1986

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Report Number:
86-616

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CSD-TR-616
July 1986

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Scope and Purpose

In studying the performance of a computer communication system with many queues and an intellegently allocated resource, it was found that certain conditions regarding the state of the system could be used to formulate guidelines for assessing the behaviour of the resource, improving resource utilization, and minimizing communication delay. In particular, if the resource based its service-oriented decisions on the various queue lengths, it was critical to determine exactly when its decisions were unbiased. This led to a series of problems, with the first set investigated here. We look for conditions under which a single resource (server) in a simple system will find queue lengths that are uniformly distributed (thus forcing the resource to be indifferent to queue lengths). In doing so, we present simple probabilistic proofs characterising a class of single-server queues (see [4]). Additionally, it is shown how a queue can be designed so that equilibrium queue lengths are partly (in a given range) uniformly distributed, a design that was essential to the original communication problem.

Abstract

In certain multiqueue systems, a resourceful server may decide on which queue to attend to next based on criteria that may include the equilibrium queue lengths of some or all queues. The server's decision will be unaffected by queue length only when the distribution of queue length is uniform. In studying this problem, we present first the simple case of the M/GI/1 queue to determine conditions under which equilibrium queue lengths can either be uniformly distributed, or have components that are uniform, and also present a method for designing such a queueing process. It is shown that there is an inherent connection between probability distributions $\{k_j\}$ of the number of arriving customers during a service that are geometric, and equilibrium queue length distributions $\{p_j\}$ that are geometric. In particular, we present a simple probabilistic proof via a recurrence equation that says that the M/M/1 queue is the only M/GI/1 queue with both $\{k_j\}$ and $\{p_j\}$ geometric.

1. Introduction

An interesting problem in queueing theory is the relationship between arrival and service processes that interact to yield certain steady-state distributions. For example, in a single server queue, how can one determine customer interarrival and service-time distributions so that given steady state queue length and waiting time distributions are obtained, for a stable queue ? The problem in all its generality is not simple because of certain computational problems that arise with the use of general distributions. In this paper we restrict our attention to a small class of M/GI/1 queueing models. Our motivation lies in describing transition matrices of finite or infinite M/GI/1 queueing chains that lead to uniform and geometric steady-state distributions. Though we know much about the M/GI/1 queue at present, results that yield qualitative insights into its behaviour with simple techniques are scarce. Thus, the results presented in this paper are *probabilistic* and computationally simple, with a focus on understanding characteristics of simple M/GI/1 processes.

We begin by mentioning a problem that motivated this work. In modelling a communication system with two queues and one server, it was necessary to determine the equilibrium queue length of one queue so that the server could make use of such information in scheduling visits to a second queue. At an extreme, the behaviour of the server in scheduling such visits could be completely indifferent to the contents of the first queue. At steady-state, this would happen only if the equilibrium distribution of the first queue was the uniform distribution. Naturally, it makes little sense to talk about an unbounded queue with such a property. Since the queues found in computer and communications systems are usually buffered, it remains to determine the kinds of M/GI/1/K queueing chains that can lead to a uniform queue length distribution.

Such problems as the one just posed are known as *inverse* problems [1]. In this particular case, we attempt to determine the characteristics of M/GI/1 processes that lead to equilibrium queue length distributions that are uniform, geometric, or combinations of these. In the case of the M/GI/1 queue it is known (see Whitt [2], Karr [3]) that the equilibrium queue-length distribution completely characterises the arrival and service processes up to a scale factor. The same can be said of the equilibrium delay distribution. This is due to the particular structure of the transition matrix. In what follows, we examine the transition matrices of special M/GI/1 queueing chains to determine the conditions under which they lead to certain distributions.

Definition. A discrete distribution $\{a_i\}_{i=0}^{\infty}$ on the nonnegative integers satisfying, $0 < a_0 < 1$, $a_i = a_0\beta^i$, with $1 \leq i \leq \infty$, and $\beta=(1-a_0)$ is a *geometric distribution* with parameter a_0, denoted by $G(a_0, \infty)$. A discrete distribution $\{b_i\}_{i=0}^{\infty}$ with $b_i = a_i$ for $0 \leq i \leq (m-2)$, $m > 1$, and $b_{m-1} = (1 - \sum_{i=0}^{m-2} a_i) > 0$, is a truncate of a geometric distribution, with exactly m terms, denoted by $G(a_0, m)$.

2. Equilibrium Queue Lengths

In an M/GI/1 queue, let X_n be a random variable representing the number of customers remaining in the queue as the n^{th} customer departs from the system. Then $\{X_n\}$ is a well-known Markov chain (see for example, Gross and Harris [4]). The arrival rate of customers to the system is taken to be λ, $\lambda > 0$, and the service-time distribution is $B(\cdot)$. The probability that j customers arrive during an arbitrary customer's service is given by

$$k_j = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \, dB(t) \tag{1}$$

where for the sake of generality the Stieltjes integral is used. As a convention, we henceforth take $\{k_j\}$ to mean $\{k_j\}_{j=0}^{\infty}$. Clearly, $\{k_j\}$ is a distribution on the nonnegative integers. Throughout the paper, we focus our attention on the class of M/GI/1 and M/GI/1/K chains for which $\{k_j\}$ has special properties, such as being distributed as $G(\alpha, \infty)$ or $G(\alpha, N)$, for $0 < \alpha < 1$. Note that when the queueing chain is finite (i.e., the queue size is restricted), of size N, $N > 1$, the probability transition matrix of the truncate takes the form

$$P = \begin{bmatrix} k_0 & k_1 & k_2 & \cdots & 1-\sum_{j=0}^{N-2} k_j \\ k_0 & k_1 & k_2 & \cdots & 1-\sum_{j=0}^{N-2} k_j \\ 0 & k_0 & k_1 & \cdots & 1-\sum_{j=0}^{N-3} k_j \\ 0 & 0 & k_0 & \cdots & 1-\sum_{j=0}^{N-4} k_j \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1-k_0 \end{bmatrix}$$

We first present a characterisation theorem for M/GI/1 queues which effectively says that equilibrium queue length distributions that are uniform can be obtained only under very special conditions, i.e., $\{k_j\}$ is geometrically distributed, the geometric parameter is $\frac{1}{2}$, *and* the queueing chain is finite. We restrict our attention to the case where the capacity of the queue is greater than 1. In the special case where the capacity of the queue is 1, results may be obtained by treating the system as a special case of the M/GI/K/K queue.

