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1985

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Wojciech Szpankowski
Purdue University, spa@cs.purdue.edu

Report Number:
85-550

Szpankowski, Wojciech, "Bounds for Queue Lengths in a Contention Packet Broadcast System" (1985).
Department of Computer Science Technical Reports. Paper 470.
<https://docs.lib.purdue.edu/cstech/470>

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CONTENTION PACKET BROADCAST SYSTEM**

Wojciech Szpankowski

CSD-TR-550
October 1985

BOUNDS FOR QUEUE LENGTHS IN A CONTENTION PACKET BROADCAST SYSTEM

*Wojciech Szpankowski**
Purdue University
Department of Computer Sciences
West Lafayette, INDIANA 47907

Abstract

A finite number of users communicating through a broadcast channel is considered. Each user has a buffer of infinite capacity, and a user randomly accesses the channel (ALOHA-type protocol). Moreover, only one packet per user might be sent in an access time. Both symmetric and asymmetric models are considered, that is, we assume either indistinguishable or distinguishable users. An exact analysis is now not available, therefore, based on some algebraic studies we shall present some lower and some upper bounds for the average queue lengths. These bounds are quite tight for a wide range of input parameters in the symmetric case. In the asymmetric case the bounds are "good" for light and heavy input traffic. In addition, stability conditions for the system will be presented.

1. INTRODUCTION

A queueing system containing some dependent discrete-time queues is analyzed. The system consists of M users transmitting fixed-length packets to each other through a common shared channel. Packets generated at users are buffered until the channel is made available to the users. The capacity of a user's buffer is unlimited. The channel time is divided into slots of the size corresponding to a packet transmission time and transmissions must start at the beginning of a slot. The key problem for such systems is multiple access of the shared channel. Many access protocols have been proposed, but the analysis has been mainly restricted to unbuffered users [12]. We shall assume throughout the paper random access protocol, however, buffered symmetric and asymmetric users are considered.

In recent years the analysis of multiaccess systems with finite number of buffered users has

been addressed in some papers. Saadawi and Ephremides [5] proposed an iterative approximation analyzing so called user and system Markov chains. Sidi and Segall [6], [7] found an explicit expression for the mean delay, but they restricted the analysis either to two identical users [6] or to so-called structured priority multi-access systems. Lately, Takagi and Kleinrock [11] presented the diffusion approximation for the system, while Hofri [1] proposed the exact solution of the system if a reservation protocol and exhaustive service is assumed. On the other hand, Tsybakov and Mikhaikov [13] found a simple upper bound, while Szpankowski [8] found upper and lower bounds for buffered symmetric ALOHA system (see also Takagi [10]). This paper directly extends the results from [8] to buffered asymmetric case. Based on some algebraic considerations, we shall present some lower and some upper bounds for the average queue lengths. The stability (ergodicity) problem will also be covered in this paper.

2. MODEL FORMULATION

In this section we formulate the model of the system in terms of a multiqueue problem. Let us consider M dependent queues competing for access to a server. We assume that time is slotted and a packet must start its transmission at the beginning of a slot, while duration of a slot corresponds to a packet transmission time. The queue lengths of the i -th buffer at the beginning of the k -th slot we denote by N_i^k , $i = 1, 2, \dots, M$, $k = 0, 1, \dots$. Let also X_i^k represent the input traffic to i -th buffer, that is, the number of packets introduced to the i -th buffer during the k -th slot. Therefore, the M -th dimensional random variable (N_1^k, \dots, N_M^k) determines the state process of the system for $k = 0, 1, \dots$. We denote also by Z_i^k , $i = 1, 2, \dots, M$, $k = 0, 1, \dots$, a control variable (access rights) for the i -th queue, which takes the values of zero or one. A random sequence $(Z_1^k, Z_2^k, \dots, Z_M^k)$ controls the access of queues to the server. Then, the queue lengths (N_1^k, \dots, N_M^k) satisfy the following stochastic equations:

$$\begin{aligned}
 N_1^{k+1} &= \{N_1^k - Z_1^k [1 - \sum_{j=2}^M Z_j^k \chi(N_j^k)]^+\}^+ + X_1^k \\
 &\dots\dots\dots \\
 N_m^{k+1} &= \{N_m^k - Z_m^k [1 - \sum_{\substack{j=1 \\ j \neq m}}^M Z_j^k \chi(N_j^k)]^+\}^+ + X_m^k
 \end{aligned} \tag{1}$$

where $a^+ = \max\{0, a\}$, while $\chi(n) = 0$ and $\chi(n) = 1$ otherwise. The above equations describe a multiqueue system with non-exhaustive service, that is, when one packet per user is served during an access time. Various access schemes are modeled by (1) depending on the interpretation of the control variables (Z_1^k, \dots, Z_M^k) . In general, Z_i^k may be a function of the queue lengths (N_1^k, \dots, N_M^k) , the other control variables $Z_j^k, j \neq i$ or it may depend on the arrival process (X_1^k, \dots, X_M^k) .

