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Abstract

This paper is concerned with the numerical solution of the American option valuation problem formulated as a parabolic free boundary/initial value model. For this we introduce and analyze several front-tracking finite difference methods and compare them with the commonly used binomial and linear complementarity techniques. The numerical experiments performed indicate that the front-tracking methods considered are efficient alternatives for approximating simultaneously the option value and optimal exercise boundary functions associated with the valuation problem.

1 Introduction

The seminal work of Black and Scholes [2] has contributed significantly to the mathematical formulation and solution of the option valuation problem. Throughout we employ the Black-Scholes model to formulate the American option valuation problem. Assuming that the price of the option is a function of the underlying asset and the time to the expiration, and under the condition that there exists a risk free replicating portofolio which duplicates the returns of the option, the Black-Scholes partial differential equation (PDE) model is as follows

$$\frac{\partial V(S,\tau)}{\partial \tau} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,\tau)}{\partial S^2} + (r-\delta)S\frac{\partial V(S,\tau)}{\partial S} - rV(S,\tau) = 0 \qquad (1.1)$$

$$S \in [0,\infty), \quad \text{and} \quad \tau \in [0,T]$$

where $V(S,\tau)$ is the price of the option at time τ, S is the price of the underlying asset, τ is the time from the initiation of the option, T is the duration of the

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option, σ is the volatility of the underlying asset, δ is the continuous dividend yield, and r is the risk free interest rate. One other parameter that does not appear directly in the equation is the strike price denoted by E. Depending on the boundary conditions and the terminal value of the option (i.e. the option *payoff*) both call and put options as well as a variety of other more complicated option products can be priced. A rigorous and detailed presentation as well as a thorough analysis of the assumptions inherent in this mathematical model and its derivation can be found in [5]. In this paper we are particularly interested in the numerical valuation of the American call on a dividend paying asset modeled by (1.1). For this we consider a number of the so-called front-tracking finite difference methods to approximate the above Black-Scholes model. These methods can be differentiated with respect to the finite difference discretizations used for the time derivatives. Front-tracking techniques have been successfully employed in the context of the Stefan problem [4]. They are characterized by the fact that they simultaneously find the value and free boundary functions. For comparison purposes we have implemented two commonly used solution approaches to the American valuation problem, the *binomial* [3] and *linear complementarity* methods [6]. A number of numerical experiments were performed under different input values and discretization parameters. The numerical data obtained indicate that the front-tracking approach to the American option valuation problem preserves its qualitative and quantitative characteristics and that it is an efficient alternative to solving the problem in single and multi-dimensional settings.

The paper is organized in seven sections. Section 2 presents the mathematical model governing the valuation of an American call on a dividend paying asset. Section 3 defines a front-tracking model for the American call problem. Section 4 formulates several explicit and implicit finite difference schemes for approximating the front-tracking model. The skeleton of the front-tracking algorithm implemented is defined in Section 5. Section 6 lists the results of the numerical experiments performed for all the front-tracking schemes introduced and their comparison with the binomial and linear complementarity algorithms. Finally, Section 7 summarizes the contribution of this paper.

2 American Call on a dividend paying asset

The pricing of an American call on a dividend paying asset with explicit reference to the free boundary can be described by the parabolic initial/boundary value problem [7]

$$\frac{\partial c(S,t)}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + (r-\delta) S \frac{\partial c(S,t)}{\partial S} - rc, \qquad (2.1)$$
$$S \in [0, f_S(t)), \quad t \in [0,T]$$

with initial condition

$$c(S,0) = \max(S-E, 0), \quad S \in [0, f_S(0)) \qquad (2.2)$$

and boundary conditions

$$c(0,t) = 0, \tag{2.3}$$

$$c(f_S(t),t) = f_S(t) - E, \qquad \frac{\partial c(f_S(t),t)}{\partial S} = 1 \tag{2.4}$$

where c denotes the value of the American call option and f_S the optimal exercise boundary.

