A factorization method for elliptic BVP

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Abstract

The technique of space invariant embedding, applied to a second order boundary value problem defined in a cylindrical domain, gives rise to a system of uncoupled first order initial value problems that includes a non-linear Riccati equation on a unbounded functional operator. We present a method to justify this equation that uses a parabolic regularization of the original problem.
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1 Introduction

The invariant embedding method was firstly proposed by Bellman [1]. Henry and Ramos [3] adapted this method to the case of the Poisson equation in an n-dimensional cylindrical domain, arising to a decoupled system of first-order initial value problems. They applied a spatial invariant embedding, using the coordinate along the axis of the cylinder. The final system includes a Riccati equation satisfied by the Dirichlet to Neumann operator on the section of the cylinder. Since this operator in unbounded, a sense must be given to the nonlinear term of the equation. Based on the framework introduced by J.L. Lions [7], for deriving the optimal feedback for optimal control problems of parabolic equations, Henry and Ramos [3] gave a complete justification for this equation, using a Galerkin method. They also showed that the factorization of the boundary value problem can be viewed as an infinite dimensional generalization of the block Gauss LU factorization. Other approaches to justify this Riccati equation can be found in [2, 4].

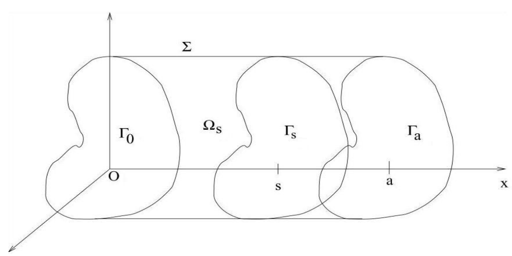
In this work, we exploit the relationship with a control problem in order to propose a regularization of the Riccati equation. This type of regularization will allow us to generalize the invariant embedding method to more general domains.

2 Factorization by invariant embedding

Let Ω be the cylinder $]0,a[\times\mathcal{O}$ in $\mathbb{R}^n$, where $\mathcal{O}$ is a smooth bounded open set in $\mathbb{R}^{n-1}$. We also consider $\Sigma =]0,a[\times\partial\mathcal{O}$ to be the "lateral boundary" of the domain, and $\Gamma_0 = \{0\}\times\mathcal{O}$, $\Gamma_a = \{a\}\times\mathcal{O}$ the "faces" of the domain. We denote the elements of $\mathbb{R}^N$ by $(x_1, x_2, \ldots, x_N) = (x, y)$, where $x = x_1$ and $y = (x_2, \ldots, x_N)$ to stress the particular role of x_1. In Ω, we consider the Poisson problem:

$$(\mathcal{P}_0)\begin{cases} -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \Delta_y u = f, \quad \text{in } \Omega \\ u_{|\Sigma} = 0, \quad u_{|\Gamma_a} = u_1 \\ -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0 \end{cases}$$

where $f \in L^2(\Omega)$, $u_0 \in (H_{00}^{1/2}(\mathcal{O}_0))'$ and $u_1 \in H_{00}^{1/2}(\mathcal{O}_a)$ (see [8] for the definition of these spaces). Then, this problem has a unique solution in $X = L^2(0,a;H_0^1(\mathcal{O})) \cap H^1(0,a;L^2(\mathcal{O}))$ [3].



Now we use the technique of invariant embedding, introduced by Bellman in [1], in order to embed the problem $(\mathcal{P}_0)$ in a family of similar problems $(\mathcal{P}_{s,h})$. For each $s \in]0,a]$, the problem $(\mathcal{P}_{s,h})$ is defined over the subcylinders $]0,s[\times\mathcal{O}$, and we impose the Dirichlet boundary condition $u_{|\Gamma_s} = h$, where $\Gamma_s = \{s\}\times\mathcal{O}$:

$$(\mathcal{P}_{s,h})\begin{cases} -\Delta u = -\frac{\partial^2 u}{\partial x^2} - \Delta_y u = f, \quad \text{in } \Omega_s \\ u_{|\Sigma} = 0, \\ u_{|\Gamma_s} = h, \quad -\dfrac{\partial u}{\partial x}|_{\Gamma_0} = u_0. \end{cases}$$

Obviously, $(\mathcal{P}_{s,h})$ reduces to $(\mathcal{P}_0)$, when $s = a$ and $h = u_1$.

