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Automatic Parameterization of Rational Curves and Surfaces II: Cubics and Cubicoids

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Abstract

Cubicoids (degree 3 surfaces) always have a parameterization in terms of rational functions, (a polynomial divided by another). On the other hand cubics (degree 3 plane curves) do not always have a rational parameterization. However they always have a parameterization of the type which allows a single square root of rational functions. In this paper we describe algorithms to obtain rational and special parametric equations for the cubics and cubicoids, given the implicit equations. These algorithms have been implemented on a VAX-8600 using VAXIMA.

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1. Introduction

Rational curves and surfaces can be represented by implicit or parametric equations. A rational cubic (degree three) curve has an implicit equation given by $C(x,y) = ax^3 + by^3 + cx^2y + dxy^2 + ex^2 + fy^2 + gxy + hx + iy + j = 0$, and rational parametric equations given by $x = u(t)/w(t)$ and $y = v(t)/w(t)$, where u, v and w are no more than cubic polynomials. Further a rational cubicoid (degree three) surface has an implicit equation given by $C(x,y,z) = ax^3 + by^3 + cz^3 + dx^2y + ex^2z + fxy^2 + gy^2z + hxz^2 + iyz^2 + jxyz + kx^2 + ly^2 + mz^2 + nxy + oxz + pyz + qx + ry + sz + t = 0$, with corresponding rational parametric equations $x = u(s,t)/q(s,t)$, $y = v(s,t)/q(s,t)$, and $z = w(s,t)/q(s,t)$. The rational parametric form of representing a surface allows greater ease for transformation and shape control, Tiller (1983), Mortenson (1985). The implicit form is preferred for testing whether a point is above, on, or below the surface, where above and below is determined relative to the direction of the surface normal. As both forms have their inherent advantages it becomes crucial to be able to go efficiently from one form to the other, especially when surfaces of an object are automatically generated in one of the two representations.

Cubicoids (degree 3 surfaces) are rational†, that is, have a parameterization in terms of rational functions (ratio of two polynomials). On the other hand cubics (degree 3 plane curves) are not all rational. However they always have a parameterization of the type which allows a single square root of rational functions. In § 2 and § 3 of this paper we describe algorithms to obtain rational and special parametric equations for the cubics and cubicoids, given the implicit equations. Polynomial parameterizations are also obtained whenever they exist for the cubics and cubicoids. Higher degree curves and surfaces in general are not rational.

The reverse problem of converting from parametric to implicit equations, called implicitization has been considered computationally by various authors in the past, see Collins (1971) and Sederberg et. al., (1985). However as yet no correct closed form solution is known for implicitizing rational surfaces or in general, implicitizing higher dimension parametric algebraic varieties.

† Except possibly the cubic cone and the cubic cylinders with nonsingular cubic generating curves.

2. Cubics

Geometric Viewpoint:

The idea of parametrizing a conic was to fix a point on the conic and take lines through that point, which intersects the conic in only one additional point, Abhyankar and Bajaj (1986a). The conic was thus rationally parametrized by a pencil of lines with parameter t corresponding to the slope of the lines. A cubic curve is a curve which intersect most lines in three points. However if we consider a *singular* cubic curve then lines through the singular point, (a double point), give a rational parameterization for the cubic curve as again these lines of slope t intersect the cubic in only one additional point. Such is not the case for *non−singular* cubic curves and thus they correspond to the cubics which do not have a rational parameterization.

Algebraic Method

A plane cubic curve is given by

$$C(x, y) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fy^2 + gxy + hx + iy + j$$

Make it nonregular in y (by eliminating the y^3 term through a coordinate transformation). If there exists a real point at infinity then linear transformations suffice. Recall that, points at infinity are given by the degree form of $C(x,y)$, (terms of highest degree). For conics, we have real points at infinity only in the case of parabolas and hyperbolas. However for cubics, we are lucky as all cubics have a real point at infinity. The reason being: the degree form always has a real root as it is of degree 3 and complex roots occur in conjugate pairs. The degree form on dehomogenizing ($y = 1$), gives $f(x) = ax^3 + bx^2 + cx + d$ which always has a real root (if $a \neq 0$). When a is zero, the cubic $C(x, y)$ is already nonregular in x. Thus to make $C(x,y)$ nonregular in y we may use a linear transformation given by

$$\begin{aligned} x &\rightarrow \alpha\bar{x} + \beta\bar{y} \\ y &\rightarrow \gamma\bar{x} + \delta\bar{y} \end{aligned} \qquad \text{.....} \quad (2.1)$$