THEOREM 1

The equilibrium queue length distribution $\{p_j\}$, with $p_j = \lim_{n\to\infty} P(X_n = j) = 1/N$, $0 \le j \le N-1$, will exist and coincide with a uniform distribution on $\{0,1,\ldots,N-1\}$ if and only if $\{k_j\}$ is $G(\frac{1}{2}, N)$, for $N > 1$, where N is the capacity of the queue.

PROOF

Assuming that a steady-state distribution, $p_j = \lim_{n\to\infty} Pr\ \{X_n = j\}$, for $j \ge 0$, exists is equivalent to assuming that the *traffic intensity* [4] ρ satisfies

$$\rho = \sum_{j=0}^{\infty} j\, k_j < 1 \tag{2}$$

since (2) is a necessary condition for an M/GI/1 queueing chain to be ergodic.

Assume that $\{p_j\}$ is a uniform distribution. It is known [3] that $\{p_j\}$ will be uniform only when the transition probability matrix for the chain $\{X_n\}$ is doubly stochastic. It follows that

$$k_{j-1} + \sum_{i=0}^{j-1} k_i = 1, \qquad j \geq 1, \quad \text{and} \tag{3}$$

$$k_j + \sum_{i=0}^{j} k_i = 1, \qquad j \geq 0 \tag{4}$$

On subtracting (4) from (3), we obtain

$$k_j = (\frac{1}{2}) k_{j-1}, \quad j \geq 1 \tag{5}$$

Since $\{k_j\}$ defined by (5) is an infinite geometric series, it sums to one under the condition that $k_0 = 1 - 1/2$, thus giving the parameter of the distribution $\{k_j\}$ as $k_0 = 1/2$. For this value of k_0, the quantity ρ evaluates to 1, thus violating (2). Equivalently, the corresponding M/GI/1 chain cannot have an equilibrium distribution. But, for every integer m, $m > 1$, we have that $\sum_{j=0}^{m} k_0 (\frac{1}{2})^j < 1$, ensuring that a distribution $\{k_j\}$ with representation $G(\frac{1}{2}, m)$ can be used to define the $m \times m$ probability transition matrix of a corresponding M/GI/1 chain. Consequently, the Markov chain yielding a uniform equilibrium distribution must be a finite chain.

Conversely, assume that $\{k_j\}$ is $G(\frac{1}{2}, m)$, for $m > 1$. The m terms of $\{k_j\}$ can be used to define an $m \times m$ probability transition matrix of an M/GI/1/m chain. Since the queueing chain is finite, an equilibrium queue length distribution exists. Since the columns of the matrix must sum to unity, it readily follows that the doubly stochastic matrix must possess an invariant vector whose components are all equal. Hence, the equilibrium distribution is uniform on $\{0, 1, \ldots, m-1\}$.

□

The above theorem tells us that if a queuing process (possibly within a system of queues) is to be constructed so that the steady-state queue length distribution is to be uniform, then

(a) a buffer of size N, $N > 0$, must be used to limit the queue, and

(b) given a fixed arrival rate λ, $\lambda > 0$, the service-time distribution must be chosen so that

$$k_m = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^m}{m!} dB(t) = (\frac{1}{2})^{m+1} \tag{6}$$

for $0 \le m \le N-2$.

As long as the M/GI/1 queue has finite capacity, it is possible to choose arrival and service processes that ensure a uniform distribution for equilibrium queue length. The task of obtaining a distribution $B(\cdot)$ that satisfies (6) is equivalent to the Stieltjes moment problem [6]. This result can be generalised to obtain interesting equilibrium queues by solving the latter problem [5]. The *height* of the uniform distribution obtained is just the reciprocal of the maximum number of customers that can be in the queue at a departure instant, i.e., the system's capacity less one. Thus, the height of the equilibrium distribution falls linearly as a function of the system capacity. Figure 1 illustrates this behaviour for finite M/GI/1 queues with $\{k_j\}$ having representations $G(\frac{1}{2}, 4)$ and $G(\frac{1}{2}, 8)$, respectively. In the case where $\{k_j\}$ is $G(\frac{1}{2}, 4)$, the probability transition matrix of the chain is given by

$$P_4 = \begin{bmatrix} 0.5 & 0.25 & 0.125 & 0.125 \\ 0.5 & 0.25 & 0.125 & 0.125 \\ & 0.5 & 0.25 & 0.25 \\ & & 0.5 & 0.5 \end{bmatrix}$$

It would seem natural to ask what other properties of the equilibrium queue length distribution can be established when $\{k_j\}$ is a geometric series. For example, could $\{k_j\}$ be an infinite geometric distribution that caused $\{p_j\}$ to be a non-geometric probability distribution? Interestingly enough, the answer is in the negative. However, it is possible to force $\{k_j\}$ be of type

$G(\alpha_1, i)$ for $j < i$, and of type $G(\alpha_2, \infty)$ for $j \geq i$, with $\alpha_1 > 0$, $\alpha_2 > 0$, $i > 1$, to obtain interesting equilibrium queue length distributions. We present one such construction as a design problem in this paper. More general constructions and some extensions are discussed in [5]. Some generalisations of this approach are mentioned later. Questions in this spirit possess the flavour of qualitative analysis and are important in that they allow for an understanding of system behaviour with the aid of relatively simple methods, possibly even allowing for glimpses of equilibrium behaviour in more complicated systems.

