

Otherwise, N is a literal l that is not $\mathcal{P}_n$-positive, but such that $\left\{{}^t\!/\!_x\right\}l$ is $\mathcal{P}'_n$-positive.

  - If $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}l \models_{\mathcal{T}}$, then we build

$$\mathsf{cut}_1\frac{\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}l \models_{\mathcal{T}} \qquad \left\{{}^t\!/\!_x\right\}\Gamma, \left\{{}^t\!/\!_x\right\}l^\perp \vdash^{\mathcal{P}'_n}}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n}}$$

    where the right premiss is proved as follows:
    Notice that the assumed derivation of $\Gamma \vdash^{\mathcal{P}_n} l$ necessarily contains a sub-derivation concluding $\Gamma, l^\perp \vdash^{\mathcal{P}_n;l^\perp}$, and applying the induction hypothesis on this yields a derivation of $\left\{{}^t\!/\!_x\right\}\Gamma, \left\{{}^t\!/\!_x\right\}l^\perp \vdash^{\mathcal{P}'_n}$.

  - Assume now that $\mathcal{R} :=\ \mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}l$ is semantically consistent. We build

$$\mathsf{Init}_1\frac{}{\left\{{}^t\!/\!_x\right\}\Gamma \vdash^{\mathcal{P}'_n} [\left\{{}^t\!/\!_x\right\}l]}$$

    and we have to prove the side-condition $\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma), \left\{{}^t\!/\!_x\right\}l^\perp \models_{\mathcal{T}}$.
    This is trivial if $\left\{{}^t\!/\!_x\right\}l \in \left\{{}^t\!/\!_x\right\}\Gamma$ (as $\left\{{}^t\!/\!_x\right\}l$ is $\mathcal{P}'_n$-positive).
    If on the contrary $\left\{{}^t\!/\!_x\right\}l \notin \left\{{}^t\!/\!_x\right\}\Gamma$, then we get it from the assumed safety of $(\left\{{}^t\!/\!_x\right\}\Gamma, \mathcal{P}'_n)$, applied to $\Gamma' :=\ \Gamma$, to the semantically consistent set $\mathcal{R}$, and to the $\mathcal{P}'_n$-positive literal $\left\{{}^t\!/\!_x\right\}l$. To apply the safety property, we note that $\mathcal{R}, \left\{{}^t\!/\!_x\right\}l^\perp \models_{\mathcal{T}}$ and that

$$\mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma) \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P}'_n}(\left\{{}^t\!/\!_x\right\}\Gamma) \cup \mathsf{U}^{\downarrow}_{\mathcal{P}'_n}$$

    provided we have $\left\{{}^t\!/\!_x\right\}l \in \mathsf{U}^{\downarrow}_{\mathcal{P}'_n}$.
    We prove that $l \in \mathsf{U}_{\mathcal{P}'_n}$ as follows:
    First notice that $l \in \mathsf{U}_{\mathcal{P}}$, otherwise l would be $\mathcal{P}$-negative and so would be $\left\{{}^t\!/\!_x\right\}l$ (since $x \notin \mathsf{FV}(\mathcal{P})$). Then notice that $\left\{{}^t\!/\!_x\right\}l$ must be $\mathcal{P}$-positive, since it is $\mathcal{P}'_n$-positive but $\left\{{}^t\!/\!_x\right\}l \notin \left\{{}^t\!/\!_x\right\}\Gamma$. Therefore $l \neq \left\{{}^t\!/\!_x\right\}l$, so $x \in \mathsf{FV}(l)$, and finally we get $l \in \mathsf{U}_{\mathcal{P}'_n}$, since none of the literals $(\left\{{}^t\!/\!_x\right\}l_k)_{1\leq k\leq n}$ have x as a free variable.

□

5.3 More general cuts

Theorem 17 (cut_3, cut_4 and cut_5) The following rules are admissible in $\mathsf{LK}^p(\mathcal{T})$:[4]

$$(\mathsf{cut}_3)\frac{\Gamma \vdash^{\mathcal{P}} [A] \quad \Gamma \vdash^{\mathcal{P}} A^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}$$

$$(\mathsf{cut}_4)\frac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta} \qquad (\mathsf{cut}_5)\frac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [B]}{\Gamma \vdash^{\mathcal{P}} [B] \text{ or } \Gamma \vdash^{\mathcal{P}}}$$

[4] The admissibility of cut_5 means that if $\Gamma \vdash^{\mathcal{P}} N$ and $\Gamma, N \vdash^{\mathcal{P};N} [B]$ are derivable in $\mathsf{LK}^p(\mathcal{T})$ then either $\Gamma \vdash^{\mathcal{P}} [B]$ or $\Gamma \vdash^{\mathcal{P}}$ is derivable in $\mathsf{LK}^p(\mathcal{T})$.

where

- N is assumed to not be $\mathcal{P}$-positive in cut_4 and cut_5;
- the sequent $\Gamma \vdash^{\mathcal{P}} \Delta$ in cut_3 and cut_4, and the pair $(\Gamma, \mathcal{P})$ in cut_5, are all assumed to be safe.

Proof: By simultaneous induction on the following lexicographical measure:

- the size of the cut-formula (A or N)
- the fact that the cut-formula (A or N) is positive or negative
  (if of equal size, a positive formula is considered smaller than a negative formula)
- the height of the derivation of the right premiss

Weakenings and contractions (as they are admissible in the system) are implicitly used throughout this proof.

In order to eliminate cut_3, we analyse which rule is used to prove the left premiss. We then use invertibility of the negative phase so that the last rule used in the right premiss is its dual one.

- $(\wedge^+)$

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} [A] \quad \Gamma \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [A \wedge^+ B]} \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} A^\perp, B^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} A \vee^- B, \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3$$

reduces to

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [B] \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} [A] \quad \Gamma \vdash^{\mathcal{P}} A^\perp, B^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} B^\perp, \Delta}\ \mathsf{cut}_3}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3$$

- $(\vee^+)$

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]} \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} A_1^\perp, \Delta \quad \Gamma \vdash^{\mathcal{P}} A_2^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \wedge^- A_2, \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3$$

reduces to

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [A_i] \qquad \Gamma \vdash^{\mathcal{P}} A_i^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3$$

- $(\exists\ )$

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} [\{^t/_x\}A]}{\Gamma \vdash^{\mathcal{P}} [\exists x A]} \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} A^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} (\forall x A^\perp), \Delta}\ x \notin \mathsf{FV}(\Gamma, \Delta, \mathcal{P})}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3$$

reduces to

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [\{^t/_x\}A] \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} A^\perp, \Delta}{\Gamma \vdash^{\mathcal{P}} (\{^t/_x\}A^\perp), \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3$$

using Lemma 16 (admissibility of instantiation) with $n = 0$, noticing that $x \notin \mathsf{FV}(\mathcal{P})$ and that $\Gamma \vdash^{\mathcal{P}} (\{^t/_x\}A^\perp), \Delta$ is safe (since $\Gamma \vdash^{\mathcal{P}} \Delta$ is safe).[5]

- $(\top^+)$

$$\dfrac{\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]} \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \perp^-, \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\ \mathsf{cut}_3 \qquad \text{reduces to} \qquad \Gamma \vdash^{\mathcal{P}} \Delta$$

[5] Using α-conversion, we can also pick x such that $x \notin \mathsf{FV}(t)$.

- (Init_1)

$$\dfrac{\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp} \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [p]} \qquad \dfrac{\Gamma, p \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} (p^{\perp}), \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_3 \quad \text{reduces to} \quad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp} \models_{\mathcal{T}} \qquad \Gamma, p \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_1$$

  with $p \in \mathcal{P}$.

- $(\mathsf{Release})$

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]} \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} \Delta}{\Gamma \vdash^{\mathcal{P}} (N^{\perp}), \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_3 \quad \text{reduces to} \quad \dfrac{\Gamma \vdash^{\mathcal{P}} N \qquad \Gamma, N \vdash^{\mathcal{P};N} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_4$$

  where N is not $\mathcal{P}$-positive. We will describe below how cut_4 is reduced.

In order to reduce cut_4, we analyse which rule is used to prove the right premiss.

- $(\wedge^-)$

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} B, \Delta \quad \Gamma, N \vdash^{\mathcal{P};N} C, \Delta}{\Gamma, N \vdash^{\mathcal{P};N} B \wedge^- C, \Delta}}{\Gamma \vdash^{\mathcal{P}} B \wedge^- C, \Delta}\,\mathsf{cut}_4$$

  reduces to

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} B, \Delta}{\Gamma \vdash^{\mathcal{P}} B, \Delta}\,\mathsf{cut}_4 \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} C, \Delta}{\Gamma \vdash^{\mathcal{P}} C, \Delta}\,\mathsf{cut}_4}{\Gamma \vdash^{\mathcal{P}} B \wedge^- C, \Delta}$$

- $(\vee^-)$

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} B, C, \Delta}{\Gamma, N \vdash^{\mathcal{P};N} B \vee^- C, \Delta}}{\Gamma \vdash^{\mathcal{P}} B \vee^- C, \Delta}\,\mathsf{cut}_4 \quad \text{reduces to} \quad \dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} B, C, \Delta}{\Gamma \vdash^{\mathcal{P}} B, C, \Delta}\,\mathsf{cut}_4}{\Gamma \vdash^{\mathcal{P}} B \vee^- C, \Delta}$$

- $(\forall\ )$

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} B, \Delta}{\Gamma, N \vdash^{\mathcal{P};N} \forall x B, \Delta}}{\Gamma \vdash^{\mathcal{P}} \forall x B, \Delta}\,\mathsf{cut}_4 \quad \text{reduces to} \quad \dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} B, \Delta}{\Gamma \vdash^{\mathcal{P}} B, \Delta}\,\mathsf{cut}_4}{\Gamma \vdash^{\mathcal{P}} \forall x B, \Delta}$$

- $(\perp^-)$

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} \Delta}{\Gamma, N \vdash^{\mathcal{P};N} \perp^-, \Delta}}{\Gamma \vdash^{\mathcal{P}} \perp^-, \Delta}\,\mathsf{cut}_4 \quad \text{reduces to} \quad \dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_4}{\Gamma \vdash^{\mathcal{P}} \perp^-, \Delta}$$

- (Store)

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, N, B^{\perp} \vdash^{\mathcal{P};N;B^{\perp}} \Delta}{\Gamma, N \vdash^{\mathcal{P};N} B, \Delta}}{\Gamma \vdash^{\mathcal{P}} B, \Delta}\,\mathsf{cut}_4 \quad \text{reduces to} \quad \dfrac{\dfrac{\Gamma, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} N \quad \Gamma, N, B^{\perp} \vdash^{\mathcal{P};B^{\perp};N} \Delta}{\Gamma, B^{\perp} \vdash^{\mathcal{P};B^{\perp}} \Delta}\,\mathsf{cut}_4}{\Gamma \vdash^{\mathcal{P}} B, \Delta}$$

  whose left branch is closed by using
  - possibly the admissibility of (Pol) (if $B \in \mathsf{U}_{\mathcal{P}}$), so as to get $\Gamma, B^{\perp} \vdash^{\mathcal{P}} N$,
  - then the admissibility of (W_l) (on $B^{\perp}$), to get to the provable premiss $\Gamma \vdash^{\mathcal{P}} N$;

  whose right branch is the same as the provable $\Gamma, N, B^{\perp} \vdash^{\mathcal{P};N;B^{\perp}} \Delta$ unless $B = N \in \mathsf{U}_{\mathcal{P}}$, in which case the commutation $\mathcal{P}; B^{\perp}; N = \mathcal{P}; N; B^{\perp}$ does not hold. In this last case, we build

$$(\mathsf{W}_r)\frac{\Gamma \vdash^{\mathcal{P}} B}{\Gamma \vdash^{\mathcal{P}} B, \Delta}$$