To make the $\bar{y}^3$ term to be zero, we set its coefficient to $(a\beta^3 + b\beta^2 + c\beta + d) = 0$ by taking $\delta = 1$, which we may. Now the transformed cubic, in a somewhat rearranged fashion, is given by

$$C(\bar{x},\bar{y}) = (u\bar{x} + v)\bar{y}^2 + (p\bar{x}^2 + q\bar{x} + r)\bar{y} + (k\bar{x}^3 + l\bar{x}^2 + m\bar{x} + n)$$

which is the usual quadratic equation. Using the old Indian method of Shreedharacharya (5th century), of solving the quadratic equation, ("multiply by 4 times the coefficient of the square term and add the square of the coefficient of the unknown, and the rest follows"), we obtain

$$4(u\bar{x}+v)^2\bar{y}^2 + 4(u\bar{x}+v)(p\bar{x}^2+q\bar{x}+r)\bar{y} + 4(u\bar{x}+v)(k\bar{x}^3+l\bar{x}^2+m\bar{x}+n) = 0$$

which on completing the square becomes

$$[2(u\bar{x}+v)\bar{y} + (p\bar{x}^2+q\bar{x}+s)]^2 = (p\bar{x}^2+q\bar{x}+s)^2 - 4(u\bar{x}+v)(k\bar{x}^3+l\bar{x}^2+m\bar{x}+n)$$

If we let

$$y^* = [2(u\bar{x}+v)\bar{y} + (p\bar{x}^2+q\bar{x}+s)] \quad \ldots\ldots (2.2)$$

then equation (2.2) becomes of the type

$$y^{*2} = g(\bar{x}), \qquad deg.\ g(\bar{x}) \le 4 \quad \ldots\ldots (2.3)$$

We only need to analyze if we can obtain a parametrization for $\bar{x}$ and y^* for then using transformations (2.1) and (2.2) we obtain the parameterization for x and y. To do this we consider several cases as follows: $g(\bar{x})$ has only one distinct root, $g(\bar{x})$ has two distinct roots...etc. In the case of multiple roots, we may use the following general method to get rid of them.

Suppose

$$y^{*2} = [\prod_{i=1}^{d}(\bar{x}-\mu_i)^2]\Omega(\bar{x}) \qquad d = 1 \ \ or \ \ 2$$

so each root μ_i occurs an even number of times and $\Omega(x)$ has no multiple roots. Then if we let

$$y^{**} = \left[\frac{y^*}{\prod_{i=1}^{d}(\bar{x}-\mu_i)}\right] \quad \ldots\ldots (2.4)$$

then equation (2.3) reduces to

$$y^{**2} = \Omega(\bar{x}) \quad \ldots\ldots (2.5)$$

If $deg.\ \Omega(\bar{x}) \le 2$, then the above equation (2.5) is a conic and a rational parametrization is always possible, Abhyankar and Bajaj (1986a). This then, together with transformations (2.1), (2.2) and (2.4), gives a parameterization for x and y of the original curve. Otherwise, $g(\bar{x})$ has either 3 or 4 distinct roots, and a rational parametrization is not possible. Further, it can be proved that these are the only cases for which the cubic curve does not have a rational parametrization, Abhyankar and Bajaj (1986b). However, by solving the above equation (2.5), quadratic in y^{**}, we always have a parameterization for the cubic of the type that allows a single square root of rational functions. The rational parameterization obtained is global, and of degree at most 3 with the parameter t ranging from $(-\infty,\infty)$ and spanning the entire curve.

3. Cubicoids

Geometric Viewpoint

If we intersect a cubic surface with a plane we get a cubic curve in general. However if we intersect it with a tangent plane then something special happens, namely, we get a *singular* cubic curve or a reducible curve (either a straight line and a conic, or three straight lines). In general we obtain a singular cubic curve as there are only a finite number of real straight lines on a cubic surface, see Henderson (1911). In either case the intersection curve is rational and can be parameterized by a single parameter, say s.