It turns out that if $\{k_j\}$ is geometric and $k_0 \neq \frac{1}{2}$, then if the queueing chain is ergodic, the equilibrium distribution must be a geometric distribution. This yields a characterisation result for the M/M/1 queue. In the next lemma, a recurrence equation for M/GI/1 queues is presented, to be used in the subsequent theorem.

LEMMA 1

If $\{k_j\}$ is distributed as $G(\alpha, \infty)$, and $\frac{1}{2} < \alpha < 1$, the equilibrium queue length distribution $\{p_j\}$ of the corresponding M/GI/1 queue is given by the recurrence $p_{n+1} = (\frac{1}{\alpha})p_n - (\frac{1-\alpha}{\alpha})p_{n-1}$, for $n \geq 1$, with $p_0 = 1 - (\frac{1-\alpha}{\alpha})$. If $0 \leq \alpha \leq 1/2$, an equilibrium queue length distribution does not exist.

PROOF.

The quantity ρ is given by

$$\rho = \sum_{j=0}^{\infty} j\, k_j = \sum_{j=0}^{\infty} j\, \alpha (1-\alpha)^j = \frac{1-\alpha}{\alpha}$$

from where the ergodicity condition $0 < \rho < 1$ can be seen to be equivalent to $1/2 < \alpha < 1$. Thus, if $1/2 < \alpha < 1$, then the equilibrium queue length distribution $\{p_j\}$ of the corresponding M/GI/1 queue exists, and is given by the recurrence

$$p_{n+1} = \frac{1}{k_0}\left[p_n - p_0 k_n - \sum_{m=1}^{n} p_m k_{n-m+1}\right] \tag{7}$$

from which, with $\alpha = k_0$, $k_n = k_0(1-\alpha)^n$, we obtain

$$p_{n+1} = \frac{1}{\alpha}\left[p_n - p_0\alpha(1-\alpha)^n - \sum_{m=1}^{n} p_m \alpha(1-\alpha)^{n-m+1}\right] \tag{8}$$

for $n \geq 0$. Similarly,

$$p_n = \frac{1}{\alpha}\left[p_{n-1} - p_0\alpha(1-\alpha)^{n-1} - \sum_{m=1}^{n-1} p_m \alpha(1-\alpha)^{n-m}\right] \tag{9}$$

for $n \geq 1$. From (8) and (9) is obtained the second order recurrence

$$p_{n+1} = (\frac{1}{\alpha})p_n - (\frac{1-\alpha}{\alpha})p_{n-1} \tag{10}$$

for $n \geq 1$, with $p_0 = 1 - \rho = 1-(1-\alpha)/\alpha$, and $p_1 = p_0(1/k_0 - 1)$.

□

In the next theorem, which characterises the M/M/1 queue, it is shown that the equilibrium queue-length distribution of an unbounded M/GI/1 queue is geometric if and only if $\{k_j\}$ is geometric and infinite, with parameter in the interval (1/2,1).

THEOREM 2

For $1/2 < \alpha < 1$, the distribution $\{k_j\}$ has the representation $G(\alpha, \infty)$ if and only if the equilibrium queue length distribution $\{p_j\}$ of the corresponding M/GI/1 queue has the representation $G([\frac{1-\alpha}{\alpha}], \infty)$.

PROOF

Assume that $\{k_j\}$ is $G(\alpha, \infty)$, and that $1/2 < \alpha < 1$. From lemma 2 it follows that $\rho < 1$, so that an equilibrium queue-length distribution does exist. Also from lemma 2 we have that $p_0 = 1 - \frac{1-\alpha}{\alpha}$,

and

$$p_{n+1} = (\frac{1}{\alpha})p_n - (\frac{1-\alpha}{\alpha})p_{n-1}$$

for $n \geq 1$, yielding the characteristic equation [7]

$$\alpha z^2 - z + (1-\alpha) = 0 \tag{11}$$

which possesses the roots $z = 1$ and $z = (1-\alpha)/\alpha$. Thus, the general solution to the recurrence in (10) is given by

$$p_n = c_1 + c_2 [\frac{(1-\alpha)}{\alpha}]^n \tag{12}$$

In satisfying boundary conditions, we require that

$$c_1 + c_2 = 1 - \rho \tag{13}$$

since we know that $p_0 = 1-\rho$ for M/GI/1 queues, and

$$\sum_{n=0}^{\infty} [c_1 \alpha_1^n + c_2 \alpha_2^n] = 1 \tag{14}$$

since the probabilities must sum to one. From (13) is obtained

$$c_1 + c_2 = 1 - \frac{(1-\alpha)}{\alpha}$$

and from (14) is obtained

$$c_1 + c_2 \frac{\alpha}{(2\alpha-1)} = 1$$

solving which yields $c_1 = 0$, and

$$c_2 = \frac{(2\alpha-1)}{\alpha} = 1 - \frac{(1-\alpha)}{\alpha}$$

On making the substitutions for c_1 and c_2 in (12), with $\rho = \frac{(1-\alpha)}{\alpha}$, one obtains

$$p_n = (1-\rho)\rho^n \tag{15}$$

for $n \geq 0$, which is $G(\rho, \infty)$, and is precisely the equilibrium distribution of an M/M/1 queue.

Conversely, assume that (15) holds, with $\rho = \frac{(1-\alpha)}{\alpha}$, and $0 < \rho < 1$. It follows that $1/2 < \alpha < 1$. The distribution $\{k_j\}$ defining the probability transition matrix for the corresponding M/GI/1 queueing chain must satisfy the recurrence of lemma 1, i.e.,

$$k_0 p_{n+1} = p_n - p_0 k_n - p_1 k_n - \sum_{m=2}^{n} p_m k_{n-m+1} \tag{16}$$

for $n > 0$. Given the truth of (15), this is otherwise expressed as

$$k_n (p_0 + p_1) = p_n - k_0 p_{n+1} - \sum_{m=2}^{n} (1-\rho)\rho^m k_{n-m+1} \tag{17}$$

for $n > 0$. On comparing (17) with a version of itself with n replaced by $(n-1)$, multiplying this new version with ρ, and then subtracting it from (17), we obtain the recurrence

$$(p_0+p_1) k_n - \rho (p_0+p_1) k_{n-1} + (1-\rho)\rho^2 k_{n-1} = 0 \tag{18}$$

for $n > 1$, which is easily simplified to yield

$$k_n = \frac{\rho}{1+\rho} k_{n-1} \tag{19}$$

for $n > 1$. Since $\{k_j\}$ is a probability distribution on the nonnegative integers, it is clear that (19) defines a geometric series with first term $k_0 = \frac{1}{1+\rho}$ and common ratio $\frac{\rho}{1+\rho}$. Thus $\{k_j\}$ has the representation $G(\alpha, \infty)$, with $\alpha = \frac{1}{1+\rho}$.