- (Init_2) when $N \notin \mathsf{U}_{\mathcal{P}}$, in which case $\mathcal{P};N = \mathcal{P}$ and $\mathsf{lit}_{\mathcal{P}}(\Gamma, N) = \mathsf{lit}_{\mathcal{P}}(\Gamma)$ (since $N \notin \mathcal{P}$ either):

$$\frac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma, N \vdash^{\mathcal{P};N}}}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_4 \qquad \text{reduces to} \qquad \frac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}}}$$

- (Init_2) when $N \in \mathsf{U}_{\mathcal{P}}$, in which case $\mathsf{lit}_{\mathcal{P};N}(\Gamma, N) = \mathsf{lit}_{\mathcal{P}}(\Gamma), N$:

$$\frac{\dfrac{\Gamma, N^{\perp} \vdash^{\mathcal{P}, N^{\perp}}}{\Gamma \vdash^{\mathcal{P}} N} \qquad \dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), N \models_{\mathcal{T}}}{\Gamma, N \vdash^{\mathcal{P};N}}}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_4 \qquad \text{reduces to} \qquad \frac{\mathsf{lit}_{\mathcal{P}, N^{\perp}}(\Gamma), N \models_{\mathcal{T}} \qquad \Gamma, N^{\perp} \vdash^{\mathcal{P}, N^{\perp}}}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_1$$

  since $\mathsf{lit}_{\mathcal{P}}(\Gamma), N \models_{\mathcal{T}}$ implies $\mathsf{lit}_{\mathcal{P}, N^{\perp}}(\Gamma), N \models_{\mathcal{T}}$.
- (Select) on formula $N^{\perp}$

$$\frac{\Gamma \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} [N^{\perp}]}{\Gamma, N \vdash^{\mathcal{P};N}}}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_4 \qquad \text{reduces to} \qquad \frac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [N^{\perp}]}{\Gamma \vdash^{\mathcal{P}} [N^{\perp}]}\mathsf{cut}_5 \qquad \Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_3$$

$$\text{or to} \qquad \frac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [N^{\perp}]}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_5$$

  depending on the outcome of cut_5
- (Select) on a formula P that is not $\mathcal{P};N$-negative

$$\frac{\Gamma, P^{\perp} \vdash^{\mathcal{P}} N \qquad \dfrac{\Gamma, P^{\perp}, N \vdash^{\mathcal{P};N} [P]}{\Gamma, P^{\perp}, N \vdash^{\mathcal{P};N}}}{\Gamma, P^{\perp} \vdash^{\mathcal{P}}}\mathsf{cut}_4 \qquad \text{reduces to} \qquad \frac{\dfrac{\Gamma, P^{\perp} \vdash^{\mathcal{P}} N \quad \Gamma, P^{\perp}, N \vdash^{\mathcal{P};N} [P]}{\Gamma, P^{\perp} \vdash^{\mathcal{P}} [P]}\mathsf{cut}_5}{\Gamma, P^{\perp} \vdash^{\mathcal{P}}}$$

$$\text{or to} \qquad \frac{\Gamma, P^{\perp} \vdash^{\mathcal{P}} N \quad \Gamma, P^{\perp}, N \vdash^{\mathcal{P};N} [N^{\perp}]}{\Gamma, P^{\perp} \vdash^{\mathcal{P}}}\mathsf{cut}_5$$

  depending on the outcome of cut_5

We have reduced all cases of cut_4; we now reduce the cases for cut_5 (again, by case analysis on the last rule used to prove the right premiss).

- $(\wedge^+)$ We are given

$$\Gamma \vdash^{\mathcal{P}} N \quad \text{and} \quad \frac{\Gamma, N \vdash^{\mathcal{P};N} [B_1] \quad \Gamma, N \vdash^{\mathcal{P};N} [B_2]}{\Gamma, N \vdash^{\mathcal{P};N} [B_1 \wedge^+ B_2]}$$

  and by cut_5 we want to derive either $\Gamma \vdash^{\mathcal{P}} [B_1 \wedge^+ B_2]$ or $\Gamma \vdash^{\mathcal{P}}$.
  If we can, we build

$$\frac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [B_1]}{\Gamma \vdash^{\mathcal{P}} [B_1]}\mathsf{cut}_5 \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [B_2]}{\Gamma \vdash^{\mathcal{P}} [B_2]}\mathsf{cut}_5}{\Gamma \vdash^{\mathcal{P}} [B_1 \wedge^+ B_2]}$$

  Otherwise we build

$$\frac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [B_i]}{\Gamma \vdash^{\mathcal{P}}}\mathsf{cut}_5$$

  where i is (one of) the premiss(es) for which cut_5 produces a proof of $\Gamma \vdash^{\mathcal{P}}$.
- $(\vee^+)$ We are given

$$\Gamma \vdash^{\mathcal{P}} N \quad \text{and} \quad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} [B_i]}{\Gamma, N \vdash^{\mathcal{P};N} [B_1 \vee^+ B_2]}$$

and by cut_5 we want to derive either $\Gamma \vdash^{\mathcal{P}} [B_1 \vee^+ B_2]$ or $\Gamma \vdash^{\mathcal{P}}$.
If we can, we build

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [B_i]}{\Gamma \vdash^{\mathcal{P}} [B_i]}\mathsf{cut}_5}{\Gamma \vdash^{\mathcal{P}} [B_1 \vee^+ B_2]}$$

Otherwise we build

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [B_i]}{\Gamma \vdash^{\mathcal{P}}}$$

- $(\exists\ )$ We are given

$$\Gamma \vdash^{\mathcal{P}} N \quad \text{and} \quad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} [\{{}^{t}\!/\!{}_{x}\} B]}{\Gamma, N \vdash^{\mathcal{P};N} [\exists x B]}$$

  and by cut_5 we want to derive either $\Gamma \vdash^{\mathcal{P}} [\exists x B]$ or $\Gamma \vdash^{\mathcal{P}}$.
  If we can, we build

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [\{{}^{t}\!/\!{}_{x}\} B]}{\Gamma \vdash^{\mathcal{P}} [\{{}^{t}\!/\!{}_{x}\} B]}\mathsf{cut}_5}{\Gamma \vdash^{\mathcal{P}} [\exists x B]}$$

  Otherwise we build

$$\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} [\{{}^{t}\!/\!{}_{x}\} B]}{\Gamma \vdash^{\mathcal{P}}}$$

- $(\top^+)$ We are given

$$\Gamma \vdash^{\mathcal{P}} N \quad \text{and} \quad \dfrac{}{\Gamma, N \vdash^{\mathcal{P};N} [\top^+]}$$

  and by cut_5 we want to derive either $\Gamma \vdash^{\mathcal{P}} [\top^+]$ or $\Gamma \vdash^{\mathcal{P}}$.
  We build

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]}$$

- $(\mathsf{Release})$ We are given:

$$\Gamma \vdash^{\mathcal{P}} N \qquad \text{and} \qquad \dfrac{\Gamma, N \vdash^{\mathcal{P};N} N'}{\Gamma, N \vdash^{\mathcal{P};N} [N']}$$

  where N' is not $\mathcal{P};N$-positive;
  and by cut_5 we want to derive either $\Gamma \vdash^{\mathcal{P}} [N']$ or $\Gamma \vdash^{\mathcal{P}}$.
  We build

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} N \quad \Gamma, N \vdash^{\mathcal{P};N} N'}{\Gamma \vdash^{\mathcal{P}} N'}\mathsf{cut}_4}{\Gamma \vdash^{\mathcal{P}} [N']}$$

  since N' is not $\mathcal{P}$-positive.
- (Init_1) We are given:

$$\Gamma \vdash^{\mathcal{P}} N \qquad \text{and} \qquad \dfrac{\mathsf{lit}_{\mathcal{P};N}(\Gamma, N), p^{\perp} \models_{\mathcal{T}}}{\Gamma, N \vdash^{\mathcal{P};N} [p]}$$

  with $p \in \mathcal{P}; N$,
  and by cut_5 we want to derive either $\Gamma \vdash^{\mathcal{P}} [p]$ or $\Gamma \vdash^{\mathcal{P}}$.
  If N is $\mathcal{P}$-negative
  then $\mathcal{P}; N = \mathcal{P}$ and p is $\mathcal{P}$-positive. So $\mathsf{lit}_{\mathcal{P};N}(\Gamma, N), p^{\perp} = \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp}$ and we build

$$(\mathsf{Init}_1)\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]}$$

If $N \in \mathsf{U}_{\mathcal{P}}$ $(\mathsf{lit}_{\mathcal{P};N}(\Gamma, N), p^{\perp} = \mathsf{lit}_{\mathcal{P}}(\Gamma), N, p^{\perp})$

– if $p = N$ then we build

$$\frac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]}$$

as N is not $\mathcal{P}$-positive;

– if $p \neq N$ then p is $\mathcal{P}$-positive

1. if $\mathsf{lit}_{\mathcal{P}}(\Gamma), N \models_{\mathcal{T}}$
   then applying invertibility of $(\mathsf{Store}^{=})$ on $\Gamma \vdash^{\mathcal{P}} N$ gives $\Gamma, N^{\perp} \vdash^{\mathcal{P}}$ and we build:

$$\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma), N \models_{\mathcal{T}} \quad \Gamma, N^{\perp} \vdash^{\mathcal{P}}}{\Gamma \vdash^{\mathcal{P}}}\,\mathsf{cut}_1$$

2. if $\mathsf{lit}_{\mathcal{P}}(\Gamma), N \not\models_{\mathcal{T}}$
   then $\mathcal{R} := \mathsf{lit}_{\mathcal{P}}(\Gamma), N$ is a set of literals satisfying $\mathsf{lit}_{\mathcal{P}}(\Gamma) \subseteq \mathcal{R} \subseteq \mathsf{lit}_{\mathcal{P}}(\Gamma) \cup \mathsf{U}_{\mathcal{P}}$ (since $N \in \mathsf{U}_{\mathcal{P}}$) and $\mathcal{R}, p^{\perp} \models_{\mathcal{T}}$.
   Hence we get $\mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp} \models_{\mathcal{T}}$ as well, since $(\Gamma, \mathcal{P})$ is assumed to be safe.
   We can finally build

$$(\mathsf{Init}_1)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp} \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [p]}$$

□

Theorem 18 ($\mathbf{cut}_6$, $\mathbf{cut}_7$, and $\mathbf{cut}_8$) The following rules are admissible in $\mathsf{LK}(\mathcal{T})$.

$$\frac{\Gamma \vdash^{\mathcal{P}} N, \Delta \quad \Gamma, N \vdash^{\mathcal{P};N} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_6 \qquad \frac{\Gamma \vdash^{\mathcal{P}} A, \Delta \quad \Gamma \vdash^{\mathcal{P}} A^{\perp}, \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_7 \qquad \frac{\Gamma, l \vdash^{\mathcal{P};l} \Delta \quad \Gamma, l^{\perp} \vdash^{\mathcal{P};l^{\perp}} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_8$$

Proof: cut_6 is proved admissible by induction on the multiset Δ: the base case is the admissibility of cut_4, and the other cases just require the inversion of the connectives in Δ (using $(\mathsf{Store}^{=})$ instead of (Store), to avoid modifying the polarisation set).

For cut_7, we can assume without loss of generality (swapping A and $A^{\perp}$) that A is not $\mathcal{P}$-positive. Applying inversion on $\Gamma \vdash^{\mathcal{P}} A^{\perp}, \Delta$ gives a proof of $\Gamma, A \vdash^{\mathcal{P};A} \Delta$, and cut_7 is then obtained by cut_6:

$$\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta \quad \Gamma, A \vdash^{\mathcal{P};A} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_6$$

cut_8 is obtained as follows:

$$\frac{\dfrac{\Gamma, l^{\perp} \vdash^{\mathcal{P};l^{\perp}} \Delta}{\Gamma \vdash^{\mathcal{P}} l, \Delta} \quad \dfrac{\Gamma, l \vdash^{\mathcal{P};l} \Delta}{\Gamma \vdash^{\mathcal{P}} l^{\perp}, \Delta}}{\Gamma \vdash^{\mathcal{P}} \Delta}\,\mathsf{cut}_7$$

□

6 Changing the polarity of connectives

In this section, we show that changing the polarity of connectives does not change provability in $\mathsf{LK}^p(\mathcal{T})$. To prove this property of the $\mathsf{LK}^p(\mathcal{T})$ system, we genealise it into a new system $\mathsf{LK}^+(\mathcal{T})$.