Now to obtain a rational parametrization of the cubicoid we take a simple point on it. The intersection of the tangent plane at this simple point with the cubicoid gives a singular cubic curve, which can be rationally parameterized by the method of § 2. Next consider a variable point s on this singular cubic curve. Then consider the tangent plane to the cubic surface at that point. We again get a singular cubic curve (or a reducible curve) as the intersection, which can be parametrized by another parameter t. Thus we get a parametrization of the cubic surface by two parameters s and t. Given values of s and t they uniquely define a point on the surface. Hence a cubic surface can be parametrized starting with a simple point. However, with s and t as parameters on singular cubic curves on the cubicoid, any point (x, y, z) on the surface can be shown to correspond to *six* pairs of (s, t), giving a *6-fold* parameterization or a *6-fold* covering of the plane. If reducible curves (conics and straight lines) are obtained as the intersection with the tangent planes, and parameters s and t are chosen on them, one could obtain a lower fold parameterization.

To ensure obtaining say a 1–*fold* parameterization of the cubicoid we need to generate two different rational curves on its surface. Let t and τ correspond to independent parameterizations of the two chosen rational curves. The set of unique lines defined by endpoints t and τ (a variable point t on one rational curve and a variable point τ on the other), intersect the cubic surface in one additional point giving a rational parameterization of the cubicoid. Here a point (x,y,z) on the cubic surface can be seen to correspond to a single pair (t,τ) yielding a *1-fold* parameterization or a *1-fold* covering of the plane.

One method of obtaining two different rational curves on the cubic surface is to repeat the above method for two different simple points on the cubicoid thereby obtaining two different singular rational cubic curves. Alternatively, one can generate two non-intersecting straight lines from the twenty seven lines on a cubic surface, see Henderson (1911). This using essentially the above method of intersections with tangent planes to the cubic surface. However, differing with the second step of the earlier method, the variable point s is now chosen appropriately such that the tangent plane to the surface at this points yields a reducible intersection with the cubic surface. Each value of s can yield one or three straight lines lying on the same tangent plane. However two specific values of s are chosen to yield two non-intersecting straight lines of the cubicoid lying on different tangent planes.

Algebraic Method

A general cubicoid (degree three) surface has an implicit equation given by

$$C(x,y,z) = ax^3 + by^3 + cz^3 + dx^2y + ex^2z + fxy^2 + gy^2z + hxz^2 + iyz^2 + jxyz$$
$$+ kx^2 + ly^2 + mz^2 + nxy + oxz + pyz + qx + ry + sz + t = 0$$

Take a simple point (x_0,y_0,z_0) on it. Most points on the cubicoid are simple, so this is not a problem. Bring the simple point to the origin by a simple translation $x = x' + x_0$, $y = y' + y_0$ and $z = z' + z_0$.

$$C(x', y', z') = a'x' + b'y' + c'z' + \cdots \quad terms \ \ of \ \ higher \ \ degree.$$

Next rotate the tangent plane to the surface at the origin, given by the order form (terms of lowest degree), to the $z = 0$ plane. This by using a simple rotation, $x' = x$, $y' = y$ and $z' = \frac{1}{c'}z - \frac{a'}{c'}x - \frac{b'}{c'}y$ which gives

$$C(x, y, z) = z + [f_2(x, y) + f_1(x, y)z + f_0z^2]$$

$$+ [g_3(x,y) + g_2(x,y)z + g_1(x,y)z^2 + g_0z^3]$$

where $f_i(x,y)$ and $g_i(x,y)$ are appropriate terms of degree i. Its intersection with the tangent plane $z = 0$ is simply,

$$f_2(x,y) + g_3(x,y) = 0 \qquad ..... \ (3.1)$$

which is either a reducible curve or a cubic curve with a double point at the origin. In all cases the curve (3.1) can be rationally parameterized with a single independent parameter s and rational functions K and L

$$\begin{aligned} x &= K(s) \\ y &= L(s) \\ z &= 0 \end{aligned}$$

Now bring a general point specified by parameter s on this parameterized curve to the origin. This by a simple translation

$$\begin{aligned} x &= \bar{x} + K(s) \\ y &= \bar{y} + L(s) \\ z &= \bar{z} \end{aligned} \qquad ..... \ (3.2)$$

Since this point also lies on the cubic surface $C(x,y,z)$, the surface equation with zero constant term, is given by

$$C(\bar{x},\bar{y},\bar{z}) = \bar{a}(s)\bar{x} + \bar{b}(s)\bar{y} + \bar{c}(s)\bar{z} + \cdots \ \textit{terms of higher degree.}$$

