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ULTIMATE STABILITY CONDITIONS FOR SOME MULTIDIMENSIONAL DISTRIBUTED SYSTEMS

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Abstract

Multidimensional systems with dependent components are useful representations for processes of interest in the fields of computer science and computer communications. Such systems can function properly if and only if they are stable. Until now, the question of ascertaining the stability of such systems in general has been often open, mainly due to difficulties with dependence and non-Markovian behaviour. In this paper we derive very general stability conditions for a class of distributed systems. These criteria say, as expected, that the average input rate cannot exceed a so called modified service rate, introduced in the paper, in order to assure stability of the system. The main results are applied to systems such as token passing rings, coupled-processor systems, buffered ALOHA systems with slotted and unslotted channels, and buffered multiaccess systems with conflict resolution algorithms.

1. INTRODUCTION

A distributed system can be viewed as a *multidimensional*, not necessarily Markovian stochastic process over a large (typically infinite) state-space. A fundamental issue in the design of a such a system is its *stability*, loosely defined as its ability to possess required properties in the presence of some disturbances. Important examples of such distributed systems are local area networks (eg., Ethernet, FDDI ring, token ring), multiprocessor systems (e.g., concurrent execution of tasks on multiprocessors), distributed computations (cooperative problem solving by sets of distributed processors), etc. More general and thus more important examples are multidimensional queuing systems with applications which include backoff analysis for multiaccess channels (Hastad, Leighton and Rogoff [HLR87]), channels with exponential backoff (Goodman *et al.* [GGM85],[GGM88], Aldous [ALD87]), bin packing problem (Floyd and Karp [FlK86],

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Courcoubetis and Weber [CoW86]), dynamic data structures in computer systems (Kirschenhofer, Prodinger and Szpankowski [KPS89]), stability involving directed acyclic graphs (Tsitsiklis *et al.* [TPH86]), stability in data base systems with concurrent processing (Courcoubetis *et al.* [CRS87]), etc. From these examples it is clear that the stability problem is of considerable importance to the computer science community.

Stability is an intrinsic part of the performance of a system and properly precedes any analysis of a system. Heuristically speaking, a unidimensional system is said to be stable if the rate at which traffic (i.e., customers, messages, items, requests, transactions) enters the system is *slower* than the rate at which this traffic can be handled (i.e., served, transmitted, stored, responded to, processed) by the system (see for example, Floyd and Karp [FlK86]). Such a condition tends to guarantee that the expected delay of an arbitrary traffic item remains bounded. There is no ready generalization of this conditions in the multidimensional case. More generally, our interest is in the existence of asymptotically stationary distributions for performance measures of interest (eg., number of bins, delay of a message, number of nodes in a dynamic tree, size of a database) in the multidimensional case.

A major difficulty that has plagued researchers in stability over the years is that in moving from unidimensional to multidimensional systems, there is no simple way to generalize the unidimensional definition of stability (cf. [FGL77, MaM81, HAJ82, SZP88, RaE89], see also [BOR76, BOR78, CRS87]). In the unidimensional case, the system tends to drift in a clear fashion either upwards or downwards, so that some sense of directionality indicates the stability or instability of the system. In the multidimensional setting, different components of a process move in different directions, some upwards, some downwards, so that a *net* effect that yields a stability condition is not easily to be had. In this paper, we get around this problem by introducing a unified model for distributed systems. The unified model allows us to ascertain stability conditions, determine quantities critical for stability, get a handle on required computations for stability, and verify whether a given distributed system is stable.

Before we present our main contribution to the stability problem, we briefly discuss a history of stability criteria for stochastic models that have been influenced by the rapid growth in the development of distributed systems. We can group relevant papers into three categories: ergodicity conditions for Markov processes (Markovian approach), stability criteria for non-Markovian processes (non-Markovian approach), and stability analyses for some specific stochastic systems, such as token passing rings, ALOHA systems, exponential back-off protocols, data base systems, data structures, etc. In the first category, we restrict our attention to Markov chains and focus on the classification of states in such a process, i.e., ergodicity and nonergodicity problems. The first seminal paper introducing easily verifiable ergodicity conditions for Markov chains with a countable number of states was due to Foster [FOS53] (the so called Lyapunov test function method). Under his influence, in 1969 Pakes derived the so called Pakes' Lemma [PAK69], a result which is probably the most often used in establishing stability for a one-dimensional Markov chain. Later, Tweedie in [TWE76, TWE81, TWE83] (and many other papers of his own or with his collaborators) extended Foster's criteria to uncountable Markov chains. Another line of research is visible in the papers of Malyshev [MAL72], Mensikov [MEN74], and Malyshev and Mensikov [MaM81]. Although they have been able to present, for some particular cases, sufficient and necessary conditions for ergodicity of a multidimensional Markov chain, unfortunately their criteria are very difficult to verify in practice, except for two-dimensional Markov chains. In the latter case, however, we should mention a contribution recently reported by Rozenkrantz [ROS89], and Vaninskii and Lazareva in [VaL88]. These authors relaxed the assumption of bounded jumps required by Malyshev in [MAL72]. On the other hand, Hajek in [HAJ82] studied bounds of exponential type for the first-hitting time and occupation times of a real-valued random sequence. These bounds present a flexible technique for providing stability of processes frequently encountered in the control of queues (e.g., geometric ergodicity for a certain two dimensional Markov chain which arises in the decentralized control of a multiaccess system; see also [MIK89]). In [SZP88] Szpankowski introduced some other criteria for multidimensional Markov chains . Finally, in 1979 Kaplan [KAP79] initiated studies in (computable) criteria for the nonergodicity of Markov chains. This work was extended in the research of Sennott *et al.* [SHT83], Szpankowski [SZP85], and Szpankowski and Rego [SzR88].

Another approach was adopted by Loynes in [LOY62] who derived stability conditions for a non-Markovian stochastic process, arising in the analysis of the G|G|s queue. He proved that the ergodicity condition of a Markovian queue (GI|GI|s) is identical to the stability condition for a non-Markovian queue (i.e., G|G|s where the arrival and service processes may be dependent), and reduces to the intuitively clear condition that the input rate must be smaller than the output rate. His work was extended by Borovkov in [BOR76, BOR78].

The third category of research in stability problems is motivated by the proliferation of distributed systems and distributed computing environments. Authors of papers in this category have studied stability conditions arising in the analysis of particular systems. For example, Kuehn [KUE79] presented stability criteria for a class of token passing systems, however, *without (formal) proof.* Watson observed in [WAT84] that in the performance evaluation of a token passing ring, "it is convenient to derive stability conditions ... (without proof)". Our studies are motivated by this fact and our intention is to fill in this gap. Other stability criteria are met in the analyses of coupled-processor systems [FaI79]. Unfortunately, the analysis of [FaI79] is restricted to the two users case, and based on rather sophisticated tools, namely the Riemann-Hilbert problem approach.

A large class of stability problems arises in the evaluation of multiaccess protocols with buffered or unbuffered (unit-capacity) users. The ergodicity condition for slotted buffered ALOHA systems was initiated by Tsybakov and Mikhailov [TsM79]. This research was continued by Saadawi and Ephremides, and Rao and Ephremides [SaE81, RaE89], Tsybakov [TSY85], Tsybakov and Bakirov [TsB84], Szpankowski [SZP88], Sharma [SHA89] and Falin [FAL88]. Finally, analysis of exponential back-off algorithms gave another impulse to stability problems (see [ALD87, ROS84, KEL85]). The contribution of computer scientists to that problem is well established in two excellent papers by Goodman *et al.* [GGM85, GGM88] and Hastad *et al.* [HLR87] (for other contributions from computer science community see [TPH86, CoW86, CRS87, FIK86, KPS88]).

In this paper, we adopt the approaches taken in the second and the third categories. Our motivation comes from the token passing ring and the ALOHA system. We use Loynes' idea (and its extension proposed by Borovkov [BOR78]) who showed that the natural stability condition for the G|G|1 queue remains true when service times form a dependent stationary process. In the setting of distributed queues with a single server, this leads easily to general stability conditions in terms of "modified service times" for an individual queue (Section 2). These conditions involve a technical stationarity requirement: a set of sufficient conditions, designed to be easily checkable, are developed in Section 2. This set of conditions applies to many practical systems such as token passing rings (Section 3.1), coupled processor systems (Section 3.2), buffered ALOHA systems with slotted and unslotted channels (Section 3.3), and buffered multiaccess systems with conflict resolution algorithms (Section 3.4), previously treated by disparate methods.

2. MAIN RESULTS

In this section we present our main contribution to the stability of *multidimensional* distributed systems. The systems under considerations are often described by a multidimensional process $\mathbf{N}^t = (N_1^t, \ldots, N_M^t)$ where time t is discrete (continuous), and the i-th component N_i^t belongs to a countable state space $\mathcal{C}$. By stability of such process we mean that the distribution of $\mathbf{N}^t$ as $t \to \infty$ *exists* and the distribution is *honest*. In other words, $\mathbf{N}^t$ is stable if for

$\mathbf{x} \in \mathcal{G}^M$, where $\mathcal{G}$ is a set of nonnegative integers, the following holds

$$\lim_{t \to \infty} Pr\{\mathbf{N}^t < \mathbf{x}\} = F(\mathbf{x}) \quad \text{and} \quad \lim_{\mathbf{x} \to \infty} F(\mathbf{x}) = 1 \tag{2.1a}$$

where $F(\mathbf{x})$ is the limiting distribution function, and by $\mathbf{x} \to \infty$ we understand that $x_j \to \infty$ for all $j \in \mathcal{M} = \{1, , \ldots, M\}$. If a weaker condition holds, namely,

$$\lim_{\mathbf{x} \to \infty} \liminf_{t \to \infty} Pr\{\mathbf{N}^t < \mathbf{x}\} = 1 \tag{2.1b}$$

then the process is called *substable* [LOY62]. Otherwise, the system is unstable (for more details see [LOY62, BOR76, BOR78]). The relationship between stability and substability is of course that a stable sequence is necessary substable, and a substable sequence is stable if the distribution function tends to a limit. For example, if $\mathbf{N}^t$ is an aperiodic and irreducible Markov chain, then substability *is equivalent* to stability, since a limiting distribution exists (it may be degenerate) for any such Markov chain.

For simplicity of further considerations we shall concentrate on a generic distributed model which is used throughout this section to describe a large class of computer and communication systems. Let us consider a distributed system with M users that require the use of a single scarce resource. In queueing terminology, we say that customers (messages) from M queues compete for access to a single server. Each queue has an infinite capacity. The arrival process to the j-th queue is Poisson with parameter λ_j, $j \in \mathcal{M}$. Messages arriving to the j-th queue possess independent lengths that form an i.i.d random sequence with distribution function $H_j(\cdot)$. The average message length is denoted by h_j, and the first two moments of the service times are assumed to be finite. The server works in a distributed fashion. While visiting the j-th nonempty queue, the server removes at most one message at time. A server may visit a queue and *not* remove a message from the queue (e.g., the user is "down" or the algorithm does not allow the user to transmit, as is done in the ALOHA system [FGL77, SaE81, SZP86, RaE89]. To avoid confusion, we coin the term *successful visit* if the server visits a queue and either removes a message or the queue is empty.