Definition 9 ($\mathbf{LK}^+(\mathcal{T})$) The sequent calculus $\mathsf{LK}^+(\mathcal{T})$ manipulates one kind of sequent:

$$\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\Delta \qquad \text{where } \mathcal{X} ::= \bullet \mid A$$

Here, $\mathcal{P}$ is a polarisation set, Γ is a multiset of literals and $\mathcal{P}$-negative formulae, Δ is a multiset of formulae, and $\mathcal{X}$ is said to be in the *focus* of the sequent.

The rules of $\mathsf{LK}^+(\mathcal{T})$, given in Figure 2, are again of three kinds: synchronous rules, asynchronous rules, and structural rules. ※

Synchronous rules

$$(\wedge^+)\frac{\Gamma \vdash^{\mathcal{P}} [A]\Delta \qquad \Gamma \vdash^{\mathcal{P}} [B]\Delta}{\Gamma \vdash^{\mathcal{P}} [A\wedge^+ B]\Delta} \qquad (\vee^+)\frac{\Gamma \vdash^{\mathcal{P}} [A_i]\Delta}{\Gamma \vdash^{\mathcal{P}} [A_1\vee^+ A_2]\Delta} \qquad (\exists)\frac{\Gamma \vdash^{\mathcal{P}} [\{\,^t\!/_x\}A]\Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x A]\Delta}$$

$$(\top^+)\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]\Delta} \qquad (\mathsf{Init}_1)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [l]\Delta}\ l \text{ is } \mathcal{P}\text{-positive} \qquad (\mathsf{Release})\frac{\Gamma \vdash^{\mathcal{P}} [\bullet]N}{\Gamma \vdash^{\mathcal{P}} [N]}\ N \text{ not } \mathcal{P}\text{-positive}$$

Asynchronous rules

$$(\wedge^-)\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A,\Delta \qquad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B,\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B,\Delta} \qquad (\vee^-)\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A_1,A_2,\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A_1\vee^- A_2,\Delta} \qquad (\forall)\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A,\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A),\Delta}\ x \notin \mathsf{FV}(\Gamma,\mathcal{X},\Delta,\mathcal{P})$$

$$(\bot^-)\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\bot^-,\Delta} \qquad (\top^-)\frac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\top^-} \qquad (\mathsf{Store})\frac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A,\Delta}\ A \text{ literal or } \mathcal{P}\text{-positive}$$

Structural rules

$$(\mathsf{Select})\frac{\Gamma, P^\perp \vdash^{\mathcal{P}} [P]\Delta}{\Gamma, P^\perp \vdash^{\mathcal{P}} [\bullet]\Delta}\ P \text{ not } \mathcal{P}\text{-negative} \qquad (\mathsf{Init}_2)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [\bullet]\Delta}$$

Figure 2: System $\mathsf{LK}^+(\mathcal{T})$

Remark 19 The $\mathsf{LK}^+(\mathcal{T})$ system is an extension system of $\mathsf{LK}^p(\mathcal{T})$: the $\mathsf{LK}^p(\mathcal{T})$ system is the fragment of $\mathsf{LK}^+(\mathcal{T})$ where every sequent $\Gamma, P^\perp \vdash^{\mathcal{P}} [\bullet]\Delta$ is requested to have either $\mathcal{X} = \bullet$ or Δ is empty. In terms of bottom-up proof-search, this only restricts the structural rules to the case where Δ is empty.

As in $\mathsf{LK}^p(\mathcal{T})$, (left-)weakening and (left-)contraction are height-preserving admissible in $\mathsf{LK}^+(\mathcal{T})$.

We can now prove a new version of identity:

Lemma 20 (Identities) For all $\mathcal{P}$, A, Δ, the sequent $\vdash^{\mathcal{P}} [A^\perp]A,\Delta$ is provable in $\mathsf{LK}^+(\mathcal{T})$.

Proof: By induction on A using an extended but well-founded order on formulae:
a formula is smaller than another one when

- either it contains fewer connectives
- or the number of connectives is equal, neither formulae are literals, and the former formula is negative and the latter is positive.

We now treat all possible shapes for the formula A:

- $A = A_1\wedge^- A_2$

$$\dfrac{\dfrac{\vdash^{\mathcal{P}} [A_1{}^\perp]A_1,\Delta}{\vdash^{\mathcal{P}} [A_1{}^\perp\vee^+ A_2{}^\perp]A_1,\Delta} \qquad \dfrac{\vdash^{\mathcal{P}} [A_2{}^\perp]A_2,\Delta}{\vdash^{\mathcal{P}} [A_1{}^\perp\vee^+ A_2{}^\perp]A_2,\Delta}}{\vdash^{\mathcal{P}} [A_1{}^\perp\vee^+ A_2{}^\perp]A_1\wedge^- A_2,\Delta}$$

  We can complete the proof on the left-hand side by applying the induction hypothesis on A_1 and on the right-hand side by applying the induction hypothesis on A_2.

- $A = A_1\vee^- A_2$

$$\dfrac{\dfrac{\vdash^{\mathcal{P}} [A_1{}^\perp]A_1,A_2,\Delta \quad \vdash^{\mathcal{P}} [A_2{}^\perp]A_1,A_2,\Delta}{\vdash^{\mathcal{P}} [A_1{}^\perp\wedge^+ A_2{}^\perp]A_1,A_2,\Delta}}{\vdash^{\mathcal{P}} [A_1{}^\perp\wedge^+ A_2{}^\perp]A_1\vee^- A_2,\Delta}$$

We can complete the proof on the left-hand side by applying the induction hypothesis on A_1 and on the right-hand side by applying the induction hypothesis on A_2.

- $A = \forall x A$

$$\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} [A^\perp]A, \Delta}{\vdash^{\mathcal{P}} [\{t/x\}A^\perp]A, \Delta}\text{ choosing t=x}}{\vdash^{\mathcal{P}} [\exists x A^\perp]A, \Delta}}{\vdash^{\mathcal{P}} [\exists x A^\perp]\forall x A, \Delta}\, x \notin \mathsf{FV}(\exists x A^\perp, \Delta)$$

We can complete the proof by applying the induction hypothesis on A.

- $A = \bot^-$

$$\dfrac{}{\vdash^{\mathcal{P}} [\top^+]\bot -, \Delta}\,\top^+$$

- $A = p^\perp$, with p not being $\mathcal{P}$-negative:

$$\dfrac{\dfrac{}{p \vdash^{\mathcal{P};p} [p]\Delta}}{\vdash^{\mathcal{P}} [p]p^\perp, \Delta}$$

as p is then $\mathcal{P};p$-positive.

- $A = P$ where P is $\mathcal{P}$-positive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} [P]P^\perp}{P^\perp \vdash^{\mathcal{P}'} [P]P^\perp}}{P^\perp \vdash^{\mathcal{P}'} [\bullet]P^\perp}}{P^\perp \vdash^{\mathcal{P}'} [P^\perp]}}{P^\perp, \Delta^\perp \vdash^{\mathcal{P}'} [P^\perp]}}{P^\perp \vdash^{\mathcal{P}} [P^\perp]\Delta}}{\vdash^{\mathcal{P}} [P^\perp]P, \Delta}$$

If P is a literal, we complete the proof with the case just above. If it is not a literal, then P is smaller than $P^\perp$ and we complete the proof by applying the induction hypothesis on P.

□

We now want to show that all asynchronous rules are invertible in $\mathsf{LK}^+(\mathcal{T})$. We first start with the following lemma:

Lemma 21 (Generalised (Init) and negative Select)
The following rules are height-preserving admissible in $\mathsf{LK}^+(\mathcal{T})$:

$$(\mathsf{Init})\dfrac{\mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\Delta} \qquad (\mathsf{Select}^-)\dfrac{\Gamma \vdash^{\mathcal{P};l^\perp} [l]\Delta}{\Gamma \vdash^{\mathcal{P};l^\perp} [\bullet]\Delta}$$

where $l^\perp \in \Gamma$ and it is not $\mathcal{P}$-negative in (Select^-).

Proof: For each rule, by induction on the proof of the premiss.
For (Init):

- if it is obtained by $(\wedge^-), (\vee^-), (\forall), (\bot^-)$, we can straightforwardly use the induction hypothesis on the premiss(es), and if it is $(\top^-)$ it is trivial;
- if it is obtained by

$$\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]\Delta'}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \Delta'}$$

then we can use the induction hypothesis on the premiss as $\mathsf{lit}_{\mathcal{P};A^\perp}(\Gamma, A^\perp), \mathsf{lit}_{\mathcal{L}}(\Delta'^\perp) = \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(A^\perp, \Delta'^\perp)$;

- the last possible way to obtain it is with $\Delta = \emptyset$ and

$$\frac{\Gamma \vdash^{\mathcal{P}} [\bullet]N}{\Gamma \vdash^{\mathcal{P}} [N]}$$

for some N that is not $\mathcal{P}$-positive, and we conclude with (Init_2).

For (Select^-), first notice that l is $\mathcal{P};l^\perp$-negative, and then:

- if again it is obtained by $(\wedge^-), (\vee^-), (\forall), (\bot^-)$, we can straightforwardly use the induction hypothesis on the premiss(es), and if it is $(\top^-)$ it is trivial;
- if it is obtained by

$$\frac{\Gamma, A^\perp \vdash^{\mathcal{P};l^\perp;A^\perp} [l]\Delta'}{\Gamma \vdash^{\mathcal{P};l^\perp} [l]A, \Delta'}$$

then we can use the induction hypothesis on the premiss, if A is not $l^\perp$ (so that $\mathcal{P};l^\perp;A^\perp = \mathcal{P};A^\perp;l^\perp$ and $l^\perp$ is not $\mathcal{P};A^\perp$-negative); if $A = l^\perp$, then we build

$$(\mathsf{Init}_2)\frac{\mathsf{lit}_{\mathcal{P};l^\perp}(\Gamma), \mathsf{lit}_{\mathcal{L}}(A^\perp, {\Delta'}^\perp) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P};l^\perp} [\bullet]A, \Delta'}$$

as $A \in \mathsf{lit}_{\mathcal{P};l^\perp}(\Gamma)$.
- the last possible way to obtain it is with $\Delta = \emptyset$ and

$$\cfrac{\cfrac{\Gamma, l^\perp \vdash^{\mathcal{P};l^\perp} [\bullet]}{\Gamma \vdash^{\mathcal{P};l^\perp} [\bullet]l}}{\Gamma \vdash^{\mathcal{P};l^\perp} [l]}$$

and we conclude with the height-preserving admissibility of contraction.

□

We can now state and prove the invertibility of asynchronous rules:

Lemma 22 (Invertibility of asynchronous rules)
All asynchronous rules are height-preserving invertible in $\mathsf{LK}^+(\mathcal{T})$.

Proof: By induction on the derivation proving the conclusion of the asynchronous rule considered.

