Purdue University

Purdue e-Pubs

Department of Computer Science Technical Reports

Department of Computer Science

1977

The Complexity of Numerical Methods for Elliptic Partial Differential Equations

Elias N. Houstis
Purdue University, enh@cs.purdue.edu

Report Number:
77-253

The Complexity of Numerical Methods for Elliptic Partial Differential Equations

E. N. Houstis
Department of Computer Sciences
Purdue University
West Lafayette, IN 47907

CSD-TR 253

November 1977

The Complexity of Numerical Methods for Elliptic Partial Differential Equations

E.N.Houstis *

Abstract

We consider three Ritz-Galerkin procedures with Hermite bicubic, bicubic spline and linear triangular elements for approximating the solution of self-adjoint elliptic partial differential equations and a Collocation with Hermite bicubics method for general linear elliptic equations defined on general two dimensional domains with mixed boundary conditions.We systematically evaluate these methods by applying them to a sample set of problems while measuring various performance criteria.The test data suggest that Collocation is the most efficient method for general use.

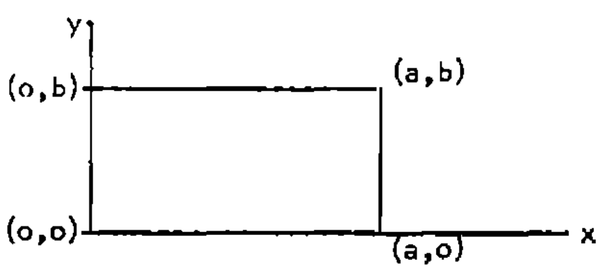
*Department of Computer Sciences Purdue University,West Lafayette, Indiana 47907

1. Introduction. In this paper, we consider three Ritz-Galerkin procedures with Hermite bicubic, bicubic spline and linear triangular elements for approximating the solution of self-adjoint elliptic partial differential equations and a Collocation with Hermite bicubics method applied to general linear elliptic equations defined on two-dimensional domains with mixed boundary conditions.

The four finite element procedures are described in Section 2-7. In Section 8 we study the structure of the linear algebraic systems for the determination of the approximate solution obtained by the mention of finite element methods. In Section 9 we deal with the direct solution of such systems. The collocation equations for rectangular domains are solved with a profile, a sparse and an almost block diagonal Gauss elimination scheme with partial pivoting for unsymmetric band matrices. In Section 10 we present a comparison of the considered finite element methods over a test set of eight problems used by Houstis, et. al. in [4].

The principal conclusion is that collocation is the most efficient method for general use. The Galerkin with bicubic splines for rectangular domains turns to be competitive to collocation for self-adjoint problems with simple functions in the differential operator and high accuracy requirements.

2. <u>The piecewise bicubic Hermite element.</u> Given the one-dimensional mesh $\Delta_x = \{a = x_0 < x_1 < \dots < x_N = b\}$, let $H(\Delta_x)$ be the space of piecewise cubic polynomials with respect to Δ_x which are continuously differentiable in $[a,b]$. We will denote by $H_0(\Delta_x)$ the set of functions $p \in H(\Delta_x)$ which satisfy the boundary conditions $p(a) = p(b) = 0$. Given the mesh $\Delta_y = \{c = y_0 < y_1 < \dots < y_M = d\}$ the space $H(\Delta_y)$ is defined analogously. In order to introduce a representation of a bicubic rectangular Hermite element we consider 8 one-dimensional functions.



