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Robert E. Lynch
Purdue University, rel@cs.purdue.edu

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We construct solutions of

(1a) $$G_{0,0}(\alpha) = 0, \qquad \mathbf{L}_\alpha G_{j,k}(\alpha) = \begin{cases} 1 & \text{if } j = k = 0, \\ 0 & \text{otherwise}, \end{cases}$$

(1b) first differences of $G_{j,k}(\alpha)$ vanish as $j^2 + k^2 \to \infty$,

where $\mathbf{L}_\alpha$ is the 9-point difference operator defined by

$$\mathbf{L}_\alpha U_{j,k} = (2\alpha - 4)U_{j,k} + (1-\alpha)[U_{j-1,k} + U_{j+1,k} + U_{j,k-1} + U_{j,k+1}]$$
$$+ (\alpha/2)[U_{j-1,k-1} + U_{j+1,k-1} + U_{j+1,k+1} + U_{j-1,k+1}],$$

or in stencil form by

(2) $$\mathbf{L}_\alpha U_{j,k} = \left\{ (1-\alpha) \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} + \frac{\alpha}{2} \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 0 & -4 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \right\} U_{j,k}.$$

The standard ('5-point star') formula of discrete potential theory is obtained with $\alpha = 0$.

We also derive the asymptotic result

(3) $$2\pi G_{j,k}(\alpha) \sim \log R + \gamma + \frac{1}{2}\log\left(\frac{8}{1-\alpha}\right) + (3\alpha - 1)\frac{\cos 4\sigma}{12R^2} + \frac{(90\alpha^2 - 18)\cos 4\sigma - (225\alpha^2 - 150\alpha + 25)\cos 8\sigma}{240R^4},$$

where $R^2 = j^2 + k^2$, $\gamma = 0.5772156649\ldots$ is Euler's constant, and $\sigma = \arctan(k/j)$.

For smooth u with $u_{j,k} = u(jh, kh)$,

$$h^{-2}\mathbf{L}_\alpha u_{j,k} = \nabla^2 u_{j,k} + O(h^2)$$

* A minor modification of the text of this report, without the Appendix and with only selected items from the Tables, appears as "Fundamental solutions of nine-point discrete Laplacians', Appl. Numer. Math. 10 (1992), 325-334.

The only value of α which yields a higher order of accuracy for the Laplace or the Poisson equation is $\alpha = 1/3$; this gives the optimal 9-point discrete Laplacian:

$$\mathbf{L}_{2/3}U_{j,k} = \frac{1}{6}\begin{array}{|c|c|c|}\hline 1 & 4 & 1 \\ \hline 4 & -20 & 4 \\ \hline 1 & 4 & 1 \\ \hline \end{array} U_{j,k}$$

for which

$$h^{-2}\mathbf{L}_{1/3}u_{j,k} = \nabla^2 u_{j,k} + \frac{h^2}{12}\nabla^4 u_{j,k} + \frac{h^4}{360}[\nabla^6 u_{j,k} + 2\frac{\partial^4}{\partial x^2 \partial y^2}\nabla^2 u_{j,k}] + O(h^6)$$

(for application to the Poisson equation, see Birkhoff-Lynch [84, p. 92]). For $\alpha = 2/3$, $\mathbf{L}_\alpha$ is Pólya's blinear finite element approximation of the Laplacian (see Birkhoff-Lynch [84, pp. 190–191]).

Values of $G(0)$ *and* $G(1/3)$. For $\alpha = 0$, the solution of (1a–b) is well-known: see McCrea-Whipple [40] (are Stöhr [50, III], Sobolev [52], Duffin [56], Duffin-Shelly [58], and van der Pol [59]). These authors (as do we) first obtain values of G, by evaluating integrals with (j,k) at mesh points along a straight line, and then employ the difference equation and symmetry to obtain values at other mesh points in the plane. Duffin [59] used the fact that discrete harmonic functions satisfy discrete Cauchy-Riemann equations to extend values from a line to the plane; we do not know if the concept of 'discrete harmonic function' can be generalized to apply to solutions of 9-point discrete Laplacians and accomplish a similar extension of values to the plane.

Some values near the origin are given in Table 1 ; Table 2 lists them accurate to 5 digits.

We are unaware of published solutions $G(\alpha)$ with α different from zero. Tables 3 and 4 give results for $\alpha = 1/3$ from our general analysis.

Analysis. As can be verified by direct substitution, a solution of (1a) can be written as

$$G_{j,k}(\alpha) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{(e^{ijx+iky}-1)\,dx\,dy}{2\alpha - 4 + 2\{(1-\alpha)[\cos x + \cos y] + \alpha\cos x\cos y\}} \tag{4}$$

where the denominator in the integrand is equal to $(\mathbf{L}_\alpha e^{ijx+iky})/e^{ijx+iky}$. It is a consequence of the asymptotic result (3) that (4) satisfies the boundary conditions (1b). Following Duffin [59, p. 348], we note that the double integral is absolutely convergent and thus it is permissible to evaluate it as an iterated integral.

For the case $\alpha = 0$, of the standard 5-point star, (4) is equivalent to the expressions given by the sources cited above; it is also used as an example in de Boor-Höllig-Riemenschneider [89] who discuss more general difference operators. There, and in some other sources, the 'diagonal' values of $G(0)$ are given; these are

$$G_{j,j}(0) = \frac{1}{\pi}(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2|j|-1}), \qquad j \pm 1, \pm 2, \ldots. \tag{5}$$

From them and $G_{0,0}(0) = 0$, all the other values can be computed by use of the difference equation and symmetry.

Instead of constructing the diagonal values $G_{j,j}(\alpha)$, we construct values along the axis: $G_{0,k}(\alpha)$ (for $\alpha = 0$ Duffin [59] also starts from such values). We note that the integration with respect to x in (4) can be carried out explicitly. With the substitutions $w = e^{ix}$, $dx = dw/iw$, that integral is equal to

$$\mathcal{I} = \int_0^{2\pi} \frac{Ce^{ijx} - 1}{2A - 2B\cos x}\, dx = \frac{1}{-Bi} \oint \frac{Cw^j - 1}{w^2 - (2A/B)w + 1}\, dw, \tag{6}$$

where the contour is around the unit circle: $w = e^{ix}$. The zeros of the denominator in the integrand are

$$w_- = \frac{A - \sqrt{A^2 - B^2}}{B}, \qquad \text{and} \qquad w_+ = 1/w_-.$$

The simple pole at w_- is the only singularity inside the circle, whence

$$\mathcal{I} = \pi \frac{C\{[A - \sqrt{A^2 - B^2}]/B\}^j - 1}{\sqrt{A^2 - B^2}}. \tag{7}$$

However, A, B, and C, involve y, the other variable of integration in (4). Possibly minor modifications of the transformations used by Stöhr [50] for the case $\alpha = 0$ can be employed to complete the integration. We are content to determine the integral for $j = 0$ and, having the values $G_{0,k}(\alpha)$, to compute all the other values by using the difference equation and symmetry. For $j = 0$, (7) can be obtained from (4) with Dwight [60, p. 218, #858.524].