- Inversion of $A\wedge^- B$: by case analysis on the last rule actually used
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, C\wedge^- D, \Delta}$$
    By induction hypothesis we get
    $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C\wedge^- D, \Delta} \text{ and } \frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, C\wedge^- D, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, C\vee^- D, \Delta}$$
    By induction hypothesis we get
    $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C\vee^- D, \Delta} \text{ and } \frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, C\vee^- D, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, C, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, (\forall x C), \Delta}\, x \notin \mathsf{FV}(\Gamma, \mathcal{X}, \Delta, A\wedge^- B)$$
    By induction hypothesis we get
    $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, (\forall x C), \Delta}\, x \notin \mathsf{FV}(\Gamma, \mathcal{X}, \Delta, A) \text{ and } \frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, C, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, (\forall x C), \Delta}\, x \notin \mathsf{FV}(\Gamma, \mathcal{X}, \Delta, B)$$
  - $$\frac{\Gamma, C^\perp \vdash^{\mathcal{P};C^\perp} [\mathcal{X}]A\wedge^- B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\wedge^- B, C, \Delta}\, C \text{ literal or } \mathcal{P}\text{-positive}$$
    By induction hypothesis we get

$$\dfrac{\Gamma, C^{\perp} \vdash^{\mathcal{P}} [\mathcal{X}]A, \Delta}{\Gamma \vdash^{\mathcal{P};C^{\perp}} [\mathcal{X}]A, C, \Delta}\ C \text{ literal or } \mathcal{P}\text{-positive} \quad \text{and} \quad \dfrac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} [\mathcal{X}]B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, C, \Delta}\ C \text{ literal or } \mathcal{P}\text{-positive}$$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A \wedge^{-} B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A \wedge^{-} B, \bot^{-}, \Delta}$

By induction hypothesis we get

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \bot^{-}, \Delta} \quad \text{and} \quad \dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, \bot^{-}, \Delta}$$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A \wedge^{-} B, \top^{-}, \Delta}$

We get

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \top^{-}, \Delta} \quad \text{and} \quad \dfrac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, \top^{-}, \Delta}$$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [C]A \wedge^{-} B, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]A \wedge^{-} B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C \wedge^{+} D,]A \wedge^{-} B, \Delta}$

By induction hypothesis we get

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [C]A, \Delta \qquad \Gamma \vdash^{\mathcal{P}} [D]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [C \wedge^{+} D]A, \Delta} \quad \text{and} \quad \dfrac{\Gamma \vdash^{\mathcal{P}} [C]B, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C \wedge^{+} D]B, \Delta}$$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [C_i]A \wedge^{-} B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1 \vee^{+} C_2]A \wedge^{-} B, \Delta}$

By induction hypothesis we get

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [C_i]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1 \vee^{+} C_2]A, \Delta} \quad \text{and} \quad \dfrac{\Gamma \vdash^{\mathcal{P}} [C_i]B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1 \vee^{+} C_2]B, \Delta}$$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} C]A \wedge^{-} B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x C]A \wedge^{-} B, \Delta}$

By induction hypothesis we get

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} C]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x C]A, \Delta} \quad \text{and} \quad \dfrac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} C]B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x C]B, \Delta}$$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^{+}]A \wedge^{-} B, \Delta}$

We get

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^{+}]A, \Delta} \quad \text{and} \quad \dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^{+}]B, \Delta}$$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]A \wedge^{-} B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp}, \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$ with p being $\mathcal{P}$-positive

We get

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]A, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp}, \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}} \quad \text{and} \quad \dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp}, \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]A \wedge^{-} B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$

We get

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}} \quad \text{and} \quad \dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [P]A \wedge^{-} B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]A \wedge^{-} B, \Delta}$ where $P^{\perp} \in \Gamma$ is not $\mathcal{P}$-positive

By induction hypothesis we get

$$\dfrac{\Gamma \vdash^{\mathcal{P}} [P]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta} \quad \text{and} \quad \dfrac{\Gamma \vdash^{\mathcal{P}} [P]B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]B, \Delta}$$

- Inversion of $A\vee^- B$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, C\wedge^- D, \Delta}$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, C\wedge^- D, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, C\vee^- D, \Delta}$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, C\vee^- D, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, C, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, (\forall x C), \Delta}\ x \notin \mathsf{FV}(\Gamma, \mathcal{X}, A\vee^- B, \Delta)$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, C, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}], A, B, (\forall x C), \Delta}\ x \notin \mathsf{FV}(\Gamma, \mathcal{X}, A, B, \Delta)$$
  - $$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} [\mathcal{X}]A\vee^- B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, C, \Delta}$$ C literal or $\mathcal{P}$-positive
    By induction hypothesis we get $$\frac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} [\mathcal{X}]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, C, \Delta}$$ C literal or $\mathcal{P}$-positive
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, \perp^-, \Delta}$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, \perp^-, \Delta}$$
  - $$\frac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A\vee^- B, \top^-, \Delta}$$
    We get $$\frac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, \top^-, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [C]A\vee^- B, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]A\vee^- B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C\wedge^+ D]A\vee^- B, \Delta}$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [C]A, B, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C\wedge^+ D]A, B, C\wedge^- D, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [C_i]A\vee^- B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1\vee^+ C_2]A\vee^- B, \Delta}$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [C_i]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1\vee^+ C_2]A, B, \Delta}$$
  - $$\frac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/_{x}\right\}C]A\vee^- B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x C]A\vee^- B, \Delta}$$
    By induction hypothesis we get $$\frac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/_{x}\right\}C]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x C]A, B, \Delta}$$
  - $$\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]A\vee^- B, \Delta}$$
    We get $$\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]A, B, \Delta}$$
  - $$\frac{}{\Gamma \vdash^{\mathcal{P}} [p]A\vee^- B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp}, \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$$ with p being $\mathcal{P}$-positive
    We get $$\frac{}{\Gamma \vdash^{\mathcal{P}} [p]A, B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp}, \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$$
  - $$\frac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]A\vee^- B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$$

We get $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, B, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [P]A{\vee^-}B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]A{\vee^-}B, \Delta}$ where $P^{\perp} \in \Gamma$ is not $\mathcal{P}$-positive

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [P]A, B, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, B, \Delta}$

- Inversion of $\forall x A$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), C{\wedge^-}D, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C{\wedge^-}D, \Delta}\ x \notin \mathsf{FV}(\Gamma, \mathcal{X}, \Delta)$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), C{\vee^-}D, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C{\vee^-}D, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), (\forall y D), \Delta}\ y \notin \mathsf{FV}(\Gamma, \mathcal{X}, (\forall x A), \Delta)$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, (\forall y D), \Delta}\ y \notin \mathsf{FV}(\Gamma, \mathcal{X}, A, \Delta)$

– $\dfrac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} [\mathcal{X}](\forall x A), \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), C, \Delta}$ C literal or $\mathcal{P}$-positive

By induction hypothesis we get $\dfrac{\Gamma, C^{\perp} \vdash^{\mathcal{P};C^{\perp}} [\mathcal{X}]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, \Delta}$ C literal or $\mathcal{P}$-positive

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), \bot^-, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \bot^-, \Delta}$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x A), \top^-, \Delta}$

We get $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \top^-, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [C](\forall x A), \Delta \quad \Gamma \vdash^{\mathcal{P}} [D](\forall x A), \Delta}{\Gamma \vdash^{\mathcal{P}} [C{\wedge^+}D](\forall x A), \Delta}$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [C]A, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [C{\wedge^+}D]A, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [C_i](\forall x A), \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1{\vee^+}C_2](\forall x A), \Delta}$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [C_i]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1{\vee^+}C_2]A, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} D](\forall x A), \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x D](\forall x A), \Delta}$

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} D]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x D]A, \Delta}$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^+](\forall x A), C, \Delta}$

We get $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]A, \Delta}$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p](\forall x A), \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$ with p being $\mathcal{P}$-positive

We get $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]A, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet](\forall x A), \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$

We get $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [P](\forall x A), \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet](\forall x A), \Delta}$ where $P^\perp \in \Gamma$ is not $\mathcal{P}$-positive

By induction hypothesis we get $\dfrac{\Gamma \vdash^{\mathcal{P}} [P]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta}$

- Inversion of storing a literal or $\mathcal{P}$-positive formulae A

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C \wedge^- D, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]C, \Delta \quad \Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]D, \Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]C \wedge^- D, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, C \vee^- D, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]C, D, \Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]C \vee^- D, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, (\forall x D), \Delta}\ x \notin \mathsf{FV}(\Gamma, \mathcal{X}, A, \Delta)$

By induction hypothesis we get $\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]D, \Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}](\forall x D), \Delta}\ x \notin \mathsf{FV}(\Gamma, A^\perp, \mathcal{X}, \Delta)$

– $\dfrac{\Gamma, B^\perp \vdash^{\mathcal{P};B^\perp} [\mathcal{X}]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, B, \Delta}$ B literal or $\mathcal{P}$-positive

We build $\dfrac{\Gamma, A^\perp, B^\perp \vdash^{\mathcal{P};A^\perp;B^\perp} [\mathcal{X}]\Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]B, \Delta}$ B literal or $\mathcal{P}$-positive

proving the premiss using the induction hypothesis in case $\mathcal{P}; B^\perp; A^\perp = \mathcal{P}; A^\perp; B^\perp$, which holds unless $A = B^\perp$ and $A \in \mathsf{U}_{\mathcal{P}}$.
In that case we have $\mathcal{P}; A^\perp = \mathcal{P}, A^\perp$, and we prove $\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]B, \Delta$ with (Init) (Lemma 21), as $\mathsf{lit}_{\mathcal{P};A^\perp}(\Gamma, A^\perp), \mathsf{lit}_{\mathcal{L}}(B^\perp, \Delta^\perp) \models_{\mathcal{T}}$.

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \bot^-, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]\Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]\bot^-, \Delta}$

– $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]A, \top^-, \Delta}$

We get $\dfrac{}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\mathcal{X}]\top^-, \Delta}$

– $\dfrac{\Gamma \vdash^{\mathcal{P}} [C]A, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [C \wedge^+ D]A, \Delta}$

By induction hypothesis we get $\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P}} [C]\Delta \quad \Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [D]\Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [C \wedge^+ D]\Delta}$

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [C_i]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [C_1 \vee^+ C_2]A, \Delta}$$

By induction hypothesis we get $$\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [C_i]\Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [C_1 \vee^+ C_2]\Delta}$$

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\{\not{t}_x\} D]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x D]A, \Delta}$$

By induction hypothesis we get $$\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\{\not{t}_x\} D]\Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\exists x D]\Delta}$$

– $$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]A, \Delta}$$

We get $$\dfrac{}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\top^+]\Delta}$$

– $$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]A, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^\perp, \mathsf{lit}_{\mathcal{L}}(A^\perp, \Delta^\perp) \models_{\mathcal{T}}$$ with p being $\mathcal{P}$-negative

We get $$\dfrac{}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [p]\Delta}\ \mathsf{lit}_{\mathcal{P};A^\perp}(\Gamma, A^\perp), p^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$$

as p is also $\mathcal{P}; A^\perp$-positive.

– $$\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(A^\perp, \Delta^\perp) \models_{\mathcal{T}}$$

We get $$\dfrac{}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\bullet]\Delta}\ \mathsf{lit}_{\mathcal{P};A^\perp}(\Gamma, A^\perp), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$$

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [P]A, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta}$$ where $P^\perp \in \Gamma$ is not $\mathcal{P}$-positive

By induction hypothesis we get $$\dfrac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [P]\Delta}{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} [\bullet]\Delta}$$

using either (Select) or (Select^-) depending on whether P is $\mathcal{P}; A^\perp$-negative.