$s = x/a$ and $0 \le s \le 1$

$t = y/b$ and $0 \le t \le 1$

$$B_{x1} = 1 - 3s^2 + 2s^3 \qquad B_{y1} = 1 - 3t^2 + 2t^3$$
$$B_{x2} = s^2(3-2s) \qquad B_{y2} = t^2(3-2t)$$
$$B_{x3} = as(s-1)^2 \qquad B_{y3} = bt(t-1)^2$$
$$B_{x4} = as^2(s-1) \qquad B_{y4} = bt^2(t-1)$$

Then the bicubic rectangular element is defined by

$$\begin{aligned} U(x,y) = {} & B_{x1} B_{y1} u_1 + B_{x2} B_{y1} u_2 + B_{x2} B_{y2} u_3 + B_{x1} B_{y2} u_4 \\ & + B_{x3} B_{y1} \sigma_{x1} + B_{x4} B_{y1} \sigma_{x2} + B_{x4} B_{y2} \sigma_{x3} + B_{x3} B_{y2} \sigma_{x4} \\ & + B_{x1} B_{y3} \sigma_{y1} + B_{x2} B_{y3} \sigma_{y2} + B_{x2} B_{y4} \sigma_{y3} + B_{x1} B_{y4} \sigma_{y4} \\ & + B_{x3} B_{y3} \tau_{xy1} + B_{x4} B_{y3} \tau_{xy2} + B_{x4} B_{y4} \tau_{xy3} + B_{x3} B_{y4} \tau_{xy4} \end{aligned}$$

where u_i = value at the point i

σ_{xi}, σ_{yi} = x and y derivatives at the point i

τ_{xyi} = xy (cross) derivative at the point i.

We denote by $B_i(x,y)$, i = 1, 16 the 16 basis functions in the above representation; i.e.

$$B_1 \equiv B_{x1} B_{x1},\quad B_2 \equiv B_{x3} B_{y1},\ \ldots,\ B_{13} \equiv B_{x1} B_{y2},\ \ldots,\ B_{16} \equiv B_{x3} B_{y4}.$$

3. The piecewise bicubic Spline Element. Let $S_0(\Delta_x)$ be the space of functions s(x) which are cubic polynomials in each subinterval $[x_i x_{i+1}]$, twice continuously differentiable in [a,b], and satisfy the boundary conditions s(a) = s(b) = 0. We choose the B-spline basis for the piecewise polynomial spare $S_0(\Delta_x)$ and denote them by $\{\phi_i(x)\}_{i=0}^{N}$. The graph of $\phi_i(x)$ is

The space $S_0(\Delta_y)$ and the corresponding basis $\{\phi_j(y)\}_{j=0}^{N}$ are defined analogously.

Then the bicubic spline is defined in each subrectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ by

$$U(x,y) = \sum_{k=i-3}^{i} \sum_{\ell=j-3}^{j} \alpha_{k\ell}\, \phi_k(x)\phi_\ell(y) \quad .$$

We denote $B_m(x,y) \equiv \phi_k(x)\phi_\ell(y)$ for $m = k + (n+1)\ell+1$, $0 \le k,\ell \le N$, $\rho = \Delta_x \times \Delta_y$ and $S_0(\rho)$ the space of bicubic splines respresented by

$$s(x,y) = \sum_{m=1}^{(N+1)^2} \beta_m B_m(x,y) \quad .$$

4. <u>Collocation with Hermite bicubic elements</u>. This method is used for approximating the solution $u(x,y)$ of the linear elliptic boundary value problem

(4.1) $Lu \equiv \alpha(x,y)u_{xx} + 2\beta(x,y)u_{xy} + \gamma(x,y)u_{yy} + \delta(x,y)u_x + \epsilon(x,y)u_y + \zeta(x,y)u$
$= f(x,y)$ defined on a general domain Ω and subject to mixed type boundary conditions

(4.2) $Bu \equiv a(x,y)u_x + b(x,y)u_y + c(x,y)u = g(x,y)$ on $\partial\Omega \equiv$ boundary of Ω.

This method consists of five components:

(i) Partition: A rectangular grid is placed over the domain Ω. Rectangular elements whose center is not inside the domain are discarded.

(ii) Approximation space: the Hermite bicubics

(iii) Operator discretization: Each bicubic element satisfies the differential equation exactly at the four Gauss points of the rectangular element. For elements that overlap the boundary the four Gauss points were projected in the portion of the element inside the domain.