In this paper, for the simplicity of the analysis, we assume random access scheme. In other words, the control variables satisfy the following conditions:

- (i) for each $1 \leq i \leq M$ the random variables $\{Z_i^k, 0 \leq k < \infty\}$ do not depend on $\{X_i^k, 0 \leq k < \infty\}$ and $\{N_i^k, 0 \leq k < \infty\}$.
- (ii) the $Z_i^k, 1 \leq i \leq M, k = 0, 1, \dots$ are statistically independent and for each $k = 0, 1, \dots$,

$$Pr\{Z_i^k = 1\} = r_i ; \quad Pr\{Z_i^k = 0\} = 1 - r_i = \bar{r}_i \quad (2)$$

- (iii) for each $1 \leq i \leq M$ the random variables $\{X_i^k, 0 \leq k < \infty\}$ are i.i.d. with mean λ_i , variance σ_i^2 and generating function $H_i(z)$.

In a symmetric case all users are indistinguishable, that is, $H_1(z) = \dots = H_M(z) = H(z)$ and $r_1 = r_2 = \dots = r_M = r$.

The interpretation of the probability r_i is obvious. It is the probability of transmitting a packet by the i -th user. Moreover, if $Z_i^k = 1$ and any other user with nonempty buffer sends packet ($Z_j^k = 0$ for $N_j^k > 0$), then successful transmission takes place; otherwise the i -th user is involved in a collision.

The M -dimensional stochastic process $\mathbf{N}^k = (N_1^k, \dots, N_M^k) k = 0, 1, \dots$, under the above assumptions becomes M -dimensional Markov chain. We shall study the generating function of \mathbf{N}^k in a steady-state, i.e.,

$$G(\mathbf{z}) \stackrel{def}{=} G(z_1, \dots, z_M) = \lim_{k \rightarrow \infty} E \left\{ \prod_{j=1}^M z_j^{N_j^k} \right\}$$

where $\mathbf{z} = (z_1, \dots, z_M)$. In the further part of this paper, we shall omit a time-index k if steady-

state is considered.

3. ANALYSIS - ASYMMETRIC CASE

In this section the generating function $G(z)$ will be found as a function of some unknown boundary functions. Then, the average queue lengths will be studied and upper as well as lower bounds will be determined. We shall also establish sufficient conditions for ergodicity of the M -dimensional Markov chain. All derivations are done here for a general asymmetric system, however, in Section 4 a symmetric case will be considered.

3.1 Generating function

Let us start with some notations. We assume that users are numbered from 1 to M and U denotes the set of all users, that is, $U = \{1, 2, \dots, M\}$. Throughout the paper a k -combination without repetition of U is used, therefore, by C_k we define a set of such k -combinations, i.e.,

$$C_k \stackrel{def}{=} \{\{i_1, i_2, \dots, i_k\} : i_j \in U, 1 \leq j \leq k ; i_j \neq i_k \text{ iff } j \neq k\}$$

An element of C_k is denoted by I , i.e., $I = \{i_1, \dots, i_k\} \in C_k$, while complement of I in U is represented by $\{j_1, j_2, \dots, j_{M-k}\} = U - I$, $I \in C_k$. Moreover, if $I = \{i_1, \dots, i_k\} \in C_k$ then $I - \{i_j\} = \{i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k\} \in C_{k-1}$, $1 \leq j \leq k$, and

$$C_k - \{n_1, \dots, n_h\} \stackrel{def}{=} \{\{i_1, \dots, i_k\} \in C_k : i_j \in U - \{n_1, \dots, n_h\}, 1 \leq j \leq k\}$$

Finally, to express the generating function $G(z)$ in a readable form we shall widely use the following notation: let $I = \{i_1, \dots, i_k\} \in C_k$, and $\{j_1, \dots, j_{M-k}\} \in U - I$, then

$$r^I \bar{r}^{U-I} \stackrel{def}{=} r_{i_1} \dots r_{i_k} \bar{r}_{j_1} \dots \bar{r}_{j_{M-k}} \quad (3a)$$

$$G(O^I, z^{U-I}) \stackrel{def}{=} G(z) \mid_{z_i = z_i = \dots = z_k = 0} \quad (3b)$$

$$G(O^I, 1^{U-I}) \stackrel{def}{=} G(z) \mid_{z_i = \dots = z_k = 0 ; z_j = \dots = z_{M-k} = 1} \quad (3c)$$

In particular, if $I = \{i\}$, then instead of (3a) we shall write $r_i \bar{r}^{M-1} = r_i \prod_{\substack{j=1 \\ j \neq i}}^M \bar{r}_j$. The function $G(O^I,$

z^{U-I}) will be called *boundary function*, while $G(O^I, 1^{U-I})$ *boundary value*.

Under the above notation in Appendix A we prove that the generating function $G(z)$ has a form:

$$G(z) = \prod_{j=1}^M H_j(z_j) \frac{L(z)}{M(z)} \quad (4a)$$

where

$$L(z) = \sum_{k=1}^M \sum_{I \in C_k} r^I \bar{r}^{U-I} [G(0^I, z^{U-I}) \sum_{j=1}^k (z_{i_j} - 1) \prod_{\substack{l=1 \\ l \neq i_j}}^M z_l - \chi(k-1) \sum_{j=1}^k G(0^{I-\{i_j\}}, z^{U-I \cup \{i_j\}}) (z_{i_j} - 1) \prod_{\substack{l=1 \\ l \neq i_j}}^M z_l] \quad (4b)$$

$$M(z) = \prod_{j=1}^M z_j - \prod_{j=1}^M H_j(z_j) \left[\prod_{j=1}^M z_j \left(1 - \sum_{k=1}^M r_k \bar{r}^{M-1} \right) + \sum_{k=1}^M r_k \bar{r}^{M-1} \prod_{\substack{l=1 \\ l \neq k}}^M z_l \right] \quad (4c)$$

The boundary functions $G(0^I, z^{U-I})$, $I \in C_k$, $1 \leq k \leq M$ are unknown in (4). We do not determine them (even for $M=2$ the problem is still unsolved), but some properties of the system will be studied based on Eqs.(4).