The details of the mathematical model of the above generic system are as follows. For a queue, say j, let $\tau_{j,n}$ $n = 0, 1, \ldots,$ denote the end of the n-th successful visit of the server. We define the n-th cycle time $C_{j,n}$, as $C_{j,n} = \tau_{j,n+1} - \tau_{j,n}$. In addition, we define a so called *modified service time*. For that purpose, we choose from the sequence $\tau_{j,n}$ of successful visits a subsequence τ_{j,n_k}, $k = 0, 1, \ldots,$ such that at a time τ_{j,n_k} the j-th queue is *nonempty*, i.e., $N_j^t > 0$ for $t = \tau_{j,n_k}$ (the queue is nonempty *after* the service). We further denote this sequence of successful visits to the j-th nonempty queue as $\tau_{j,k}^*$. Then, the modified service time is defined as

$$C_{j,k}^* = \tau_{j,n_k+1} - \tau_{j,n_k} \quad k = 0, 1, \ldots, \tag{2.2a}$$

that is, during the time $C_{j,k}^*$ *exactly* one message is removed from the j-th queue, and hence $C_{j,k}^*$ may be interpreted as a new *modified service time*. Note that at time τ_{j,n_k+1} the queue may or may not be empty . If the queue is empty at a successful visit time $\tau_{j,n}$ (i.e., $n \neq n_k$), then the time elapsed until the next successful visit of the server is called a *vacation time*. More specifically, we define the l-th vacation time $V_{j,l}$, where $l = 0, 1, \cdots$, as

$$V_{j,l} = \tau_{j,n_l+1} - \tau_{j,n_l} \qquad \text{for} \qquad n_l \neq n_k \tag{2.2b}$$

where at time τ_{j,n_l} the queue is empty, i.e., $N_j^t = 0$ for $t = \tau_{j,n_l}$. Naturally, if a customer arrives during a vacation it cannot be served until the end of this vacation.

In order to illustrate the above definitions, we show in Figure 1 a time diagram for one isolated queue of the distributed system. It is not difficult to conclude that this queue behaves as an M|G|1 (*not* M|G|I1 !) queue with vacation [DOS85], and with the modified service times and vacations possible dependent on the input process. Indeed, in a queueing system with vacation it is assumed that a single server of walking type serves, each time it visits a nonempty queue, one customer for a service time S_n, and then takes a rest period R_n. If the queue is empty when the server returns, then the server takes off for a vacation period V_n. Any isolated queue in our generic distributed model works exactly in this manner. For example, the modified service time C_n^* is equal to $S_n + R_n$. Note that in our distributed system the server visits other queues during the rest time R_n or the vacation time V_n.

In summary, the evolution of the j-th queue in our generic model can be described by a stochastic equation

$$N_j^{\tau_{j,n+1}} = [N_j^{\tau_{j,n}} - 1]^+ + X_j(\tau_{j,n+1}, \tau_{j,n}) \quad n = 0, 1, \ldots, \tag{2.3}$$

where $X_j(\tau_{j,n+1}, \tau_{j,n})$ stands for the number of new arrivals to the j-th queue during the cycle time $\mathbf{C}_{j,n} = (\tau_{j,n+1}, \tau_{j,n})$. Note that the length of the cycle $\mathbf{C}_{j,n} = (\tau_{j,n+1}, \tau_{j,n})$ is equal either to the modified service time $\mathbf{C}_{j,k}^*$ (if the queue is nonempty at time $\tau_{j,n}$) or to the vacation time $\mathbf{V}_{j,l}$ (if the queue is empty at time $\tau_{j,n}$). In general the distribution of $X_j(\tau_{j,n+1}, \tau_{j,n})$ depends on whether the interval $(\tau_{j,n+1}, \tau_{j,n})$ is the service time or the vacation time. The next two examples specify possible distributed algorithms for server behavior and illustrates the definition of $\mathbf{C}_{j,n}$, $\mathbf{C}_{j,k}^*$ and $\mathbf{V}_{j,l}$.

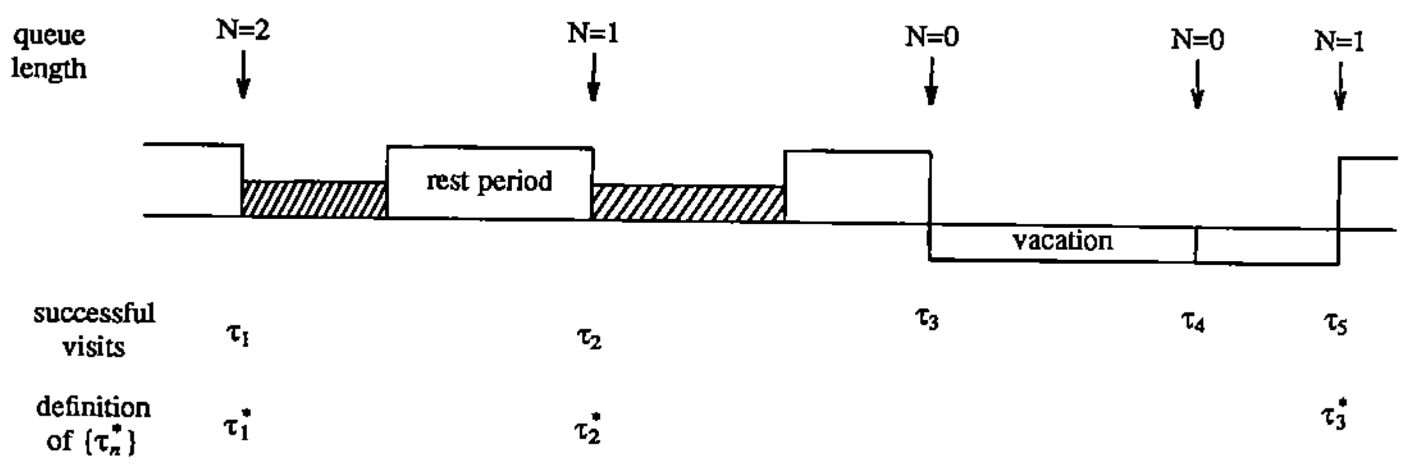

Figure 1. Illustration of $\{\tau_n\}_{n=0}^{\infty}$ and $\{\tau_{n_k}^*\}_{k=0}^{\infty}$ in an M|G|1 queue with vacation.

EXAMPLE 2.1. *Token passing ring* [KUE79, WAT84].

In this system, M queues (users) are handled by a single token (server), which visits the queues in a cyclic order. It is assumed that a *walking time*, W_j, is required to switch from queue j to $(j+1)$ mod M. More specifically, when the server visits the j-th queue, it serves at most one customer, then walks in time W_j to the $(j+1)$-st queue, etc. The sequence $\mathbf{C}_{j,n}$ is defined as the sequence of time intervals which have elapsed between two consecutive visits of the server to the j-th queue. The vacation $\mathbf{V}_{j,l}$ is the time the server is away from the j-th *empty* queue, and the modified service time $\mathbf{C}^*_{j,k}$ represents the period of time the token is away from the j-th *nonempty* queue. Figure 1 shows a typical behavior of such a queue.

EXAMPLE 2.2. *Buffered ALOHA system* [TsM79, SaE81, SZP86, RaE89].

There are M distributed users, each having an infinite buffer for storing fixed-length packets. The packets are transmitted through a broadcast channel. The channel is slotted, and a slot duration is equal to a packet transmission time. Each nonempty user transmits a packet with a probability r_i in a slot, where $i \in \mathcal{M}$. If two or more users transmit simultaneously, then a collision occurs and the packets must be retransmitted in the future. When exactly one packet is transmitted in a slot, then a successful transmission takes place. Referring to our multiqueue model, we say that the server (channel) visits all queues simultaneously at the end of each slot. However, a successful visit occurs if and only if, a successful transmission takes place or the queue is empty (see Figure 2, where empty and dashed boxes represent successful transmission and collisions, respectively). The end of a successful transmission or the end of a slot in which a new customer arrives to an empty queue is denoted by $\tau^*_{j,k}$. Therefore, $\mathbf{C}^*_{j,k}$ is the time between the end of a successful transmission or the end of a slot in which a newly arrived customer found the queue empty, and the end of the next successful transmission. As suggested by Figure 2 the vacation time falls into the idle time, so any queue in this system can be interpreted as a synchronized (slotted) M|G|1 queue *without* vacations, but with *dependent* service times. □

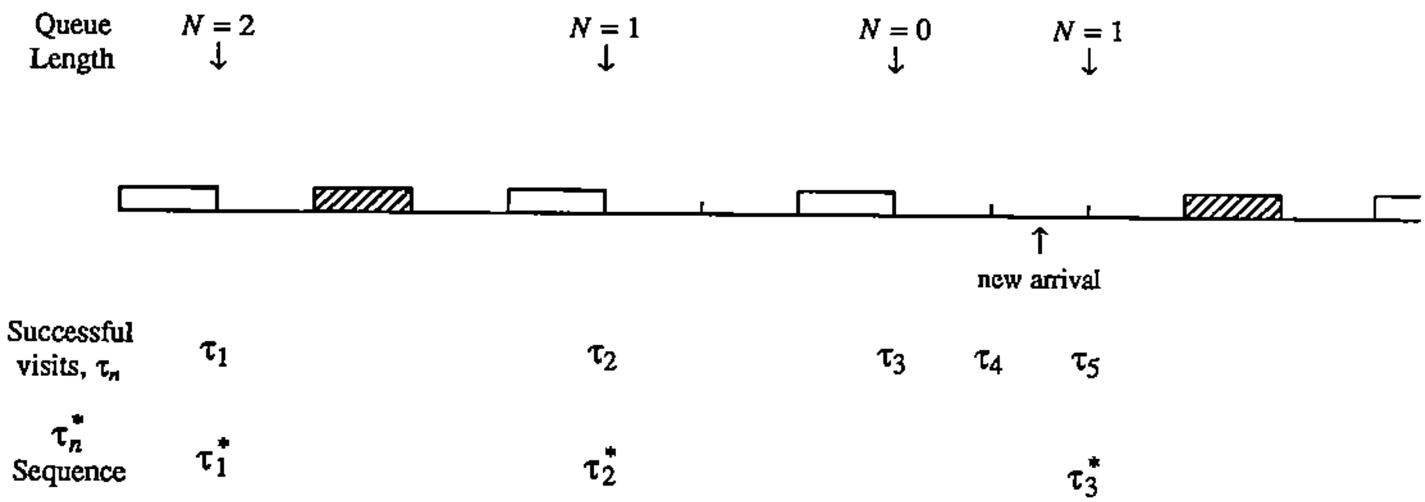

Figure 2. Interpretation of successful visits in slotted ALOHA systems

Now we are ready to present our main results. The basic idea is as follows. At first, we prove that for stability of $\mathbf{N}^t$, it is required that every component N_i^t, $i \in \mathcal{M}$ of $\mathbf{N}^t$ is stable. This *isolation lemma* allows us to consider every non-Markovian queue N_i^t in an isolation. For a single general G|G|1 queue, Loynes [LOY62] and Borovkov [BOR78] proved sufficient and necessary conditions for stability of a single queue (and we shall extend it below to a single queue with vacation), and this together with the isolation lemma is used to derive sufficient and necessary stability conditions for the multidimensional Markov chain $\mathbf{N}^t$.

We start with "isolation" lemmas that allow us to study every (non-Markovian) component N_i^t of $\mathbf{N}^t$ separately.

Lemma 1. If for all $j \in \mathcal{M}$, the one dimensional process N_j^t is stable (substable), then the M-dimensional process $\mathbf{N}^t = (N_1^t, N_2^1, \ldots, N_M^t)$ is substable.

Proof. Since each component of the process $\mathbf{N}^t$ is stable, then by definition (2.1) for all $j \in \mathcal{M}$

$$\lim_{x_j \to \infty} \lim_{t \to \infty} Pr\{N_j^t > x_j\} = 0$$

But

$$1 \geq \lim_{x \to \infty} \lim_{t \to \infty} Pr\{N_j^t \leq x_j \text{ ,for } j = 1, 2, ..., M\} \geq 1 - \sum_{j=1}^{M} \lim_{x_j \to \infty} \lim_{t \to \infty} Pr\{N_j^t > x_j\} = 1$$

Thus

$$\lim_{\mathbf{x} \to \infty} \lim_{t \to \infty} Pr\{\mathbf{N}^t < \mathbf{x}\} = 1$$

and $\mathbf{N}^t$ is substable by (2.1b). If $\mathbf{N}^t$ is a Markov chain, then substability implies stability. ■

Lemma 2. If for some j, say j^*, $N_{j^*}^t$ is unstable, then $\mathbf{N}^t$ is also unstable.