(iv) Discretization of boundary conditions: The boundary conditions are interpolated at a selected set of boundary points (see [4]). If the domain is a rectangle and the problem has homogeneous Dirichlet or Neumann boundary conditions, then the Hermite bicubics were selected to satisfy the boundary conditions.

(v) Equation solution: The linear system is solved by these direct equation solvers based on Gauss elimination. A description of the equation solution algorithms will be given in Section 7.

The error analysis of this method for rectangular regions is given by Houstis in [3]. The computer implementation of the above described Collocation method used for the numerical experimentation is due to Houstis and Rice [5].

5. Ritz-Galerkin with Hermite bicubic elements. This method is used to approximate the solution $u(x,y)$ of the self-adjoint boundary value problem.

(5.1) $Lu \equiv -D_x(p(x,y)D_y u) - D_y(q(x,y)D_y n) + c(x,y)n = f(x,y)$ on a rectangular domain subject to homogeneous boundary conditions.

(5.2) $u(x,y) = 0$ or $\frac{\partial u}{\partial n} = 0$ or $\partial\Omega$. The functions p, q, c and f are assumed to be smooth and to satisfy

(5.3) $p(x,y) \geqq \gamma$, $q(x,y) \geqq \gamma$, $c(x,y) \geqq 0$ on Ω for some positive constant γ.

The method consists of the following components.

(i) Grid: rectangular

(ii) Approximation space: the Hermite bicubics which satisfy boundary conditions (5.2).

(iii) Operator discretization: In each element E of the partition we have the Galerkin equations

$$\sum_{i=1}^{16} \alpha_i \iint_E \{p\, D_x B_i\, D_x B_j + q\, D_y B_i\, D_y B_j + c\, B_i\, B_j\}\, dxdy = \iint_E f\, B_j\, dxdy$$

(iv) Equation solution: The local equations are assembled by the direct stiffness method to form the global matrix. The equations are solved by profile Gauss elimination for symmetric positive definite matrices.

For an error analysis of the above method see [7]. The computer implementation of this method used for experimentation is due to Houstis. A nine-point Gaussian quadrature scheme is used to compute the coefficients of the Galerkin equations.

6. <u>Ritz-Galerkin with bicubic Spline elements.</u> This method can be used to approximate the solution of (5.1), (5.2). It consists of the same components as the Ritz-Galerkin with Hermite bicubics where B_i's in the third component are the B-splines. The Galerkin equations are solved by a sparse Gauss elimination algorithm for symmetric positive definite matrices. This method is studied in [2]. Its computer implementation used is due to Eisenstat and Schultz.

7. <u>Ritz-Galerkin with triangular linear elements.</u> This method has been implemented to approximate the solution of (5.1) over a general two-dimensional domains provided the solution is known on a part of the boundary. It consists from the same components as the above described Ritz-Galerkin methods. The Galerkin equations are solved by a Gauss elimination algorithm for symmetric band positive definite matrices. A four-point Gauss quadrature scheme is used to compute the coefficients of the Galerkin equations. The implementation is due to Houstis.

8. The Structure of matrices of the four finite element methods.

The local nature of the basis functions, used for the representation of the approximate solution in the three finite element methods considered, dominates the structure of the finite element equations. In the case of Hermite cubics, the one-dimensional basis functions

(8.1) have support contained in at most two contiguous subintervals and

(8.2) at most four basis have support in any subinterval $[x_i, x_{i+1}]$.

In the case of cubic B-splines each basis function

(8.3) has support contained in at most four contiguous subintervals and

(8.4) at most four basis functions have support in any subinterval $[x_i, x_{i+1}]$.

Because of properties (8.1), (8.2) each collocation equation has 16 non-zero elements. The equations which correspond to collocation points associated with each element have the same structure. Thus the system of Collocation equations has an almost diagonal structure with 2N+6 ($H_0(\Delta_x \times \Delta_y)$) or 4N+12 ($H(\Delta_x \times \Delta_y)$) half bandwidth for rectangular domains.