Notice that we have converted (1.1) into a forward parabolic PDE by applying the transformation $t = T - \tau$. Thus, the payoff of the option is taken as the initial value of the problem. For $S \in [f_S(t), \infty)$ the value of the call is equal to the payoff function. Moreover, the complete call value is given as

$$c_{complete}(S,t) = \begin{cases} c(S,t) & \text{if} \quad S \in [0, f_S(t)), \\ \max(S-E,0) & \text{if} \quad S \in [f_S(t), \infty) \end{cases} \tag{2.5}$$

Equation (2.5) makes explicit that the American call has an optimal exercise boundary, $f_S(t)$, which indicates whether the option should be held or exercised at time t.

A number of researchers have proposed numerical solutions to the above problem, most of which are based on the linear complementarity formulation of the free boundary problem [6]. This formulation makes no explicit reference to the free boundary which can be obtained in a postmortem fashion. Front-tracking methods on the other hand are based on the explicit approximation of the free boundary during the numerical solution of the problem and simultaneously provide the pair of functions satisfying the complete free boundary problem without any need for postprocessing.

3 A Front-Tracking Model for the American Call Problem

The challenge in a front-tracking method for the free boundary problem is to come up with an auxiliary equation that will help in "tracking" the free boundary at each marching step through time. In [1] and [6] the behavior of the free boundary close to the expiration date is analyzed and some approximations are suggested. In this paper, we are introducing a procedure to estimate the free boundary for the complete duration of the option.

In order to develop this procedure we first formulate the American call model (2.1) onto a rectangular domain $[0,1] \times [0,T]$ by introducing the new space variable

$$x = \frac{S}{f_S(t)}. \tag{3.1}$$

This "front-fixing" transformation was first introduced by Landau and applied in the context of finite difference methods by Crank [4]. If we denote the transformed value function by C and apply the transformation to equation (2.1) and the corresponding initial/boundary conditions, we obtain

$$\frac{\partial C}{\partial t} = \left((r-\delta) + \frac{1}{f_S(t)}\frac{df_S(t)}{dt}\right)x\frac{\partial C}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} - rC \tag{3.2}$$

subject to initial condition

$$C(x,0) = \max(xf_S(0) - E, 0), \quad x \in [0, f_S(0)) \tag{3.3}$$

and boundary conditions

$$C(0,t) = 0, \tag{3.4}$$
$$C(1,t)|_{S=f_S(t)} = f_S(t) - E, \qquad \frac{\partial C(1,t)}{\partial x}|_{S=f_S(t)} = f_S(t).$$

Notice that the above PDE model is defined in terms of the unknown free boundary function $f_S(t)$. For its determination we observe that at $x=1 \quad (S = f_S(t))$ the following relations hold

$$\frac{\partial C(x,t)}{\partial x}|_{x=1} = f_S(t), \tag{3.5}$$
$$C(x,t)|_{x=1} = f_S(t) - E, \tag{3.6}$$
$$\frac{\partial C}{\partial t}|_{x=1} = \frac{\partial (f_S(t) - E)}{\partial t} = \frac{df_S(t)}{dt}. \tag{3.7}$$

Moreover, if we evaluate (3.2) at $x=1 \quad (S = f_S(t))$ then we obtain

$$\frac{df_S(t)}{dt} = rE - \delta f_S(t) + \frac{1}{2}\sigma^2\frac{\partial^2 C}{\partial x^2}|_{x=1}. \tag{3.8}$$

Notice that equation (3.8) is an ordinary differential equation (ODE) that tracks the free boundary and it is coupled with equation (3.2) with respect to the free boundary. In order to solve (3.8) we need an initial condition. It can be obtained by evaluating (2.1) at $t=0, \quad S = f_S(0)$ so as to be consistent with the payoff function and the given boundary conditions. It can be shown that $f_S(0) = \frac{rE}{\delta}$. In fact, a more detailed analysis presented in [6] demonstrates that this condition is true asymptotically as we get closer to the expiration. Specifically, the free boundary satisfies this initial condition as time approaches the expiration. At exactly the expiration ($t=0$) it equals the strike price of the call. This is true since (3.2) holds with $t \in (0,T]$, while at $t=0$ the price is given by the payoff function. Our numerical results indicate that the jump of the free boundary, very close to expiration, does not affect the numerical solution.