- Inversion of ($\perp^-$)

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, C \wedge^- D, \Delta}$$

By induction hypothesis we get $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]C, \Delta \quad \Gamma \vdash^{\mathcal{P}} [\mathcal{X}]D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]C \wedge^- D, \Delta}$$

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, C \vee^- D, \Delta}$$

By induction hypothesis we get $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]C, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]C \vee^- D, \Delta}$$

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, (\forall x D), \Delta}\ x \notin \mathsf{FV}(\Gamma, \mathcal{X}, \Delta)$$

By induction hypothesis we get $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]D, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}](\forall x D), \Delta}\ x \notin \mathsf{FV}(\Gamma, \mathcal{X}, \Delta)$$

– $$\dfrac{\Gamma, B^\perp \vdash^{\mathcal{P};B^\perp} [\mathcal{X}]\perp^-, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, B, \Delta}$$ B literal or $\mathcal{P}$-positive

By induction hypothesis we get $$\dfrac{\Gamma, B^\perp \vdash^{\mathcal{P};B^\perp} [\mathcal{X}]\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]B, \Delta}$$ B literal or $\mathcal{P}$-positive

– $$\dfrac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, \Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\perp^-, \perp^-, \Delta}$$

By induction hypothesis we get
$$\frac{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\Delta}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\bot^-, \Delta}$$

– $\frac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\bot^-, \top^-, \Delta}$

We get
$$\frac{}{\Gamma \vdash^{\mathcal{P}} [\mathcal{X}]\top^-, \Delta}$$

– $\frac{\Gamma \vdash^{\mathcal{P}} [C]\bot^-, \Delta \quad \Gamma \vdash^{\mathcal{P}} [D]\bot^-, \Delta}{\Gamma \vdash^{\mathcal{P}} [C\wedge^+ D]\bot^-, \Delta}$

By induction hypothesis we get
$$\frac{\Gamma \vdash^{\mathcal{P}} [C]\Delta \quad \Gamma \vdash^{\mathcal{P}} [D]\Delta}{\Gamma \vdash^{\mathcal{P}} [C\wedge^+ D]\Delta}$$

– $\frac{\Gamma \vdash^{\mathcal{P}} [C_i]\Delta}{\Gamma \vdash^{\mathcal{P}} [C_1\vee^+ C_2]\bot^-, \Delta}$

By induction hypothesis we get
$$\frac{\Gamma \vdash^{\mathcal{P}} [C_i]\Delta}{\Gamma \vdash^{\mathcal{P}} [C_1\vee^+ C_2]\Delta}$$

– $\frac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} D]\bot^-, \Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x D]\bot^-, \Delta}$

By induction hypothesis we get
$$\frac{\Gamma \vdash^{\mathcal{P}} [\left\{{}^{t}\!/\!_{x}\right\} D]\Delta}{\Gamma \vdash^{\mathcal{P}} [\exists x D]\Delta}$$

– $\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]\bot^-, \Delta}$

We get
$$\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]\Delta}$$

– $\frac{}{\Gamma \vdash^{\mathcal{P}} [p]\bot^-, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$ with p being $\mathcal{P}$-positive

By induction hypothesis we get
$$\frac{}{\Gamma, A^\perp \vdash^{\mathcal{P}} [p]\Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^\perp, \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$$

– $\frac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]\bot^-, \Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$

By induction hypothesis we get
$$\frac{}{\Gamma, A^\perp \vdash^{\mathcal{P}} [\bullet]\Delta}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^\perp) \models_{\mathcal{T}}$$

– $\frac{\Gamma \vdash^{\mathcal{P}} [P]\bot^-, \Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]\bot^-, \Delta}$ where $P^\perp \in \Gamma$ is not $\mathcal{P}$-positive

By induction hypothesis we get
$$\frac{\Gamma \vdash^{\mathcal{P}} [P]\Delta}{\Gamma \vdash^{\mathcal{P}} [\bullet]\Delta}$$

- Inversion of $\top^-$: nothing to do.

□

Now that we have proved the invertibility of asynchronous rules, we can use it to transform any proof of $\mathsf{LK}^+(\mathcal{T})$ into a proof of $\mathsf{LK}^p(\mathcal{T})$.

Lemma 23 (Encoding $\mathsf{LK}^+(\mathcal{T})$ in $\mathsf{LK}^p(\mathcal{T})$)

1. If $\Gamma \vdash^{\mathcal{P}} [A]$ is provable in $\mathsf{LK}^+(\mathcal{T})$, then $\Gamma \vdash^{\mathcal{P}} [A]$ is provable in $\mathsf{LK}^p(\mathcal{T})$.
2. If $\Gamma \vdash^{\mathcal{P}} [\bullet]\Delta$ is provable in $\mathsf{LK}^+(\mathcal{T})$, then $\Gamma \vdash^{\mathcal{P}} \Delta$ is provable in $\mathsf{LK}^p(\mathcal{T})$.

Proof: By simultaneous induction on the assumed derivation.

1. For the first item we get, by case analysis on the last rule of the derivation:
   - $\frac{\Gamma \vdash^{\mathcal{P}} [A_1] \quad \Gamma \vdash^{\mathcal{P}} [A_2]}{\Gamma \vdash^{\mathcal{P}} [A_1\wedge^+ A_2]}$ with $A = A_1\wedge^+ A_2$.

     The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [A_1]$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} [A_1]$ and the induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [A_2]$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} [A_2]$. We get:

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_1] \quad \Gamma \vdash^{\mathcal{P}} [A_2]}{\Gamma \vdash^{\mathcal{P}} [A_1 \wedge^+ A_2]}$$

- $\dfrac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]}$ with $A = A_1 \vee^+ A_2$.
  The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [A_i]$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} [A_i]$. We get:

$$\frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]}$$

- $\dfrac{\Gamma \vdash^{\mathcal{P}} [\{t/x\}A]}{\Gamma \vdash^{\mathcal{P}} [\exists x A]}$ with $A = \exists x A$.
  The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\{t/x\}A]$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} [\{t/x\}A]$. We get:

$$\frac{\Gamma \vdash^{\mathcal{P}} [\{t/x\}A]}{\Gamma \vdash^{\mathcal{P}} [\exists x A]}$$

- $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [p]}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp} \models_{\mathcal{T}}$ with $A = p$ where p is a $\mathcal{P}$-positive literal.
  We can perform the same step in $\mathsf{LK}^p(\mathcal{T})$:

$$\frac{}{\Gamma \vdash^{\mathcal{P}} [p]}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), p^{\perp} \models_{\mathcal{T}}$$

- $\dfrac{\Gamma \vdash^{\mathcal{P}} [\bullet]N}{\Gamma \vdash^{\mathcal{P}} [N]}$ with $A = N$ and N is not $\mathcal{P}$-positive.
  The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]N$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} N$. We get:

$$\frac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]}$$

2. For the second item, we use the height-preserving invertibility of the asynchronous rules, so that we can assume without loss of generality that if Δ is not empty then the last rule of the derivation decomposes one of its formulae.
   - $\dfrac{\Gamma \vdash^{\mathcal{P}} [\bullet]A_1, \Delta_1 \quad \Gamma \vdash^{\mathcal{P}} [\bullet]A_2, \Delta_1}{\Gamma \vdash^{\mathcal{P}} [\bullet]A_1 \wedge^- A_2, \Delta_1}$ with $\Delta = A_1 \wedge^- A_2, \Delta_1$.
     The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]A_1, \Delta_1$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A_1, \Delta_1$ and the induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]A_2, \Delta_2$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A_2, \Delta_2$. We get:

$$\frac{\Gamma \vdash^{\mathcal{P}} A_1, \Delta_1 \quad \Gamma \vdash^{\mathcal{P}} A_2, \Delta_1}{\Gamma \vdash^{\mathcal{P}} A_1 \wedge^- A_2, \Delta_1}$$

   - $\dfrac{\Gamma \vdash^{\mathcal{P}} [\bullet]A_1, A_2, \Delta_1}{\Gamma \vdash^{\mathcal{P}} [\bullet]A_1 \vee^- A_2, \Delta_1}$ with $\Delta = A_1 \vee^- A_2, \Delta_1$.
     The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]A_1, A_2, \Delta_1$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A_1, A_2, \Delta_1$ and we get:

$$\frac{\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta_1}{\Gamma \vdash^{\mathcal{P}} A_1 \vee^- A_2, \Delta_1}$$

- $\dfrac{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta_1}{\Gamma \vdash^{\mathcal{P}} [\bullet]\forall x A, \Delta_1}\, x \notin \mathsf{FV}(\Gamma, \Delta_1)$ with $\Delta = \forall x A, \Delta_1$.
The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]A, \Delta_1$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A, \Delta_1$. We get:

$$\dfrac{\Gamma \vdash^{\mathcal{P}} A, \Delta_1}{\Gamma \vdash^{\mathcal{P}} \forall x A, \Delta_1}\, x \notin \mathsf{FV}(\Gamma, \Delta_1)$$

- $\dfrac{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} [\bullet]\Delta_1}{\Gamma \vdash^{\mathcal{P}} [\bullet]A, \Delta_1}$ with $\Delta = A, \Delta_1$ and A is a literal or is $\mathcal{P}$-positive.
The induction hypothesis on $\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]\Delta_1$ gives $\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}}_{\mathsf{LK}^p(\mathcal{T})} \Delta_1$. We get:

$$\dfrac{\Gamma, A^{\perp} \vdash^{\mathcal{P};A^{\perp}} \Delta_1}{\Gamma \vdash^{\mathcal{P}} A, \Delta_1}$$

- $\dfrac{\Gamma \vdash^{\mathcal{P}} [\bullet]\Delta_1}{\Gamma \vdash^{\mathcal{P}} [\bullet]\bot^-, \Delta_1}$ with $\Delta = \bot^-, \Delta_1$.
The induction hypothesis on $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [\bullet]\Delta_1$ gives $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} \Delta_1$. We get:

$$\dfrac{\Gamma \vdash^{\mathcal{P}} \Delta_1}{\Gamma \vdash^{\mathcal{P}} \bot^-, \Delta_1}$$

- $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]\top^-, \Delta_1}$ with $\Delta = \top^-, \Delta_1$.
We get:

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}} \top^-, \Delta_1}$$

- $\dfrac{\Gamma, P^{\perp} \vdash^{\mathcal{P}} [P]\Delta}{\Gamma, P^{\perp} \vdash^{\mathcal{P}} [\bullet]\Delta}$ where P is not $\mathcal{P}$-negative.
As already mentioned, we can assume without loss of generality that Δ is empty. The induction hypothesis on $\Gamma, P^{\perp} \vdash^{\mathcal{P}}_{\mathsf{LK}^+(\mathcal{T})} [P]$ gives $\Gamma, P^{\perp} \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} [P]$. We get:

$$\dfrac{\Gamma, P^{\perp} \vdash^{\mathcal{P}} [P]}{\Gamma, P^{\perp} \vdash^{\mathcal{P}}}$$

- $\dfrac{}{\Gamma \vdash^{\mathcal{P}} [\bullet]\Delta}\, \mathsf{lit}_{\mathcal{P}}(\Gamma), \mathsf{lit}_{\mathcal{L}}(\Delta^{\perp}) \models_{\mathcal{T}}$
As already mentioned, we can assume without loss of generality that Δ is empty. We get:

$$\dfrac{}{\Gamma \vdash^{\mathcal{P}}}\, \mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}$$

□

Lemma 24 We have:

1. $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} {\top^+}^{\perp}, \top^-$, and
2. $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} {\top^-}^{\perp}, \top^+$, and
3. $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} (A \wedge^+ B)^{\perp}, (A \wedge^- B)$, and
4. $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} (A \wedge^- B)^{\perp}, (A \wedge^+ B)$, provided that sequent is safe.

Proof:

1. For the first item we get:

$$\dfrac{}{\vdash^{\mathcal{P}} {\top^+}^{\perp}, \top^-}$$

2. For the second item we get:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\top^-, \top^{+\perp} \vdash^{\mathcal{P}} [\top^+]}}{\top^-, \top^{+\perp} \vdash^{\mathcal{P}}}}{\top^- \vdash^{\mathcal{P}} \top^+}}{\vdash^{\mathcal{P}} \top^{-\perp}, \top^+}$$

3. For the third item we get:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P};A} [A^\perp]B^\perp, A}{A \vdash^{\mathcal{P};A} [A^\perp]B^\perp, A}}{A \vdash^{\mathcal{P};A} [\bullet]B^\perp, A}}{\vdash^{\mathcal{P}} [\bullet]A^\perp, B^\perp, A} \qquad \dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P};B} [B^\perp]A^\perp, B}{B \vdash^{\mathcal{P};B} [B^\perp]A^\perp, B}}{B \vdash^{\mathcal{P};B} [\bullet]A^\perp, B}}{\vdash^{\mathcal{P}} [\bullet]A^\perp, B^\perp, B}}{\vdash^{\mathcal{P}} [\bullet](A^\perp \vee^- B^\perp), (A \wedge^- B)}}{\vdash^{\mathcal{P}} [\bullet](A \wedge^+ B)^\perp, (A \wedge^- B)}}{\vdash^{\mathcal{P}} (A \wedge^+ B)^\perp, (A \wedge^- B)}\;\text{Lemma 23(2)}$$

Both left hand side and right hand side can be closed by Lemma 20.