First of all, we shall determine the generating function of the queue length in the n -th buffer, $n \in U$. Substituting in Eq.(4) $z_k = 1$ for all $k \in U - \{n\}$ one finds

$$G(z_n, 1^{U-\{n\}}) = \bar{G}(z_n) \frac{r_n \sum_{k=0}^{M-1} \sum_{I \in C_k - \{n\}} r^I \bar{r}^{U-I-\{n\}} [G(0^n, 0^I, 1^{U-I-\{n\}}) - \chi(k)G(z_n, 0^I, 1^{U-I-\{n\}})]}{r_n \bar{r}^{M-1} - \lambda_n} \quad (5a)$$

where

$$\bar{G}(z_n) = (r_n \bar{r}^{M-1} - \lambda_n) \frac{(z_n - 1)H_n(z_n)}{z_n - H_n(z_n) [(1 - r_n \bar{r}^{M-1})z_n + r_n \bar{r}^{M-1}]} \quad (5b)$$

The function $\bar{G}(z_n)$ is a generating function itself, ie. $\bar{G}(1) = 1$, and it represents the queue length in the n -th buffer under the condition that all other buffers are never empty [8] (for each $t \geq 0$, $N_i(t) > 0$, $i \in U - \{n\}$). Let us denote

$$a_n \stackrel{def}{=} r_n \bar{r}^{M-1} - \lambda_n \quad n \in U \quad (6)$$

Since $G(1^U) = 1$ and $\bar{G}(1) = 1$, then (5) implies that for all $n \in U$

$$r_n \sum_{k=0}^{M-1} \sum_{I \in C_k - [n]} r^I \bar{r}^{U-I-\{n\}} [G(0^n, 0^I, 1^{U-I-\{n\}}) - \chi(k)G(1^n, 0^I, 1^{U-I-\{n\}})] = a_n \quad (7)$$

Note that in (7) there are $2^M - 1$ unknown boundary values $G(0^I, 1^{M-I})$, $I \in C_k$, $0 \leq k \leq M-1$ and only M equations. Therefore, Eqs.(7) do not determine the boundary values, but they are useful for further considerations .

Finding in (5) the derivative with respect to z_n for $z_n = 1$ one may calculate the average queue length, EN_n , in the n -th buffer. Let us denote

$$G_n(1^n, 0^I, 1^{U-I-\{n\}}) \stackrel{def}{=} \frac{d}{dz_n} G(z_n, 0^I, 1^{U-I-\{n\}})|_{z_n=1}, \quad I \in C_k, \quad 1 \leq k \leq M \quad (8)$$

Then, after some algebra

$$EN_n = E\bar{N}_n - \frac{P_n}{a_n} \quad n \in U \quad (9a)$$

where

$$E\bar{N}_n = \frac{d}{dz_n} \bar{G}(z_n)|_{z_n=1} = \frac{\sigma_n^2 + \lambda_n - \lambda_n^2}{2a_n} = \frac{\Omega_n}{2a_n} \quad (9b)$$

$$P_n = \sum_{k=1}^{M-1} \sum_{I \in C_k - [n]} r_n r^I \bar{r}^{U-I-\{n\}} G_n(1^n, 0^I, 1^{U-I-\{n\}}) \quad (9c)$$

$$\Omega_n \stackrel{def}{=} \sigma_n^2 + \lambda_n - \lambda_n^2 \quad (9d)$$

The average queue length $E\bar{N}_n$ (Eq.(9b)) is the queue length in the n -th buffer under the condition that all other buffers are never empty. In order to find explicit formula on EN_n one must determine the derivatives of boundary functions, $G_n(1^n, 0^I, 1^{U-I-\{n\}})$, $I \in C_k$, $1 \leq k < M-1$. There is a small chance to find them, but some improvement over (9) might be achieved performing some algebra on terms in (9). The general idea is to choose some users from U and determine total average queue lengths in the chosen buffers. Then this average value is compared with the sum of average queue lengths found in (9) to reduce the number of unknown boundary values.