4 Finite Difference Approximations to the American Call Model

In the following we develop several finite difference (FD) schemes to approximate the solution of the free boundary problem (3.2) to (3.4) and the corresponding front-tracking equation (3.8). These schemes differ with respect to the approximations used for the time derivatives. We assume that the space domain is discretized in intervals of length $h = \frac{1}{N}$, where N is the resolution of the space discretization, and the time step is of length $\Delta t = \frac{T}{M}$, where M is the resolution of the time discretization. A superscript n indicates that the value of the superscripted function is taken at time step $n\Delta t$, for $n = 1(1)\{\frac{T}{M}\}$ and a subscript j indicates that the value of the subscripted function is taken at the point jh, for $j = 1(1)\{\frac{1}{N}\}$. Following we describe the selected approximations for the American call problem.

4.1 Approximations of the free boundary

Using a three point Lagrange formula we can approximate the second order partial derivative at the right boundary

$$\frac{\partial^2 C^{(n)}}{\partial x^2}|_{x=1} = \frac{3f_S^{(n)} - 4\frac{\partial C^{(n)}}{\partial x}|_{1-h} + \frac{\partial C^{(n)}}{\partial x}|_{1-2h}}{2h}. \tag{4.1}$$

A first order approximation to the free boundary is defined by the relation

$$f_S^{(n+1)} = f_S^{(n)} + \Delta t \frac{df_S^{(n)}}{dt}. \tag{4.2}$$

A fourth order approximation of $f_S^{(n+1)}$ can be obtained by using the Runge-Kutta method

$$f_S^{(n+1)} = f_S^{(n)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned} k_1 &= \Delta t(F(f_S^{(n)})), \quad k_2 = \Delta t(F(f_S^{(n)} + \frac{1}{2}k_1)), \\ k_3 &= \Delta t(F(f_S^{(n)} + \frac{1}{2}k_2)), \quad k_4 = \Delta t(F(f_S^{(n)} + k_3)), \\ F(f_S^{(n)}) &= rE - Df_S^{(n)} + \frac{1}{2}\sigma^2 \frac{\partial^2 C^{(n)}}{\partial x^2}|_{x=1}. \end{aligned} \tag{4.3}$$

Finally, the boundary condition can be evaluated by

$$C^{(n+1)}|_{x=1} = f_S^{(n+1)} - E. \tag{4.4}$$

4.2 Approximation of the American call value

Having determined the free boundary condition at each time step, the problem reduces to a standard boundary value problem which can be solved by either an explicit or an implicit difference scheme. The explicit methods considered include a fourth order Runge–Kutta approximation and a first order backward difference method. The implicit methods applied are based on a second order Crank–Nicolson scheme and a fully implicit first order backward difference method. Below we define the various approximations to the boundary value problem. Throughout, we approximate the partial derivatives with respect to the space variable involved in (3.2) as follows

$$\frac{\partial C^{(n)}}{\partial x} = \frac{C_{j+1}^{(n)} - C_{j-1}^{(n)}}{2h}, \quad \frac{\partial^2 C^{(n)}}{\partial x^2} = \frac{C_{j+1}^{(n)} - 2C_j^{(n)} + C_{j-1}^{(n)}}{h^2}. \tag{4.5}$$