4. For the fourth item, we get:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} [A^\perp]A}{\vdash^{\mathcal{P}} [A^\perp \vee^+ B^\perp]A}}{A \wedge^- B \vdash^{\mathcal{P}} [A^\perp \vee^+ B^\perp]A}}{A \wedge^- B \vdash^{\mathcal{P}} [\bullet]A}}{A \wedge^- B \vdash^{\mathcal{P}} A}\;\text{Lemma 23(2)}}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} [B^\perp]B}{\vdash^{\mathcal{P}} [A^\perp \vee^+ B^\perp]B}}{A \wedge^- B \vdash^{\mathcal{P}} [A^\perp \vee^+ B^\perp]B}}{A \wedge^- B \vdash^{\mathcal{P}} [\bullet]B}}{A \wedge^- B \vdash^{\mathcal{P}} B}\;\text{Lemma 23(2)}}{\dfrac{A \wedge^- B \vdash^{\mathcal{P}} A^\perp, B}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A^\perp, B}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} [A]A^\perp, B^\perp \quad \vdash^{\mathcal{P}} [B]A^\perp, B^\perp}{\vdash^{\mathcal{P}} [A \wedge^+ B]A^\perp, B^\perp}}{A^\perp \vee^- B^\perp \vdash^{\mathcal{P}} [A \wedge^+ B]A^\perp, B^\perp}}{\dfrac{A^\perp \vee^- B^\perp \vdash^{\mathcal{P}} [\bullet]A^\perp, B^\perp}{A^\perp \vee^- B^\perp \vdash^{\mathcal{P}} A^\perp, B^\perp}\;\text{Lemma 23(2)}}}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A^\perp, B^\perp}$$

$$\dfrac{\dfrac{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A \qquad \dfrac{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A^\perp, B \qquad (A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A^\perp, B^\perp}{(A \wedge^- B), (A^\perp \vee^- B^\perp) \vdash^{\mathcal{P}} A^\perp}\;\mathsf{cut}_7}{(A \wedge^- B), (A \wedge^+ B)^\perp \vdash^{\mathcal{P}}}\;\mathsf{cut}_7}{\vdash^{\mathcal{P}} (A \wedge^- B)^\perp, (A \wedge^+ B)}$$

All branches are closed by Lemma 20. □

Lemma 25

If $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} \Delta, C$ and $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} D, C^\perp$ then $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} \Delta, D$, provided that sequent is safe.

Proof:

$$\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} \Delta, C}{\Gamma \vdash^{\mathcal{P}} D, \Delta, C} \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} D, C^\perp}{\Gamma \vdash^{\mathcal{P}} \Delta, D, C^\perp}}{\Gamma \vdash^{\mathcal{P}} \Delta, D}\;\mathsf{cut}_7$$

□

Corollary 26 (Changing the polarity of connectives) Provided those sequents are safe,

1. If $\Gamma \vdash^{\mathcal{P}} \top^+, \Delta$ then $\Gamma \vdash^{\mathcal{P}} \top^-, \Delta$;
2. If $\Gamma \vdash^{\mathcal{P}} \top^-, \Delta$ then $\Gamma \vdash^{\mathcal{P}} \top^+, \Delta$;
3. If $\Gamma \vdash^{\mathcal{P}} \perp^+, \Delta$ then $\Gamma \vdash^{\mathcal{P}} \perp^-, \Delta$;
4. If $\Gamma \vdash^{\mathcal{P}} \perp^-, \Delta$ then $\Gamma \vdash^{\mathcal{P}} \perp^+, \Delta$;
5. If $\Gamma \vdash^{\mathcal{P}} A \wedge^+ B, \Delta$ then $\Gamma \vdash^{\mathcal{P}} A \wedge^- B, \Delta$;
6. If $\Gamma \vdash^{\mathcal{P}} A \wedge^- B, \Delta$ then $\Gamma \vdash^{\mathcal{P}} A \wedge^+ B, \Delta$;

7. If $\Gamma \vdash^{\mathcal{P}} A \vee^{+} B, \Delta$ then $\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, \Delta$;
8. If $\Gamma \vdash^{\mathcal{P}} A \vee^{-} B, \Delta$ then $\Gamma \vdash^{\mathcal{P}} A \vee^{+} B, \Delta$.

Furthermore, notice that in each implication, the safety of one sequent implies the safety of the other.

Proof:

1. By Lemma 25 and Lemma 24(1).
2. By Lemma 25 and Lemma 24(2).
3. By Lemma 25 and Lemma 24(1).
4. By Lemma 25 and Lemma 24(2).
5. By Lemma 25 and Lemma 24(3).
6. By Lemma 25 and Lemma 24(4).
7. By Lemma 25 and Lemma 24(3).
8. By Lemma 25 and Lemma 24(4).

□

We have proven that changing the polarities of the connectives that are present in a sequent, does not change the provability of that sequent in $\mathsf{LK}^p(\mathcal{T})$.

7 Completeness

$\mathsf{LK}(\mathcal{T})$ is a complete system for first-order logic modulo a theory. To show this, we review the grammar of first-order formulae and map those formulae to polarised formulae.

Definition 10 (Plain formulae) Let P_Σ^a be a sub-signature of the first-order predicate signature P_Σ such that for every predicate symbol P/n of P_Σ, P/n is in P_Σ^a if and only if $P^\perp/n$ is not in P_Σ^a.

Let $\mathcal{A}$ be the subset of $\mathcal{L}$ consisting of those literals whose predicate symbols are in P_Σ^a. Literals in $\mathcal{A}$, denoted a, a', etc, are called *atoms*.

The formulae of first-order logic, here called *plain formulae*, are given by the following grammar:

$$\text{Plain formulae} \quad A, B, \ldots \quad ::= \; a \mid A \vee B \mid A \wedge B \mid \forall x A \mid \exists x A \mid \neg A$$

where a ranges over atoms. ※

Definition 11 (ψ) Let ψ be the function that maps every plain formula to a set of formulae (in the sense of Definition 4) defined as follows:

$$\begin{array}{lcr}
\psi(a) & := & \{a\} \\
\psi(A \wedge B) & := & \{A' \wedge^{-} B', A' \wedge^{+} B' \mid A' \in \psi(A), B' \in \psi(B)\} \\
\psi(A \vee B) & := & \{A' \vee^{-} B', A' \vee^{+} B' \mid A' \in \psi(A), B' \in \psi(B)\} \\
\psi(\exists x A) & := & \{\exists x A' \mid A' \in \psi(A)\} \\
\psi(\forall x A) & := & \{\forall x A' \mid A' \in \psi(A)\} \\
\psi(\neg A) & := & \{A'^{\perp} \mid A' \in \psi(A)\} \\
\psi(\Delta, A) & := & \{\Delta', A' \mid \Delta' \in \psi(\Delta), A' \in \psi(A)\} \\
\psi(\emptyset) & := & \emptyset
\end{array}$$

※

Remark 27
1. $\psi(A) \neq \emptyset$
2. If $A' \in \psi(A)$, then $\left\{{}^{t}\!/\!{}_{x}\right\} A' \in \psi(\left\{{}^{t}\!/\!{}_{x}\right\} A')$.
3. If $C' \in \psi(\left\{{}^{t}\!/\!{}_{x}\right\} A)$, then $C' = \left\{{}^{t}\!/\!{}_{x}\right\} A'$ for some $A' \in \psi(A)$.

Notation 12 When F is a plain formula and Ψ is a set of plain formulae, $\Psi \models F$ means that Ψ entails F in first-order classical logic.

Given a theory $\mathcal{T}$ (given by a semantical inconsistency predicate), we define the set of all *theory lemmas* as

$$\Psi_{\mathcal{T}} := \{l_1 \vee \cdots \vee l_n \mid \psi(l_1)^{\perp}, \cdots, \psi(l_n)^{\perp} \models_{\mathcal{T}}\}$$

We generalise the notation $\models_{\mathcal{T}}$ to write $\Psi \models_{\mathcal{T}} F$ when $\Psi_{\mathcal{T}}, \Psi \models F$, in which case we say that F is a *semantical consequence* of Ψ.

Notation 13 In the rest of this section we will use the notation $A \wedge^? B$ (resp. $A \vee^? B$) to ambiguously represent either $A \wedge^+ B$ or $A \wedge^- B$ (resp. $A \vee^+ B$ or $A \vee^- B$). This will make the proofs more compact, noticing that Corollary 26(2) and 26(4) respectively imply the admissibility in $\mathsf{LK}^p(\mathcal{T})$ of

$$\frac{\Gamma \vdash^{\mathcal{P}} \Delta, A \wedge^- B}{\Gamma \vdash^{\mathcal{P}} \Delta, A \wedge^? B} \qquad \frac{\Gamma \vdash^{\mathcal{P}} \Delta, A \vee^- B}{\Gamma \vdash^{\mathcal{P}} \Delta, A \vee^? B}$$

provided the sequents are safe (and note that safety of the conclusion entails safety of the premiss).

Lemma 28 (Equivalence between different polarisations)

For all $A', A'' \in \psi(A)$, we have $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A', {A''}^{\perp}, \Delta$, provided the sequent is safe.

Proof: In the proof below, for any formula A, the notations A' and A'' will systematically designate elements of $\psi(A)$.

The proof is by induction on A:

1. $A = a$

   Let $A', A'' \in \psi(a) = \{a\}$. Therefore $A' = A'' = A = a$.

$$\frac{\dfrac{}{\Gamma, \psi^{\perp}(a), \psi(a), \Gamma' \vdash^{\mathcal{P}'}}\,(\mathsf{Id}_2)}{\Gamma \vdash^{\mathcal{P}} \psi(a), \psi^{\perp}(a), \Delta}$$

2. $A = A_1 \wedge A_2$

   Let $A'_1, A''_1 \in \psi(A_1)$, $A'_2, A''_2 \in \psi(A_2)$ and $A' = A'_1 \wedge^? A'_2$, $A'' = A''_1 \wedge^? A''_2$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash^{\mathcal{P}} A'_1, {A''_1}^{\perp}, \Delta}{\Gamma \vdash^{\mathcal{P}} A'_1, {A''_1}^{\perp}, {A''_2}^{\perp}, \Delta} \qquad \dfrac{\Gamma \vdash^{\mathcal{P}} A'_2, {A''_2}^{\perp}, \Delta}{\Gamma \vdash^{\mathcal{P}} A'_2, {A''_1}^{\perp}, {A''_2}^{\perp}, \Delta}}{\Gamma \vdash^{\mathcal{P}} A'_1 \wedge^- A'_2, {A''_1}^{\perp} \vee^- {A''_2}^{\perp}, \Delta}}{\Gamma \vdash^{\mathcal{P}} A', {A''_1}^{\perp} \vee^- {A''_2}^{\perp}, \Delta}}{\Gamma \vdash^{\mathcal{P}} A', {A''}^{\perp}, \Delta}$$

   We can complete the proof on the left-hand side by applying the induction hypothesis on A_1 and on the right-hand side by applying the induction hypothesis on A_2.

3. $A = A_1 \vee A_2$

   By symmetry, using the previous case.