Proof. Since $N_{j^*}^t$ is unstable, hence by (2.1)

$$\lim_{x_{j^*} \to \infty} \lim_{t \to \infty} \inf Pr\{N_{j^*}^t < x_{j^*}\} < 1$$

Then

$$\lim_{x_{j^*} \to \infty} \lim_{t \to \infty} \inf Pr\{\mathbf{N}^t < \mathbf{x}\} \leq \lim_{x_{j^*} \to \infty} \lim_{t \to \infty} \inf Pr\{N_{j^*}^t < x_{j^*}\} < 1$$

which proves Lemma 2. ■

Let us now assume that $\mathbf{N}^t = (N_1^t, N_2^t, \ldots, N_M^t)$ represents queue lengths in our generic queueing model. By the isolation lemmas, we know that $\mathbf{N}^t$ is stable if and only if every queue N_j^t is stable. We note that the process N_j^t describing the queue length in the j-th buffer is *not Markovian*. In particular, the interarrival times $\{A_n\}_{n=0}^{\infty}$ and service times $\{S_n\}_{n=0}^{\infty}$ might not be i.i.d., and in addition $\{S_n\}$ may depend on $\{A_n\}$. What can be said about stability of such a general G|G|1 queue? In 1962 Loynes proved the following result.

Theorem 1a [Loynes 1962]. Let the pair $\{A_n, S_n\}_{n=0}^{\infty}$ be a strictly stationary and ergodic (metrically transitive) process. We denote by EA and ES the average interarrival time and service time. Then the following holds

(i) if $EA < ES$, then the G|G|1 queue is stable in the sense of definition (2.1a),

(ii) if $EA > ES$, then the G|G|1 queue is unstable,

(iii) if $EA = ES$ then the queue may be stable, substable or unstable. If $\{S_n\}$ and $\{A_n\}$ are independent of each other, and one of them is formed of non-constant mutually independent random variables, then the queue is unstable. ■

For some distributed systems an isolated queue may be a single queue with a vacation, as discussed above. Then, we need an extension of Theorem 1a proved below.

Theorem 1b. Consider a GI|G|1 queue with vacation, and let the hypotheses of Theorem 1a hold except that the input process forms a renewal process. In addition, we assume that the vacation sequence $\{V_n\}$ (which may depend on the arrival and service times) can be upper bounded by a strictly stationary process $\{\bar{V}_n\}$ with a finite mean that is independent of the service times $\{S_n\}$ and the interarrival times $\{A_n\}$. Then

(i) $EA < ES$ implies substability of the system

(ii) $EA > ES$ implies instability of the system.

Proof. Doshi in [DOS85] has shown, using sample path arguments, that the waiting time W_k in the G|G|1 queue with vacation and the waiting time w_k in the G|G|1 without vacation are related by the following stochastic formula $W_k = w_k + D_k$ where $D_k = \sum_{i=1}^{\kappa} V_i - \sum_{i=1}^{k} I_i$. In the latter formula, I_i represents the i-th idle time in G|G|1 queue without vacation, and κ is the appropriate index of a busy period that the k-th arrival falls in (for details see [DOS85]). From Theorem 1a we know that w_k converges to a stationary distribution if $EA < ES$. So to prove substability of W_k it suffices to show that D_k is bounded in the probability sense, that is, $\lim_{k\to\infty} \liminf_{x\to\infty} Pr\{D_k < x\} = 1$. But, from the above assumptions $V_k \underset{st}{\le} \bar{V}_k$, hence $D_k \underset{st}{\le} \bar{D}_k$, and $\{\bar{V}_k\}$ (where $\underset{st}{\le}$ means *stochastically smaller* [STO83]) are independent of the arrival and service processes. So, under strict stationarity of $\{\bar{V}_k\}$ by [DOS85, Theorem 1, Hypothesis H2] we show that $D_k \overset{d}{\to} D$ in distribution. Noting that $Pr\{D_k < x\} \ge Pr\{\bar{D}_k < x\}$ and $\lim_{k\to\infty} Pr\{\bar{D}_k < \infty\} = 1$, we prove immediately the substability of W_k for the GI|G|1 queue with vacation, and this proves condition (i). For part (ii) we really do not need any additional assumptions regarding the vacation times. Indeed, if $EA > ES$, then by Theorem 1a w_k is unstable, hence W_k is also unstable. ■

In the view of Theorems 1a and 1b we adopt the following assumptions regarding our generic model.

A1. The sequence $\{\mathbf{C}^*_{j,k}\}$ is a strictly stationary (ergodic) random sequence with average $C^*_j \stackrel{def}{=} E\{\mathbf{C}^*_{j,k}\}^\dagger$

A2. The evolution of the system up to time t is independent of the arrival process in (t, ∞).

A3. The sequence $\{ \mathbf{V}_{j,l} \}$ of vacation times is bounded from the above by a strictly stationary sequence $\{\overline{\mathbf{V}}_{j,l}\}$ independent of the modified service time and with finite mean.

Then our first main result is given below.

PROPOSITION. Under assumptions A1, A2 and A3, the process $\mathbf{N}^t$ satisfying (2.3) is substable if

$$\lambda_j C^*_j < 1 \quad \text{for all} \quad j \in \mathcal{M} \tag{2.4a}$$

and is unstable if

$$\lambda_j C^*_j > 1 \quad \text{for at least one} \quad j \in \mathcal{M} \tag{2.4b}$$

Proof. Let us concentrate on one queue, say $j = 1$. From the description of our model, we know that (2.3) holds for $j = 1$, whence $\mathbf{C}^*_{1,k}$, as defined in (2.2a), can be interpreted as a (modified) service time in an M|G|1 queue with vacation $\mathbf{V}_{1,l}$, as defined in (2.2b). The process represented by (2.3) is *not* Markovian. Assume, however, for a moment that (2.3) represents a Markov chain. Then by Pakes' Lemma [PAK69] and Kaplan's theorem [KAP79] such a queue is stable if and only if $E\{X_1(\tau_{n+1}, \tau^*_n)\} < 1$. But with A2, we have $E\{X_1(\tau_{n+1}, \tau^*_n)\} = \lambda_1 C^*_1$, as required in (2.4). Fortunately, under assumption A1, Theorem 1a

† Note that "boldface" $\mathbf{C}^*_{j,k}$ denotes a random sequence while "romanface" C^*_j denotes *the average* of $\mathbf{C}^*_{j,k}$.

shows that for the M|G|1 queue without vacation, (2.4a) is sufficient for stability, and (2.4b) is sufficient for instability of (2.3) (for $j=1$), even when the service times are dependent, as long as A1 holds. If, in addition, A3 holds, then by Theorem 1b the same conditions (2.4a) and (2.4b) are sufficient for the stability and instability respectively of an M|G|1 queue with vacation. Finally, by the isolation Lemma 1, condition (2.4a) is sufficient for substability of $\mathbf{N}'$, and by Lemma 2 condition (2.4b) is sufficient for instability. ■

In some applications the assumptions A1 and A3 regarding strict stationarity of the modified service times and vacation times are too strong. Therefore, we may apply a result of Borovkov [BOR76] who extended Theorem 1a, proving that strict stationarity of the interarrival times and service times in a single G|G|1 queue can be replaced by asymptotic stationarity [BOR76, p. 12]. As in [BOR76] a sequence $\{S_{n,k}=S_{n+k}\ , k>0\}$ is *asymptotically stationary* if it converges as a process as $n\to\infty$ to a strictly stationary sequence $\{S^k\ ,k>0\}$. Roughly speaking, this means that for all sufficiently large n the joint distribution $Pr\{S_n=m_1\ , S_{k+1}=m_2\ , \ldots,\ S_{n+k}=m_k\}$ exists. Also in other words, the process $\{S_{n,k}=S_{n+k}\}$ defined for $k>-n+1$ describes a process with the event n (e.g., the n-th arrival) chosen as central, converges in distribution to some limit process $\{S^k\ , k>0\}$. We note here that for asymptotic stationarity of a Markov sequence N' it suffices that that one-dimensional limiting distributions exist. With this definition in mind, we can relax our assumptions A1 and A3, and adopt the following two modified postulates

A1′ The sequence of the modified service times $\{\mathbf{C}^*_{j,k}\}$ is *asymptotically stationary.*

A3′ The upper bounded sequence of vacations $\{\overline{\mathbf{V}}_{j,l}\}$ is *asymptotically stationary.*

Then the following corollary to the Proposition can be established

Corollary 1. If A1′ and A3′ replace assumptions A1 and A3 in our Proposition, then the thesis (2.4) of the theorem holds. ■

While stability criteria (2.4) appear to be simple, complications arise when one attempts to compute the average modified service time C_j^* for a particular system since this quantity usually depends on input rates λ_k , $k \neq j$, and the conditional behavior of some subsystems of the system (see next section). Nevertheless, the criteria (2.4) establish the ultimate goal we need to achieve in order to prove stability. If, for some reason, C_j^* is difficult to compute, the Proposition and Corollary 1 can be used to derive sufficient conditions for stability and sufficient conditions for instability. Indeed, let us assume we can bound the average C_j^* by $\underline{C}_j^*$ from below, and by $\overline{C}_j^*$ from above, that is, $\underline{C}_j^* \le C_j^* \le \overline{C}_j^*$. Then $\lambda_j \overline{C}_j^* < 1$ for all $j \in \mathcal{M}$ implies that $\lambda_j C_j^* < 1$, hence stability. On the other hand, if for some j, $\lambda_j \underline{C}_j^* > 1$, then $\lambda_j C_j^* > 1$, and instability follows.

Corollary 2 . Let $\underline{C}_j^* \le C_j^* \le \overline{C}_j^*$. (i) If for all $j \in \mathcal{M}$

$$\lambda_j \overline{C}_j^* < 1 \tag{2.5}$$

then the system is stable.

(ii) If for some $j \in \mathcal{M}$

$$\lambda_j \underline{C}_j^* > 1 \tag{2.6}$$

then the system is unstable. ■

It must be stressed, however, that verifying stationarity assumptions A1 and A3 can lead to major difficulties in assessing stability of some computer communication systems. Therefore, we present below a set of conditions which are sufficient to verify (asymptotic) stationarity of the modified service times $\mathbf{C}_{j,k}^*$ and vacation times $\mathbf{V}_{j,l}$. More precisely, we shall show that replacing assumptions (A1) and (A3) by some other hypotheses, which are easier to verify in practice, leads also to stability condition (2.4). In particular, it turns out that to establish the fact that condition (2.4a) is *necessary* for stability of N^t is a rather easy task, and this can be done under a fairly general hypothesis. Indeed, let us assume the following.

(A′) There exist *continuous* vector-functions $\mathbf{f}(\cdot)$ and $\mathbf{g}(\cdot)$ such that the modified service time and the vacation time can be represented for every n as $\mathbf{C}^*_{j,k+n} = \mathbf{f}(\mathbf{N}^{t+n}, \mathbf{X}^{t+n})$ and $\mathbf{V}_{j,l+n} = \mathbf{g}(\mathbf{N}^{t+n}, \mathbf{X}^{t+n})$ where $\mathbf{X}^t$ is a stationary sequence.

Then, the next theorem provides easily checkable necessary conditions for stability of $\mathbf{N}^t$.

Theorem 2. If $\mathbf{N}^t$ is *stable*, and assumptions A2 and (A′) hold, then $\lambda_j C_j^* \le 1$ for all $j = 1, 2, \ldots, M$.