Total queue lengths for two chosen users

Let us choose two users, say n and m , $n, m \in U$. The generating function of $N_{\Sigma}^{n,m} \stackrel{def}{=} N_n + N_m$ is $G(z_n, z_m, 1^{U-\{n,m\}})|_{z_n=z_m=z}$ where $|z| < 1$, that is, in (4a) we substitute

$z_n = z_m = z$, $z_i = 1$ for $i \in U - \{n, m\}$. Finding the derivative with respect to z for $z = 1$ one obtains the following formula for the average queue length of $N_{\Sigma}^{n,m}$:

$$EN_{\Sigma}^{n,m} = EN_{\Sigma}^{n,m} \frac{P_n + P_m}{a_n + a_m} + \frac{R_{n,m}^1 + R_{n,m}^2}{a_n + a_m} \quad (10)$$

where

$$R_{n,m}^1 = \sum_{k=1}^{M-1} \sum_{I \in C_{k-1} - \{n,m\}} r^I \bar{r}_m \bar{r}^{U-I-\{m\}} [G_n(1^n, 0^m, 0^I, 1^{U-I-\{m\}}) - \chi(k-1)G_n(1^n, 1^m, 0^I, 1^{U-I-\{m\}})]$$

$$R_{n,m}^2 = \sum_{k=1}^{M-1} \sum_{I \in C_{k-1} - \{n,m\}} r^I \bar{r}_n \bar{r}^{U-I-\{n\}} [G_m(0^n, 1^m, 0^I, 1^{U-I-\{n\}}) - \chi(k-1)G_m(1^n, 1^m, 0^I, 1^{U-I-\{n\}})]$$

$$EN_{\Sigma}^{n,m} = \frac{\Omega_n + \Omega_m - 2\lambda_n \lambda_m}{2(a_n + a_m)}$$

and P_k , Ω_k , $k \in \{n, m\}$ are defined in (9c) and (9d), respectively. Note now that $EN_{\Sigma}^{n,m} = EN_n + EN_m$, where $EN_{\Sigma}^{n,m}$ is given by (10) while EN_n and EN_m by (9). Comparing both sides of the above equation one finds a formula for a weighted sum of P_n and P_m as a function of known values EN_n , EN_m , $EN_{\Sigma}^{n,m}$ and unknown values $R_{n,m}^1$ and $R_{n,m}^2$. Since P_n and P_m is a function of EN_n and EN_m respectively, we finally obtain a weighted sum for EN_n and EN_m . Let us denote for all $n, m \in U$, $n \neq m$

$$d_n(m) \stackrel{def}{=} a_n \bar{r}_n + a_m \bar{r}_n \quad (11)$$

Then, after some tedious algebra we find,

$$r_m d_n(m) EN_n + r_n d_m(n) EN_m = \frac{\bar{r}_n r_m \Omega_n + r_n \bar{r}_m \Omega_m - 2\lambda_n \lambda_m r_n r_m}{2} - S_{n,m} \quad (12)$$

where

$$S_{n,m} = r_n r_m \sum_{k=1}^{M-2} \sum_{I \in C_k - \{n,m\}} r^I \bar{r}^{U-I-\{n,m\}} [\bar{r}_n G_n(1^n, 1^m, 0^I, 1^{U-I-\{n,m\}}) + \bar{r}_m G_m(1^n, 1^m, 0^I, 1^{U-I-\{n,m\}})]$$

Let us notice that for $M = 2$ we completely eliminate the unknowns involved in $S_{n,m}$, and then

$$r_2 d_1(2) EN_1 + r_1 d_2(1) EN_2 = \frac{\bar{r}_1 r_2 \Omega_1 + r_1 \bar{r}_2 \Omega_2 - 2\lambda_1 \lambda_2 r_1 r_2}{2} \quad (13)$$

Unfortunately, there are still in (13) two unknown values EN_n and EN_m and only one equation, but (13) (as well as (12)) is a significant improvement in comparison with Eq.(9). For symmetric

case, as we shall see, we get in this case an exact formula for EN_n .

Total queue length for all users

The above idea might be used to find a weighted sum for $k \leq M$ chosen users. However, since the computations become more and more complex we restrict our considerations to the case $k = M$, that is, we derive below a weighted sum for average queue lengths of all users.

Let $N_\Sigma = \sum_{i=1}^M N_i$. Then the average value of N_Σ , EN_Σ , is the derivative of $G(z, z, \dots, z)$ at

$z = 1$. After some algebra one shows that

$$EN_\Sigma = EN_\Sigma + R_\Sigma \quad (14)$$

where

$$R_\Sigma = \frac{\sum_{n=1}^M \sum_{k=1}^{M-1} \sum_{l \in C_k - \{n\}} G_n(1^n, 0^l, 1^{U-l-\{n\}}) [k \cdot r^l \bar{r}^{U-l} - \sum_{l \in U-l} r_l r^l \bar{r}^{U-l-\{l\}}]}{\sum_{n=1}^M a_n}$$

$$EN_\Sigma = \frac{\sum_{n=1}^M \Omega_n - 2 \sum_{1 \leq k < l \leq M} \lambda_k \lambda_l}{2 \sum_{n=1}^M a_n}$$

As in (10) the derivative of boundary functions, $G_n(1^n, 0^l, 1^{U-l-\{n\}})$, are unknown and we do not determine them.