□

An alternate method of showing that $\{k_j\}$ is $G(\alpha, \infty)$ is to use the fact that such M/GI/1 matrices characteristically satisfy [2]

$$k_j = \mu \int_0^\infty \frac{e^{-(\lambda+\mu)t} (\lambda t)^j}{j!} \, dt \tag{20}$$

for $j \geq 0$, and from this is obtained

$$k_j = \frac{\lambda^j \mu}{(\lambda+\mu)^{j+1}}$$
$$= \left(\frac{\rho}{1+\rho}\right)^{j+1} \left(\frac{1}{\rho}\right) = (1-\alpha)^j \alpha$$

where $\mu > 0$ is the mean of the service-time distribution defined in (1). It follows that $\{k_j\}$ is $G(\alpha, \infty)$.

At this stage, it is clear that the equilibrium queue length distribution $\{p_j\}$ is going to be geometric as long as $\{k_j\}$ is geometric, with parameter α satisfying $1/2 < \alpha < 1$. As $\alpha \to 1/2$, the rate of decay of the (geometric) equilibrium queue length distribution decreases rapidly, *almost* tending to a uniform distribution. Thus, the steady-state queue lengths get longer and longer, with every kind of queue length occurring with nearly the same probability. As α gets arbitrarily close to 1/2, the queue length distribution gets arbitrarily close to a uniform distribution. However, this peculiar property is meaningless when $\alpha = 1/2$, because it forces the queue length distribution to degenerate to a situation where it does not exist. In queueing terminology, we arrive at $\rho = 1$, yielding a chain which is recurrent, and thus one that cannot possibly possess an equilibrium distribution.

The behaviour described above is graphically illustrated in Figures 2a through 2d. In Fig 2a, $\alpha = 0.85$ (i.e., $\rho = 0.176$) thus forcing small equilibrium queue lengths with larger probabilities. Of some interest is the fact that as ρ decreases to 0, the circles and the squares in the figure get closer and closer together, until ρ finally reaches zero, when every circle is made to reside in a single corresponding square. So as ρ decreases to 0, the $\{p_j\}$ and $\{k_j\}$ distributions converge to the same geometric distribution. As ρ approaches 1/2, larger equilibrium queue lengths occur with larger probabilities, as shown in Fig 2b. The rate of decay of $\{p_j\}$ increases rapidly, but never does get quite as high as the decay rate of $\{k_j\}$. Nevertheless, there is a balanced interplay between the two geometric distributions.

In Fig 2c, $\alpha = 0.501$ (i.e., $\rho = 0.996$), with $\{k_j\}$ falling exceedingly fast, but $\{p_j\}$ falling very gently. This is made even more visible in Fig 2d, where the same graph is displayed over a smaller range. In fact, for $\alpha = 1/2 + \varepsilon$, with $0 < \varepsilon < 0.001$, the line representing $\{p_j\}$ approaches the horizontal as ε approaches 0, thus making arbitrarily long queues almost as likely to occur as extremely small queues.

3. Uniformisation of Queue Lengths

A subject of some interest is the design of queues. More specifically, is it possible to find arrival rates and service-time combinations that yield a desired steady-state queue length distribution in an M/GI/1 queue? Though the answer is not simple, such combinations can be found. In general the task involves the construction of special distribution functions via solutions to the Stieltjes moment problems. In very special cases, however, simple solutions are possible. An example is presented in the following.

Suppose that we wished to construct an M/GI/1 queueing process such that for a given integer L, $L > 0$, the steady state queue-length distribution $\{p_j\}$ satisfies

$$p_j = \begin{cases} c & 0 \le j \le L \\ f(\{c, p_0, \ldots, p_{j-1}, k_0, \ldots, k_{j-1}\}) & j > L \end{cases} \tag{21}$$

where $f(\cdot, \ldots, \cdot)$ is a function involving the recurrence in (7), c is defined by L, and $L > 0$ may be chosen arbitrarily. The expression in (21) requires that $c = 1/(L+1)$ and $\{p_j\}$ be uniform on $\{0, 1, \ldots, L\}$. Applying theorem 1, this can be done by choosing $k_m = \frac{1}{2} k_{m-1}$, for $1 < m < L$, with $k_0 = \frac{1}{2}$. Since $\{k_j\}$ must be a distribution function function on the nonnegative integers, we require that $\sum_{j=0}^{\infty} k_j = 1$. If we choose $k_j = \alpha \beta^{j-L}$ for $\alpha > 0$, $\beta > 0$, $j \ge L$, this requirement translates into

$$\sum_{j=0}^{L-1} (\frac{1}{2})^{j+1} + \sum_{j=L}^{\infty} \alpha \beta^{j-L} = 1 \tag{22}$$

which yields the relation

$$\alpha = \frac{(1-\beta)}{2^L} \tag{23}$$

for each fixed L, $L > 0$. Equation (23) tells us how α is computed for any value of L, given β. The next step is to determine the range of possible values that β can take while still keeping the system stable. Recall that for a stable system, we require $\rho = \sum_{j=0}^{\infty} j\,k_j < 1$. This is simplified to give