Proof. Since $\mathbf{N}^t$ is stable, hence $\mathbf{N}^{t+n}$ converges in distribution to a stationary process $\mathbf{N}^t$ as $n \to \infty$. Then, by the continuous mapping theorem (cf. [BIL86]), we obtain $\mathbf{C}^*_{j,k+n} \xrightarrow{d} \mathbf{C}^*_j(k) = f(\mathbf{N}^t, \mathbf{X}^t)$, where $\mathbf{C}^*_j(k)$ is a strictly stationary process. In the same manner we prove asymptotic stationarity for $\overline{\mathbf{V}}_{j,l}$. Then, postulates A1′ and A3′ hold, and the theorem follows form Corollary 1. ■

Unfortunately, the reverse to Theorem 2 is much harder to prove. To see this let us look at one queue, say $j=1$. We now must show that $\lambda_1 \cdot C_1^* < 1$ implies stability of this queue without explicitly referring to strict (asymptotic) stationarity of the modified service time $\mathbf{C}^*_{1,k}$ and vacation time $\overline{\mathbf{V}}_{1,l}$. The difficulties arise because that queue may be stable even if the other queues are unstable. So, $\mathbf{N}^t$ might not possess a stationary distribution, or even the limit of $\mathbf{N}^t$ as t tends to infinity might not exist. To avoid these types of problems we simply strengthen our assumption (A′). First of all, we modify (A′) in such a way that we restrict the possible class of processes $\mathbf{C}^*_{j,k}$ and $\mathbf{V}_{j,l}$. We shall assume that the modified service time and vacation time *do not* depend directly on the actual values of the queue lengths N_i^t, $i = 1, 2, \ldots, M$, $i \ne j$, but they are rather a function of whether a queue is empty or not. More precisely,

(A) Let $\mathbf{Y}^t = (Y_1^t, Y_2^t, \ldots, Y_M^t)$ where $Y_j^t = \chi(N_j^t)$ with $\chi(0) = 0$ and $\chi(x) = 1$ for $x > 0$. Then the modified service time $\mathbf{C}^*_{j,k}$ and the vacation time $\mathbf{V}_{j,l}$ can be represented as $\mathbf{C}^*_{j,k} = f(\mathbf{Y}^t, \mathbf{X}^t, t = \tau_{i,n}, i \in \mathcal{M})$ and $\mathbf{V}_{j,l} = g(\mathbf{Y}^t, \mathbf{X}^t, t = \tau_{i,n}, i \in \mathcal{M})$, where the process

$\mathbf{X}^t$ is a *stationary* sequence, and the functions $f(\cdot)$ and $g(\cdot)$ are continuous mappings. For example, in the token passing ring, one shows that $\mathbf{C}^*_{j,k} = S^k_j + W^k_0 + \sum_{l=1,\, l\neq j}^{M} Y^{t_l}_l \cdot S^k_l$ for some $t_l = \tau_{i,n}$, $i \in \mathcal{M}$, where S^k_j, W^k_0 are *stationary* random sequences representing the service times at the j-th station and the total walking time, respectively. Secondly, to assure that certain limits exist we postulate that $\mathbf{N}^t$ imbedded at the instant of successful visits is a Markov chain. In fact, to simplify notation we assume

(B) $\mathbf{N}^t$ is an aperiodic irreducible Markov chain.

These two assumptions are not yet sufficient to establish stability. As explained, to prove sufficient conditions for stability we must show that the sequence $\mathbf{Y}^t$ converges in distribution to a stationary one (more precisely, joint distributions of $\mathbf{Y}^t$ must converge, but under (A) and (B) one shows that the above suffices). This has to be proved in the the case when some queues are stable while the others are unstable. To formulate it more rigorously, let us partition the set of all queues $\mathcal{M} = \{1, 2, \ldots, M\}$ into two disjoint sets, namely, a set of stable queues $\mathcal{S}$, and a set of unstable queues $\mathcal{U}$, i.e., $\mathcal{M} = \mathcal{S} \cup \mathcal{U}$. In the same manner, we partition the processes $\mathbf{N}^t$ and $\mathbf{Y}^t$, that is, $\mathbf{N}^t = (\mathbf{N}_{\mathcal{U}}{}^t, \mathbf{N}_{\mathcal{S}}{}^t)$ and $\mathbf{Y}^t = (\mathbf{Y}_{\mathcal{U}}{}^t, \mathbf{Y}_{\mathcal{S}}{}^t)$. In addition, to describe all possible states of the process $\mathbf{Y}^t$, we introduce an M dimensional zero-one vector $\mathbf{z} = (z_1, \ldots, z_M)$ such that for every $j \in \mathcal{M}$ the j-th component z_j is either zero or one, i.e., $z_j \in \{0, 1\}$. The set of all zero-one M-tuples is denoted as Θ_M, i.e.,

$$\Theta_M = \{\mathbf{z}\colon \mathbf{z} = (z_1, \ldots, z_M),\ z_j \in \{0, 1\},\quad j \in \mathcal{M}\} \tag{2.7}$$

Moreover, any vector $\mathbf{z}$ can be partitioned as $\mathbf{z} = (\mathbf{z}_{\mathcal{U}}, \mathbf{z}_{\mathcal{S}})$, where $\mathbf{z}_{\mathcal{U}}$ represents states of the process $\mathbf{Y}_{\mathcal{U}}{}^t$ while $\mathbf{z}_{\mathcal{S}}$ describes states for $\mathbf{Y}_{\mathcal{S}}{}^t$. By $\mathbf{0} = (\mathbf{0}_{\mathcal{U}}, \mathbf{0}_{\mathcal{S}})$ and $\mathbf{1} = (\mathbf{1}_{\mathcal{U}}, \mathbf{1}_{\mathcal{S}})$ we mean the all-zeros and all-ones vectors, respectively. To prove a sufficient condition for stability we need to show that $\lim_{t\to\infty} Pr\{\mathbf{Y}^t = \mathbf{z}\}$ exists. Naturally, in the presence of (B) $\lim_{t\to\infty} Pr\{\mathbf{Y}^t = \mathbf{0}\} = 0$, but if any component of $\mathbf{z}$ becomes one, then difficulties arise. Nevertheless, assumptions (A) and (B) imply some interesting properties which are summarized in the next lemma.

Lemma 3. If $\mathcal{U} \neq \varnothing$ ($\varnothing$ represents the empty set), and (B) holds, then

$$\lim_{t\to\infty} Pr\{\mathbf{N}_{\mathcal{U}}{}^t = \mathbf{0}_{\mathcal{U}}\} = 0 \tag{2.8a}$$

and

$$\lim_{t\to\infty} Pr\{\mathbf{Y}^t = (\mathbf{0}_{\mathcal{U}}, \mathbf{z}_{\mathcal{S}})\} = 0 \tag{2.8b}$$

Proof. For the simplicity of notation, we assume that $\mathcal{U} = \{1, 2, \ldots, L\}$ and $1 \le L < M$. Since $\mathbf{N}^t$ is an M-dimensional Markov chain, hence $\lim_{t\to\infty} Pr\{\mathbf{N}^t = \mathbf{k} = (k_1, k_2, \ldots, k_M)\} = 0$ Now, using this fact and the stability of the M-th queue we prove that $\lim_{t\to\infty} Pr\{N_1^t = k_1, \ldots, N_{M-1}^t = k_{M-1}\} = 0$ (note that the $M-1$ dimensional process is *not* Markovian). Indeed, for any positive κ and for $M \in \mathcal{S}$ we obtain

$$\lim_{t\to\infty} Pr\{N_1^t = k_1, \ldots, N_{M-1}^t = k_{M-1}\} = \lim_{t\to\infty} \sum_{j=1}^{\infty} Pr\{N_1^t = k_1, \ldots, N_M^t = j\} =$$

$$\sum_{j=1}^{\kappa} \lim_{t\to\infty} Pr\{N_1^t = k_1, \ldots, N_M^t = j\} + \lim_{t\to\infty} Pr\{N_1^t = k_1, \ldots, N_M^t > \kappa\} \le \lim_{t\to\infty} Pr\{N_M^t > \kappa\}$$

But, $\lim_{\kappa\to\infty} \lim_{t\to\infty} Pr\{N_M^t > \kappa\} = 0$ since the M-th queue is stable (see (2.1a)). This implies $\lim_{t\to\infty} Pr\{N_1^t = k_1, \ldots, N_{M-1}^t = k_{M-1}\} = 0$ as desired. Repeating $(M-L)$ times the same type of "tightness" argument applied to stable queues, we finally prove (2.8a). This immediately implies (2.8) since $Pr\{\mathbf{Y}^t = (\mathbf{0}_{\mathcal{U}}, \mathbf{z}_{\mathcal{S}})\} \le Pr\{\mathbf{Y}_{\mathcal{U}}{}^t = \mathbf{0}_{\mathcal{U}}\} = 0$. ■

In order to establish easily verifiable conditions for stability, using assumptions (A) and (B), we need, however, to prove that $\lim_{t\to\infty} Pr\{\mathbf{Y}^t = \mathbf{z}\}$ exists for *every* vector $\mathbf{z} \in \Theta_M$. One possible solution is to adopt one more assumption, namely

(C) Let $\mathcal{U} \neq \varnothing$ and $\mathcal{U} \neq \mathcal{M}$. Then, for every $k \in \mathcal{U}$ we assume the following

$$\lim_{t\to\infty} Pr\{N_k^t = 0\} = 0 \tag{2.9}$$

Then, one proves

Theorem 3. If (A), (B) and (C) hold, then condition (2.4a) is sufficient for stability of the process $\mathbf{N}^t$.

Proof. We must prove that $\lim_{t\to\infty} Pr\{\mathbf{Y}^t = \mathbf{z}\}$ exists for every $\mathbf{z} \in \Theta_M$. Then our theorem follows from Corollary 1. The value of the probability $Pr\{\mathbf{Y}^t = (\mathbf{z}_{\mathcal{U}}, \mathbf{z}_{\mathcal{S}})\}$ depends on whether $\mathbf{z}_{\mathcal{U}} = \mathbf{1}_{\mathcal{U}}$ or not. If $\mathbf{z}_{\mathcal{U}} \neq \mathbf{1}_{\mathcal{U}}$, then there exists a $k \in \mathcal{U}$ such that $z_k = 0$. Arguments similar to the ones used in the proof of (2.8b) immediately show that $\lim_{t\to\infty} Pr\{\mathbf{Y}^t = \mathbf{z}\} = 0$ in this case. So, now we turn to the case $\mathbf{z}_{\mathcal{U}} = \mathbf{1}_{\mathcal{U}}$. Then,

$$Pr\{\mathbf{Y}_{\mathcal{S}} = \mathbf{z}_{\mathcal{S}}\} - \sum_{k \in \mathcal{U}} Pr\{Y_k^t = 0\} \leq Pr\{\mathbf{Y}^t = (\mathbf{1}_{\mathcal{U}}, \mathbf{z}_{\mathcal{S}})\} \leq Pr\{\mathbf{Y}_{\mathcal{S}}{}^t = \mathbf{z}_{\mathcal{S}}\}$$

But, condition (2.9) from assumption (C) implies that the LHS of the above is equal to the RHS, so $\lim_{t\to\infty} Pr\{\mathbf{Y}^t = (\mathbf{1}_{\mathcal{U}}, \mathbf{z}_{\mathcal{S}})\} = \lim_{t\to\infty} Pr\{\mathbf{Y}_{\mathcal{S}}{}^t = \mathbf{z}_{\mathcal{S}}\}$, and the latter limit exists since $\mathbf{Y}_{\mathcal{S}}{}^t$ is a stable process. ■

There are, however, situations when checking condition (2.9) in assumption (C) is rather troublesome. Therefore, we suggest yet another approach, which ideally applies to the systems we plan to study in this section. We replace assumption (C) by a more restrictive one, namely

(C′) Let for every $k \in \mathcal{U}$ be defined a modified queue length $\overline{N}_k^t$ such that the k-th queue be never empty, that is, $N_k^t \geq 1$ for every $t = 0, 1, \cdots$. Then,

$$\{N_k^t\}_{k \in \mathcal{U}} \leq \{\overline{N}_k^t\}_{k \in \mathcal{U}} \tag{2.10}$$

where $\leq$ means "stochastically smaller" [STO83]. In addition, we assume that the remaining queues form an $|\mathcal{S}|$-dimensional Markov chain denoted by $\overline{\mathbf{N}}_{\mathcal{S}}{}^t = \{\overline{N}_i^t\}_{i \in \mathcal{S}}$ and

$$\{N_i^t\}_{i \in \mathcal{S}} \leq \{\overline{N}_i^t\}_{i \in \mathcal{S}} \tag{2.11}$$

Also, for the process $\overline{\mathbf{N}}^t = (\overline{\mathbf{N}}_{\mathcal{U}}{}^t, \overline{\mathbf{N}}_{\mathcal{S}}{}^t)$ we denote by $\widetilde{C}_j^*$ the average modified service time. We assume

$$C_j^* \le \widetilde{C}_j^* \tag{2.12}$$

Note that (2.11) implies condition (2.9) required for Theorem 3. Since $\mathbf{N}' \le \overline{\mathbf{N}}'$, then stability of $\overline{\mathbf{N}}'$ implies stability of the original process $\mathbf{N}'$ [SZP88]. Together with (2.12) we immediately find the following conclusion.