We now determine $G_{0,k}(\alpha)$ from (6). Set $t = u + iv = e^{iy}$. Since

$$A = \alpha - 2 + (1-\alpha)\cos y, \qquad \text{and} \qquad B = \alpha - 1 - \alpha\cos y,$$

we have

$$\begin{aligned} A^2 - B^2 &= [3 - 2\alpha - (1-2\alpha)\cos y](1-\cos y) = -\frac{(1-2\alpha)}{4t^2}(t - D_\alpha)(1/D_\alpha - t)(t-1)^2 \\ &= -\frac{(1-2\alpha)}{4t^2}(-t^2 + b_\alpha t - 1)(t-1)^2, \end{aligned}$$

where

$$b_\alpha = \frac{6-4\alpha}{1-2\alpha} \qquad \text{and} \qquad D_\alpha = \frac{b_\alpha}{2} - \frac{\sqrt{8(1-\alpha)}}{1-2\alpha}.$$

Some specific cases of interest include:

$$D_0 = 3 - 2\sqrt{2} = 0.17157; \quad D_{1/3} = 7 - 4\sqrt{3} = 0.071797; \quad D_{2/3} = -5 + 2\sqrt{6} = -0.10102;$$

$$\text{for } \alpha = 1/2: \qquad A^2 - B^2 = 2(1-\cos x) = -(t-1)^2/t;$$

$$D_\alpha \text{ is positive for } \alpha < 1/2; \qquad D_{1/2} = 0; \qquad D_\alpha \text{ is negative for } 1/2 < \alpha \le 1;$$

$$D_\alpha \text{ is complex and } |D_\alpha| = 1 \text{ for } \alpha > 1.$$

For $\alpha < 1/2$, and $1/2 < \alpha < 1$, (4) becomes (see (6) and (7))

(8)
$$\begin{aligned} G_{0,k}(\alpha) &= -\frac{1}{2\pi}\int_{t=e^{i0+}}^{t=e^{2\pi-}} \frac{t^k - 1}{t-1}\,\frac{1}{\sqrt{(1-2\alpha)(t-D_\alpha)(1/D_\alpha - t)}}\,dt \\ &= -\frac{1}{2\pi}\int_{t=e^{i0+}}^{t=e^{2\pi-}} \frac{1 + t + t^2 + \cdots + t^{k-1}}{\sqrt{(1-2\alpha)(t-D_\alpha)(1/D_\alpha - t)}}\,dt \end{aligned}$$

Because $G_{0,0}(\alpha) = 0$, it follows that for $k = 0, 1, \ldots,$

(9)
$$G_{0,k+1}(\alpha) = G_{0,k}(\alpha) - \frac{1}{2\pi}\int_{t=e^{i0+}}^{t=e^{2\pi-}} \frac{t^k\,dt}{\sqrt{(1-2\alpha)(t-D_\alpha)(1/D_\alpha - t)}}.$$

With reference to Figure 1(a) which illustrates the situation for $\alpha < 1/2$ (when $0 < D_\alpha < 1$) and Figure 1(b) which illustrates the situation for $1/2 < \alpha < 1$ (when $-1 < D_\alpha < 0$), the path of integration in (9) is along the arc abc of the unit circle. The integrand has branch points at D_α and $1/D_\alpha$, as indicated by 'bullets' in the figure; the dashed line indicates the branch cut we take between these points, beginning at D_α and going to the right along the real axis to $1/D_\alpha$. The integral is equal to zero along the closed contour abcdefa which does not enclose any singularities.

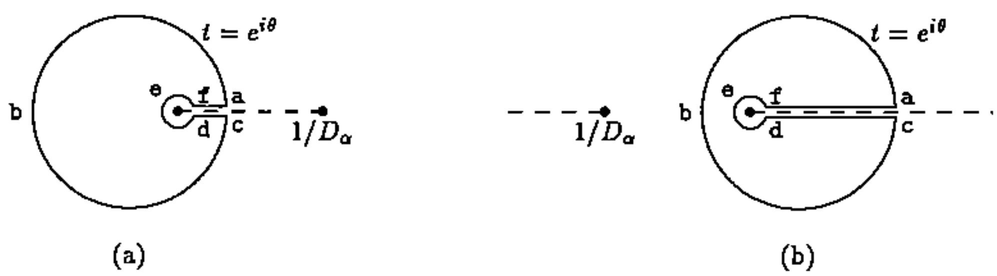

Figure 1.

With $t = D_\alpha + \rho e^{i\theta}$, $dt = \rho\, i\, e^{i\theta}\, d\theta$, and $\rho \ll 1$, one finds that the integrand is $O(\sqrt{\rho})$; hence the contribution along the arc def tends to zero as $\rho \downarrow 0$.

With $t = D_\alpha + \rho e^{i\theta}$, $dt = e^{i\theta}\, d\rho$, the integrals along the intervals cd and fa are

$$-\int_{\rho=0}^{1-D_\alpha} \frac{(D_\alpha + \rho e^{i2\pi})^k}{\sqrt{\rho}\, e^{i\pi}(1/D_\alpha - D_\alpha - \rho e^{i2\pi})}\, e^{i2\pi}\, d\rho, \qquad \text{and} \qquad -\int_{\rho=1-D_\alpha}^{0} \frac{(D_\alpha + \rho)^k}{\sqrt{\rho}\,(1/D_\alpha - D_\alpha - \rho)}\, d\rho,$$

respectively. Consequently, (9) becomes

$$G_{0,k+1}(\alpha) = G_{0,k}(\alpha) + \frac{1}{\pi} \int_{u=D_\alpha}^{1} \frac{u^k}{\sqrt{(1-2\alpha)(u - D_\alpha)(1/D_\alpha - u)}}\, du. \tag{10}$$

Let $\mathcal{I}_k(\alpha)$ denote the integral in (10):

$$\mathcal{I}_k(\alpha) = \frac{1}{\pi} \int_{u=D_\alpha}^{1} \frac{u^k}{\sqrt{U}}\, du \qquad \text{where} \quad U = (1-2\alpha)(-u^2 + b_\alpha u - 1). \tag{11}$$