4.2.1 First Order Backward Difference Explicit Scheme

In this scheme we approximate (3.2) by the difference equation

$$\frac{C_j^{(n+1)} - C_j^{(n)}}{\Delta t} = \frac{1}{2} j\left((r-\delta) + \frac{1}{f_S^{(n)}}\frac{df_S^{(n)}}{dt}\right)(C_{j+1}^{(n)} - C_{j-1}^{(n)}) \tag{4.6}$$

$$+ \frac{1}{2}\sigma^2 j^2 (C_{j+1}^{(n)} - 2C_j^{(n)} + C_{j-1}^{(n)}) - rC_j^{(n)} \tag{4.7}$$

and obtain the explicit equation

$$C_j^{(n+1)} = a_j C_{j-1}^{(n)} + b_j C_j^{(n)} + c_j C_{j+1}^{(n)} \tag{4.8}$$

where

$$a_j = \frac{1}{2}\Delta t\left(\frac{1}{2}\sigma^2 j^2 - \left((r-\delta) + \frac{1}{f_S^{(n)}}\frac{df_S^{(n)}}{dt}\right)\right), \tag{4.9}$$

$$b_j = 1 - \Delta t(\sigma^2 j^2 - r), \tag{4.10}$$

$$c_j = \frac{1}{2}\Delta t\left(\frac{1}{2}\sigma^2 j^2 + \left((r-\delta) + \frac{1}{f_S^{(n)}}\frac{df_S^{(n)}}{dt}\right)\right). \tag{4.11}$$

Equations (4.8) can be solved for each point in space to determine the call value at time step $n+1$.

4.2.2 Fourth Order Runge-Kutta Explicit Scheme

In this scheme we replace (3.2) by

$$V(C^{(n)}) = \left((r-\delta) + f_S^{(n)}\frac{df_S^{(n)}}{dt}\right)x\frac{\partial C^{(n)}}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C^{(n)}}{\partial x^2} - rC^{(n)} \tag{4.12}$$

and apply a fourth order Runge–Kutta scheme

$$C^{(n+1)} = C^{(n)} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \tag{4.13}$$

where

$$k_1 = \Delta t(V(C^{(n)})), \quad k_2 = \Delta t(V(C^{(n)} + \frac{1}{2}k_1)),$$
$$k_3 = \Delta t(V(C^{(n)} + \frac{1}{2}k_2)), \quad k_4 = \Delta t(V(C^{(n)} + k_3)),.$$

Equations (4.13) can be solved for each point in space to determine the call value at time step $n+1$.

4.2.3 First Order Implicit Backward Difference Scheme

In this scheme we replace (3.2) by

$$\begin{aligned}\frac{C_j^{(n+1)} - C_j^{(n)}}{\Delta t} &= \frac{1}{2}j\left((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}\right)(C_{j+1}^{(n+1)} - C_{j-1}^{(n+1)}) \\ &+ \frac{1}{2}\sigma^2 j^2 (C_{j+1}^{(n+1)} - 2C_j^{(n+1)} + C_{j-1}^{(n+1)}) - rC_j^{(n+1)}\end{aligned} \tag{4.14}$$

and obtain the linear system of equations

$$a_j C_{j-1}^{(n+1)} + b_j C_j^{(n+1)} + c_j C_{j+1}^{(n+1)} = C_j^{(n)} \tag{4.15}$$

where

$$a_j = \frac{1}{2}\Delta t\left(j((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}) - \sigma^2 j^2\right), \tag{4.16}$$

$$b_j = 1 + \Delta t\left((\sigma^2 j^2 + r\right), \tag{4.17}$$

$$c_j = -\frac{1}{2}\Delta t\left(j((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}) + \sigma^2 j^2\right). \tag{4.18}$$

which can be solved directly or iteratively.