Corollary 3. Assume that (A), (B) and (C′) hold. If $\lambda_j \widetilde{C}_j^* < 1$ for all $j = 1, 2, \ldots, M$, then the original process $\mathbf{N}'$ is stable. ■

If the process $\overline{\mathbf{N}}'$ bounds the original process $\mathbf{N}'$ very tightly, then we may expect that for some systems $C_j^* = \widetilde{C}_j^*$. In such a case the condition (2.4a) is sufficient for stability of $\mathbf{N}'$.

In conclusion, we briefly summarize our approach for proving stability conditions for a Markovian systems analyzed in the next section. At first, Theorem 2 is applied, that is, it is assumed that the process $\mathbf{N}'$ is stable and, in the presence of the Markovian assumption (B), one can construct a stationary version of the process by selecting an appropriate initial condition. Next, the average modified service time C_j^* for every j is computed, and necessary conditions are established. We note that for M=2 (and for systems analyzed in Section 3 also for M=3) our Lemma 3 implies assumption (C). So by Theorem 2 the necessary conditions are automatically the sufficient ones. For M>3 we apply some other tricks. In particular, we shall try to show that $\widetilde{C}_j^*$ introduced in assumption (C′) is equal to the average value of the original modified service time C_j^*.

Remarks 1

(i) Our Proposition does not solve the stability problem in the case where $\lambda_j C_j^* = 1$ for some j. Loynes showed that in such a case a queue may be stable or unstable. However, if the interarrival times and modified service times are independent of each other, Corollary 1 in [LOY62, p. 508] proves that $\lambda_j\ C_j^* = 1$ leads to "pure" instability (see our Theorem 1a (iii)).

(ii) Several generalizations of the main Proposition are possible as a consequence of Loynes' [LOY62] and Borovkov's [BOR78] results. First, however, we note that strictly speaking, assumption A2 is redundant for an application of Loynes' result. Nevertheless, we have adopted it to get a simple expression for the criteria (2.4), and we need it for our applications in Section 3. Regarding extension of the Proposition, we can safely replace the Poisson arrival process with any renewal process which is independent of the service times process. Secondly, we can relax the one packet at a time assumption in our model. Let us postulate that whenever the server visits a nonempty queue, say j, it removes at most s_j messages. Then we can look at the j-th queue as an M|G|s_j queue with an appropriate modification in the definition of $\mathbf{C}^*_{j,k}$. From Corollary 1 to Theorem 8 in [LOY62], it follows that our system is substable if

$$\lambda_j \, C^*_j < s_j \tag{2.13}$$

for all $j \in \mathcal{M}$ and unstable if $\lambda_j C^*_j > s_j$ for at least one j. Thirdly, we can replace a one server model by a *many* server model. Then, the Proposition holds if one assumes that C^*_j stands for the average time between two successful visits of *any* server under the assumption that the j-th queue is nonempty at the beginning of $\mathbf{C}^*_{j,k}$.

(iii) Let us assume that the process $\mathbf{N}^t$ defined at the instant of successful visits, $\mathbf{N}^{\tau_k}$, is an M-dimensional Markov chain. Is it possible to derive stability conditions (2.4) from a standard ergodicity criteria for Markov chains, that is, through the Lyapunov test function approach [FOS53, SZP88, TWE76]? The answer seems to be *no*. Indeed, for such multidimensional Markov chains components of the drift vector $\mathbf{d}(\mathbf{n}) = E\{\mathbf{N}^{t+1} - \mathbf{N}^t \mid \mathbf{N}^t = \mathbf{n}\} = (d_1(\mathbf{n}), \ldots, d_M(\mathbf{n}))$ are equal either to $d_j(\mathbf{n}) = \lambda_j \, C^*_j - 1$ if $n_j > 0$ or to $d_j(\mathbf{n}) = \lambda_j EV_j$ if $n_j = 0$. But, in the latter case $d_j(\mathbf{n}) > 0$ for infinitely many states, namely those in the set $A = \{\mathbf{n} = (n_1, \ldots, n_M)\colon n_j = 0\}$ This causes formidable difficulties in applying the Lyapunov test function approach (cf. [FOS53, MaM81, SZP88, TWE76, TWE81]) to a stability analysis of the multidimensional Markov chain $\mathbf{N}^t$ solved in our Proposition.

3. APPLICATIONS TO SOME DISTRIBUTED SYSTEMS

In this section, we use our Proposition and Corollary 1 to establish stability criteria for such distributed systems as token passing rings (Sec. 3.1), coupled-processor systems (Sec. 3.2), buffered ALOHA systems (Sec. 3.3), and multiaccess systems with conflict resolution algorithms (Sec. 3.4).

3.1 Token passing ring [KUE79, WAT84]

In this section we analyze the token passing ring system described in Example 2.1. Briefly, we recall that the system consists of M users each containing an infinite capacity buffer. A server (token) visits all queues in a cyclic order. The average transmission time (service time) is denoted by h_j, $j \in \mathcal{M}$, and the walking time required to switch from queue j to $j+1$ mod M, is denoted by W_j. It is easy to see that an isolated queue is a queue with vacation. To verify assumption A3 we simply upper bound the vacation time in, say, the j-th queue by a vacation sequence $\{\mathbf{V}_{j,l}\}$ defined in a modified system that force all other queues to be nonempty when the j-th queue goes for a vacation. Naturally, the upper bounded sequence is a renewal process, hence hypotheses of Theorem 1b hold, and one can apply our Proposition. Having this in mind, we first attack necessary conditions for stability by applying Theorem 2. The system is described by an M-dimensional process $\mathbf{N}^t = (N_1^t, \ldots, N_M^t)$ where N_j^t is the queue length at the j-th user at time t. In general, $\mathbf{N}^t$ is *not* a Markov chain, but $\mathbf{N}^t$ becomes a Markov chain if one imbeds the process at the token scan instants of all queues. Naturally, assumption (A') is satisfied, so we can refer to our Theorem 2, and assume that the process $\mathbf{N}^t$ is stationary.

To evaluate C_j^*, we need a little bit of notation. As before, Θ_M is defined as follows

$$\Theta_M = \{\mathbf{z} : \mathbf{z} = (z_1, \ldots, z_M),\ z_j \in \{0, 1\},\ j \in \mathcal{M}\} \tag{3.1}$$

In addition, $\mathbf{z}^{(j)} \in \Theta_{M-1}$, denotes an $(M-1)$-tuple with the j-th coordinate missing, that is,

$$\mathbf{z}^{(j)} = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_M) \in \Theta_{M-1} \tag{3.2}$$

Finally, since only empty and nonempty buffers are important for stability we adopt the following definition

$$P(\mathbf{z}^{(j)}) = Pr\{\, Y_k^t = z_k, \quad k \in \mathcal{M} - \{j\} \mid N_j^{\tau_{j,k}^*} > 0\} \tag{3.3}$$

where $\tau_{j,k}^* < t < \tau_{j,n_k+1}^*$. For example, for $M = 3$ and $\mathbf{z}^{(2)} = (1,0)$ $P(\mathbf{z}^{(2)}) = P_2(1,0) = Pr\{N_1^t \geq 1, N_3^t = 0 \mid N_2^{\tau_{2,k}^*} > 0\}$, and this represents the conditional probability that the first buffer is nonempty, while the third is empty. We emphasize here the fact that $P(\mathbf{z}^{(j)})$ does *not* depend upon the time t since the process $Y_k^t = \chi(N_k^t)$ is stationary by selecting an appropriate initial distribution (such a distribution exists since by our assumption the process is a stable one).

By the above , the average of the modified service time C_j^* for the j-th user is

$$C_j^* = \sum_{\mathbf{z}^{(j)} \in \Theta_{M-1}} P(\mathbf{z}^{(j)}) \sum_{\substack{k=1 \\ k \neq j}}^{M} [\chi(z_k) h_k + w_k] + w_j + h_j \tag{3.4a}$$

and $w_i = EW_i$ with $w_0 = \sum_{i=1}^{M} w_i$. Note also that

$$\sum_{\{z^{(j)}: z_k = 1\}} P(z^{(j)}) = Pr\{N_k^t \geq 1 \mid N_j^{\tau_j^{\cdot}\mu} > 0\} \tag{3.4b}$$

So, after grouping all probabilities with the coefficient h_k, $k \neq j$, one finds

$$C_j^* = w_0 + h_j + \sum_{\substack{k=1 \\ k \neq j}}^{M} h_k \, Pr\{N_k^t \geq 1 \mid N_j^{\tau_j^{\cdot}\mu} > 0\}$$

Now, for a stable (and unstable) system, by flow balance arguments [KUE79, WAT84], the following holds

$$Pr\{N_k^t \geq 1 \mid N_j^t > 0\} = \min\{1, \lambda_k C_j^*\} \tag{3.5}$$

Hence, after some manipulation, one proves

$$C_j^* = \frac{w_0 + h_j}{1 - \rho_0 + \rho_j} \tag{3.6}$$

where $\rho_j = \lambda_j h_j$ and $\rho_0 = \sum_{j=1}^{M} \rho_j$. By Theorem 2, $\lambda_j C_j^* \leq 1$ for all $j \in \mathcal{M}$ is necessary for stability of the system, if one understands C_j^* in the sense of (3.6).

To establish sufficient conditions for stability of the token passing ring we appeal to our Corollary 3. It is an easy task to check that all three required assumptions (A), (B) and (C′) are satisfied. We focus on one queue, say the j-th one, and we allow the other queues to be stable and unstable. The upper bounding process $\overline{N}^t$ is defined in the same manner as in (C′), that is, the unstable queues are assumed to be *never empty*. Then, the remaining queues form a stable Markov chain which is postulated to be stationary by selecting an appropriate initial distribution. In the upper bounding system the average modified service time $\tilde{C}_j^*$ is given by formula (3.4a) with the probability $P(z^{(j)})$ replaced by the probability $\overline{P}(z^{(j)})$ defined in a natural way. Now, summing up the probabilities $\overline{P}(z^{(j)})$ over all $\{z^{(j)}: z_k = 1\}$ we obtain, as in (3.4b), $Pr\{\overline{N}_k^t \geq 1 \mid \overline{N}_j^{\tau_j^{\cdot}\mu} > 0\}$, which also becomes

$$Pr\{\tilde{N}_k^{\prime} \geq 1 \mid \tilde{N}_j^{\tau_j \mu} > 0\} = \min\{1, \lambda_k \tilde{C}_j^*\} \tag{3.5a}$$

as in (3.5) for the original system. Therefore, the modified service time $\tilde{C}_j^*$ in the upper bounded system reduces, after some elementary algebra, to formula (3.6), and hence $\tilde{C}_j^* = C_j^*$ for every partition of the set $\mathcal{M}$ into stable queues $\mathcal{S}$ and unstable queues $\mathcal{U}$. This proves that the following condition

$$\lambda_j < \frac{1-\rho_0+\rho_j}{w_0+h_j} \qquad j \in \mathcal{M} \tag{3.7}$$

is sufficient and necessary for stability of the system (except when there is an equality in (3.7), however, it is reasonable to conjecture that the system is unstable in this case too).

Derivation of (3.7) crucially depends on (3.4b) and (3.5). Indeed, the conditional service time, i.e., $\sum_{k=1,\neq j}^{M} [\chi(z_k)h_k + w_k]$ in (3.4a) is a linear combination of h_k, and after regrouping, the probability associated with h_k reduces to (3.4b), which can be easily computed from (3.5). If the above properties are not satisfied, then a closed-form formula for the modified service time is hard to find. This is illustrated in the next example.