The same procedure as above is needed to determine a weighted sum of the average queue lengths in all buffers. Since $EN_\Sigma = \sum_{i=1}^M EN_i$ then by (9) and (14) we find initially a weighted sum for P_n , $n = 1, 2, \dots, M$ and then noting that P_n is a function of EN_n we derive a formula for a weighted sum for EN_n , $n = 1, 2, \dots, M$. To express it in a compact form let us introduce some more notations. If x_1, x_2, \dots, x_M is a sequence of real values, then we denote

$$\begin{aligned} x_\Pi &= \prod_{i=1}^M x_i, & x_\Pi(n) &= \prod_{i=1}^M x_i \\ x_\Sigma &= \sum_{i=1}^M x_i, & x_\Sigma(n) &= \sum_{\substack{i=1 \\ i \neq n}}^M x_i \end{aligned}$$

Then, we prove that

$$\sum_{n=1}^M [a_n \bar{r}_n + a_{\Sigma}(n) r_n] EN_n / r_n = \sum_{n=1}^M \frac{\bar{r}_n \Omega_n - r_n \lambda_n \sum_{i \in U - \{n\}} \lambda_i}{2r_n} - S_{\Sigma} \quad (15)$$

where

$$S_{\Sigma} = \sum_{n=1}^M \sum_{k=2}^{M-1} \sum_{I \in C_k - [n]} \left[\sum_{j=1}^k G_n(1^n, 0^{I - \{i_j\}}, 1^{U - I - \{n\} \cup \{i_j\}}) - (k-1) G_n(1^n, 0^I, 1^{U - I - \{n\}}) \right]$$

Eq.(15) is synonymous to Eq.(12) and the advantages of these equations will be obvious in the further part of the paper when upper and lower bounds of the average queue lengths will be studied.

3.2 Ergodicity and some other properties

For further investigations we must derive some inequalities. Noting that for

$$I = \{i_1, \dots, i_k\} \in C_k$$

$$G(0^I, 1^{U-I}) = Pr \{N_{i_1} = 0, \dots, N_{i_k} = 0\} \leq \sum_{l=0}^{\infty} \{N_{i_1} = 0, \dots, N_{i_{j-1}} = 0, N_{i_j} = l, N_{i_{j+1}} = 0, \dots, N_{i_k} = 0\} = G(0^{I - \{i_j\}}, 1^{U - I \cup \{i_j\}})$$

we prove that for any $I = \{i_1, \dots, i_k\} \in C_k$, $1 \leq k \leq M$ and $1 \leq j \leq k$

$$0 < G(0^I, 1^{U-I}) < G(0^{I - \{i_j\}}, 1^{U - I \cup \{i_j\}}) \leq 1 \quad (16)$$

In particular, using (16) and (7) one shows that for all $n \in U$

$$a_n < r_n \bar{r}^{M-1} G(0^n, 1^{U - \{n\}}). \quad (17)$$

Applying similar considerations to the derivatives of boundary functions we easily obtain

the following inequality: for all $I = \{i_1, \dots, i_k\} \in C_k - [n]$, $n \in U$, $1 \leq k \leq M$ and $1 \leq j \leq k$

$$0 < G_n(1^n, 0^I, 1^{U - I - \{n\}}) < G_n(1^n, 0^{I - \{i_j\}}, 1^{U - I - \{n\} \cup \{i_j\}}) \leq EN_n \quad (18)$$

These inequalities will be useful in establishing bounds for the average queue lengths.

Let us now consider ergodicity (stability) conditions for the Markov chain

$N^k = (N_1^k, \dots, N_M^k)$. We prove that

Theorem 1:

Let $0 < r_n < 1$ for all $n \in U$. Then the Markov chain N^k is ergodic if for all $n, m \in U$ the following hold:

$$\begin{aligned} (a) \quad & d_n(m) > 0 \text{ and } d_m(n) > 0 \text{ if } r_n + r_m \leq 1 \\ (b) \quad & d_n(m) > 0 \text{ or } d_m(n) > 0 \text{ if } r_n + r_m > 1 \end{aligned} \tag{19}$$

where $d_n(m)$ $n, m \in U$ is defined in (11).

Proof: See Appendix B. □

As a simple conclusion of Theorem 1 we have

Corollary 1: The Markov chain N^k is ergodic if

$$a_n > 0 \text{ for all } n \in U \tag{20}$$

Proof: It follows directly from Theorem 1 (see proof in Appendix B). □

The conditions proved in the theorem are only sufficient for the ergodicity but not necessary. For more detailed considerations see [9] where some necessary conditions are established.

In the further part of this paper we shall often refer to the Theorem 1. For our convenience we shall introduce some sets. Let

$$E(n, m) = \{(\lambda_n, \lambda_m) : (19) \text{ is satisfied}\} \tag{21a}$$

$$EI(n, m) = \{(\lambda_n, \lambda_m) : d_n(m) > 0 \text{ AND } d_m(n) > 0\} \tag{21b}$$

$$EU(n, m) = \{(\lambda_n, \lambda_m) : d_n(m) > 0 \text{ OR } d_m(n) > 0\} \tag{21c}$$

$$EI(n, m) \quad \text{for } r_n + r_m \leq 1$$

$$E(n, m) = \tag{21d}$$

$$EU(n, m) \quad \text{for } r_n + r_m > 1$$

$$E = \bigcap_{\substack{n=1 \\ n \neq m}}^M \bigcap_{m=1}^M E(n, m), \quad EI = \bigcap_{\substack{n=1 \\ n \neq m}}^M \bigcap_{m=1}^M EI(n, m) \tag{21e}$$

$$E_a = \{(\lambda_1, \lambda_2, \dots, \lambda_M) : a_n > 0 \text{ for all } n \in U\} \tag{21f}$$

The theorem states that the Markov chain is ergodic if the input rates $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_M) \in \mathbb{E}$, but let us notice that for some $\lambda \notin \mathbb{E}$ the chain may be also ergodic. Moreover, for $M = 2$ conditions (19) are necessary and sufficient for ergodicity.