$$
\begin{aligned}
\sum_{j=0}^{\infty} j\,k_j &= \sum_{j=0}^{L-1} j\,k_j + \sum_{j=L}^{\infty} j\,k_j \\
&= \sum_{j=0}^{L-1} j(\frac{1}{2})^{j+1} + \sum_{j=L}^{\infty} j\,\alpha\beta^{j-L} \\
&= 1 - \frac{(L+1)}{2^L} + \frac{\alpha\beta}{(1-\beta)^2} + \frac{L\,\alpha}{(1-\beta)} \\
&= 1 - \frac{(L+1)}{2^L} + \frac{\beta}{2^L(1-\beta)} + \frac{L}{2^L}
\end{aligned} \tag{24}
$$

which together with the requirement that $\beta > 0$, yields the sufficient condition $0 < \beta < \frac{1}{2}$ for an equilibrium queue length distribution to exist. The quantity ρ is computed with the aid of (20), and $p_0 = 1 - \rho$. Additionally, from our construction, we know that $p_j = 1 - \rho$, for $1 \le j \le L$. It is now left to determine p_j, $j > L$. From the recurrence in (7) is obtained, for $n \ge L$,

$$p_{n+1} = 2\left\{ p_n - p_0\,\alpha\beta^{n-L} - \sum_{m=1}^{n-L+1} p_m\,\alpha\beta^{n-L-m+1} - \sum_{m=n-L+2}^{n} p_m(\frac{1}{2})^{n-m+2} \right\} \tag{25}$$

which after formal manipulation gives

$$p_{L+1} = 2(1-\rho)\,[1/2 - 2\alpha + \frac{1}{2^L}] \tag{26}$$

$$p_{L+2} = \frac{3}{2}p_{L+1} + 2(1-\rho)\,[\frac{1}{2^{L+1}} - 1/4 - \alpha - 2\alpha\beta] \tag{27}$$

and

$$p_{n+1} = 2\left\{ \gamma_0 p_n + \gamma_1 p_{n-1} + \gamma_2 p_{n-2} + \gamma_{3,L}\, p_{n-L+1} + \gamma_{4,L}\, p_{n-L} \right\} \tag{28}$$

for $n \ge L+2$. The constants in (28) are

$$
\begin{gathered}
\gamma_0 = \frac{\beta}{2} + 1, \quad \gamma_1 = -(\beta + \frac{1}{2}), \quad \gamma_2 = \frac{\beta}{2}, \\
\gamma_{3,L} = (\frac{1}{2})^{L+1} - \alpha, \quad \text{and} \quad \gamma_{4,L} = \frac{\alpha}{2} - \beta(\frac{1}{2})^{L+1}
\end{gathered} \tag{29}
$$

We see that even though it is impossible to have equilibrium queue length distributions that are both infinite and uniform, it is possible to have queue length distributions that are uniform in a finite interval, and either zero elsewhere (i.e., a finite queue), or tending to a geometric distribution. The geometric tendency for the tail is evident from the figures as well as from numerical verification. One plausible explanation for this is as follows. In (28), observer that for large L, the quantities $\gamma_{3,L}$ and $\gamma_{4,L}$ are negligible. Thus, for large L, (28) reduces to a recurrence in which p_{n+1} depends only on p_n, p_{n-1} and p_{n-2}. If additionally, the quantity β is very small, the effect of p_{n-2} on p_{n+1} in (28) may be ignored, consequently yielding a second-order recurrence. Under *nice* boundary conditions, this is equivalent to a geometric distribution. However, when β is not negligible, the recurrence still involves p_{n-2}, and the resulting characteristic equation for the recurrence will be a cubic polynomial. This will yield a solution that is a mixture of two exponentials, which shows itself in the tail of the distribution.

In Fig 3a is shown a queue designed so that the first three queue-length probabilities (i.e., p_0, p_1, and p_2) are all equal, but the rest are not. Note that the manner in which the queue was designed ensures that the empty queue also occurs with the same uniform probability. If more queue states are required to have the same uniform probability, this probability must fall, as is shown in Fig 3d for the case of the first four queue-length probabilities.

4. Concluding Remarks

It may seem both remarkable and yet understandable that vastly different equilibrium queue lengths can occur with almost the same probability, for M/GI/1 type queues. This *uniformisation of queue-length* arises, though, only when ρ gets arbitrarily close to 1, or when the process is subject to *heavy traffic*. Some extensions of the above example are of interest, such as when $\{k_j\}$ is made up of a "pieced-together" combination of n geometric distributions instead of just two. If some of the distributions used have a common ratio of 1/2, the resulting equilibrium queue length distribution will have corresponding uniform portions, which alternating with geometric portions, yield interesting *geo-form* distributions.