Example 3.1 *Coupled-token in a ring.*

Let us modify the system in such a way that a token when visiting the j-th nonempty station transmits a message with rate $1/h_j$, if the $(j-1)$-st mod M queue was nonempty, and with rate $1/h_j^{\prime}$ if the $(j-1)$-st mod M queue was empty. For simplicity, we assume $M = 3$. Then, an appropriate formula for C_j^*, say $j = 2$, is

$$C_2^* = w_0 + [h_1\, Pr\{N_1 \geq 1, N_3 \geq 1 | N_2 > 0\} + h_1^{\prime}\, Pr\{N_1 \geq 1, N_3 = 0 | N_2 > 0\}]$$

$$+ [h_2\, Pr\{N_1 > 0 | N_2 > 0\} + h_2^{\prime}\, Pr\{N_1 > 0 | N_2 > 0\}] + h_3\, Pr\{N_3 > 0 | N_2 > 0\}$$

The joint distribution $Pr\{N_1 \geq 1, N_3 \geq 1 | N_2 > 0\}$, is not easy to compute. This situation is even better illustrated by the buffered ALOHA system, which is discussed in Section 3.3.

Remarks 2

(i) The stability conditions (3.7) for a token passing ring with all infinite buffers have been *intuitively* derived by Kuehn [KUE79]. As it was pointed out by Watson [WAT84], it is convenient to derive such stability conditions for other modified token passing rings, based on Kuehn's analysis, however, without proof. To the authors' best knowledge, this paper provides the first proof of Kuehn's conditions. Some generalization of these conditions are possible. First, if one assumes that at most s_j messages are removed from the j-th queue, then the modified service time, C_j^*, becomes

$$C_j^* = \frac{w_0 + s_j\, h_j}{1 - \rho_0 + \rho_j} \tag{3.8}$$

and $\lambda_j\, C_j^* < s_j$ for all $j \in \mathcal{M}$ is the stability condition. Furthermore, let us assume that the j-th station is "up" with probability r_j, and "down" with probability $\bar{r}_j = 1 - r_j$. Customers can be served if and only if the station is "up". For simplicity, assume that $s_j = 1$ for $j \in \mathcal{M}$. Then, the modified service time is

$$C_j^* = \frac{w_0 + r_j\, h_j}{1 - \rho_0 + \rho_j} \tag{3.9}$$

where $\rho_j = \lambda_j\, r_j\, h_j$ and $\rho_0 = \sum_{j=1}^{M} \rho_j$, and $\lambda_j\, C_j^* < 1$ for all j is the ultimate stability condition.

(ii) It is important to understand why in the case of the token passing ring, we have been able to compute exact stability conditions, that is, to evaluate C_j^*. Note that knowing the vector $\mathbf{z}^{(j)}$ (i.e., under the condition that $[\chi(N_1^j), \ldots, \chi(N_{j-1}^j), \chi(N_{j+1}^j), \ldots, \chi(N_M^j)] = \mathbf{z}^{(j)}$) the conditional modified service time for the j-th station is a linear function of the average service times of those stations for which the buffer is nonempty. This allows us to group the joint probabilities $P(\mathbf{z}^{(j)})$ such that the coefficient at h_k is a one dimensional probability (3.5), which is easy to evaluate. If the above grouping does not work, then joint distributions appear in the expression for C_j^* and this causes additional difficulties. This is illustrated in Example 3.1.

3.2 Coupled-processors system

In [FaI79] (see also [SZP88]) Fayolle and Iasnogrodski described a coupled-processor system. A queueing model for this consists of two M|M|1 queues with infinite capacities. The service rate of each server is μ_1 and μ_2 respectively, if the queues are nonempty. If the second queue is empty, then the service rate for the first queue is μ_1^*; and reverse, the second queue serves with rate μ_2^* if the first queue is empty. To establish stability condition we apply Theorem 2 and Theorem 3 By Lemma 3, the condition (2.9) of assumption (C) required in Theorem 3 is fulfilled. Indeed, in a two-queue model at least one queue must be stable to assure stability, and this is enough to quote Lemma 3 and show that for the unstable queue

$\lim_{t\to\infty} Pr\{N_j^t = 0\} = 0$. In other words, in a two-queue model the set of unstable queues is of cardinality not larger than one, and therefore Lemma 3 can be used to verify all assumptions required in Theorem 3. In conclusion, we assume throughout this subsection that the process $\mathbf{Y}^t$ defined in assumption (A) is stationary, whence our Proposition can be directly applied. We point out also that M|M1 assumption is not relevant for our analysis, and the analysis below works without significant changes for GI|GI|1 coupled-processors system.

For stability purposes, it is convenient to deal with the modified service rate, i.e., $1/C_j^*$, $j = 1, 2$. For obvious reasons, we have

$$\frac{1}{C_1^*} = \mu_1 \, Pr\{N_2^t > 0 \mid N_1^{\tau_{1,k}^*} > 0\} + \mu_1^* \, Pr\{N_2^t = 0 \mid N_1^{\tau_{1,k}^*} > 0\} \tag{3.10}$$

$$\frac{1}{C_2^*} = \mu_2 \, Pr\{N_1^t > 0 \mid N_2^{\tau_{2,k}^*} > 0\} + \mu_2^* \, Pr\{N_1^t = 0 \mid N_2^{\tau_{2,k}^*} > 0\} \tag{3.11}$$

But, the following holds

$$Pr\{N_2^t = 0 \mid N_1^{\tau_{1,k}^*} > 0\} = \max\{\, 0,\ 1 - \lambda_2/\mu_2 \,\} \tag{3.12a}$$

$$Pr\{N_1^t = 0 \mid N_2^{\tau_{2,k}^*} > 0\} = \max\{\, 0,\ 1 - \lambda_1/\mu_1 \,\} \tag{3.12b}$$

Therefore, from (3.11), (3.12) and the Proposition, the system is stable if and only if the following inequalities

$$\lambda_1 < \mu_1^* + \frac{\lambda_2}{\mu_2}(\mu_1 - \mu_1^*) \tag{3.13a}$$

$$\lambda_2 < \mu_2^* + \frac{\lambda_1}{\mu_1}(\mu_2 - \mu_2^*) \tag{3.13b}$$

are simultaneously satisfied.

Remarks 3

(i) Conditions (3.13) coincide with the stability criteria established in [FaI79]. Note, however, that the authors of [FaI79] used the Riemann-Hilbert problem to obtain (3.13). A generalization to M coupled processors, as described in [SZP88], is possible.

(ii) Both inequalities, (3.13a) and (3.13b) must be satisfied simultaneously for establishing stability regions. Note, however, that during the course of the derivation, we have to concentrate on one queue, say the first one. Then, according to (3.12), two regions must be considered, $\lambda_2 \le \mu_2$ and $\lambda_2 > \mu_2$. In the first region (see (3.13a))

$$\lambda_1 < \mu_1^* + \frac{\lambda_2}{\mu_2} (\mu_1 - \mu_1^*) \tag{3.14a}$$

while in the second region (i.e., $\lambda_2 > \mu_2$)

$$\lambda_1 < \mu_1 \tag{3.14b}$$

Nevertheless, the union of the regions (3.14a) and (3.14b) is contained in the intersection of the regions from (3.13a) and (3.13b). If we reverse the queues and concentrate on the second queue, we obtain two inequalities similar to (3.14), that is, one as (3.13b) and the second $\lambda_2 < \mu_2$. The intersection of these regions and the one established in (3.14), coincides with (3.13).

3.3 Buffered ALOHA system [TsM79, SaE81, SZP86, RaE89]

The buffered ALOHA system was described in Example 2.2. It consists of M buffered users. The channel (server) is slotted and the duration of a slot is equal to a fixed-packet length transmission time. At the beginning of a slot, the j-th user with a nonempty buffer transmits with probability r_j, and delays transmission for one slot with probability $\overline{r}_j = 1 - r_j$. If two or more users transmit simultaneously, then a collision occurs (unsuccessful transmission) and the colliding users repeat transmission in the future according to the above described random procedure.

The system is described by an M-dimensional Markov chain $\mathbf{N}^t = (N_1^t, \ldots, N_M^t)$ where N_j^t represents the number of packets in the j-th queue at the *end* of the t-th slot, $t = 1, 2, \ldots$. We first deal with the necessary stability condition, so Theorem 2 is applied. Actually, we assume a stationary version of the stable system, and for stability we compute the average C_j^* of the modified service time, which is the average time between two successful transmissions from the j-th user (see Figure 2, and Example 2.2 for more detailed discussions). In this case, however, it is more convenient to deal with the probability of a successful transmission $P_{succ}^{(j)}$ (in a slot), instead of C_j^*. These two quantities are related by $C_j^* = 1/P_{succ}^{(j)}$. Indeed, let $\mathbf{Z}_t^{(j)}$ be a random variable, which takes on value 1 if the transmission from the j-th user is successful at the t-th slot, otherwise $\mathbf{Z}_t^{(j)} = 0$. Then, it must be true that $1 = \sum_{t=1}^{\mathbf{C}_j^*} \mathbf{Z}_t^{(j)}$. In the presence of assumption A2, by Wald's formula, one immediately proves $C_j^* = E\,\mathbf{C}_j^* = 1/E\,\mathbf{Z}_t^{(j)} = 1/P_{succ}^{(j)}$.

The probability $P_{succ}^{(j)}$ is a conditional probability of a successful transmission from the j-th user under the condition that $N_j^t > 0$. By Theorem 2, stability of the ALOHA system implies that

$$\lambda_j < P_{succ}^{(j)} \qquad \text{for all} \quad j \in \mathcal{M}. \tag{3.15}$$

In order to evaluate $P_{succ}^{(j)}$ we note that it depends only on the probabilities of emptiness of the other buffers, so notation from the previous sections is adopted here. In particular, we define the probability $P(\mathbf{z}^{(j)})$ as

$$P(\mathbf{z}^{(j)}) = Pr\{\ Y_k^t = z_k, \quad k \in \mathcal{M} - \{j\} \mid N_j^t > 0\} \tag{3.16}$$

where, as before, $Y_k^t = \chi(N_k^t)$. Then, one immediately obtains

$$P_{succ}^{(j)} = r_j \sum_{\mathbf{z}^{(j)} \in \Theta_{M-1}} P(\mathbf{z}^{(j)}) \prod_{\substack{k=1 \\ k \neq j}}^{M} (1 - r_k)^{\chi(z_k)} \tag{3.17a}$$

Noting that $\sum_{\mathbf{z}^{(j)} \in \Theta_{M-1}} P(\mathbf{z}^{(j)}) = 1$, (3.17a) can be equivalently expressed as

$$P_{succ}^{(j)} = r_j \left\{ 1 - \sum_{k=1, \neq j}^{M} \sum_{(i_1, \ldots, i_k) \in \mathcal{M} - \{j\}} (-1)^k r_{i_1}, \ldots, r_{i_k} Pr\{N_{i_1}^t \geq 1, \ldots, N_{i_k}^t \geq 1 \mid N_j^t \geq 1\} \right\} \tag{3.17b}$$

which is useful in evaluating some bounds on $P_{succ}^{(j)}$.

As long as a sufficient condition is concerned, we adopt the approach from Corollary 3. To recall, we divide the set of users $\mathcal{M} - \{j\}$ into stable and unstable subsets. For unstable users we assume that they are never empty (for example, by transmitting dummy packets). Then the system of stable queues is a Markov chain [SZP86], and the original process is upper bounded by the modified process $\overline{\mathbf{N}}^t$ [SZP88] as defined in assumption (C′). Let $\overline{P}_{succ}^{(j)}$ be the probability of success in the modified system for any partition of the user set into stable and unstable queues.

This probability is given exactly by the same formula as $P^{(j)}_{succ}$ (see (3.17)) except that the probability $P(\mathbf{z}^{(j)})$ is replaced by $\overline{P}(\mathbf{z}^{(j)})$ for the upper bounding system. In particular, Corollary 3 implies that

$$\lambda_j < \overline{P}^{(j)}_{succ} \qquad \text{for all} \quad j \in \mathcal{M} \tag{3.18}$$

is sufficient for ergodicity of the system. We shall *not* argue here whether $\overline{P}^{(j)}_{succ}$ is equal to $P^{(j)}_{succ}$ or not since none of these probabilities, as we shall see, can be easy computed. We shall use some other arguments to obtain computable stability conditions. We must, however, mention here that (3.15) is sufficient and necessary for $M=2$ and $M=3$ users due to our Theorem 3 and Lemma 3 proved in Section 2. For $M=2$ we have already shown this fact in Section 3.2, while the case $M=3$ will be discussed soon.