4.2.4 Second Order Implicit Crank-Nicolson Scheme

In this case (3.2) is approximated by

$$\begin{aligned}\frac{C_j^{(n+1)} - C_j^{(n)}}{\Delta t} &= \frac{1}{2}[\frac{1}{2}j\left((r-\delta) + \frac{1}{f_S^{(n)}}\frac{df_S^{(n)}}{dt}\right)(C_{j+1}^{(n)} - C_{j-1}^{(n)}) \\ &+ \frac{1}{2}\sigma^2 j^2 (C_{j+1}^{(n)} - 2C_j^{(n)} + C_{j-1}^{(n)}) - rC_j^{(n)}] \\ &+ \frac{1}{2}[\frac{1}{2}j\left((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}\right)(C_{j+1}^{(n+1)} - C_{j-1}^{(n+1)}) \\ &+ \frac{1}{2}\sigma^2 j^2 (C_{j+1}^{(n+1)} - 2C_j^{(n+1)} + C_{j-1}^{(n+1)}) - rC_j^{(n+1)}]\end{aligned} \tag{4.19}$$

which leads to the linear system of equations

$$a_jC_{j-1}^{(n+1)} + b_jC_j^{(n+1)} + c_jC_{j+1}^{(n+1)} = d_jC_{j-1}^{(n)} + e_jC_j^{(n)} + f_jC_{j+1}^{(n)} \tag{4.20}$$

with

$$a_j = \frac{1}{4}\Delta t\Big(j\big((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}\big) - \sigma^2 j^2\Big), \tag{4.21}$$

$$b_j = 1 + \frac{1}{2}\Delta t(\sigma^2 j^2 + r), \tag{4.22}$$

$$c_j = -\frac{1}{4}\Delta t\Big(j\big((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}\big) + \sigma^2 j^2\Big), \tag{4.23}$$

$$d_j = -\frac{1}{4}\Delta t\Big(j\big((r-\delta) + \frac{1}{f_S^{(n)}}\frac{df_S^{(n)}}{dt}\big) - \sigma^2 j^2\Big), \tag{4.24}$$

$$e_j = 1 - \frac{1}{2}\Delta t(\sigma^2 j^2 + r), \tag{4.25}$$

$$f_j = \frac{1}{4}\Delta t\Big(j\big((r-\delta) + \frac{1}{f_S^{(n+1)}}\frac{df_S^{(n+1)}}{dt}\big) + \sigma^2 j^2\Big). \tag{4.26}$$

$$\tag{4.27}$$

We can solve the linear system of equations (4.20) by using either an appropriate sparse direct method or an iterative solver.

5 Front-Tracking Algorithm for the American Call Problem

In this section we present an outline of a front-tracking algorithm for the solution of the American call on a dividend paying asset problem. The algorithm can be described in terms of the following steps:

- *INPUT*: E, σ, δ, r, T
- *COMPUTATION*:
  - Set up Initial and Boundary conditions
  - for $n = 1$ to $n = \frac{T}{M}$
    1. compute the free boundary at step n utilizing one of the two explicit schemes identified,
    2. compute the value function at step n utilizing any of the explicit or implicit schemes identified,
  - repeat steps 1 and 2, until termination criteria are satisfied

- *OUTPUT*: optimal exercise boundary f_S and option price c

It is worth noticing that the front-tracking algorithm provides to the user the option price on the complete domain for all possible asset prices and time frames. This should be compared to the binomial method which, in general, must be repeated for each asset price, and the linear complementarity approach which computes the optimal exercise boundary in a postmortem fashion.

6 Numerical Results

In this section we present a series of numerical data for the front-tracking, binomial, and linear complementarity solutions to the American call problem. The corresponding algorithms were implemented in C^{++} and executed on a Sun SPARCstation 20 using single precision.