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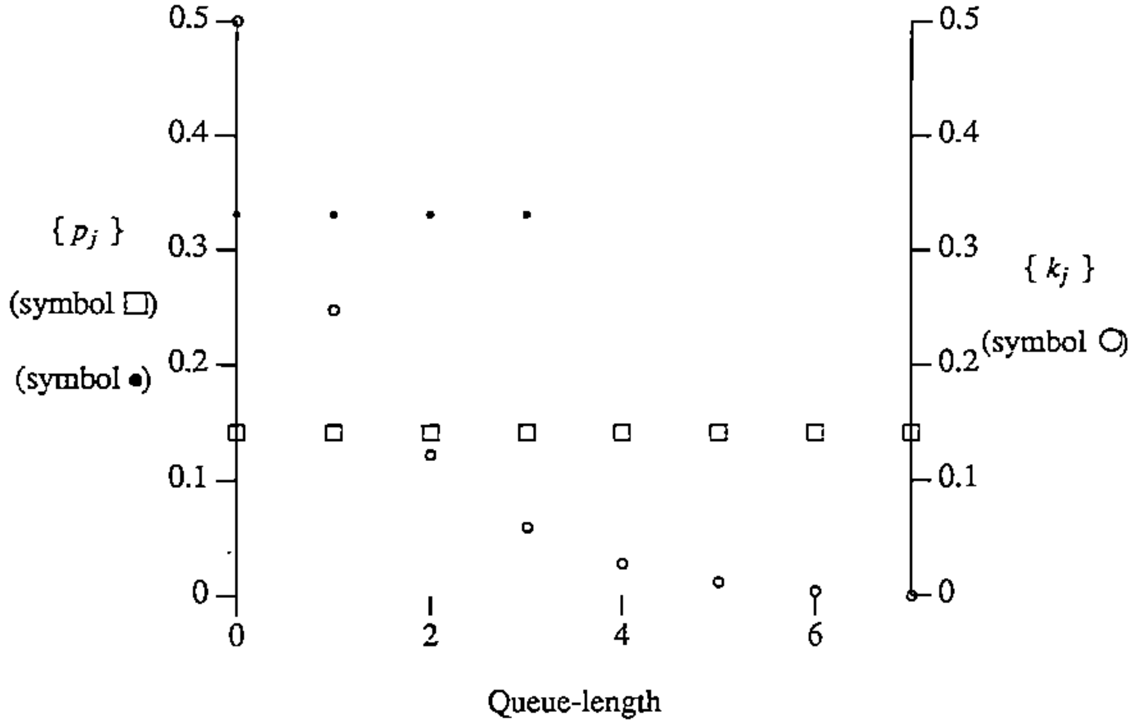

Figure 1 (Uniform distributions)

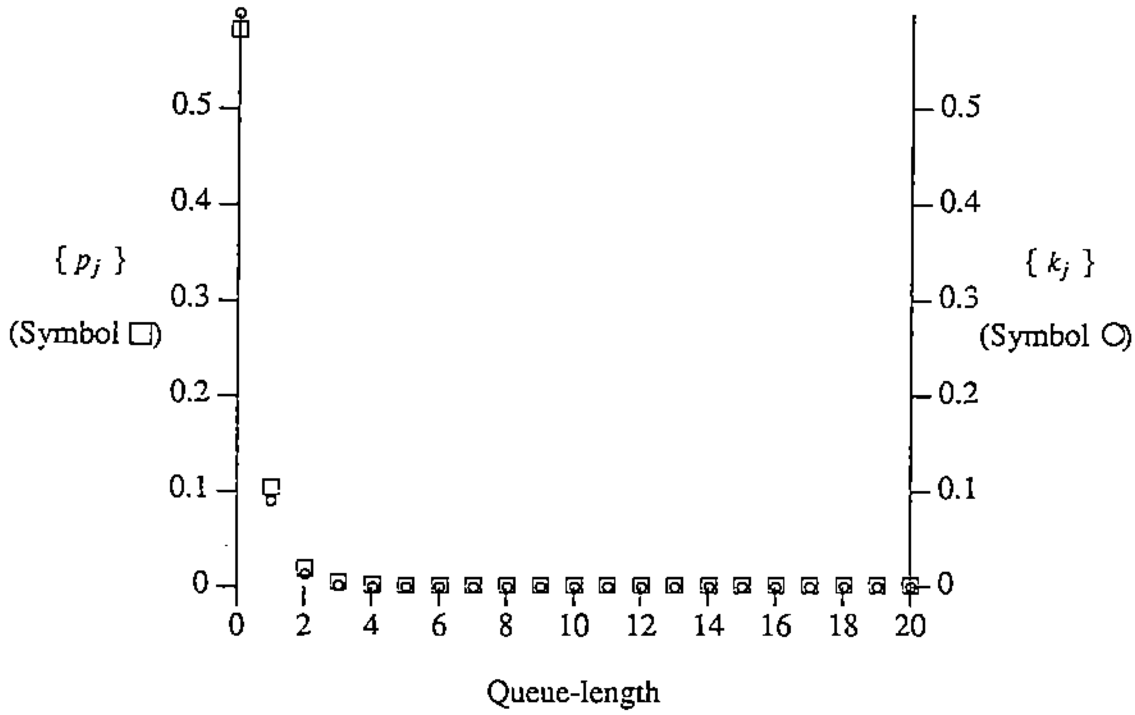

Figure 2a (Geometric distribution with $\alpha=0.85$)