Next a simple rotation

$$\begin{aligned} \bar{x} &= \hat{x} \\ \bar{y} &= \hat{y} \\ \bar{z} &= \frac{1}{\bar{c}(s)}\hat{z} - \frac{\bar{a}(s)}{\bar{c}(s)}\hat{x} - \frac{\bar{b}(s)}{\bar{c}(s)}\hat{y} \end{aligned} \qquad ..... \ (3.3)$$

makes the tangent plane to the surface at the origin to be the $\hat{z} = 0$ plane, resulting again in

$$C(\hat{x},\hat{y},\hat{z}) = \hat{z} + [\hat{f}_2(\hat{x},\hat{y}) + \hat{f}_1(\hat{x},\hat{y})\hat{z} + \hat{f}_0\hat{z}^2]$$
$$+ [\hat{g}_3(\hat{x},\hat{y}) + \hat{g}_2(\hat{x},\hat{y})\hat{z} + \hat{g}_1(\hat{x},\hat{y})\hat{z}^2 + \hat{g}_0(\hat{x},\hat{y})\hat{z}^3].$$

Its intersection with $\hat{z} = 0$ plane will give

$$\hat{f}_2(\hat{x},\hat{y}) + \hat{g}_3(\hat{x},\hat{y}) = 0 \qquad ..... \ (3.4)$$

which is a plane curve with coefficients involving s. Once again the curve (3.4) can be rationally parametrized with a single independent parameter t

$$\hat{x} = M(t)$$
$$\hat{y} = N(t) \qquad \ldots\ldots \quad (3.5)$$
$$\hat{z} = 0$$

with coefficients of rational functions M and N also involving s. Finally using (3.5) and the transformations (3.2) and (3.3) above, gives a rational parameterization of the cubic surface for the original variables x, y and z in terms of rational functions involving both parameters s and t. The rational parameterization we obtain is global, with parameters s and t both ranging from $(-\infty,\infty)$ and covering the entire surface.

Alternatively consider the plane curve given by (3.4), with coefficients involving s. For certain values of s, the plane curve is reducible which gives the lines on the cubic surface. Specifically (3.4) is reducible for those values of s for which the two polynomials $\hat{f}_2(\hat{x}, \hat{y})$ and $\hat{g}_3(\hat{x}, \hat{y})$ have a linear or quadratic common factor. One way of obtaining these s values is as follows. Consider $\hat{f}_2(\hat{x}, \hat{y}) = 0$ the homogeneous equation of degree 2 with coefficients involving s. It has two linear factors $\hat{y} = m_1(s)\hat{x}$ and $\hat{y} = m_2(s)\hat{x}$. Substituting either of these into the homogeneous equation $g_3(\hat{x}, \hat{y}) = 0$ yields a cubic equation of the form $p(s)\hat{x}^3 = 0$ where $p(s)$ is a function of s. Specific solutions s of the equation $p(s) = 0$ can easily be obtained by using known methods for obtaining roots of polynomial equations, see Buchberger et. al. (1982). With (3.2) and (3.3) and for two appropriate choices of s one obtains the linear parametric equations of two distinct lines L_1 and L_2 on the cubic surface, viz., $x_1 = t, y_1 = a_1 t + b_1, z_1 = c_1 t + d_1$ and $x_2 = \tau, y_2 = a_2\tau + b_2, z_2 = c_2\tau + d_2$.

Next consider the straight line passing through a point on each of the two distinct lines L_1 and L_2. This in space is given by two equations

$$\frac{z - z_1}{x - x_1} = \frac{z_2 - z_1}{x_2 - x_1} \qquad \ldots\ldots \quad (3.6)$$

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \qquad \ldots\ldots \quad (3.7)$$

and defines a family of straight lines for varying t and τ. Substituting for y and z in terms of x in the equation of the cubic surface $C(x,y,z) = 0$ yields a cubic equation in x with coefficients in t

and τ, viz., $g(x,t,\tau)=0$. However both $x=t$ and $x=\tau$ satisfy this equation and thus $\frac{g(x,t,\tau)}{(x-t)(x-\tau)}=0$ is linear in x yielding x as a rational function of t and τ. Together with (3.6) and (3.7) this yields a rational parameterization of the cubic surface in terms of the independent parameters t and τ.

4. Conclusion

For surfaces of degree higher than three no rational parametric forms exist in general, although parameterizable subclasses can be identified. For low degree curves and surfaces, in this paper and in Abhyankar and Bajaj (1987a) procedures have been developed and implemented for parameterizing implicit forms. Various computational issues in extending this approach to parameterize planar curves of higher degree are discussed in Abhyankar and Bajaj (1987b). Currently efforts are being made to obtain explicit parameterizations of special families of quartic surfaces and surfaces of higher degree which would prove useful for representing blending surfaces.

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