Given the values of $\mathcal{I}_0$ and $\mathcal{I}_1$, the other values of $\mathcal{I}_k$ can be found by a recurrence relation. Because

$$\frac{d}{du} u^k \sqrt{U} = k u^{k-1}\sqrt{U} + \frac{(1-2\alpha)(-2u^{k+1} + b_\alpha u^k)}{2\sqrt{U}}$$
$$= (1-2\alpha)\frac{-(k+1)u^{k+1} + (k+1/2)u^k - k u^{k-1}}{\sqrt{U}}$$

we have

$$\begin{aligned} \mathcal{I}_{k+1} &= \frac{1}{k+1}\left\{(k+1/2)b_\alpha \mathcal{I}_k - k\mathcal{I}_{k-1} - \frac{1}{1-2\alpha}[u^k\sqrt{U}]\Big|_{u=D_\alpha}^{1}\right\} \\ &= \frac{1}{k+1}\left[(k+1/2)b_\alpha \mathcal{I}_k - k\mathcal{I}_{k-1} - \frac{2}{\pi(1-2\alpha)}\right]. \end{aligned} \tag{12}$$

When $\alpha < 1/2$ we have

$$\begin{aligned} \mathcal{I}_0(\alpha) &= \frac{1}{\pi\sqrt{1-2\alpha}} \int_{u=D_\alpha}^{1} \frac{1}{\sqrt{-u^2 + b_\alpha u - 1}}\, du = -\frac{1}{\pi\sqrt{1-2\alpha}}\left[\arcsin \frac{b_\alpha - 2u}{\sqrt{{b_\alpha}^2 - 4}}\right]\Bigg|_{u=D_\alpha}^{u=1} \\ &= \frac{1}{\pi\sqrt{1-2\alpha}}\left[\frac{\pi}{2} - \arcsin \frac{1}{\sqrt{2(1-\alpha)}}\right], \quad \text{for } \alpha < 1/2. \end{aligned} \tag{13}$$

To justify this choice from Dwight [60, p. 75, #380.001], we note that the argument of the square root in the arcsine is positive:

$${b_\alpha}^2 - 4 = (b_\alpha - 2)(b_\alpha + 2) = \frac{32(1-\alpha)}{(1-2\alpha)^2} > 0 \quad \text{for } \alpha < 1/2.$$

When u varies from D_α to unity, the argument of the arcsine decreases from

$$\frac{2\sqrt{8(1-\alpha)}/(1-2\alpha)}{\sqrt{32(1-\alpha)/(1-2\alpha)}} = \frac{1}{2}\sqrt{2} \qquad \text{to} \qquad \frac{1}{\sqrt{2(1-\alpha)}},$$

so the argument is positive and less than unity for $D_\alpha \le u \le 1$ and $\alpha < 1/2$.

For $1/2 < \alpha < 1$, the coefficient $(2\alpha - 1)$ of u^2 in U in (11) is positive, and Dwight [60, p. 75, #380.001], yields

$$\begin{aligned} \mathcal{I}_0(\alpha) &= \frac{1}{\pi\sqrt{2\alpha-1}} \int_{u=D_\alpha}^{1} \frac{1}{\sqrt{u^2 - b_\alpha u + 1}}\, du = \frac{\log[2\sqrt{u^2 - b_\alpha u + 1} + 2u - b_\alpha]\big|_{u=D_\alpha}^{u=1}}{\pi\sqrt{2\alpha - 1}} \\ &= \frac{1}{\pi\sqrt{2\alpha-1}} \log\left[\frac{\sqrt{2\alpha-1}+1}{\sqrt{2(1-\alpha)}}\right], \quad \text{for } 1/2 < \alpha < 1. \end{aligned} \tag{14}$$

We can use either (13) or (15) to evaluate the limit as α tends to 1/2. Thus with $\alpha = 1/2+\beta$,

$$\mathcal{I}_0(1/2+\beta) = \frac{1}{\pi\sqrt{2\beta}} \log\left[\frac{\sqrt{2\beta}+1}{\sqrt{1-\beta/2}}\right] = \frac{\log[1+\sqrt{2\beta}+\beta/4+\cdots]}{\pi\sqrt{2\beta}} = \frac{1}{\pi}+O(\sqrt{\beta}).$$

In summary, we have

$$G_{0,1}(\alpha) = \mathcal{I}_0(\alpha) = \begin{cases} \dfrac{1}{\pi\sqrt{1-2\alpha}}\left[\dfrac{\pi}{2} - \arcsin\dfrac{1}{\sqrt{2(1-\alpha)}}\right], & \text{if } \alpha < 1/2, \\ \dfrac{1}{\pi}, & \text{if } \alpha = 1/2, \\ \dfrac{1}{\pi\sqrt{2\alpha-1}} \log\left[\dfrac{\sqrt{2\alpha-1}+1}{\sqrt{2(1-\alpha)}}\right], & \text{if } 1/2 < \alpha < 1. \end{cases} \tag{16}$$

Finally, from Dwight [60, p. 75, 380.011] we obtain

$$\begin{aligned} \mathcal{I}_1(\alpha) &= \frac{1}{\pi\sqrt{1-2\alpha}} \int\limits_{u=D_\alpha}^{1} \frac{u}{\sqrt{U}}\,du = \frac{b_\alpha}{2}\mathcal{I}_0(\alpha) - \frac{\sqrt{U}}{\sqrt{\pi(1-2\alpha)}}\Bigg|_{u=D_\alpha}^{u=1} \\ &= \frac{b_\alpha}{2}\mathcal{I}_0(\alpha) - \frac{2}{\pi(1-2\alpha)}. \end{aligned} \tag{17}$$

Asymptotic expansion. McCrea-Whipple [40, (9.6)] present the result (in our notation: their G is 4 times our G)

$$2\pi G_{j,k}(0) \sim A_{j,k}^{(0)} = \log R + \gamma + \frac{3}{2}\log 2, \tag{18}$$

where γ is Euler's constant $R^2 = j^2+k^2$, and McCrea-Whipple give the next term of the expansion as $o(1/j)$. Here and below we use $A_{j,k}^{(m)}(\alpha)$ to denote the asymptotic expansion of $G_{j,k}(\alpha)$ through terms of order $1/R^m$. McCrea-Whipple's $A_{j,k}^{(0)}(0)$ is independent of angle and depends only on the distance R from the origin.