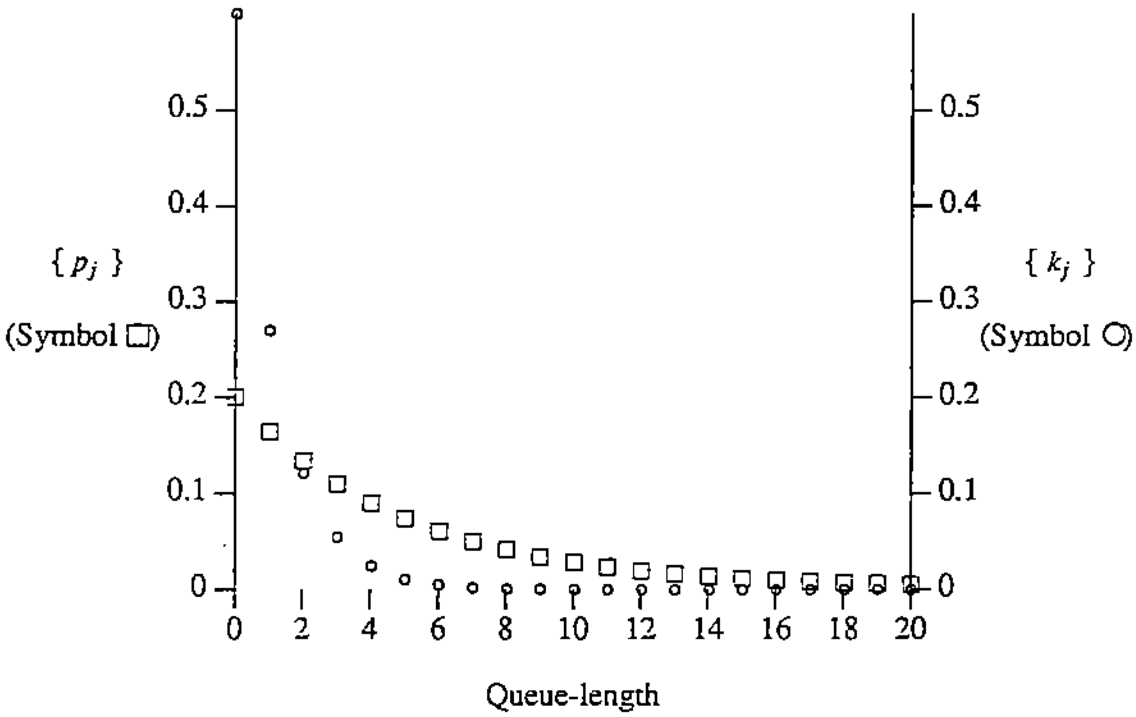

Figure 2b (Geometric distribution with $\alpha=0.55$)

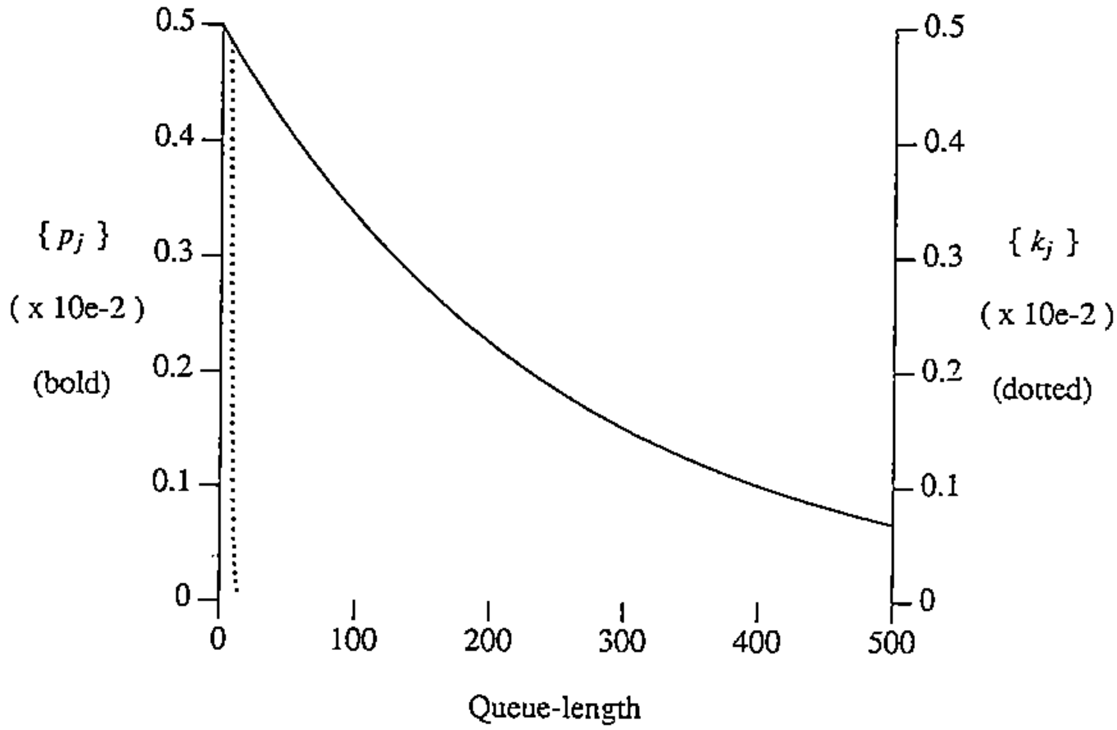

Figure 2c (Geometric/Uniform distribution with α=0.501)

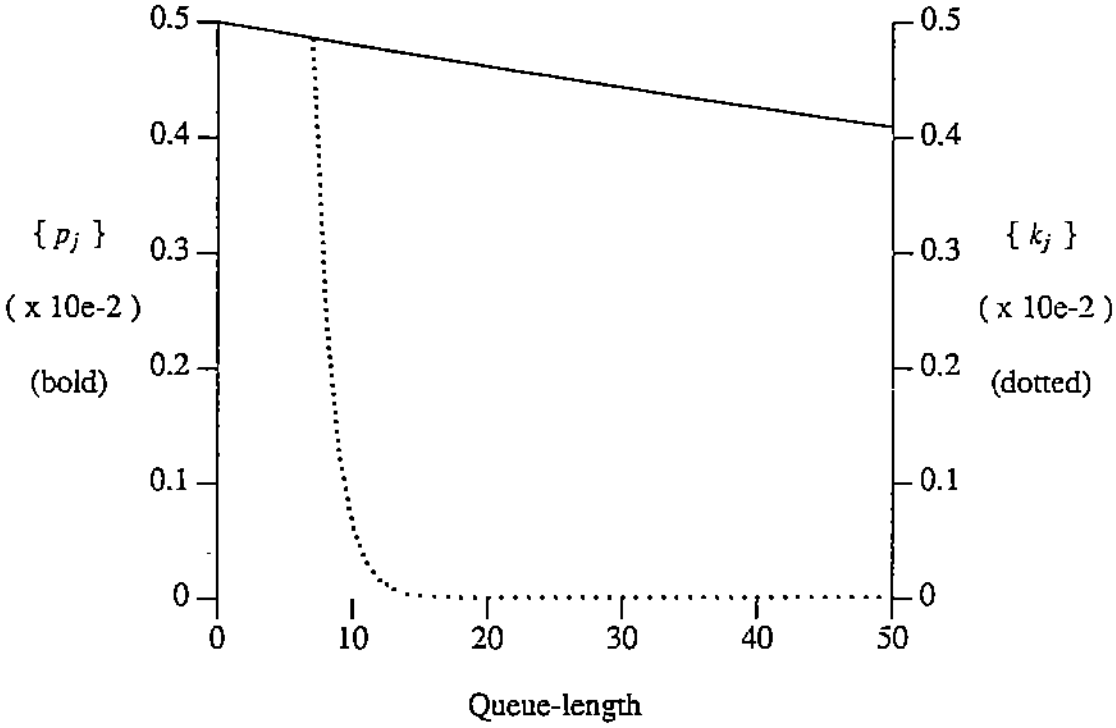

Figure 2d (Geometric/Uniform distribution with α=0.501)