Let us first concentrate on an $M=2$ users ALOHA system. We know that (3.15) is sufficient and necessary for stability in this case.. In particular, (3.17a) implies

$$\lambda_1 < P^{(1)}_{succ} = r_1\{P_1(0) + \overline{r}_2\, P_1(1)\} \tag{3.19}$$

where $P_1(0) = 1 - P_1(1) = Pr\{N_2^t = 0 \mid N_1^t > 0\}$. Since the first buffer is nonempty, this probability can be easily computed from statistical equilibrium arguments, that is,

$$P_1(0) = \max\,\{0,\ 1 - \frac{\lambda_2}{r_2(1-r_1)}\ \}$$

Two cases must be considered: (i) $\lambda_2 < r_2\,\overline{r}_1$ and (ii) $\lambda_2 > r_2\,\overline{r}_1$. In the first case, the second queue is stable (precisely: conditionally stable), while in the second case, the second queue is unstable. For $\lambda_2 < r_2\,\overline{r}_1$, (3.19) implies

$$P^{(1)}_{succ} = r_1(1 - \lambda_2/\overline{r}_1) \tag{3.20a}$$

while for $\lambda_2 > r_2\,\overline{r}_1$

$$P^{(1)}_{succ} = r_1\,\overline{r}_2 \tag{3.20b}$$

Reversing the queues, one immediately obtains

$$P^{(2)}_{succ} = r_2(1 - \lambda_1/\overline{r}_2) \text{ for } \lambda_1 < r_1\,\overline{r}_2 \tag{3.21a}$$

$$P_{succ}^{(2)} = r_2 \bar{r}_1 \quad \text{for } \lambda_1 \ge r_1 \bar{r}_2 \tag{3.21b}$$

The intersection of both regions (3.20) and (3.21), can be rewritten as

$$\lambda_1 < r_1(1 - \lambda_2/\bar{r}_1) \tag{3.22a}$$

$$\lambda_2 < r_2(1 - \lambda_1/\bar{r}_2) \tag{3.22b}$$

where both conditions (3.22a) and (3.22b), must be simultaneously satisfied.

Now we consider the case of $M = 3$ users which is by far the more difficult. We focus our attention on the first user. Then (3.15) and (3.17) imply

$$\lambda_1 < P_{succ}^{(1)} = r_1[P_1(0,0) + \bar{r}_2 P_1(1,0) + \bar{r}_3 P_1(0,1) + \bar{r}_1 \bar{r}_2 P_1(1,1)] =$$
$$r_1[1 - r_2 Pr\{N_2 > 0 | N_1 > 0\} - r_3 Pr\{N_3 > 1 | N_1 > 0\} + r_2 r_3 P_1(1,1)\} \tag{3.23}$$

where the notation was explained earlier. We consider three cases (i) both queues, the second and the third, are unstable (i.e., λ_2 and λ_3 are "large"), (ii) either the second or the third queue is stable and the other unstable and (iii) both queues are stable (λ_1 and λ_3 are "small"). The third case is the most difficult to analyze. In the second case we apply Lemma 3 to verify assumption (C) required in Theorem 3. For the first case we need to be careful in applying Theorem 3 since $P(0,1)$ and $P(1,0)$ may not exist in such a (unstable) situation. Nevertheless, we upper bound the system in this case by the one defined in assumption (C') (see also discussion above), and then $\bar{P}_1(0,0) = \bar{P}_1(1,0) = \bar{P}_1(1,1) = 0$, and $\bar{P}_1(1,1) = 1$. Hence by (3.23) and (3.18)

$$\lambda_1 < r_1 \bar{r}_2 \bar{r}_3 \tag{3.24a}$$

is sufficient for ergodicity. We shall soon see that this case is, in fact, entirely covered by the second case, which is discussed next.

In the second case, we can safely apply Theorem 3, so (3.23) is sufficient and necessary for stability of the system. Let us assume that the third queue is unstable and the second queue is stable. Then $P_1(0,0) = P_1(0,1) = 0$ and $P_1(1,1) = 1 - P(1,0) =$

$Pr\{N_3^1 > 0 | N_1^1 > 0\} = \lambda_3 / r_3 \bar{r}_1 \bar{r}_2$. Moreover, (3.23) implies

$$\lambda_1 < r_1 \bar{r}_2 \left[1 - \frac{\lambda_3}{\bar{r}_1 \bar{r}_2} \right] \tag{3.24b}$$

and reversing the condition imposed on the second and third queue, one obtains

$$\lambda_1 < r_1 \bar{r}_3 \left[1 - \frac{\lambda_2}{\bar{r}_1 \bar{r}_2} \right] \tag{3.24c}$$

In the third case, we *must* compute the joint probabilities $P_1(0, 0)$, $P_1(1, 0)$, $P_1(0, 1)$ and $P_1(1, 1)$. Note that these probabilities are estimated under the condition that the first queue is nonempty. There is a relationship between these probabilities, that is, $P_1(1, 0)$, $P_1(0,1)$, and $P_1(1,1)$ can be expressed as a function of $P_1(0,0)$. The latter probability can be, on the other hand, computed as in [NAI85], where Nain solved (exactly) a two-user buffered ALOHA system. Let $F_1(x, y)$ denote the generating function of (N_2^1, N_3^1) under the condition $N_1^1 > 0$. Then, with a minor modification, it is proved in [NAI85] (see also [SZP86]) that

$$\lambda_2 = \bar{r}_1 r_2 \bar{r}_3 [1 - F_1(0, 1)] + \bar{r}_1 r_2 \bar{r}_3 [F_1(1, 0) - F_1(0, 0)] \tag{3.25a}$$

$$\lambda_3 = \bar{r}_1 \bar{r}_2 r_3 [1 - F_1(1, 0)] + \bar{r}_1 r_2 r_3 [F_1(0, 1) - F_1(0, 0)] \tag{3.25b}$$

Noting that $P_1(1, 0) = F_1(1, 0) - F_1(0, 0)$, $P_1(0, 1) = F_1(0, 1) - F_1(0, 0)$, $P_1(1, 1) = 1 - F_1(0, 1) - F_1(1, 0) + F_1(0, 0)$, and taking into account (3.25) and (3.23), we have

$$\lambda_1 < P_{succ}^{(1)} = r_1 \left\{ 1 - \frac{\lambda_2 \bar{r}_2 / \bar{r}_1 + \lambda_3 \bar{r}_3 / \bar{r}_1 + r_2 r_3 [P_1(0, 0) - 1]}{1 - r_2 - r_3} \right\} \tag{3.26}$$

The probability $P_1(0, 0)$ is computed in [NAI85] using the method of the Riemann-Hilbert reduction problem (see (4.10) in [NAI85])), where either $(r_2 + r_3 \neq 1)$

$$P_1(0, 0) = \left(1 - \frac{\lambda_2}{\bar{r}_3 \bar{r}_1} - \frac{\lambda_3}{r_3 \bar{r}_1} \right) \exp \left[\frac{\gamma(1)}{2\pi i} \right] \int_{|t|=1} \frac{\log g(t)}{t[t - \gamma(1)]} \, dt \tag{3.27a}$$

or

$$P_1(0, 0) = \left(1 - \frac{\lambda_2}{r_2 \bar{r}_1} - \frac{\lambda_3}{\bar{r}_2 \bar{r}_1} \right) \exp \left[\frac{\gamma(1)}{2\pi i} \right] \int_{|t|=1} \frac{\log g_1(t)}{t[t - \gamma(1)]} \, dt \tag{3.27b}$$

depending on whether $P_1(0, 0)$ is computed from $F_1(0, y)$ or $F_1(x, 0)$. Naturally, $F_1(x, 0)|_{x=0} = F(0\ y)|_{y=0} = P_1(0, 0)$, however, the first term in (3.27) can be expressed in two different ways as shown in (3.27). The region of validity of (3.27a) and (3.27b) is defined in [NAI85]. In (3.27), $\gamma(x)|_{x=1}$ is the inverse of a conformal mapping of a unit circle onto a curve L_x defined in [NAI85] (see [NAI85], p. 54 and Lemma 4.1). The functions $g(t)$ and $g_1(t)$ are defined in [NAI85], too (see [NAI85], p. 58).

Formula (3.26) is valid only for those λ_2 and λ_3 which assure $P_1(0, 0) > 0$ (see the first term in (3.27)). As in the case of $M = 2$, the stability region for $M = 3$ is determined by the intersection of (3.26a) and

$$\lambda_2 < P_{succ}^{(2)} = r_2 \left\{ 1 - \frac{\lambda_1 \bar{r}_1/\bar{r}_2 + \lambda_3 \bar{r}_3/\bar{r}_2 + r_1 r_3 [P_2(0, 0) - 1]}{1 - r_1 - r_3} \right\} \tag{3.26a}$$

$$\lambda_3 < P_{succ}^{(3)} = r_3 \left\{ 1 - \frac{\lambda_1 \bar{r}_1/\bar{r}_3 + \lambda_2 \bar{r}_2/\bar{r}_3 + r_1 r_2 [P_3(0, 0) - 1]}{1 - r_1 - r_2} \right\} \tag{3.26b}$$

where $P_2(0, 0)$ and $P_3(0, 0)$ have the same pattern as (3.27). Note that, with (3.27a), the following three points belong to the boundary of the stability region : $\omega = (\lambda_1, \lambda_2, \lambda_3) = (r_1\bar{r}_2\bar{r}_3\ ,\ \bar{r}_1 r_2 \bar{r}_3\ ,\ \bar{r}_1\bar{r}_2 r_3)$, $A = (r_1\bar{r}_2\ ,\ \bar{r}_1 r_2\ , 0)$, $B = (r_1, 0, 0)$. With (3.27b), one proves that ω, B and $C = (r_1\bar{r}_3\ , 0, \bar{r}_1 r_3)$ belong to the boundary region, too. Using (3.26b), (3.26c) and an appropriate formula on $P_1(0, 0)$ and $P_3(0, 0)$ we can also show that ω belongs to the boundary region, together with $D = (0, 0, r_3)$, $E(0, r_2\bar{r}_3\ ,\ \bar{r}_2 r_3)$ and $F = (0, r_2\ , 0)$. Figure 3 presents boundary lines of the stability region for $M = 3$ with points ω, A, B, C, D, E and F explicitly shown.