Duffin-Shaffer [60] derived the complete asymptotic expansion of the double Fourier Transforms of molifications of functions of the form $r^q x^m y^n + F_1(x,y)$, where $r^2 = x^2+y^2$, where m and n are nonnegative integers, and where F_1 has partial derivatives of all orders. As an example, they consider the discrete Green's Function of the standard 5 point approximation of the Laplacian, i.e., (4) with $\alpha = 0$. Duffin-Shaffer [60, Theorem 4] derive for $\alpha = 0$:

$$2\pi G_{j,k}(0) \sim A_{j,k}^{(2)}(0) = \log R + \gamma + \frac{3}{2}\log 2 - \frac{\cos 4\sigma}{12R^2}, \tag{19}$$

where $\sigma = \arctan(k/j)$. The $O(1/R^2)$ term includes an angular dependence, having the 4-fold symmetry of the square mesh.

We now present some new results.

McCrea-Whipple [40] obtained (18) by starting with the sum (5) defining the diagonal values $G_{j,j}(0)$ and using the definition of Euler's constant. The complete asymptotic expansion for this sum is easily obtained from the expansion of the logarithmic derivative of the Gamma function. Combine (6.3.2) and (6.3.18) of Abramowitz-Stegun [64, pp. 258–259] to get

$$1 + 1/2 + \cdots 1/(j-1) \sim \log j + \gamma - \frac{1}{2j} - \frac{1}{12j^2} + \frac{1}{120j^4} - \cdots - \frac{B_{2m}}{2mj^{2m}} - \cdots, \tag{20}$$

where B_m is the m-th Bernoulli number. Write this equation with $2j$ in place of j and subtract from the result one-half of (20), set $j = R\sqrt{2}$, and simplify to obtain

$$2\pi G_{j,j}(0) \sim \log R + \frac{3}{2}\log 2 + \frac{1}{12R^2} - \frac{7}{240R^4} + \cdots + \frac{B_{2m}(2^{2m-1}-1)}{2^{2m+1}mR^{2m}} + \cdots. \tag{21}$$

When $j = k$ so that $\sigma = \pi/4$, Duffin-Schaffer's (19) agree with the first three terms of (21).

Next we consider the asysmptotic expansion of $G_{j,k}(\alpha)$.

From the analysis of Duffin-Shaffer [60], a straightforward calculation yields the the asymptotic expansion to as many terms as one wants for general $\alpha < 1$; in particular through terms $O(1/R^4)$:

$$\begin{aligned} 2\pi G_{j,k}(\alpha) \sim 2\pi A^{(4)}_{j,k}(\alpha) = \log\, R + \gamma + \frac{1}{2}\log\left(\frac{8}{1-\alpha}\right) + \frac{(3\alpha-1)\cos 4\sigma}{12R^2} \\ + \frac{(90\alpha^2-18)\cos 4\sigma - (225\alpha^2 - 150\alpha + 25)\cos 8\sigma}{240R^4}. \end{aligned} \tag{22}$$

The next term in the asymptotic expansion is $O(1/R^6)$. For $\alpha = 0$ and $j = k$, this agrees with the comparable terms in (21).

Note that for the optimal 9-point discretization of the Laplacian ($\alpha = 1/3$), the coefficient of R^{-2} in (22) is zero.

We outline how this expression is obtained (more details are given in Lynch [90]). The function F used in the derivation of Duffin-Shaffer [60, p. 594] is replaced with

$$F(x, y; \alpha) = -\left[2\alpha - 4 + 2\{(1-\alpha)[\cos x + \cos y] + \alpha \cos x \cos y\}\right]^{-1} \tag{23}$$

and from (7) above, their double integral I_2 reduces to

$$(24)\qquad \begin{aligned} I_2 &= \int_{\epsilon}^{\pi} dy \int_{0}^{\pi} 4F(x,y;\alpha)\,dx = \int_{\epsilon}^{\pi} \frac{dy}{\sqrt{[3-2\alpha-(1-2\alpha)\cos y](1-\cos y)}} \\ &= 2\pi \int_{0}^{V(\epsilon)} \frac{dV}{1-V^2} = 2\pi[-\log \epsilon + \frac{1}{2}\log\left(\frac{8}{1-\alpha}\right) + O(\epsilon^2)]. \end{aligned}$$

where the change of variable

$$(25)\qquad V(y;\alpha) = \left(\frac{1+\cos y}{3-2\alpha-(1-2\alpha)\cos y}\right)^{1/2},$$

is a slight modification of the ingenuous change $V(y;0)$ used by Duffin-Shaffer. The rest of their analysis (pp. 594–595) remains unchanged to get

$$2\pi G_{j,k}(\alpha) \sim 2\pi A_{j,k}^{(0)}(\alpha) = \log R + \gamma + \frac{1}{2}\log\left(\frac{8}{1-\alpha}\right).$$

The other terms of the asymptotic expansion are obtained from the Taylor series representation

$$F_1(r\cos\theta, r\sin\theta;\alpha) = -F(r\cos\theta, r\sin\theta;\alpha) - r^{-2} = \Lambda_0(\theta) + \Lambda_1(\theta)r^2 + \Lambda_2(\theta)r^4 + \cdots.$$

and the use of Duffin-Shaffer [60, Theorem 2; see also pp. 595–596].

Relative errors

$$\mathcal{R}(\alpha;m) = \frac{G_{j,k}(\alpha) - A_{j,k}^{(m)}(\alpha)}{G_{j,k}(\alpha)}$$

are listed in Tables 8 – 10 and Tables 13 – 14

Computational instability. The computation of values $G_{j,k}(\alpha)$, beginning along the k-axis and progressing stepwise into the first quadrant, is inherently unstable because one is thereby solving an elliptic difference equation as an 'initial value' problem. Thus, with a fixed precision arithmetic, roundoff error eventually becomes larger than the values of G.

Similarly, to get accurate values using expressions such as in Table 3 , one needs to use greater precision arithmetic the greater the distance from the origin. For example, extending the table, one finds

$$G_{10,0}(1/3) = 1985408996\sqrt{3}/3 - 126039765042/35\pi$$

$$= 1146276418.2921045\ldots - 1146276417.6360281\ldots = 0.6560764\ldots,$$

and thus, 10 significant digits are lost when the subtraction is carried out. Likewise, when $G_{10,10}(1/3)$ is evaluated from table entries, the difference between two numbers each equal to about 2×10^{17} is formed to yield a value which is equal to about 0.7; 17 significant digits are lost.

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Appendix 1

Although the derivation of the asymptotic expression (22) is a direct consequence of results of Duffin-Shaffer [60], it took us a considerable time to derive and check the equations. Here we record some of the intermediate steps.

First consider their integral I_2 (see (24)).