Each entry of the system of Ritz-Galerkin (Hermite bicubics) equations is the sum of integrals over 4 contiguous rectangular elements. Besides, each equation has at most 36 non-zero elements. The system of Galerkin (Hermite bicubics) equations for problem (5.1), (5.2) is symmetric positive definite with 2N+6 ($H_0(\rho)$) half bandwidth.

Finally, because of properties (8.3)(8.4) each entry of the Galerkin (bicubic spline) system is the sum of integrals over 16 contiguous rectangular elements. It is symmetric and positive definite with 3N+7 ($S_0(p)$) half bandwidth and 49 non-zero elements per equation.

9. The direct solution of the three linear Finite Element systems.

For the solution of Ritz-Galerkin (Hermite bicubics) a profile Gauss elimination algorithm for symmetric positive definite matrices without pivoting is used. The Ritz-Galerkin (bicubic spline) system of equations is solved by a sparse Gauss elimination scheme.

For the system of Collocation (Hermite bicubics) equations three equation solvers were applied. The first is a profile Gauss elimination algorithm (BNBSOL) for unsymmetric band matrices, (stored in band storage mode) with row pivoting and taking into account the zeroes in the system. The second is a sparse Gauss elimination algorithm (NSPIV) with column pivoting (see [6]). The coefficient matrix of Collocation equations A is stored by means of three vectors which contain the non-zero elements of A row by row, the column number and the position of the first element of the ith row of A in the previous two vectors. Finally, the third scheme (SLVBLK) used is a Gauss elimination with rowpivoting for solving almost block diagonal linear systems (see [1]). The matrix is stored in blocks in one-dimensional array together with four vectors containing an index pointing the starting of ith block, the number of rows, the number of columns of each block, the number of steps of the Gauss algorithm to be performed on the ith block.

The Collocation (Hermite bicubics) and Galerkin (Hermite bicubics) were compared by Houstis, et. al. in [4]. In Table 2 we present the solution of an elliptic boundary value problem (see [4]) by the four finite element procedures described in this paper.

The data in Table 2 indicate that collocation with Hermite bicubics requires the least execution time for generating equations and that Collocation is faster than the other considered for the element methods. In Table 3 we observe that the profile Gauss elimination scheme BNDSOL is more efficient for moderate-size systems of collocation equations.

10. Test Results. In this section, we present a comparison of the finite element procedures considered above over a set of eight test problems used by Houstis, et. al. in [4]. We measure equation formation and solution time in seconds. The maximum error is calculated for each mesh. These results are shown in Tables 4-11. All computations were performed on a CDC 6500 in single precision arithmetic.

The data in Table 2 indicate the superiority of Collocation (C^1) for operators with expensive functions. The results in Tables 4, 6, 7 show that collocation (C^1) is more efficient than Galerkin (C^2) for simple operators and moderate accuracy (1 to 5 digits correct). The superiority of Collocation (C^1) over Galerkin (C^0) for curved boundaries is demonstrated in Tables 9, 10. Finally, Tables 8, 11 show that Galerkin (C^0) is more efficient than collocation (C^1) and Galerkin (C^2) only for low accuracy (1 digit correct) and non-smooth solutions. These results turn out to be compatible with those obtained in [4].

REFERENCES

[1] Carl de Boor and R. Weiss [1976], SOLVEBLOK: A package for solving almost block diagonal linear systems, with applications to spline approximation and the numerical solution of ordinary differential equations, MRC Technical Summary Report #1625.

[2] S. C. Eisenstat and M. H. Schultz [1973], The complexity of Partial Differential Equations in "Complexity of Sequential and Parallel Numerical Algorithms," (J. F. Traub, ed.), Academic Press, pp. 271-282.

[3] E. N. Houstis [1976], Collocation methods for linear Elliptic problems, to appear in Mathematics of Computation.

[4] E. N. Houstis, R. E. Lynch, T. S. Papatheodorou, and J. R. Rice [1977], Evaluation of numerical methods for Elliptic partial differential equations, accepted for publication in the Journal of Computational Physics.