The next theorem gives sufficient condition for finiteness of the average queue length in a buffer.

Theorem 2: The average queue length in the n -th buffer, EN_n , is finite if $a_n > 0$.

Proof. Let \bar{N}_n^k be a one-dimensional Markov chain representing queue length in the n -th buffer under the condition that all other buffers are never empty, i.e., $N_i^k > 0$ for $k = 0, 1, \dots$, and $i \in U - \{n\}$. Then, it is obvious that N_n^k is stochastically smaller than \bar{N}_n^k , $\{N_n^k, k \geq 0\} \leq_{st} \{\bar{N}_n^k, k \geq 0\}$ [13] [9]. Moreover, $EN_n \leq E\bar{N}_n$ and $E\bar{N}_n$ is given by Eq.(9b). But $E\bar{N}_n$ is finite if $a_n > 0$, and that finishes the proof. □

The condition $a_n > 0$ is not necessary. To show it let us consider a system consisting of two users. When $\lambda_2 \rightarrow 0$, then EN_1 is finite for $\lambda_1 < r_1 - \epsilon$, where ϵ is a small number such that $\epsilon \rightarrow 0$ as $\lambda \rightarrow 0$ (see Fig.B1 in Appendix B).

3.3 Lower and upper bounds

We shall derive in this section some lower and upper bounds based on Eqs.(9)-(11) and inequality (18). The l -th lower and the l -th upper bound will be denoted by $E_l N_n$, $E_l \bar{N}_n$, $n \in U$, respectively.

Let us start with Eq.(9). Because of (18) one finds for all $n \in U$

$$0 < P_n < EN_n \sum_{k=1}^{M-1} \sum_{l \in C_k - \{n\}} r_l^l r_n^l \bar{r}^{U-l\{n\}} = EN_n r_n (1 - \bar{r}^{U-\{n\}}) \quad (22)$$

and the LHS of (22) is used for upper bound while the RHS of (22) for lower bound. Substituting

(22) in (9) one immediately obtains

Corollary 2:

$$(i) \text{ for } \lambda_n < r_n \quad EN_n > \frac{\Omega_n}{2(r_n - \lambda_n)} \stackrel{def}{=} E_1 N_n \quad (23)$$

$$(ii) \text{ for } a_n > 0 \quad EN_n < \frac{\Omega_n}{2a_n} \stackrel{def}{=} E_1 \tilde{N}_n \quad (24)$$

□

Let us now consider Eq.(12) and denote the first term of (12) as

$$\Omega_{nm} \stackrel{def}{=} \frac{\bar{r}_n r_m \Omega_n + r_n \bar{r}_m \Omega_m - 2\lambda_n \lambda_m r_n r_m}{2}$$

Moreover, because of (18) we find the following inequalities for the second term of (12):

$$0 < S_{nm} < r_n r_m (1 - q_{nm})(\bar{r}_n EN_n + \bar{r}_m EN_m) \quad (25a)$$

where q_{nm} is defined as

$$q_{nm} \stackrel{def}{=} \prod_{\substack{j=1 \\ j \neq n, m}}^M (1 - r_j) \quad (25b)$$

To find the next upper and lower bounds we substitute in LHS of (12) EN_m either by $E_1 N_m$ or $E_1 \tilde{N}_m$ and we use (25a). Then we prove

Corollary 3:

(i) For $(\lambda_n, \lambda_m) \in EI(n, m)$ and $n, m \in U$

$$EN_n < \frac{(r_m - \lambda_m)[\bar{r}_n \Omega_n - 2\lambda_n \lambda_m r_n] + r_n \Omega_m [\bar{r}_m (1 - q_{nm}) + \lambda_n]}{2(r_m - \lambda_m)(r_n \bar{r}_n q_{nm} - \bar{r}_n \lambda_n - r_n \lambda_m)} \stackrel{def}{=} E_2 \tilde{N}_n(m) \quad (26)$$

(ii) For $\lambda = (\lambda_1, \dots, \lambda_M) \in EI$

$$EN_n < \min_m E_2 \tilde{N}_n(m) \stackrel{def}{=} E_2 \tilde{N}_n \quad (27)$$

(iii) For $a_m > 0$ and $r_n \bar{r}_n - \bar{r}_n \lambda_n - r_n \lambda_m > 0$ and $r_m \bar{r}_m - r_m \lambda_n - \bar{r}_m \lambda_m > 0$

$$EN_n > \frac{(r_m \bar{r}^{M-1} - \lambda_m)(\bar{r}_n \Omega_n - 2\lambda_n \lambda_m r_n) - r_n \Omega_m [\bar{r}_m(1 - q_{nm}) - \lambda_n]}{2(r_m \bar{r}^{M-1} - \lambda_m)(r_n \bar{r}_n - \bar{r}_n \lambda_n - r_n \lambda_m)} \stackrel{def}{=} E_2 N(n) \quad (28)$$