From (25) one obtains

$$dV = \frac{2(\alpha-1)\sin y}{[(2\alpha-1)\cos y - 2\alpha+3]^{3/2}\,[1+\cos y]^{1/2}}\,dy$$

and

$$1 - V^2 = \frac{2(\alpha-1)(1-\cos y)}{(2\alpha-1)\cos y - 2\alpha + 3}.$$

Using $\sin y = \sqrt{1+\cos y}\sqrt{1-\cos y}$, one gets

$$\frac{dV}{1-V^2} = -\frac{dy}{\sqrt{[3-2\alpha-(1-2\alpha)\cos y](1-\cos y)}}.$$

For the limits, one finds that $V=0$ when $y=\pi$ and when $y=\epsilon \ll 1$

$$\begin{aligned} V(\epsilon) &= \left[\frac{2-\epsilon^2/2+\cdots}{2+(1-2\alpha)\epsilon^2/2+\cdots}\right]^{1/2} \\ &= 1-\epsilon^2/8-(1-2\alpha)\epsilon^2/8+\cdots = 1-2(1-\alpha)\epsilon^2/8+\cdots. \end{aligned}$$

Finally, we have

$$\begin{aligned} \int_0^{V(\epsilon)} \frac{dV}{1-V^2} &= \frac{1}{2}\left[-\log(1-V)+\log(1+V)\right]\Big|_0^{V(\epsilon)} \\ &= \frac{1}{2}\log\left(\frac{4}{1-\alpha}\right) - \log\epsilon + \frac{1}{2}\log 2 + O(\epsilon^2) \end{aligned}$$

from which (25) follows.

Next we consider the terms in the asymptotic expansion which depend of the angle between an axis and a mesh point (j,k).

With $r^2 = x^2 + y^2$, $c = \cos\theta$, and $s = \sin\theta$, and F as in (23), the first few terms of the Taylor series exansion of $F_1 = -F - 1/r^2$ are (after use of elementary trig identities)

$$
\begin{aligned}
F_1(rc, rs; \alpha) &= \frac{1}{12}[1 + (6\alpha - 2)c^2 s^2] \\
&\quad + \frac{r^2}{720}[3 + (30\alpha - 14)c^2 s^2 + (20 - 120\alpha + 180\alpha^2)c^4 s^4] + \cdots \qquad (1) \\
&= \Lambda_0(\theta) + \Lambda_1(\theta) r^2 + \cdots .
\end{aligned}
$$

The Fourier transforms of molified $1/12$ and $3r^2/720$, which appear in $\Lambda_0(\theta)$ $\Lambda_1(\theta)r^2$, are 'completely asymptotic to zero' (Duffin-Shaffer [60, p. 582]) and can be ignored.

Set $R^2 = j^2 + k^2$, $\sigma = \arctan k/j$, $C = \cos\sigma$, and $S = \sin\sigma$. It is a consequence of Duffin-Shaffer [60, Theorem 2 (see also paragraph 2 of p. 596)]) that the Fourier Transform T of molified $c^2s^2 = x^2y^2/r^4$ is [Theorem 2 with $q = -4$, $n = 1$]

$$
T(c^2 s^2) = T\left(\frac{x^2 y^2}{r^4}\right) \sim \frac{1}{4}\frac{\partial^4}{\partial x^2\,\partial y^2} R^2 \log R = -\frac{(C^4 - 6C^2S^2 + S^4)}{2} = -\frac{\cos(4\sigma)}{2}.
$$

Similarly [Theorem 2 with $q = -2$, $n = 0$]

$$
T(r^2 c^2 s^2) = T\left(\frac{x^2 y^2}{r^2}\right) \sim -\frac{\partial^4}{\partial x^2\,\partial y^2} \log R = -\frac{6\cos 4\sigma}{R^4},
$$

and [Theorem 2 with $q = -6$, $n = 2$]

$$
T(r^2 c^4 s^4) = T\left(\frac{x^4 y^4}{r^6}\right) \sim -\frac{1}{64}\frac{\partial^8}{\partial x^4\,\partial y^4} R^4 \log R = \frac{15\cos 8\sigma - 6\cos 4\sigma}{R^4}.
$$

Appendix 2: Tables

Table 1, Parts 1 and 2: Formulas for $G_{j,k}(0)$

Table 2: Numerical values of $G_{j,k}(0)$

Table 3, Parts 1 through 5: Formulas for $G_{j,k}(1/3)$

Table 4: Numerical values of $G_{j,k}(1/3)$

Table 5 : Numerical values of error $G_{j,k}(0) - A^{(0)}_{j,k}(0)$

Table 6 : Numerical values of error $G_{j,k}(0) - A^{(2)}_{j,k}(0)$

Table 7 : Numerical values of error $G_{j,k}(0) - A^{(4)}_{j,k}(0)$

Table 8 : Relative Error $\mathcal{R}(0;0) = [G_{j,k}(0) - A^{(0)}_{j,k}(0)]/G_{j,k}(0)$

Table 9 : Relative Error $\mathcal{R}(0;2) = [G_{j,k}(0) - A^{(2)}_{j,k}(0)]/G_{j,k}(0)$

Table 10 : Relative Error $\mathcal{R}(0;4) = [G_{j,k}(0) - A^{(4)}_{j,k}(0)]/G_{j,k}(0)$

Table 11 : Numerical values of error $G_{j,k}(1/3) - A^{(0)}_{j,k}(1/3) = G_{j,k}(1/3) - A^{(2)}_{j,k}(1/3)$

Table 12 : Numerical values of error $G_{j,k}(1/3) - A^{(4)}_{j,k}(1/3)$

Table 13 : Numerical values of realative error $\mathcal{R}(1/3;0) = \mathcal{R}(1/3;2) = (G_{j,k}(0) - A^{(0)}_{j,k}(0))/G_{j,k}(0) = (G_{j,k}(0) - A^{(2)}_{j,k}(0))/G_{j,k}(0)$

Table 14 : Numerical values of realative error $\mathcal{R}(1/3;4) = (G_{j,k}(1/3) - A^{(4)}_{j,k}(1/3))/G_{j,k}(1/3)$

$k=5$	*	*	*	*	*	$\frac{563}{315\pi}$
$k=4$	*	*	*	*	$\frac{176}{105\pi}$	$\frac{20}{21\pi}+\frac{1}{4}$
$k=3$	*	*	*	$\frac{23}{15\pi}$	$\frac{12}{5\pi}-\frac{1}{4}$	$\frac{499}{35\pi}-4$
$k=2$	*	*	$\frac{4}{3\pi}$	$\frac{2}{3\pi}+\frac{1}{4}$	$3-\frac{118}{15\pi}$	$\frac{97}{4}-\frac{1118}{15\pi}$
$k=1$	*	$\frac{1}{\pi}$	$\frac{2}{\pi}-\frac{1}{4}$	$\frac{23}{3\pi}-2$	$\frac{40}{\pi}-\frac{49}{4}$	$\frac{3323}{15\pi}-70$
$k=0$	0	$\frac{1}{4}$	$1-\frac{2}{\pi}$	$\frac{17}{4}-\frac{12}{\pi}$	$20-\frac{184}{3\pi}$	$\frac{401}{4}-\frac{940}{3\pi}$
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$