[5] E. N. Houstis and J. R. Rice [1977], Software for linear Elliptic problems on general two dimensional domains, "Advances in computer methods for Partial Differential Equations--II," (R. Vichnevetsky, ed.) published by IMACS (AICA).

[6] A. H. Sherman [1976], Algorithms for Sparse Gaussian Elimination with Partial Pivoting, to appear.

[7] M. H. Schultz [1973], Spline Analysis, Prentice-Hall, Inc.

Table 1 Data indicating the structure of Galerkin and Collocation equations based on piecewise polynomial approximations for a N x N mesh of rectangular elements.

	GALERKIN			COLLOCATION
	Linear	Hermite Cubics	Cubic Splines	Hermite Cubics
Number of Equations	$(N-1)^2$	$4N^2$	$(N+1)^2$	$4N^2$
Half bandwidth	N + 3	2N+6	3N+7	2N+6
Sparsity*	5	36	49	16

*Sparsity is the maximum number of nonzero elements per row.

Table 2 Data for solving $u_{xx} + u_{yy} - [100 + \cos(3\pi x) + \sin(2\pi y)]u = f$ on unit square with u taken as $[5.4-\cos(4\pi x)]\sin(\pi x)(y^2-y)[5.4-\cos(4\pi y)]*[1/(1+\phi^4)-1/2]$

$$\phi = 4(x-.5)^2 + 4(y-.5)^2$$

METHOD: GALERKIN based on Hermite bicubics (C^1)

N	Number of Equations	Half Bandwidth	Matrix Formation	Profile Gauss Elimination Solution	Maximum Error
3	36	12	4.463	.059	3.09E-01
4	64	14	7.865	.204	5.39E-02
5	100	16	12.377	.532	4.78E-03
6	144	18	17.695	1.164	8.40E-03
7	196	20	23.996	2.112	3.13E-03
8	256	22	31.384	3.666	6.60E-03
9	324	24	39.98	5.835	4.50E-03

METHOD: GALERKIN based on bicubic splines (C^2)

N	Number of Equations	Half Bandwidth	Matrix Formation	Sparse Gauss Solution	Maximum Error
2	9	Full	.196	.008	7.669E-01
3	16	Full	.485	.030	1.098E+00
4	25	19	.920	.075	1.585E-01
5	36	22	1.469	.169	4.032E-01
6	49	25	2.159	.287	1.540E-01
7	64	28	2.977	.494	6.443E-02
8	81	31	3.961	.793	3.588E-02
9	100	34	5.048	1.180	3.171E-02
10	121	37	6.232	1.722	2.168E-02

METHOD: COLLOCATION based on Hermite bicubics

N	Number of Equations	Half Bandwidth	Matrix Formation	Profile Gauss Solution	Maximum Error
2	16	10	.082	.139	8.48E-01
3	36	12	.189	.19	2.10E-01
4	64	14	.335	.463	1.31E-01
5	100	16	.518	.921	3.31E-02
6	144	18	.776	1.710	2.68E-02
8	256	22	1.367	4.405	1.25E-02
9	324	24	1.714	6.663	6.88E-03

Table 3 Data indicating Collocation equation solution times for BNDSOL, NSPIV, SLVBLK

	SLVBLK		NSPIV		BNDSOL	
N	Matrix Formation	Equation Solution	Matrix Formation	Equation Solution	Matrix Formation	Equation Solution
2	.033	.036	.036	.054	.036	.061
3	.089	.151	.081	.216	.086	.199
4	.178	.419	.143	.584	.159	.477
5	.308	.924	.223	1.266	.255	.963
6	.485	1.775	.322	2.391	.368	1.739
7	.724	3.042	.443	4.055	.5	2.836
8					.645	4.451

Table 4. Data for solving $(e^{xy}u_x)_x + (e^{-xy}u_y)_y - \frac{u}{1+x+y} = f$ on unit square with u taken as $e^{xy}\sin(\pi x)\sin(\pi y)$.