Generalization for $M > 3$ is very intricate, since we need to estimate $P(\mathbf{z}^{(j)})$. In this case, however, some bounds are easy to obtain from (3.17), (3.18) and Corollary 2. Since we deal with $P_{succ}^{(j)}$ instead of C_j^* the Corollary 2 and Corollary 3 read: if $\underline{P}_{succ}^{(j)} \le \tilde{P}_{succ}^{(j)} \le P_{succ}^{(j)} \le \bar{P}_{succ}^{(j)}$, then $\lambda_j < \underline{P}_{succ}^{(j)}$ for $j \in \mathcal{M}$, is sufficient for stability and $\lambda_j \ge \bar{P}_{succ}^{(j)}$ for some j, is sufficient

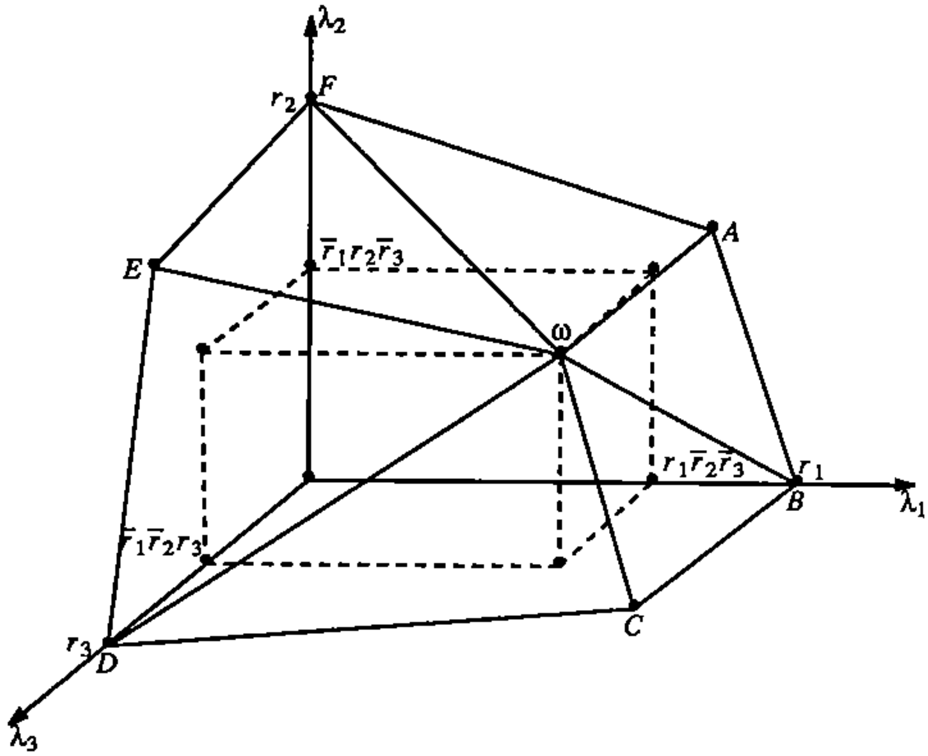

Figure 3. Stability region for $M = 3$ users in slotted ALOHA system.

for instability. In particular, by (3.17a) and $\sum_{z^{(j)} \in \Theta_{M-1}} P(z^{(j)}) = 1$ for all $z^{(j)}$ we have

$$\prod_{\substack{k=1 \\ k \neq j}}^{M} (1 - r_k)^{\chi(z_k)} \geq \prod_{\substack{k=1 \\ k \neq j}}^{M} (1 - r_k) \tag{3.28}$$

one immediately proves that

$$\lambda_j < r_j \prod_{\substack{k=1 \\ k \neq j}}^{M} (1 - r_k) \quad j = 1, \ldots, M \tag{3.29}$$

is sufficient for stability. On the other hand, since $\prod_{\substack{k=1 \\ k \neq j}}^{M} (1 - r_k)^{\chi(z_k)} \leq 1$ we prove that

$$\lambda_j \geq r_j \tag{3.30}$$

for some j is sufficient for instability of the ALOHA system (see also [SZP88]).

To obtain more sophisticated bounds, we need a better estimate of the probability $P(z^{(j)})$. Let us mention here one possibility (for a more sophisticated approach see [RaE89]). We use (3.17b) instead of (3.17a), and let the probability in (3.17b) be denoted as

$$Pr\{N_{i_1}^t \geq 1, \ldots, N_{i_k}^t \geq 1 \mid N_j^t \geq 1\} = P_j(1^{i_1}, \ldots, 1^{i_k}) \tag{3.31a}$$

Let also for $i_l \in \mathcal{M} - \{j\}$

$$P_j(1^{i_l}) = 1 - P_j(0^{i_l}) = Pr\{N_{i_l}^t \geq 1 \mid N_j^t \geq 1\} \tag{3.31b}$$

Note that

$$P_j(1^n) - \sum_{\substack{l=1 \\ l \neq n}}^{k} P_j(0^{i_l}) = 1 - \sum_{l=1}^{k} P_j(0^l) \leq P_j(1^{i_1}, \ldots, 1^{i_k}) \leq P_j(1^{i_l}) \tag{3.32}$$

for some $n, l \in \{1, 2, \ldots, k\}$. The probabilities $P_j(1^{i_l})$ can be estimated using the dominance arguments presented in [SZP88]. For example

$$\frac{\lambda_l}{r_l} \leq P_j(1^l) \leq \frac{\lambda_l}{r_l \prod_{k=1, \neq l}^{M} (1 - r_k)} \tag{3.33}$$

To obtain the LHS of (3.33), it was assumed that all buffers except the j-th are always empty, while in the RHS of (3.33), we postulate that all buffers except the j-th are always nonempty. Using (3.17b), (3.32) and (3.33) upper and lower bounds on $P_{succ}^{(j)}$ can be obtained, whence bounds on the stability region.

Finally, let us apply our Proposition and Corollary 2 to the buffered *unslotted* ALOHA system [SAN80, TsB84, TSY85]. In this case, the channel is not slotted, and a nonempty user can transmit whenever it wishes. There is, however, a restriction. Each nonempty user, say j-th, chooses instants $t_1^{(j)}$, $t_2^{(j)}$ to transmit, where $\tau_k^{(j)} = t_{k+1}^{(j)} - t_k^{(j)}$ is a *stationary renewal* process with the average $E\tau_k^{(j)} = 1/\mu_j$ and distribution function $H_j(t)$. In addition, we assume that $\tau_k^{(j)} \geq T$ where T is the packet transmission time. This condition assures that suicide transmissions do not occur [SAN80, TsB84]. Let C_j^* and $P_{succ}^{(j)}$ denote the average time between two successful transmissions and probability of successful transmission for the j-th user, respectively, assuming the j-th user is nonempty. We proved before that $C_j^* = 1/P_{succ}^{(j)}$, and we focus here only on some bounds of the stability region, that is, we find $\underline{P}_{succ}^{(j)}$ and $\overline{P}_{succ}^{(j)}$ such that $\underline{P}_{succ}^{(j)} \le \tilde{P}_{succ}^{(j)} \le P_{succ}^{(j)} \le \overline{P}_{succ}^{(j)}$. To assure the LHS of the last inequalities (sufficient conditions for stability), we assume that all buffers are never empty. Then, using arguments from [SAN80] and Corollary 2, we immediately prove that

$$\lambda_j < \underline{P}_{succ}^{(j)} = \mu_j \prod_{\substack{k=1 \\ k \ne j}}^{M} \int_T^{\infty} [1 - H_k(x + T)]\, dx \tag{3.34}$$

for $j \in \mathcal{M}$ is sufficient for stability. On the other hand, assuming that all buffers except the j-th are always empty, we estimate $P_{succ}^{(j)} \le \overline{P}_{succ}^{(j)} = \mu_j$, whence

$$\lambda_j \ge \mu_j \tag{3.35}$$

for some j is sufficient for instability of the system.

Remarks 4

(i) The ergodicity analysis of the buffered slotted ALOHA system was initiated by Tsybakov and Mikhailov [TsM79] who obtained our bound (3.29). Their method, however, was essentially different from ours. They use a mixture of dominance and probabilistic arguments. In particular, using Malyshev's condition [MAL72], they established the exact stability condition for $M = 2$ users. Note, however, that Malyshev's criteria apply only to bounded arrival processes, and therefore, it does not work in the case of the Poisson arrival process, i.e., the case we have considered in (3.22). Actually, very recently Rosenkrantz [ROS89], and Vaninskii and Lazareva [VaL88], extended Malyshev's condition to an unbounded input stream, so (3.22) can be eventually obtained form [ROS89, VaL88]. Rao and Ephremides [RaE89] also provided stability conditions for $M = 2$ using dominance arguments, however, a Bernoulli input was assumed. To the best of the authors' knowledge, the stability conditions for $M = 3$ are new.

(ii) Pure dominance arguments have been used in Szpankowski [SZP88] to prove (3.29) and (3.30). In [SZP88] the Lyapunov test function method is used to prove some other bounds on the ergodicity and nonergodicity regions. To obtain more sophisticated bounds, we must evaluate the joint distribution of the empty-nonempty probabilities. Using sophisticated dominance arguments, Rao and Ephremides [RaE89] established such bounds, whence a subset of the ergodicity region (the best bounds up to date for *not* "very asymmetric" ALOHA systems). Finally, Sharma in [SHA89] and Fallin in [FAL88] proved bounds of the form (3.29) for stationary nonindependent input.

(iii) The unslotted buffered ALOHA system, from the ergodicity view point, was analyzed in [TsB84] (see also [TSY85]). Tsybakov and Bakirov proved our result (3.34), but more sophisticated arguments, similar to those in [TsM79], were used.

3.4 A multiaccess buffered system with conflict resolution algorithms [CAP79, MAS81]

In this section we shall analyze *blocked* conflict resolution algorithms (CRA) with a finite number, say M, of buffered users. The previous analyses of CRA have been restricted to an infinite number of unit-capacity users (e.g., see [CAP79, FIM85, MAS81, SZP87]). We focus on the so called blocked stack-type CRA, either modified or non-modified. The description of these algorithms can be found by the reader in [FIM85, MAS81, SZP87]. In short, the channel is slotted and users with nonempty buffers transmit fixed-length packets at the beginning of the slot. If two or more users transmit, then a collision occurs and a divide-and-conquer algorithm is used to resolve it. Let us assume that at the beginning of the t-th slot, an initial collision occurs and $B^t = n$ nonempty users are involved. Then each user from that group flips an unfair coin, and with probability p, it transmits in the next slot, while with probability $1-p$ the user *does not* transmit, and delays its transmission until those users who transmitted in the second slot (the first slot contains the initial collision) resolve the conflict. All users who are not involved in the initial collision are blocked, and can send packets only after the entire collision is resolved. During a *conflict resolution interval* (CRI) only one packet can be transmitted from a nonempty buffer. This is called the non-modified blocked stack algorithm or the non-modified Capetanakis-Tsybakov-Mikhailov algorithm [MAS81]. In the case of modified CRA some collisions are avoided by noting that if after a collision, there is an empty slot, then the next slot must definitely contain a collision. We can "skip " over that step (wasteful slot) by allowing all involved users to flip a coin again (see [FIM85, MAS81]).

To derive a stability condition, let us define $\mathbf{C}_{j,k}^{*}$ as the length of the k-th conflict resolution session (i.e., the number of slots required to solve the initial collision) under the assumption that the j-th user is nonempty. In this section we consider only necessary conditions for stability or equivalently sufficient conditions for instability. We assume that the process is stationary, and then compute the average modified service time as follows

$$C_j^* = E\ \mathbf{C}_{j,k}^* = \sum_{n=0}^{M-1} \pi_n\ E\{\mathbf{C}_j^* \mid B = n+1\} \tag{3.36}$$

where π_n is the steady-state probability that at the beginning of the CRI, there are exactly n nonempty buffers except the j-th one, and $E\{\mathbf{C}_j^* \mid B = n\}$ is the conditional average session length, subject to the initial conflict of multiplicity n. In [CAP79, MAS81, SZP87] it is proved that there exist such constants α_u, α_l, β_u, β_l that for modified and non-modified CRA

$$\alpha_l\ n + \beta_l \le E\ \{\mathbf{C}_j^* \mid B = n\} \le \alpha_u\ n + \beta_u \tag{3.37}$$

for all $n \ge 0$. Then, the above implies

$$\alpha_l + \beta_l + \sum_{n=0}^{M-1} n\pi_n \le E\ \mathbf{C}_j^* \le \alpha_u + \beta_u + \sum_{n=0}^{M-1} n\pi_n \tag{3.38}$$

But, $\sum_{n=0}^{M-1} n\pi_n$ denotes the average number of nonempty buffers at the beginning of a CRI. Hence for the stable system, by work conservative arguments, one easily proves

$$\sum_{n=1}^{M-1} n\pi_n = C_j^* \sum_{\substack{k=1 \\ k\ne j}}^{M} \lambda_k \tag{3.39}$$

Combining (3.38) and (3.39) and referring to Corollary 2, we prove that

$$\lambda_j > \frac{1}{\alpha_l + \beta_l} - \frac{\alpha_l}{\alpha_l + \beta_l} \sum_{\substack{k=1 \\ k\ne j}}^{M} \lambda_k\ , \tag{3.40}$$

is sufficient for instability. To prove a sufficient condition for stability one must consider the RHS of (3.38) and modify equality (3.39) in such a way that only stable queues are involved in the sums in (3.39). The details are left to the reader.

The constant α_u, β_u, α_l and β_l are easy to obtain from the previous analysis of the unbuffered system with conflict resolution algorithms (see [FIM85, MAS81, SZP87]). For example, from [MAS81, SZP87] for $p = 0.5$, it is proved that in the case of non-modified blocked CRA $\alpha_u = 2.8867$, $\beta_u = 1.2336$, $\alpha_l = 2.881$, $\beta_l = -1.8867$.

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