Table 1 , part 1: Fundamental Solution $G_{j,k}(0)$

$k=5$					
$k=4$	$5-\frac{626}{45\pi}$				
$k=3$	$\frac{13462}{105\pi}-\frac{161}{4}$	$\frac{327143}{315\pi}-330$			
$k=2$	$168-\frac{18412}{35\pi}$	$\frac{4321}{4}-\frac{71230}{21\pi}$	$6671-\frac{6601046}{315\pi}$		
$k=1$	$\frac{1234}{\pi}-\frac{1569}{4}$	$\frac{721937}{105\pi}-2188$	$\frac{191776}{5\pi}-\frac{48833}{4}$	$\frac{6734979}{315\pi}-68244$	
$k=0$	$521-\frac{24526}{15\pi}$	$\frac{11073}{4}-\frac{130424}{15\pi}$	$14928-\frac{4924064}{105\pi}$	$\frac{325441}{4}-\frac{1789192}{7\pi}$	$447001-\frac{442352326}{315\pi}$
	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 1 , part 2: Fundamental Solution $G_{j,k}(0)$

$k=5$	*	*	*	*	*	0.56892					
$k=4$	*	*	*	*	0.53355	0.55315	0.57196				
$k=3$	*	*	*	0.48808	0.51394	0.53819	0.56036	0.58048			
$k=2$	*	*	0.42441	0.46221	0.49596	0.52530	0.55081	0.57318	0.59301		
$k=1$	*	0.31831	0.38662	0.44038	0.48240	0.51625	0.54440	0.56842	0.58935	0.60787	
$k=0$	0	0.25000	0.36338	0.43028	0.47699	0.51290	0.54212	0.56676	0.58808	0.60687	0.62368
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 2 . Fundamental Solution $G_{j,k}(0)$

$k=6$	*	*	*	*	*
$k=5$	*	*	*	*	*
$k=4$	*	*	*	*	$468888-\frac{541280\sqrt{3}}{3}-\frac{491280}{\pi}$
$k=3$	*	*	*	$\frac{11475}{2}-\frac{5976}{\pi}-2214\sqrt{3}$	$\frac{46824}{\pi}+\frac{102385\sqrt{3}}{6}-44460$
$k=2$	*	*	$78-\frac{84}{\pi}-\frac{88\sqrt{3}}{3}$	$\frac{516}{\pi}+\frac{1241\sqrt{3}}{6}-522$	$2820-\frac{3438}{\pi}-996\sqrt{3}$
$k=1$	*	$\frac{3}{2}-\frac{2\sqrt{3}}{3}$	$\frac{12}{\pi}+\frac{3\sqrt{3}}{2}-6$	$\frac{36}{\pi}-\frac{58\sqrt{3}}{3}+\frac{45}{2}$	$\frac{816}{\pi}-\frac{607\sqrt{3}}{6}-84$
$k=0$	0	$\frac{\sqrt{3}}{6}$	$\frac{4\sqrt{3}}{3}-\frac{6}{\pi}$	$\frac{27\sqrt{3}}{2}-\frac{72}{\pi}$	$\frac{464\sqrt{3}}{3}-\frac{840}{\pi}$
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$

Table 3 , part 1: Fundamental Solution $G_{j,k}(1/3)$

$k=6$	*	$3548089278-\frac{18577857348}{5\pi}-1365656328\sqrt{3}$
$k=5$	$\frac{80442003}{2}-\frac{4211700}{\pi}-\frac{46444498\sqrt{3}}{3}$	$\frac{1811805348}{5\pi}+\frac{799053497\sqrt{3}}{6}-346009986$
$k=4$	$\frac{4064088}{\pi}+\frac{2990043\sqrt{3}}{2}-3883092$	$27503244-\frac{144196278}{5\pi}-\frac{31737100\sqrt{3}}{3}$
$k=3$	$\frac{571437}{2}-\frac{294960}{\pi}-\frac{33228\sqrt{3}}{3}$	$\frac{88988804}{5\pi}+\frac{1239219\sqrt{3}}{2}-1639710$
$k=2$	$\frac{7668}{\pi}+\frac{39289\sqrt{3}}{6}-13782$	$63594-\frac{792948}{5\pi}-\frac{22712\sqrt{3}}{3}$
$k=1$	$\frac{8844}{\pi}-1806\sqrt{3}+\frac{627}{2}$	$\frac{592668}{5\pi}-\frac{126647\sqrt{3}}{6}-1170$
$k=0$	$\frac{11249\sqrt{3}}{6}-\frac{10200}{\pi}$	$23436\sqrt{3}-\frac{637614}{5\pi}$
	$j=5$	$j=6$

Table 3 , part 2: $G_{j,k}(1/3)$

$k=8$	*
$k=7$	$\frac{637522286979}{2} - \frac{333805695888}{\pi} - \frac{368073770402\sqrt{3}}{3}$
$k=6$	$\frac{163810235436}{5\pi} + \frac{72251260201\sqrt{3}}{6} - 312852657242$
$k=5$	$\frac{5213506893}{2} - \frac{13647204396}{5\pi} - 1003403262\sqrt{3}$
$k=4$	$\frac{913117824}{5\pi} + \frac{404669057\sqrt{3}}{6} - 174948780$
$k=3$	$\frac{17461971}{2} - \frac{41231028}{5\pi} + \frac{10576138\sqrt{3}}{3}$
$k=2$	$\frac{673443\sqrt{3}}{2} - \frac{4720308}{5\pi} - 282714$
$k=1$	$\frac{7564632}{5\pi} - \frac{841682\sqrt{3}}{3} + \frac{8733}{2}$
$k=0$	$\frac{1792225\sqrt{3}}{6} - \frac{8126832}{5\pi}$
	$j=7$