METHOD: COLLOCATION based on Hermite bicubics (C^1)

N	Matrix Formation	Profile Gauss Elimination Sol.	Maximum Error
2	.059	.061	3.17E-02
3	.137	.203	5.64E-03
4	.248	.464	1.79E-03
5	.396	.932	8.51E-04
6	.569	1.73	3.11E-04
7	.792	2.961	1.82E-04
8	1.028	4.491	1.13E-04

METHOD: GALERKIN based on bicubic splines (C^2)

N	Matrix Formation	Sparse Gauss Solutions	Maximum Error	L_2-Error
2	.175	.007	1.497E-02	5.221E-03
3	.429	.028	5.267E-03	1.353E-03
4	.811	.077	1.876E-03	4.155E-04
5	1.314	.16	7.260E-04	1.623E-04
6	1.922	.285	3.391E-04	7.672E-05
7	2.662	.507	1.792E-04	4.072E-05
8	3.54	.783	1.004E-04	2.366E-05

Table 5. Data for solving $u_{xx} + u_{yy} = f$, $u = 0$ on unit square with u taken as $3e^x e^y (x-x^2)(y-y^2)$.

METHOD: COLLOCATION based on Hermite bicubics (C^1)

N	Matrix Formation	Profile Gauss Elimination Sol.	Maximum Error
3	.086	.199	4.48E-04
4	.159	.477	1.35E-04
5	.255	.963	5.00E-05
6	.368	1.739	2.79E-05
7	.5	2.836	1.49E-05
8	.645	4.451	3.28E-05

METHOD: GALERKIN based on bicubic splines (C^2)

N	Matrix Formation	Sparse Gauss Elim. Sol.	Maximum Error	L_2-Error
2	.108	.006	3.335E-03	1.150E-03
3	.279	.029	1.045E-03	2.744E-04
4	.544	.074	3.361E-04	9.037E-05
5	.857	.163	1.597E-04	3.911E-05
6	1.296	.288	7.781E-05	1.929E-05
7	1.798	.501	4.278E-05	1.065E-05
8	2.407	.793	2.531E-05	6.327E-06
9	3.06	1.194	1.562E-05	3.996E-06
10	3.82	1.733	1.004E-05	2.645E-06

METHOD: GALERKIN based on linear triangular elements (C^0)

N	Matrix Formation	Gauss Elimin. Sol.	Maximum Error
2	.02	.003	6.433E-02
4	.082	.009	3.620E-02
8	.327	.130	9.674E-03
16	1.338	1.772	2.466E-03
32	3.035	8.305	1.100E-03

Table 6. Data for solving $u_{xx} + u_{yy} = f$, $u = 0$ on unit square with u taken as $x^{5/2}y^{5/2} - xy^{5/2} - x^{5/2}y + xy$.

METHOD: COLLOCATION based on Hermite bicubics (C^1)

N	Matrix Formation	Profile Gauss Eliminat. Sol.	Maximum Error
2	.034	.062	7.50E-05
3	.081	.213	3.20E-05
4	.146	.456	2.00E-05
5	.240	.955	1.40E-05
6	.348	1.709	9.69E-06
7	.501	2.811	7.10E-06
8	.633	4.331	5.40E-06

METHOD: GALERKIN based on bicubic Splines (C^2)

N	Matrix Formation	Profile Gauss Elimin. Sol.	Maximum Error	L_2-Error
2	.102	.008	2.650E-04	1.036E-04
3	.264	.030	8.059E-05	3.270E-05
4	.515	.074	4.191E-05	1.447E-05
5	.844	.157	2.439E-05	7.518E-06
6	1.246	.29	1.472E-05	4.409E-06
7	1.745	.498	1.019E-05	2.800E-06
8	2.321	.789	7.394E-06	1.891E-06
9	2.981	1.176	5.499E-06	1.338E-06
10	3.735	1.705	4.234E-06	9.819E-07

METHOD: GALERKIN based on linear triangular elements (C^0)

NN	Matrix Formation	Gauss Elim. Solution	Maximum Error
2	.017	.001	1.708E-02
4	.07	.008	4.801E-03
8	.284	.131	1.348E-03
16	1.179	1.791	3.401E-04
32	2.671	8.42	1.516E-04

Table 7. Data for solving $4u_{xx} + u_{yy} - 64u = f$, $u = 0$ on unit square with u taken as $4(x^2 - x)(\cos(2\pi y) - 1)$.