(iv) For $\lambda = (\lambda_1, \dots, \lambda_M) \in E_a$

$$EN_n > \max\{0, \max_m E_2 N_n(m)\} \stackrel{def}{=} E_2 N_n \quad (29)$$

□

Finally, let us consider Eq.(15) for $\lambda \in E_a$. Then, S_Σ in (15) may be bounded as below

$$0 < S_{nm} = \sum_{n=1}^M \sum_{k=2}^{M-1} \sum_{l \in C_k - \{n\}} \left[\sum_{j=1}^k G_n(1^n, 0^{l-\{ij\}}, 1^{U-l-\{n\} \cup \{ij\}}) - (k-1)G_n(1^n, 0^l, 1^{U-l-\{n\}}) \right] < \sum_{n=1}^M EN_n \bar{r}_n \sum_{k=1, k \neq n}^M r_k (1 - q_{nm}) \quad (30)$$

To derive upper and lower bounds from (15) we must evaluate $M - 1$ unknown values of the LHS of (15), namely: EN_m for $m \in U - \{n\}$. But we have just found that $E_l N_m \leq EN_m \leq E_l \bar{N}_n$, $l = 1, 2$ (Corollary 2 and 3). Using these bounds for EN_m , $m \in U - \{n\}$ in the LHS of (15) and bounding the S_Σ in (15) by (30) we evaluate RHS of (15). Denoting

$$\Omega \stackrel{def}{=} \sum_{n=1}^M \frac{\bar{r}_n \Omega_n - r_n \lambda_n \sum_{i=1, i \neq n}^M \lambda_i}{2r_n}$$

we finally obtain

Corollary 4:

For all $n \in U$ and

(i) For $\lambda \in E$

$$EN_n < \frac{r_n \left\{ \Omega - \sum_{j \in U - \{n\}} [a_j \bar{r}_j + a_\Sigma(j) r_j] E_1 N_j / r_j \right\}}{a_n \bar{r}_n + a_\Sigma(n) r_n} \stackrel{def}{=} E_3 \bar{N}_n \quad (31a)$$

(ii) For $\lambda \in EI$

$$EN_n < \frac{r_n \{ \Omega - \sum_{j \in U - \{n\}} [a_j \bar{r}_j + a_{\Sigma}(j) r_j] E_2 N_j / r_j \}}{a_n \bar{r}_n + a_{\Sigma}(n) r_n} \stackrel{def}{=} E_4 \bar{N}_n \quad (31b)$$

(iii) For $\lambda \in E_a$

$$EN_n > \frac{r_n \{ \Omega - \sum_{k \in U - \{n\}} [a_k \bar{r}_k + a_{\Sigma}(k) r_k + r_k \bar{r}_k \sum_{j \in U - \{k\}} r_j (1 - q_{kj})] E_1 \bar{N}_k / r_k \}}{a_n \bar{r}_n + a_{\Sigma}(n) r_n + r_n \bar{r}_n \sum_{k \in U - \{n\}} r_k (1 - q_{nk})} \stackrel{def}{=} E_3 \bar{N}_n \quad (2a)$$

(iv) For $\lambda \in EI$

$$EN_n > \frac{r_n \{ \Omega - \sum_{k \in U - \{n\}} [a_k \bar{r}_k + a_{\Sigma}(k) r_k + r_k \bar{r}_k \sum_{j \in U - \{k\}} r_j (1 - q_{kj})] E_2 \bar{N}_k / r_k \}}{a_n \bar{r}_n + a_{\Sigma}(n) r_n + r_n \bar{r}_n \sum_{k \in U - \{n\}} r_k (1 - q_{nk})} \stackrel{def}{=} E_4 \bar{N}_n \quad (2b)$$

□

Concluding out, we have established four upper and four lower bounds for the average queue lengths in the asymmetric buffered ALOHA - type system.

4. ANALYSIS - SYMMETRIC CASE

We assume now that users are indistinguishable, that is, $H_1(z) = H_2(z) = \dots = H_M(z) = H(z)$ and $r_1 = r_2 = \dots = r_M = r$. We denote simply by λ and σ^2 the expected input rate and variance of the input process, respectively. Symmetry means also that all queue lengths have the same distributions, what implies that all expected queue lengths are the same and equal to EN . But, what is more important, for all k -combinations $I \in C_k$ the appropriate boundary functions are equal. In particular, for all $I \in C_k - [n]$ and $n \in U$ the derivatives of the boundary functions at $z_n = 1$ are reduced to

$$G_n(1^n, 1^{U-I-\{n\}}, 0^I) = G_1(1, 1^{M-k-1}, 0^k) \quad \text{for all } I \in C_k - [n] \quad (33)$$

where the LHS of (33) denotes the derivative with respect to z_1 at $z_1 = 1$ with k other variables equal to 0 and the remaining $M-k-1$ variables equal to 1. This radically reduces previously obtained formulas. For example EN and EN_{Σ} (Eqs.(10, (14)) become

$$EN = E\bar{N} - \frac{\sum_{k=1}^{M-1} r^{k+1} \bar{r}^{M-k-1} \binom{M-1}{k} G_1(1, 1^{M-k-1}, 0^k)}{r\bar{r}^{M-1} - \lambda} \quad (34a)$$