Table 3 , part 3: Fundamental Solution $G_{j,k}(1/3)$

$k=8$	$29007966233184 - \frac{1063197554675904}{35\pi} - \frac{33495513241984\sqrt{3}}{3}$
$k=7$	$\frac{104814921206832}{35\pi} + \frac{2201426511819\sqrt{3}}{2} - 2859737730648$
$k=6$	$245947699224 - \frac{1287795020238}{5\pi} - \frac{283994189852\sqrt{3}}{3}$
$k=5$	$\frac{652517188536}{35\pi} + \frac{41096467057\sqrt{3}}{6} - 17797890216$
$k=4$	$1029676080 - \frac{7608458712}{7\pi} - 394733136\sqrt{3}$
$k=3$	$\frac{2057239656}{35\pi} + \frac{87922385\sqrt{3}}{6} - 44090712$
$k=2$	$1224456 - \frac{89769766}{5\pi} + \frac{7777604\sqrt{3}}{3}$
$k=1$	$\frac{138772800}{7\pi} - \frac{7267797\sqrt{3}}{2} - 16296$
$k=0$	$\frac{11576128\sqrt{3}}{3} - \frac{734887008}{35\pi}$
	$j=8$

Table 3 , part 4: Fundamental Solution $G_{j,k}(1/3)$

$k{=}9$	$\frac{5330168522889219}{2} - \frac{97680439347912144}{35\pi} - \frac{1025791413567606\sqrt{3}}{3}$
$k{=}8$	$\frac{1932242417187984}{7\pi} + \frac{608743524607649\sqrt{3}}{3} - 263593670884776$
$k{=}7$	$\frac{46383570998973}{2} - \frac{850022301711936}{35\pi} - \frac{26779583469442\sqrt{3}}{3}$
$k{=}6$	$\frac{12959045776836}{7\pi} + \frac{1360950893091\sqrt{3}}{2} - 1767902673330$
$k{=}5$	$\frac{225392535603}{2} - \frac{4127627044932}{35\pi} - \frac{130176231386\sqrt{3}}{3}$
$k{=}4$	$\frac{41022211488}{7\pi} + \frac{13339376065\sqrt{3}}{6} - 5716142676$
$k{=}3$	$\frac{428117805}{2} - \frac{48829140}{\pi} - 114613326\sqrt{3}$
$k{=}2$	$\frac{259085609\sqrt{3}}{6} - \frac{1530359772}{7\pi} - 5201766$
$k{=}1$	$\frac{9112006872}{35\pi} - \frac{143639986\sqrt{3}}{3} + \frac{121635}{2}$
$k{=}0$	$\frac{100685835\sqrt{3}}{2} - \frac{9587754864}{35\pi}$
	$j{=}9$

Table 3 , part 5: Fundamental Solution $G_{j,k}(1/3)$

$k{=}5$	*	*	*	*	*	0.60092					
$k{=}4$	*	*	*	*	0.56541	0.58513	0.60404				
$k{=}3$	*	*	*	0.51963	0.54577	0.57023	0.59253	0.61273			
$k{=}2$	*	*	0.45515	0.49374	0.52801	0.55757	0.58316	0.60555	0.62539		
$k{=}1$	*	0.34530	0.41779	0.47284	0.51506	0.54887	0.57695	0.60092	0.62180	0.64028	
$k{=}0$	0	0.28868	0.39954	0.46437	0.51022	0.54575	0.57477	0.59931	0.62056	0.63931	0.65608
	$j{=}0$	$j{=}1$	$j{=}2$	$j{=}3$	$j{=}4$	$j{=}5$	$j{=}6$	$j{=}7$	$j{=}8$	$j{=}9$	$j{=}10$

Table 4 . Fundamental Solution $G_{j,k}(1/3)$

$k=5$	*	*	*	*	*	2.63E−04					
$k=4$	*	*	*	*	4.10E−04	2.92E−04	1.82E−04				
$k=3$	*	*	*	7.23E−04	4.50E−04	2.27E−04	9.09E−05	1.65E−05			
$k=2$	*	*	1.59E−03	7.51E−04	2.26E−04	−1.16E−06	−8.44E−05	−1.08E−04	−1.08E−04		
$k=1$	*	5.81E−03	1.20E−03	−2.02E−04	−4.08E−04	−3.64E−04	−2.92E−04	−2.30E−04	−1.84E−04	−1.49E−04	
$k=0$	0	−7.34E−03	−4.28E−03	−1.91E−03	−9.85E−04	−5.91E−04	−3.95E−04	−2.84E−04	−2.16E−04	−1.68E−04	−1.36E−04
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 5 . Error $G_{j,k}(0) - A^{(0)}_{j,k}(0)$

$k=5$	*	*	*	*	*	3.84E−04					
$k=4$	*	*	*	*	5.11E−04	4.50E−04	3.44E−04				
$k=3$	*	*	*	6.54E−04	6.18E−04	4.38E−04	2.89E−04	1.89E−04			
$k=2$	*	*	5.98E−04	8.47E−04	5.23E−04	2.88E−04	1.61E−04	9.39E−05	5.76E−05		
$k=1$	*	−8.24E−04	1.20E−03	4.62E−04	1.44E−04	4.83E−05	1.89E−05	8.58E−06	4.45E−06	2.56E−06	
$k=0$	0	5.92E−03	−9.65E−04	−4.38E−04	−1.56E−04	−6.06E−05	−2.70E−05	−1.37E−05	−7.70E−06	−4.69E−06	−3.03E−06
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 6 . Error $G_{j,k}(0) - A^{(2)}_{j,k}(0)$

$k=5$	*	*	*	*	*	3.86E−04					
$k=4$	*	*	*	*	5.15E−04	4.50E−04	3.41E−04				
$k=3$	*	*	*	6.68E−04	6.13E−04	4.27E−04	2.81E−04	1.84E−04			
$k=2$	*	*	6.71E−04	7.97E−04	4.79E−04	2.69E−04	1.54E−04	9.22E−05	5.77E−05		
$k=1$	*	3.37E−04	5.08E−04	3.55E−04	1.45E−04	6.05E−05	2.87E−05	1.54E−05	9.07E−06	5.73E−06	
$k=0$		3.44E−02	8.17E−04	−8.59E−05	−4.51E−05	−1.50E−05	−4.96E−06	−1.80E−06	−7.42E−07	−3.47E−07	−1.77E−07
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 7 . Error $G_{j,k}(0) - A^{(4)}_{j,k}(0)$