METHOD: COLLOCATION based on Hermite bicubics (C^1)

N	Matrix Formation	Profile Gauss Elimin. Sol.	Maximum Error
2	.034	.053	5.15E-02
3	.082	.191	3.05E-02
4	.159	.46	7.89E-03
5	.239	.961	4.21E-03
6	.366	1.714	1.98E-03
7	.489	2.878	1.04E-03
8	.622	4.428	3.96E-04

METHOD: GALERKIN based on bicubic splines (C^2)

N	Matrix Formation	Sparse Gauss Elimin. Sol.	Maximum Error	L_2-Error
2	.11	.008	1.675E-02	8.020E-03
3	.285	.029	5.417E-02	2.200E-02
4	.549	.074	1.114E-02	4.566E-03
5	.923	.156	5.288E-03	1.673E-03
6	1.357	.292	2.173E-03	7.182E-04
7	1.901	.494	9.849E-04	3.650E-04
8	2.53	.791	5.570E-04	2.038E-04

Table 8. Data for solving $u_{xx} + u_{yy} = f$, $u = 0$ on the unit square with u taken as

$$10\,\phi(x) * \phi(y), \quad \phi(x) = e^{-100(x-.1)^2}(x^2-x)$$

METHOD: COLLOCATION based on Hermite bicubics (C^1) *

N	Matrix Formation	Profile Gauss Elimin. Sol.	Maximum Error
2	.063	.061	2.3E-00
3	.143	.214	5.71E-01
4	.239	.482	3.38E-01
5	.367	.968	3.20E-01
6	.536	1.720	1.59E-01
7	.719	2.814	1.03E-01
8	.946	4.39	8.16E-02
9	1.223	6.71	1.49E-02

*Uniform mesh

METHOD: GALERKIN based on linear triangular elements (C^0)

N	Matrix Formation	Gauss Elim. Solution	Maximum Error
2	.059	.000	1.439
4	.234	.008	1.888E-01
8	.921	.13	3.093E-02
16	3.718	1.775	1.891E-02
32	8.38	8.338	8.985E-03

Table 8. (continued)

METHOD: GALERKIN based on bicubic splines (C^2)

N	Matrix Formation	Sparse Gauss Elimin. Sol.	Maximum Error
2	.146	.008	6.218E-01
3	.368	.029	5.425E-01
4	.683	.075	1.906E-01
5	1.121	.156	3.261E-01
6	1.657	.294	1.365E-01
7	2.301	.493	2.289E-01
8	3.048	.779	3.086E-02
9	3.855	1.169	1.308E-01
10	4.819	1.704	4.293E-03

METHOD: COLLOCATION based on Hermite bicubics (C^1) *

N	Matrix Formation	Profile Gauss Elim. Sol.	Maximum Error
3	.127	.195	2.90E-01
4	.229	.468	3.00E-01
5	.358	.963	9.10E-02
6	.542	1.753	6.16E-02
7	.73	2.856	3.80E-02
8	.97	4.547	2.65E-02

*Non-uniform mesh

Table 9. Data for solving $u_{xx} + u_{yy} = f$, $u = g$ on Ω (Figure 1) with u taken as

$$y[(x-2)^2 + y^2-1]e^{-.0625x(x-4)(y-2)} / [(3+(x-2)^2)(3+y^2)]$$

METHOD: COLLOCATION based on Hermite bicubics (C^1)

Number of Equations	Matrix Formation	Profile Gauss Elim. Sol.	Maximum Error
56	.146	.507	2.367E-03
108	.311	1.478	9.307E-04
164	.496	3.049	2.305E-04
240	.746	5.646	1.141E-04

METHOD: GALERKIN based on linear triangular elements (C^0)

Number of Equations*	Matrix Formation	Gauss Elim. Sol.	Maximum Error
2	.095	.002	3.344E-01
17	.403	.023	1.476E-01
45	.886	.101	8.302E-o2

*Boundary conditions have been eliminated.