4. $A = \forall x A_1$

   Let $A' = \forall x A'_1$ and $A'' = \forall x A''_1$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}'} [{A''_1}^{\perp}] A''_1}{\vdash^{\mathcal{P}'} [\exists x A''_1] A''_1}}{\Gamma, \forall x A''_1 \vdash^{\mathcal{P}} [\bullet] A''_1, \Delta}}{\Gamma \vdash^{\mathcal{P}} A''_1, \exists x {A''_1}^{\perp}, \Delta}\,\text{Lemma 23(2)} \qquad \Gamma \vdash^{\mathcal{P}} A'_1, {A''_1}^{\perp}, \Delta}{\Gamma \vdash^{\mathcal{P}} A'_1, \exists x {A''_1}^{\perp}, \Delta}\,\text{Lemma 25}}{\Gamma \vdash^{\mathcal{P}} \forall x A'_1, \exists x {A''_1}^{\perp}, \Delta}$$

   We can complete the proof on the left-hand side by Lemma 20 and the right-hand side by applying the induction hypothesis on A_1.

5. $A = \exists x A_1$

   By symmetry, using the previous case.

6. $A = \neg A_1$

   Let $A', A'' \in \psi(\neg A_1)$.

   Let $A' = {A'_1}^{\perp}$ with $A'_1 \in \psi(A_1)$ and $A'' = {A''_1}^{\perp}$ with $A''_1 \in \psi(A_1)$.

   The induction hypothesis on A_1 we get: $\Gamma \vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A', {A''}^{\perp}, \Delta$ and we are done.

□

Definition 14 (Theory restricting) A polarisation set *does not restrict* the theory $\mathcal{T}$ if for all sets $\mathcal{B}$ of literals that are semantically inconsistent (i.e. $\mathcal{B} \models_{\mathcal{T}}$), there is a subset $\mathcal{B}' \subseteq \mathcal{B}$ that is already semantically inconsistent and such that at most one literal of $\mathcal{B}'$ is $\mathcal{P}$-negative. ※

Remark 29 The empty polarisation set restricts no theories.

Theorem 30 (Completeness of $\mathsf{LK}^p(\mathcal{T})$) Assume $\mathcal{P}$ does not restrict $\mathcal{T}$ and $\Delta \models_{\mathcal{T}} A$.

Then for all $A' \in \psi(A)$ and $\Delta' \in \psi(\Delta)$, we have $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A', {\Delta'}^{\perp}$, provided that sequent is safe.

Proof: We prove a slightly more general statement:
for all $A' \in \psi(A)$ and all multiset Δ' of formulae that contain an element of $\psi(\Delta)$ as a sub-multiset, we have $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A', {\Delta'}^{\perp}$, provided that sequent is safe.

We caracterise $\Delta \models_{\mathcal{T}} A$ by the derivability of the sequent $\Psi_{\mathcal{T}}, \Delta \vdash A$ in a standard natural deduction system for first-order classical logic. We write $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A$ for this derivability property.

For any formula A, the notation A' will systematically designate an element of $\psi(A)$.

The proof is by induction of $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A$, and case analysis on the last rule:

- Axiom:
$$\frac{}{\Psi_{\mathcal{T}}, \Delta \vdash A}\ A \in \Psi_{\mathcal{T}}, \Delta$$
By case analysis:
  - If $A \in \Delta$ then we prove $\vdash^{\mathcal{P}} A', {\Delta'}^{\perp}$ with $A', A'' \in \psi(A)$ and $A'' \in \Delta'$, using Lemma 28.
  - If $A \in \Psi_{\mathcal{T}}$ then A is of the form $l_1 \vee \cdots \vee l_n$ with $\psi(l_1)^{\perp}, \ldots, \psi(l_n)^{\perp} \models_{\mathcal{T}}$.
    Let $\{\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp}\}$ be a subset of $\{\psi(l_1)^{\perp}, \ldots, \psi(l_n)^{\perp}\}$ that is already semantically inconsistent and such that at most one literal is $\mathcal{P}$-negative, say possibly $\psi(l'_m)^{\perp}$.
    Let $C' \in \psi(A)$. C' is of the form $\psi(l_1) \vee^{?} \cdots \vee^{?} \psi(l_n)$.
    We build
$$\cfrac{\cfrac{\cfrac{\cfrac{\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp} \vdash^{\mathcal{P}'}}{\vdash^{\mathcal{P}} \psi(l'_1), \ldots, \psi(l'_m)}}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, \psi(l_1), \ldots, \psi(l_n)}}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, \psi(l_1) \vee^{-} \cdots \vee^{-} \psi(l_n)}}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C'}$$
    where $\mathcal{P}' := \mathcal{P}; \psi(l'_1)^{\perp}; \ldots; \psi(l'_m)^{\perp}$.
    If $\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp}$ is syntactically inconsistent, we close with Id_2.
    Otherwise
$$\mathcal{P}; \psi(l'_1)^{\perp}; \ldots; \psi(l'_{m-1})^{\perp} = \mathcal{P}, \psi(l'_1)^{\perp}, \ldots, \psi(l'_{m-1})^{\perp}$$
    as none of the $\psi(l'_i)^{\perp}$, for $1 \leq i \leq m-1$, is $\mathcal{P}$-negative. And for all i such that $1 \leq i \leq m-1$, the literal $\psi(l'_i)^{\perp}$ is $\mathcal{P}'$-positive.
    Now if $\psi(l'_m)^{\perp}$ is $\mathcal{P}'$-positive as well, we have
$$\mathsf{lit}_{\mathcal{P}'}(\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp}) = \psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp}$$
    and we can close with (Init_2).
    If $\psi(l'_m)^{\perp}$ is not $\mathcal{P}'$-positive, we simply have
$$\mathsf{lit}_{\mathcal{P}'}(\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp}) = \psi(l'_1)^{\perp}, \ldots, \psi(l'_{m-1})^{\perp}$$
    but we can still build
$$\cfrac{(\mathsf{Init}_1)\cfrac{\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp} \models_{\mathcal{T}}}{\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp} \vdash^{\mathcal{P}'} [\psi(l'_m)]}}{\psi(l'_1)^{\perp}, \ldots, \psi(l'_m)^{\perp} \vdash^{\mathcal{P}'}}$$

- And Intro:

$$\frac{\Psi_{\mathcal{T}}, \Delta \vdash A_1 \quad \Psi_{\mathcal{T}}, \Delta \vdash A_2}{\Psi_{\mathcal{T}}, \Delta \vdash A_1 \wedge A_2}$$

$A' \in \psi(A_1 \wedge A_2)$ is of the form $A'_1 \wedge^? A'_2$ with $A'_1 \in \psi(A_1)$ and $A'_2 \in \psi(A_2)$.
Since $\vdash^{\mathcal{P}} A'_1 \wedge^? A'_2, {\Delta'}^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} A'_1, {\Delta'}^{\perp}$ and $\vdash^{\mathcal{P}} A'_2, {\Delta'}^{\perp}$ are also safe, and we can apply the induction hypothesis
  - on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A_1$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A'_1, {\Delta'}^{\perp}$
  - and on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A_2$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A'_2, {\Delta'}^{\perp}$.

We build:

$$\dfrac{\dfrac{\vdash^{\mathcal{P}} A'_1, {\Delta'}^{\perp} \quad \vdash^{\mathcal{P}} A'_2, {\Delta'}^{\perp}}{\vdash^{\mathcal{P}} A'_1 \wedge^{-} A'_2, {\Delta'}^{\perp}}}{\vdash^{\mathcal{P}} A'_1 \wedge^? A'_2, {\Delta'}^{\perp}}$$

- And Elim

$$\frac{\Psi_{\mathcal{T}}, \Delta \vdash A_1 \wedge A_{-1}}{\Psi_{\mathcal{T}}, \Delta \vdash A_i}$$

with $i \in \{1, -1\}$.
Since $\psi(A_{-i}) \neq \emptyset$, let $A'_{-i} \in \psi(A_{-i})$ and $C' = A'_1 \wedge^{-} A'_{-1}$ $(C' \in \psi(A_1 \wedge A_{-1}))$.
Since $\vdash^{\mathcal{P}} A'_i, {\Delta'}^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} C', A'_i, {\Delta'}^{\perp}$ is also safe, and we can apply the induction hypothesis on $\Psi_{\mathcal{T}}, \Delta \vdash A_1 \wedge A_{-1}$ (with ${A'_i}^{\perp}, \Delta'$ and C') to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} C', A'_i, {\Delta'}^{\perp}$.
We finally get:

$$\dfrac{\dfrac{\vdash^{\mathcal{P}} C', A'_i, {\Delta'}^{\perp}}{\vdash^{\mathcal{P}} A'_i, A'_i, {\Delta'}^{\perp}}\text{ Lemma 7}}{\vdash^{\mathcal{P}} A'_i, {\Delta'}^{\perp}}\ \mathsf{C}_r$$

- Or Intro

$$\frac{\Psi_{\mathcal{T}}, \Delta \vdash A_i}{\Psi_{\mathcal{T}}, \Delta \vdash A_1 \vee A_{-1}}$$

$A' \in \psi(A_1 \vee A_{-1})$ is of the form $A'_1 \vee^? A'_{-1}$ with $A'_1 \in \psi(A_1)$ and $A'_{-1} \in \psi(A_{-1})$.
Since $\vdash^{\mathcal{P}} A'_1 \vee^? A'_{-1}, {\Delta'}^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} A'_1, A'_{-1}, {\Delta'}^{\perp}$ is also safe, and we can apply the induction hypothesis on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A_i$ (with ${A'_{-i}}^{\perp}, \Delta'$ and A'_i) to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A'_1, A'_{-1}, {\Delta'}^{\perp}$ and we build:

$$\dfrac{\dfrac{\vdash^{\mathcal{P}} A'_1, A'_{-1}, {\Delta'}^{\perp}}{\vdash^{\mathcal{P}} A'_1 \vee^{-} A'_{-1}, {\Delta'}^{\perp}}}{\vdash^{\mathcal{P}} A'_1 \vee^? A'_{-1}, {\Delta'}^{\perp}}$$

- Or Elim

$$\frac{\Psi_{\mathcal{T}}, \Delta \vdash A_1 \vee A_2 \quad \Psi_{\mathcal{T}}, \Delta, A_1 \vdash C \quad \Psi_{\mathcal{T}}, \Delta, A_2 \vdash C}{\Psi_{\mathcal{T}}, \Delta \vdash C}$$

Let $D' = A'_1 \vee^{-} A'_2$ with $A'_1 \in \psi(A_1)$ and $A'_2 \in \psi(A_2)$.
Since $\vdash^{\mathcal{P}} C', {\Delta'}^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} C', {A'_1}^{\perp}, {\Delta'}^{\perp}$ and $\vdash^{\mathcal{P}} C', {A'_2}^{\perp}, {\Delta'}^{\perp}$ and $\vdash^{\mathcal{P}} C', D', {\Delta'}^{\perp}$ are also safe, and we can apply the induction hypothesis
  - on $\Psi_{\mathcal{T}}, \Delta, A_1 \vdash_{\mathsf{FOL}} C$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} C', {A'_1}^{\perp}, {\Delta'}^{\perp}$
  - on $\Psi_{\mathcal{T}}, \Delta, A_2 \vdash_{\mathsf{FOL}} C$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} C', {A'_2}^{\perp}, {\Delta'}^{\perp}$.
  - and on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A_1 \vee A_2$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} C', D', {\Delta'}^{\perp}$.