$k=5$	*	*	*	*	*	4.63E−04					
$k=4$	*	*	*	*	7.69E−04	5.28E−04	3.19E−04				
$k=3$	*	*	*	1.48E−03	8.76E−04	4.22E−04	1.62E−04	2.84E−05			
$k=2$	*	*	3.75E−03	1.62E−03	4.56E−04	−2.20E−06	−1.53E−04	−1.88E−04	−1.83E−04		
$k=1$	*	1.82E−02	3.11E−03	−4.58E−04	−8.46E−04	−7.06E−04	−5.36E−04	−4.05E−04	−3.12E−04	−2.46E−04	
$k=0$		−2.94E−02	−1.18E−02	−4.44E−03	−2.07E−03	−1.16E−03	−7.29E−04	−5.02E−04	−3.65E−04	−2.78E−04	−2.18E−04
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 8 , Relative Error $\mathcal{R}(0;0) = [G_{j,k}(0) - A^{(0)}_{j,k}(0)]/G_{j,k}(0)$

$k=5$	*	*	*	*	*	6.75E−04					
$k=4$	*	*	*	*	9.57E−04	8.14E−04	6.02E−04				
$k=3$	*	*	*	1.34E−03	1.20E−03	8.14E−04	5.16E−04	3.26E−04			
$k=2$	*	*	1.41E−03	1.83E−03	1.05E−03	5.49E−04	2.92E−04	1.64E−04	9.71E−05		
$k=1$	*	−2.59E−03	3.11E−03	1.05E−03	2.98E−04	9.36E−05	3.46E−05	1.51E−05	7.54E−06	4.21E−06	
$k=0$		2.37E−02	−2.66E−03	−1.02E−03	−3.28E−04	−1.18E−04	−4.97E−05	−2.41E−05	−1.31E−05	−7.73E−06	−4.86E−06
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 9 , Relative Error $\mathcal{R}(0;2) = [G_{j,k}(0) - A^{(2)}_{j,k}(0)]/G_{j,k}(0)$

$k=5$	*	*	*	*	*	6.78E−04					
$k=4$	*	*	*	*	9.65E−04	8.13E−04	5.96E−04				
$k=3$	*	*	*	1.37E−03	1.19E−03	7.94E−04	5.01E−04	3.17E−04			
$k=2$	*	*	1.58E−03	1.72E−03	9.66E−04	5.13E−04	2.80E−04	1.61E−04	9.72E−05		
$k=1$	*	1.06E−03	1.32E−03	8.07E−04	3.01E−04	1.17E−04	5.28E−05	2.71E−05	1.54E−05	9.42E−06	
$k=0$		1.38E−01	2.25E−03	−2.00E−04	−9.45E−05	−2.92E−05	−9.14E−06	−3.17E−06	−1.26E−06	−5.73E−07	−2.84E−07
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 10 , Relative Error $\mathcal{R}(0;4) = [G_{j,k}(0) - A^{(4)}_{j,k}(0)]/G_{j,k}(0)$

$k=5$	*	*	*	*	*	2.03E−06					
$k=4$	*	*	*	*	4.85E−06	2.76E−06	1.39E−06				
$k=3$	*	*	*	1.44E−05	6.84E−06	2.66E−06	8.25E−07	1.37E−07			
$k=2$	*	*	5.99E−05	2.18E−05	4.84E−06	1.58E−07	−7.84E−07	−7.85E−07	−6.26E−07		
$k=1$	*	5.31E−04	1.11E−04	−2.93E−06	−8.84E−06	−5.45E−06	−3.10E−06	−1.83E−06	−1.14E−06	−7.21E−07	
$k=0$		−9.34E−04	−3.85E−04	−8.48E−05	−2.44E−05	−9.34E−06	−4.35E−06	−2.29E−06	−1.34E−06	−8.40E−07	−6.87E−07
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 11 . Error $G_{j,k}(1/3) - A^{(0)}_{j,k}(1/3) = G_{j,k}(1/3) - A^{(2)}_{j,k}(1/3)$

$k=5$	*	*	*	*	*	−8.99E−08					
$k=4$	*	*	*	*	−3.29E−07	−9.44E−08	4.71E−09				
$k=3$	*	*	*	−1.96E−06	−3.16E−07	9.84E−08	9.12E−08	6.00E−08			
$k=2$	*	*	−2.30E−05	−2.93E−07	1.13E−06	4.65E−07	1.44E−07	4.96E−08	1.36E−08		
$k=1$	*	−7.95E−04	5.11E−05	1.19E−05	1.39E−06	7.39E−08	−3.75E−08	−3.95E−08	−3.46E−08	−7.80E−09	
$k=0$		4.37E−03	−5.38E−05	−1.93E−05	−3.68E−06	−8.48E−07	−2.55E−07	−8.42E−08	−4.20E−08	−3.14E−08	−1.57E−07
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 12 . Error $G_{j,k}(1/3) - A^{(4)}_{j,k}(1/3)$

$k=5$	*	*	*	*	*	3.38E−06					
$k=4$	*	*	*	*	8.58E−06	4.71E−06	2.29E−06				
$k=3$	*	*	*	2.77E−05	1.25E−05	4.66E−06	1.39E−06	2.23E−07			
$k=2$	*	*	1.32E−04	4.42E−05	9.18E−06	2.83E−07	−1.35E−06	−1.30E−06	−1.00E−06		
$k=1$	*	1.54E−03	2.65E−04	−6.19E−06	−1.72E−05	−9.93E−06	−5.37E−06	−3.04E−06	−1.83E−06		−3.52E−07
$k=0$		−3.24E−03	−9.64E−04	−1.83E−04	−4.78E−05	−1.71E−05	−7.56E−06	−3.83E−06	−2.15E−06	−1.31E−06	−1.05E−06
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 13 . Relative Error $\mathcal{R}(1/3;0) = \mathcal{R}(1/3;2) = (G_{j,k}(0) - A^{(0)}_{j,k}(0))/G_{j,k}(0) = (G_{j,k}(0) - A^{(2)}_{j,k}(0))/G_{j,k}(0)$

$k=5$	*	*	*	*	*	−1.50E−07					
$k=4$	*	*	*	*	−5.82E−07	−1.61E−07	7.80E−09				
$k=3$	*	*	*	−3.77E−06	−5.79E−07	1.73E−07	1.54E−07	9.79E−08			
$k=2$	*	*	−5.06E−05	−5.94E−07	2.14E−06	8.34E−07	2.47E−07	8.19E−08	2.18E−08		
$k=1$	*	−2.30E−03	1.22E−04	2.52E−05	2.69E−06	1.35E−07	−6.49E−08	−6.58E−08	−5.57E−08	−1.22E−08	
$k=0$		1.51E−02	−1.35E−04	−4.16E−05	−7.22E−06	−1.55E−06	−4.43E−07	−1.41E−07	−6.76E−08	−4.91E−08	−2.39E−07
	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$

Table 14 . Relative Error $\mathcal{R}(1/3;4) = (G_{j,k}(1/3) - A^{(4)}_{j,k}(1/3))/G_{j,k}(1/3)$