Table 1 lists the call option value obtained by the front-tracking algorithm for the several time discretization schemes and asset prices. Table 2 lists the

Asset Price	Euler	Runge–Kutta	Fully Implicit	Crank–Nicolson
2.0	0.00000	0.00000	0.00000	0.00000
3.0	0.00000	0.00000	0.00000	0.00000
4.0	0.00031	0.00031	0.00031	0.00031
5.0	0.00493	0.00495	0.00493	0.00496
6.0	0.03190	0.03196	0.03209	0.03199
7.0	0.11940	0.11952	0.11975	0.11957
8.0	0.31309	0.31315	0.31352	0.31331
9.0	0.64520	0.64493	0.64561	0.64540
10.0	1.12397	1.12310	1.12429	1.12413
11.0	1.73707	1.73537	1.73728	1.73717
12.0	2.46123	2.45866	2.46134	2.46128
13.0	3.27106	3.26765	3.27111	3.27109
14.0	4.14427	4.14012	4.14428	4.14428
15.0	5.06352	5.05884	5.06352	5.06352
16.0	6.01642	6.01149	6.01642	6.01642
17.0	7.00000	7.00000	7.00000	7.00000

Table 1: The American call option value obtained by the front-tracking method for four FD discretization schemes. In all cases the free boundary is approximated with a Runge-Kutta method. The time step size and the input parameters used are: $\Delta t = 1.5 \times 10^{-4}$, $E = 10.0$, $\sigma = 0.4$, $\delta = 0.08$, $r = 0.1$, $T = 0.5$.

European and American call values for a set of input values, utilizing the binomial and the Crank–Nicolson based front-tracking algorithms respectively. As was expected, the American call is more valuable than the European equivalent. A comparison of the front-tracking algorithm with the two commonly

Asset Price	European Call	American Call
2.0	0.02588	0.02647
3.0	0.10984	0.11202
4.0	0.27233	0.27760
5.0	0.51425	0.52708
6.0	0.83190	0.85565
7.0	1.21850	1.25291
8.0	1.66037	1.72682
9.0	2.15123	2.25220
10.0	2.68620	2.82857
11.0	3.25966	3.45010
12.0	3.86564	4.11181
13.0	4.49824	4.80944
14.0	5.15200	5.53929
15.0	5.82215	6.29818
16.0	6.51598	7.08358
17.0	7.22127	7.89321

Table 2: The European and American call values for $E = 10.0$, $\sigma = 0.8$, $\delta = 0.2$, $r = 0.25$, $T = 1$. The European price has been calculated using the binomial method for 256 time steps while the American call price was obtained by applying the front-tracking algorithm with the Crank-Nicolson approximation for $\Delta t = 3.9 \times 10^{-5}$.

used American option pricing algorithms, namely the binomial and the linear complementarity, is presented in Table 3. The results obtained agree to at least two decimal digits. Table 4 presents the optimal exercise boundary for two different sets of input data, testing the variability of the front-tracking solution with respect to the input data. Table 5 lists the American call option prices obtained with the front-tracking algorithm for various times and asset prices. The results indicate that the call option price is decreasing with time and increasing with asset price, which is in agreement with its theoretical behavior. The efficiency of the methods implemented measured in seconds is reported in Table 6. The front-tracking algorithm used is based on the Crank–Nicolson approximation with $\Delta t = 1.5 \times 10^{-4}$. The binomial method is taken for 256 time steps. The linear complementarity method is based on the Crank–Nicolson approximation with $\Delta t = 7.5 \times 10^{-3}$. The data indicate that for our implementation the front-tracking algorithm is several times faster than the binomial and more than two times faster than the linear complementarity methods. Figure 1 depicts the optimal exercise boundary for an American call option as calculated by the front-tracking algorithm based on the Crank–Nicolson approximation. Figure 2 gives the plot of the solution in the complete domain for an American call as calculated by the front-tracking algorithm.