We build:

$$\dfrac{\vdash^{\mathcal{P}} D', C', \Delta'^{\perp} \qquad \dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} A_1'^{\perp}, C', \Delta'^{\perp} \quad \vdash^{\mathcal{P}} A_2'^{\perp}, C', \Delta'^{\perp}}{\vdash^{\mathcal{P}} A_1'^{\perp} \wedge^- A_2'^{\perp}, C', \Delta'^{\perp}}}{\vdash^{\mathcal{P}} A_1'^{\perp} \wedge^+ A_2'^{\perp}, C', \Delta'^{\perp}}}{\vdash^{\mathcal{P}} (A_1' \vee^- A_2')^{\perp}, C', \Delta'^{\perp}}}{\vdash^{\mathcal{P}} C', \Delta'^{\perp}}\,\mathsf{cut}_7$$

- Universal quantifier Intro

$$\dfrac{\Psi_{\mathcal{T}}, \Delta \vdash A}{\Psi_{\mathcal{T}}, \Delta \vdash \forall x A}\, x \notin \Gamma$$

$C' \in \psi(\forall x A)$ is of the form $\forall x A'$ with $A' \in \psi(A)$.
Since $\vdash^{\mathcal{P}} C', \Delta'^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} A', \Delta'^{\perp}$ is also safe, and we can apply the induction hypothesis on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} A$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A', \Delta'^{\perp}$ to get:

$$\dfrac{\vdash^{\mathcal{P}} A', \Delta'^{\perp}}{\vdash^{\mathcal{P}} \forall x A', \Delta'^{\perp}}$$

- Universal quantifier Elim

$$\dfrac{\Psi_{\mathcal{T}}, \Delta \vdash \forall x A}{\Psi_{\mathcal{T}}, \Delta \vdash \left\{\tfrac{t}{x}\right\} A}$$

$C' \in \psi(\left\{\tfrac{t}{x}\right\} A)$ is of the form $\left\{\tfrac{t}{x}\right\} A'$ with $A' \in \psi(A)$ (by Remark 27).
Since $\vdash^{\mathcal{P}} C', \Delta'^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} (\forall x A'), C', \Delta'^{\perp}$ is also safe, and we can apply the induction hypothesis on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} \forall x A$ (with $C'^{\perp}, \Delta'$ and $(\forall x A')$) to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} (\forall x A'), C', \Delta'^{\perp}$.
We build

$$\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} (\forall x A'), \left\{\tfrac{t}{x}\right\} A', \Delta'^{\perp}}{\vdash^{\mathcal{P}} A', \left\{\tfrac{t}{x}\right\} A', \Delta'^{\perp}}\,\text{Lemma 7}}{\vdash^{\mathcal{P}} \left\{\tfrac{t}{x}\right\} A', \left\{\tfrac{t}{x}\right\} A', \Delta'^{\perp}}\,\text{Lemma 16}}{\vdash^{\mathcal{P}} \left\{\tfrac{t}{x}\right\} A', \Delta'^{\perp}}\,\mathsf{C}_r$$

- Existential quantifier Intro

$$\dfrac{\Psi_{\mathcal{T}}, \Delta \vdash \left\{\tfrac{t}{x}\right\} A}{\Psi_{\mathcal{T}}, \Delta \vdash \exists x A}$$

$C' \in \psi(\exists x A)$ is of the form $\exists x A'$ with $A' \in \psi(A)$.
Let $A'_t = \left\{\tfrac{t}{x}\right\} A'$ ($A'_t \in \psi(\left\{\tfrac{t}{x}\right\} A)$ by Remark 27).
Since $\vdash^{\mathcal{P}} C', \Delta'^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} A'_t, \Delta'^{\perp}$ is also safe, and we can apply the induction hypothesis on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} \left\{\tfrac{t}{x}\right\} A$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} A'_t, \Delta'^{\perp}$.
By Lemma 25 it suffices to prove $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} \exists x A', A_t'^{\perp}$ in order to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} C', \Delta'^{\perp}$:

$$\dfrac{\dfrac{\dfrac{\vdash^{\mathcal{P}} [A'_t] A_t'^{\perp}}{\vdash^{\mathcal{P}} [\exists x A'] A_t'^{\perp}}}{\forall x A'^{\perp} \vdash^{\mathcal{P}} [\bullet] A_t'^{\perp}}}{\vdash^{\mathcal{P}} \exists x A', A_t'^{\perp}}\,\text{Lemma 23(2)}$$

We can complete the proof by applying Lemma 20.

- Existential quantifier Elim

$$\dfrac{\Psi_{\mathcal{T}}, \Delta \vdash \exists x A \quad \Gamma, \Delta, A \vdash B}{\Psi_{\mathcal{T}}, \Delta \vdash B}\, x \notin \Gamma, B$$

Let $C' = \exists x A'$ with $A' \in \psi(A)$.
Since $\vdash^{\mathcal{P}} B', \Delta'^{\perp}$ is assumed to be safe, $\vdash^{\mathcal{P}} B', C', \Delta'^{\perp}$ and $\vdash^{\mathcal{P}} B', A'^{\perp}, \Delta'^{\perp}$ are also safe, and we can apply the induction hypothesis

– on $\Psi_{\mathcal{T}}, \Delta \vdash_{\mathsf{FOL}} \exists x A$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} B', C', {\Delta'}^{\perp}$;

– on $\Gamma, \Delta, A \vdash_{\mathsf{FOL}} B$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} B', {A'}^{\perp}, {\Delta'}^{\perp}$.

We build

$$\dfrac{\vdash^{\mathcal{P}} C', B', {\Delta'}^{\perp} \qquad \dfrac{\dfrac{\vdash^{\mathcal{P}} {A'}^{\perp}, B', {\Delta'}^{\perp}}{\vdash^{\mathcal{P}} \forall x({A'}^{\perp}), B', {\Delta'}^{\perp}}}{\vdash^{\mathcal{P}} {C'}^{\perp}, B', {\Delta'}^{\perp}}}{\vdash^{\mathcal{P}} B', {\Delta'}^{\perp}}\ \mathsf{cut}_7$$

- Negation Intro

$$\dfrac{\Psi_{\mathcal{T}}, \Delta, A \vdash B \wedge \neg B}{\Psi_{\mathcal{T}}, \Delta \vdash \neg A}$$

If $C' \in \psi(\neg A)$ then ${C'}^{\perp} \in \psi(A)$. Let $D' = D'_1 \wedge^- D'_2$ with $D'_1 \in \psi(B)$ and $D'_2 \in \psi(\neg B)$. Therefore ${D'_2}^{\perp} \in \psi(B)$, $D' \in \psi(B \wedge \neg B)$ and $\Delta', {C'}^{\perp} \in \psi(\Delta, A)$.

Since $\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C'$ is assumed to be safe, $\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C', D'$ is also safe, and we can apply the induction hypothesis on $\Psi_{\mathcal{T}}, \Delta, A \vdash_{\mathsf{FOL}} B \wedge \neg B$ to get $\vdash^{\mathcal{P}}_{\mathsf{LK}^p(\mathcal{T})} {\Delta'}^{\perp}, C', D'$. We build

$$\dfrac{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C', D' \qquad \dfrac{\dfrac{\dfrac{}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C', {D'_1}^{\perp}, {D'_2}^{\perp}}\ \text{Lemma 28}}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C', {D'_1}^{\perp} \vee^- {D'_2}^{\perp}}}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C', {D'}^{\perp}}\ \text{Corollary 26(4)}}{\vdash^{\mathcal{P}} {\Delta'}^{\perp}, C'}\ \mathsf{cut}_7$$

- Negation Elimination

$$\dfrac{\Psi_{\mathcal{T}}, \Delta \vdash \neg\neg A}{\Psi_{\mathcal{T}}, \Delta \vdash A}$$

$A' \in \psi(A)$ is such that $A' \in \psi(\neg\neg A)$.

The induction hypothesis on $\Psi_{\mathcal{T}}, \Delta \vdash \neg\neg A$ gives $\vdash^{\mathcal{P}} {\Delta'}^{\perp}, A'$ and we are done.

□

8 The system used for simulation of $\mathsf{DPLL}(\mathcal{T})$

The motivation for the $\mathsf{LK}^p(\mathcal{T})$ system was to perform proof-search modulo theories, and in particular simulate $\mathsf{DPLL}(\mathcal{T})$ techniques. Therefore, we conclude this report with the actual system that we use in other works [FLM12, FGLM13] to perform the simulation:

It is the $\mathsf{LK}^p(\mathcal{T})$ system, extended with the admissible and invertible rules (Pol) and (cut_7) (or more precisely restricted versions of them), as shown in Fig 3.

Synchronous rules

$$(\wedge^+)\frac{\Gamma \vdash^{\mathcal{P}} [A] \qquad \Gamma \vdash^{\mathcal{P}} [B]}{\Gamma \vdash^{\mathcal{P}} [A \wedge^+ B]} \qquad (\vee^+)\frac{\Gamma \vdash^{\mathcal{P}} [A_i]}{\Gamma \vdash^{\mathcal{P}} [A_1 \vee^+ A_2]} \qquad (\exists)\frac{\Gamma \vdash^{\mathcal{P}} [\{{}^{t}\!/\!_{x}\} A]}{\Gamma \vdash^{\mathcal{P}} [\exists x A]}$$

$$(\top^+)\frac{}{\Gamma \vdash^{\mathcal{P}} [\top^+]} \qquad (\mathsf{Init}_1)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}} [l]}\ l \text{ is } \mathcal{P}\text{-positive} \qquad (\mathsf{Release})\frac{\Gamma \vdash^{\mathcal{P}} N}{\Gamma \vdash^{\mathcal{P}} [N]}\ N \text{ is not } \mathcal{P}\text{-positive}$$

Asynchronous rules

$$(\wedge^-)\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta \qquad \Gamma \vdash^{\mathcal{P}} B, \Delta}{\Gamma \vdash^{\mathcal{P}} A \wedge^- B, \Delta} \qquad (\vee^-)\frac{\Gamma \vdash^{\mathcal{P}} A_1, A_2, \Delta}{\Gamma \vdash^{\mathcal{P}} A_1 \vee^- A_2, \Delta} \qquad (\forall)\frac{\Gamma \vdash^{\mathcal{P}} A, \Delta}{\Gamma \vdash^{\mathcal{P}} (\forall x A), \Delta}\ x \notin \mathsf{FV}(\Gamma, \Delta, \mathcal{P})$$

$$(\bot^-)\frac{\Gamma \vdash^{\mathcal{P}} \Delta}{\Gamma \vdash^{\mathcal{P}} \Delta, \bot^-} \qquad (\top^-)\frac{}{\Gamma \vdash^{\mathcal{P}} \Delta, \top^-} \qquad (\mathsf{Store})\frac{\Gamma, A^\perp \vdash^{\mathcal{P};A^\perp} \Delta}{\Gamma \vdash^{\mathcal{P}} A, \Delta}\ \begin{array}{l} A \text{ is a literal} \\ \text{or is } \mathcal{P}\text{-positive} \end{array}$$

Structural rules

$$(\mathsf{Select})\frac{\Gamma, P^\perp \vdash^{\mathcal{P}} [P]}{\Gamma, P^\perp \vdash^{\mathcal{P}}}\ P \text{ is not } \mathcal{P}\text{-negative} \qquad (\mathsf{Init}_2)\frac{\mathsf{lit}_{\mathcal{P}}(\Gamma) \models_{\mathcal{T}}}{\Gamma \vdash^{\mathcal{P}}}$$

Admissible/Invertible rules

$$(\mathsf{Pol})\frac{\Gamma \vdash^{\mathcal{P}, l}}{\Gamma \vdash^{\mathcal{P}}}\ \mathsf{lit}_{\mathcal{P}}(\Gamma), l^\perp \models_{\mathcal{T}} \qquad \frac{\Gamma \vdash^{\mathcal{P}} l \quad \Gamma \vdash^{\mathcal{P}} l^\perp}{\Gamma \vdash^{\mathcal{P}}}\ \mathsf{cut}_7$$

where $\mathcal{P}; A := \mathcal{P}, A$ if $A \in \mathsf{U}_{\mathcal{P}}$
$\mathcal{P}; A := \mathcal{P}$ if not

Figure 3: System for the simulation of $\mathsf{DPLL}(\mathcal{T})$

References

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