Figure 1 The geometry and boundary conditions for problem in Table 9.

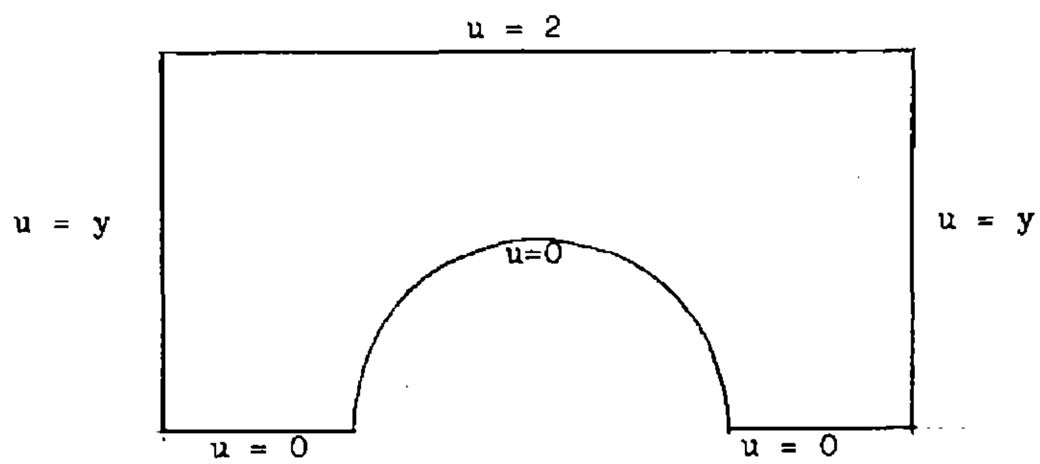


Table 10. Data for solving $u_{xx} + u_{yy} = f$, $u = g$ on an ellipse with u taken as

$$u = (e^x + e^y)/(1 + xy)$$

METHOD: COLLOCATION based on Hermite bicubics (C^1)

Number of Equations	Matrix Formation	Profile Gauss Elim. Sol.	Maximum Error
24	.048	.143	1.42E-02
56	.122	.558	7.80E-03
156	.366	2.972	3.28E-04
228	.572	5.662	2.20E-04

METHOD: GALERKIN based on linear triangular elements (C^0)

Number of Equations*	Matrix Formation	Gauss Elimin. Sol.	Maximum Error
1	.022	.001	7.001E-02
3	.042	.002	8.256E-02
8	.081	.008	4.256E-02
39	.289	.112	3.039E-02

*The Boundary conditions have been eliminated.

Table 11. Data for solving $u_{xx} + u_{yy} = f$, $u = g$ on the unit square with u taken as $\phi(x) * \phi(y)$ where $\phi(x) = U(.35) + (U(.35) - U(.65))p(x)$ is a quintic polynomial determined so that $\phi(x)$ has two continuous derivatives and $U(x)$ is unit step function.

METHOD: COLLOCATION based on Hermite bicubics (C^1)

N	Matrix Formation	Profile Gauss Elim. Sol.	Maximum Error	Maximum Error*
3	.152	.846	5.34E-01	
4	.242	1.838	1.13E-01	
5	.363	3.436	9.90E-03	
6	.505	5.79	1.51E-02	1.77E-03
7	.664	9.25	5.99E-02	
8	.845	14.19	7.03E-02	4.13E-04

*Collocation -- non-uniform mesh

METHOD: GALERKIN based on linear triangular elements (C^0)

N	Matrix Formation	Gauss Elim. Sol.	Maximum Error
2	.027	.001	2.007E-01
4	.103	.008	1.298E-01
8	.447	.134	4.828E-02
16	1.8	1.796	1.629E-02
32	4.018	8.41	3.693E-03