Asset Price	Binomial	Complementarity	Front-Tracking
2.0	0.00000	0.00000	0.00000
3.0	0.00022	0.00024	0.00026
4.0	0.00448	0.00454	0.00465
5.0	0.02996	0.03005	0.03031
6.0	0.11088	0.11070	0.11113
7.0	0.28442	0.28493	0.28538
8.0	0.57892	0.57953	0.57996
9.0	1.00452	1.00325	1.00359
10.0	1.54806	1.54890	1.54934
11.0	2.20103	2.20073	2.20090
12.0	2.93843	2.93839	2.93854
13.0	3.74255	3.74298	3.74305
14.0	4.59862	4.59789	4.59783
15.0	5.48953	5.48964	5.48947
16.0	6.40822	6.40779	6.40765
17.0	7.34447	7.34510	7.34474

Table 3: The binomial, linear complementarity and front-tracking solutions to the pricing of an American call problem for $E = 10.0$, $\sigma = 0.5$, $\delta = 0.04$, $r = 0.12$, $T = 0.5$. The price has been calculated using the binomial method for 256 time steps, the Crank-Nicolson implementation of the linear complementarity method with $\Delta t = 0.8 \times 10^{-4}$, and the Crank–Nicolson implementation of the front-tracking method for $\Delta t = 1.0 \times 10^{-4}$.

7 Conclusions

This paper has introduced and analyzed a class of front-tracking FD methods for solving the free boundary model governing the American option valuation problem. These techniques are characterized by the fact that they simultaneously compute both the price and the optimal exercise boundary functions. A number of numerical experiments performed indicate that they exhibit similar quantitative and qualitative behavior with the commonly used binomial and linear complementarity techniques. In addition, the front-tracking methods are computationally more efficient than the other two and can be easily generalized to multi-dimensional option valuation problems. The stability and convergence analysis of the front-tracking methods considered will be reported elsewhere.

Time	Call A	Call B
0.00	33.10738	23.93899
0.05	32.66230	23.33682
0.10	32.22852	22.66658
0.15	31.81007	21.91206
0.20	31.41223	21.05184
0.25	31.04194	20.05814
0.30	30.70821	18.89629
0.35	30.42245	17.52438
0.40	30.19844	15.89122
0.45	30.05208	13.93527
0.50	30.00000	12.50000

Table 4: The approximate optimal exercise boundary as calculated by the Crank-Nicolson implementation of the front-tracking algorithm with $\Delta t = 1.0 \times 10^{-4}$, for call A: $E = 10$, $\sigma = 0.5$, $\delta = 0.04$, $r = 0.12$ and $T = 0.5$ and with $\Delta t = 3.9 \times 10^{-5}$, for call B: $E = 10$, $\sigma = 0.8$, $\delta = 0.2$, $r = 0.25$ and $T = 0.5$.

Asset Price	$t = 0.0$	$t = 0.25$	$t = 0.5$	$t = 0.75$	$t = 1.0$
2.0	0.02645	0.01011	0.00169	0.00001	0.00000
3.0	0.11199	0.05876	0.01819	0.00085	0.00000
4.0	0.27754	0.17468	0.07672	0.00973	0.00000
5.0	0.52702	0.37228	0.20381	0.04801	0.00000
6.0	0.85648	0.65529	0.41627	0.14742	0.00000
7.0	1.25906	1.01991	0.72014	0.33731	0.00000
8.0	1.72674	1.45955	1.11324	0.63544	0.00000
9.0	2.25212	1.96674	1.58936	1.04614	0.00000
10.0	2.82849	2.53348	2.13997	1.56350	0.00000
11.0	3.45003	3.15292	2.75595	2.17566	1.00000
12.0	4.11174	3.81866	3.42898	2.86853	2.00000
13.0	4.80937	4.52513	4.15119	3.62815	3.00000
14.0	5.53920	5.26787	4.91580	4.44200	4.00000
15.0	6.29813	6.04265	5.71723	5.29954	5.00000
16.0	7.08353	6.84615	6.55048	6.19221	6.00000
17.0	7.89316	7.67554	7.41154	7.11328	7.00000

Table 5: The American call option prices as obtained by the front-tracking algorithm with the Crank-Nicolson approximation for $\Delta t = 3.9 \times 10^{-5}$, for various times before expiration, and for $E = 10$, $\sigma = 0.8$, $\delta = 0.2$, $r = 0.25$ and $T = 1$.