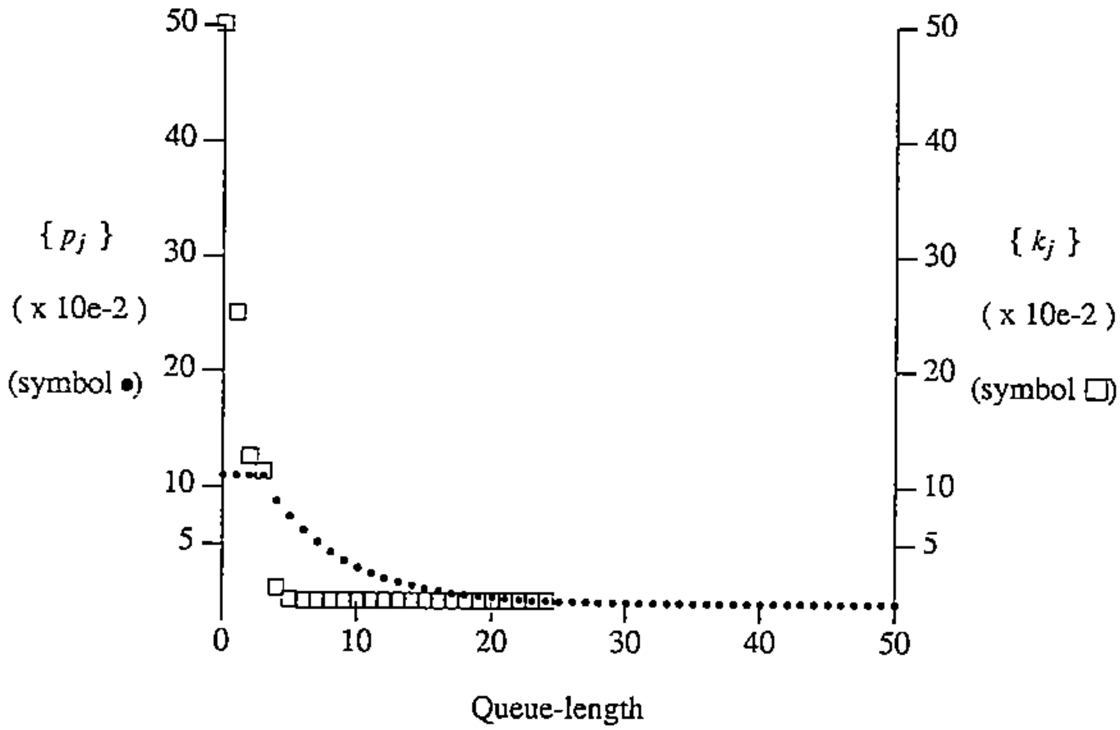

Figure 3a (Geometric-Uniform(3) combination with ρ=0.888)

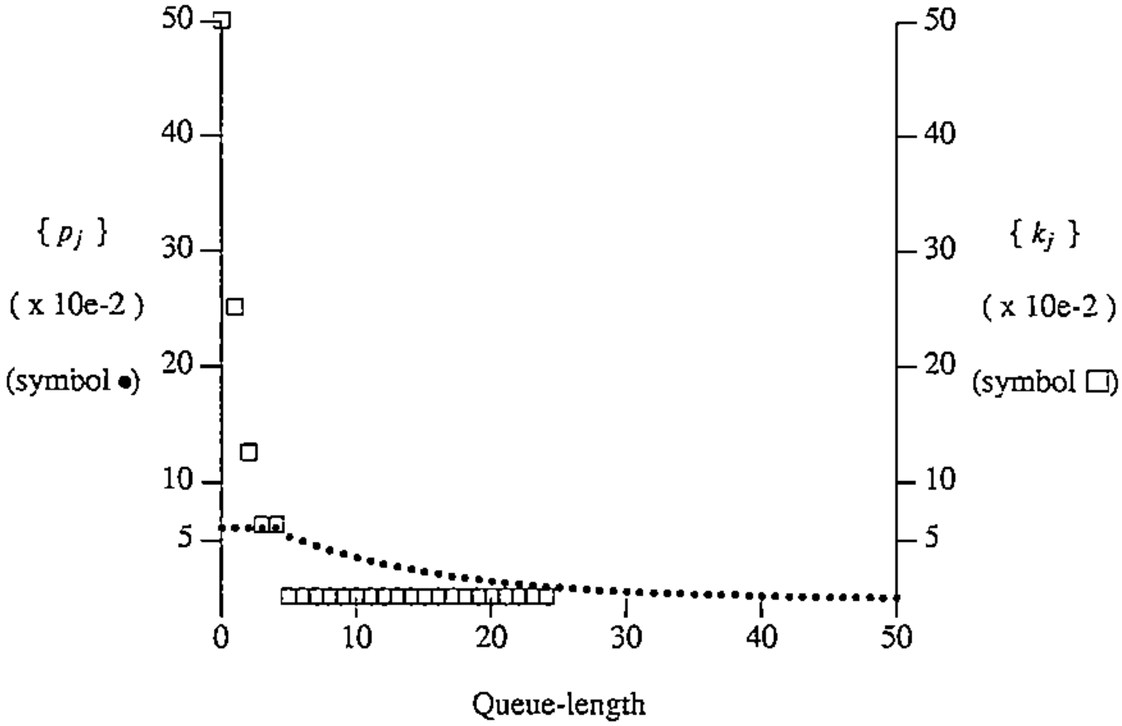

Figure 3b (Geometric-Uniform(4) combination with $\rho = 0.937$)