As in [7], we define:

Definition 2.1 *For every $s \in]0,a]$ we define the Dirichlet to Neumann (DtN) map*

P, by $P(s)h = \dfrac{\partial \gamma}{\partial x}|_{\Gamma_s}$, *where γ is the solution of*

$$\begin{cases} -\Delta\gamma = 0, \quad \textit{in } \Omega_s \\ \gamma_{|_\Sigma} = 0, \\ -\dfrac{\partial\gamma}{\partial x}|_{\Gamma_0} = 0, \quad \gamma_{|_{\Gamma_s}} = h, \end{cases}$$

and $r(s) = \dfrac{\partial \beta}{\partial x}|_{\Gamma_s}$, *where β is the solution of*

$$\begin{cases} -\Delta\beta = f, \quad \textit{in } \Omega_s \\ \beta_{|_\Sigma} = 0, \\ -\dfrac{\partial\beta}{\partial x}|_{\Gamma_0} = u_0, \quad \beta_{|_{\Gamma_s}} = 0. \end{cases}$$

For every $s \in [0,a]$, $P(s) : H_{00}^{1/2}(\mathcal{O}) \to (H_{00}^{1/2}(\mathcal{O}))'$ is a linear operator and $r(s) \in (H_{00}^{1/2}(\mathcal{O}))'$, due to the well posedness of the problem and trace properties.

By linearity of $(\mathcal{P}_{s,h})$, we have

$$\frac{\partial u}{\partial x}|_{\Gamma_s} = P(s)h + r(s), \forall s \in [0,a]$$

where $u = \gamma + \beta$.

Further, the solution of $(\mathcal{P}_0)$ is given by

$$\frac{\partial u}{\partial x}(x,y) = (P(x)u_{|_{\Gamma_x}})(y) + (r(x))(y). \tag{1}$$

We obtain, deriving the previous equation formally with respect to x,

$$\frac{\partial^2 u}{\partial x^2} = -\Delta_y u - f = \frac{\partial P}{\partial x}u + P\frac{\partial u}{\partial x} + \frac{\partial r}{\partial x},$$

and, substituting $\frac{\partial u}{\partial x}$ given by (1),

$$\left(\frac{\partial P}{\partial x} + P^2 + \Delta_y\right)u + \frac{\partial r}{\partial x} + Pr + f = 0.$$

Consequently, since u is arbitrary, we obtain the formal system:

$$\begin{cases} \dfrac{\partial P}{\partial x} + P^2 + \Delta_y = 0, \\ \dfrac{\partial r}{\partial x} + Pr = -f, \\ -\dfrac{\partial u}{\partial x} + Pu = -r. \end{cases} \tag{2}$$

Again from (1), and considering the initial condition in Γ_0, we obtain $P(0) = 0$ and $r(0) = -u_0$. The initial condition for the third equation is, naturally, $u(a) = u_1$.

From the first two equations of (2), integrating from 0 to a, we can obtain P and r. Substituting both of them in the third equation, and integrating reversely, we find u. Thus, (2) is a decoupled system. Further, the first equation of (2) is a Riccati equation on the operator P (which is an operator on functions on y, depending on x).

We can find in [3] the precise meaning of system (2):

Theorem 2.2 *The solution u of $(\mathcal{P}_0)$ is the unique solution of (2) in the sense that*

1. *the operator*

$$P \in L^\infty\big(0,a;\ \mathcal{L}(H_0^1(\mathcal{O}), L^2(\mathcal{O})) \cap \mathcal{L}(H_{00}^{1/2}(\mathcal{O}), (H_{00}^{1/2}(\mathcal{O}))') \cap \mathcal{L}(L^2(\mathcal{O}), H^{-1}(\mathcal{O}))\big)$$

   verifies the Riccati equation

$$\left(\frac{\partial P}{\partial x}h, \overline{h}\right) + (Ph, P\overline{h}) = (\nabla_y h, \nabla_y \overline{h}),\ \forall h, \overline{h} \in H_0^1(\mathcal{O}), \tag{3}$$

   with the initial condition $P(0) = 0$;

2. *the function $r \in \mathcal{C}\big([0,a]; (H_{00}^{1/2}(\mathcal{O}))'\big)$ verifies the equation*

$$\left\langle \frac{\partial r}{\partial x}, h \right\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})} + \langle Pr, h\rangle_{H^{-1}(\mathcal{O}),H_0^1(\mathcal{O})} = (-f, h),\ \forall h \in H_0^1(\mathcal{O}),$$

   with the initial condition $r(0) = -u_0$;

3. *the function $u \in X$ verifies the equation*

$$-\left(\frac{\partial u}{\partial x}, h\right) + (Pu, h) = -(r, h),\ \forall h \in L^2(\mathcal{O}),$$

   with the initial condition $u(a) = u_1$.

The operator P appearing on the first equation of (2) is not bounded, and consequently we can not give a precise meaning to the term P^2. However, in the case of a cylindrical domain, the equation (3) can be fully justified using either a Galerkin method [3], or Hilbert-Schmidt operators [4], or a semigroup approach [2]. Unfortunately neither of these methods seems to work in the case of a more general domain such as, for instance, a non cylindrical one [6]. For this reason, in the next section, we present a method based in a regularization of the problem $(\mathcal{P}_0)$ that can be directly generalized to other types of domains.