Binomial	Complementarity	Front-tracking
1.41 sec	0.41 sec	0.16 sec

Table 6: Time taken by the three indicated methods to compute the price of an American call with $E = 10$, $\sigma = 0.8$, $\delta = 0.2$, $r = 0.25$ and $T = 0.25$, and a similar level of accuracy.

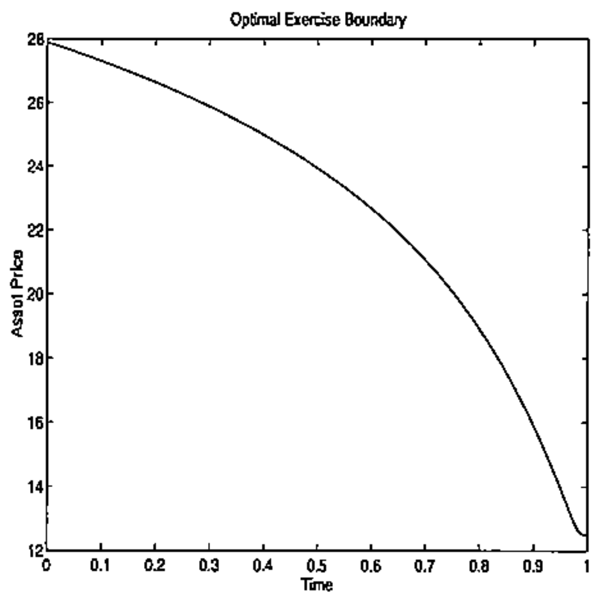

Figure 1: The optimal exercise boundary for an American call option for $E = 10$, $\sigma = 0.8$, $\delta = 0.2$, $r = 0.25$ and $T = 1$ obtained with the front-tracking Crank–Nicolson method for $\Delta t = 3.9 \times 10^{-5}$.

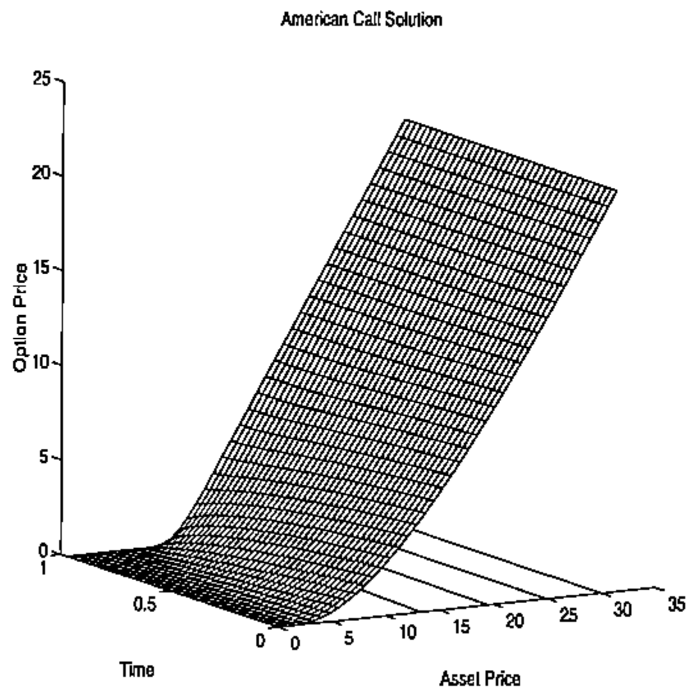

Figure 2: Plot of the American call price for the duration of its lifetime with $E = 10$, $\sigma = 0.8$, $\delta = 0.2$, $r = 0.25$ and $T = 1$ obtained by the front-tracking Crank–Nicolson method for $\Delta t = 3.9 \times 10^{-5}$.

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