3 Fourth order regularization

Next, we consider a regularization of $(\mathcal{P}_0)$ with a 4-order transversal operator:

$$(\mathcal{P}_\varepsilon) \begin{cases} -\dfrac{\partial^2 u_\varepsilon}{\partial x^2} - \Delta_y u_\varepsilon + \varepsilon \Delta_y^2 u_\varepsilon = f, \quad \text{in } \Omega \\ u_\varepsilon|_\Sigma = 0, \qquad \Delta_y u_{\varepsilon}{}_{|_\Sigma} = 0 \\ (\dfrac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon)|_{\Gamma_0} = -u_0, \quad u_\varepsilon|_{\Gamma_a} = u_1. \end{cases}$$

Considering $f \in L^2(\Omega)$, and $u_0, u_1 \in L^2(\mathcal{O})$, this problem has a unique solution in $X_\varepsilon = \{u_\varepsilon \in L^2(0,a; H_0^1(\mathcal{O})) : \dfrac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon \in L^2(\Omega)\}$. Further, the problem has also a unique solution in $Y = \{u_\varepsilon \in L^2(0,a; H_0^1(\mathcal{O})) : \dfrac{\partial u_\varepsilon}{\partial x} + \Delta_y u_\varepsilon \in L^2(0,a; H_0^1(\mathcal{O})),\ \Delta_y u_\varepsilon \in L^2(\Omega)\}$, for $f \in L^2(\Omega)$, and $u_0, u_1 \in H_0^1(\mathcal{O})$.

This problem corresponds exactly to the optimality system associated to an optimal control problem of a parabolic equation studied in [7]. Consequently, using the identity $\frac{\partial u_\varepsilon}{\partial x} = (P_\varepsilon - \sqrt{\varepsilon}\Delta_y)u_\varepsilon + r_\varepsilon$, the system can be decoupled to:

$$\begin{cases} \frac{\partial P_\varepsilon}{\partial x} + P_\varepsilon^2 - \sqrt{\varepsilon}(\Delta_y P_\varepsilon + P_\varepsilon \Delta_y) + \Delta_y = 0, \;\; P_\varepsilon(0) = 0 \\ -\frac{\partial r_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y r_\varepsilon = P_\varepsilon r_\varepsilon + f, \;\; r_\varepsilon(0) = -u_0 \\ \frac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon = P_\varepsilon u_\varepsilon + r_\varepsilon, \;\; u_\varepsilon(a) = u_1. \end{cases} \tag{4}$$

Now, we have $P_\varepsilon \in \mathcal{L}(L^2(\mathcal{O}), L^2(\mathcal{O}))$ and $P_\varepsilon \in \mathcal{L}(H_0^1(\mathcal{O}), H_0^1(\mathcal{O}))$. Also, the first equation of (4) can be written in a variational form as:

$$\big(\frac{\partial P_\varepsilon}{\partial x} h, \overline{h}\big) + (P_\varepsilon h, P_\varepsilon \overline{h}) - \sqrt{\varepsilon}(\Delta_y h, P_\varepsilon \overline{h}) - \sqrt{\varepsilon}(P_\varepsilon h, \Delta_y \overline{h}) = (\nabla_y h, \nabla_y \overline{h}), \; \forall h, \overline{h} \in H_0^1(\mathcal{O}).$$

Theorem 3.1 *Let u_ε be the solution of problem ($\mathcal{P}_\varepsilon$), and u the solution of problem ($\mathcal{P}_0$), with $u_1 \in H_{00}^{1/2}(\mathcal{O})$. Then, when ε goes to 0, we have $u_\varepsilon \to u$ in $L^2(0, a; H_0^1(\mathcal{O}))$, and $\frac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon \to \frac{\partial u}{\partial x}$ in $L^2(\Omega)$.*

As a consequence, $\left(\frac{\partial u_\varepsilon}{\partial x} + \sqrt{\varepsilon}\Delta_y u_\varepsilon\right)|_{\Gamma_s} \to \frac{\partial u}{\partial x}|_{\Gamma_s}$ in $\big(H_{00}^{1/2}(\mathcal{O}_s)\big)'$, when ε goes to 0, and we obtain the following result:

Theorem 3.2 *As $\varepsilon \to 0$, P_ε converges to P, which satisfies the Riccati equation*

$$\left(\frac{\partial P}{\partial x} h, \overline{h}\right) + (Ph, P\overline{h}) = (\nabla_y h, \nabla_y \overline{h}), \; \forall h, \overline{h} \in H_0^1(\mathcal{O}), \tag{5}$$

with the initial condition $P(0) = 0$.

We notice that equation (5) is the same as (3). This method seems to be generalizable to more general types of domains, which will allow us to complete the justification for the factorization by invariant embedding techniques presented in [5] for a star-shaped domain, and in [6] for non-cylindrical domains.

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