$$E\bar{N} = \frac{\sigma^2 + \lambda - \lambda^2}{2(r\bar{r}^{M-1} - \lambda)} \quad (34b)$$

and

$$EN_{\Sigma} = E\bar{N}_{\Sigma} + \frac{\sum_{k=1}^{M-1} r^{k+1} \bar{r}^{M-k-1} (M-k) G_1(1, 1^{M-k-1}, 0^k) \left[\binom{M-1}{k-1} - r \binom{M}{k} \right]}{r\bar{r}^{M-1} - \lambda} \quad (35a)$$

$$E\bar{N}_{\Sigma} = \frac{\sigma^2 + \lambda - M\lambda^2}{2(r\bar{r}^{M-1} - \lambda)} \quad (35b)$$

Then analogous to (12) and (15) hold, but instead of using them we derive directly from (34) and (35) an appropriate ‘improvement’. Hence, comparing (34) and (35) and noting that $EN_{\Sigma} = M EN$ we find

$$\sum_{k=1}^{M-1} k r^k \bar{r}^{M-k-1} \binom{M-1}{k} G_1(1, 1^{M-k-1}, 0^k) = \frac{(M-1)\sigma^2 + (M-1)\lambda}{2} \quad (36)$$

The LHS of (36) is ‘almost’ the unknown denominator in (34) except the factor k in (36). Divide now (36) by a real number α , $0 < \alpha \leq M-1$ (we call α a *splitting factor*) and note that $k/\alpha = 1 + (k-\alpha)/\alpha$, that is, we split the LHS sum of (36) into two sums, the first being exactly the denominator of (34). Hence, by (36) and the above after some algebra we obtain our final result

$$EN = \frac{(\sigma^2 + \lambda)[1 - r(M-1)/\alpha] - \lambda^2}{2(r\bar{r}^{M-1} - \lambda)} + \frac{\sum_{k=1}^{M-1} r^{k+1} \bar{r}^{M-k-1} \frac{k-\alpha}{\alpha} \binom{M-1}{k} G_1(1, 1^{M-k-1}, 0^k)}{r\bar{r}^{M-1} - \lambda} \quad (37)$$

where $0 < \alpha \leq M-1$. This equation is analogous to (12) and (15) (asymmetric case). In particular, for $M=2$ either (37) or (13) implies that (see also [7])

$$EN = \frac{\bar{r}(\sigma^2 + \lambda) - \lambda^2}{2(r\bar{r} - \lambda)}, \quad M=2 \quad (38)$$

Moreover, stability conditions are reduced to:

Corollary 5: Symmetric system is ergodic if and only if

$$\lambda < r\bar{r}^{M-1} \quad (39)$$

Proof. Sufficient condition follows from Corollary 1, while necessary condition is more complex to prove. However, using so-called *unbounded random walk approach* we showed in [9] that $\lambda \geq r\bar{r}^{M-1}$ is sufficient condition for nonergodicity of the system (see also [13]).

□

Let us now derive lower and upper bound for the average queue length. Previously obtained formula might be applied here, however, formula (37) gives us a chance to derive more sophisticated bounds. Therefore, let us for simplicity assume that the splitting factor α is an integer in a range $[1, M-1]$. Then, dividing the second term of (37) into two components we find

$$EN = \frac{(\sigma^2 + \lambda)[1 - r(M-1)/\alpha] - \lambda^2}{2(r\bar{r}^{M-1} - \lambda)} + \frac{\sum_{k=1}^{\alpha-1} \frac{k-\alpha}{\alpha} r^{k+1} \bar{r}^{M-k-1} \binom{M-1}{k} G_1(1, 1^{M-k-1}, 0^k)}{r\bar{r}^{M-1} - \lambda} + \frac{\sum_{k=\alpha+1}^{M-1} \frac{k-\alpha}{\alpha} r^{k+1} \bar{r}^{M-k-1} \binom{M-1}{k} G_1(1, 1^{M-k-1}, 0^k)}{r\bar{r}^{M-1} - \lambda} \quad (40)$$

Note now that for fixed α the second term is positive while the third one is negative. Therefore to find lower bound we use for $1 \leq k \leq \alpha-1$ the inequality $G_1(1, 1^{M-k-1}, 0^k) < EN$ while for $k \geq \alpha+1$ we assume that $G_1(1, 1^{M-k-1}, 0^k) > 0$. Then

Corollary 6: For all $\alpha = 1, 2, \dots, M-1$

$$EN \geq \frac{(\sigma^2 + \lambda)[1 - r(M-1)/\alpha] - \lambda^2}{2\left\{ \sum_{k=0}^{\alpha-1} \binom{M-1}{k} \frac{\alpha-k}{k} r^{k+1} \bar{r}^{M-k-1} - \lambda \right\}} \stackrel{def}{=} E_S^{\alpha} N \quad (41)$$

where subscript S in RHS of (41) stands for symmetric.

□

Note also, that $E_S^{\alpha} N$ is positive only for $r \leq \alpha/(M-1)$. Two values of α are very attractive, namely $\alpha = 1$ and $\alpha = M-1$. Then

$$E_S^1 N = \frac{(\sigma^2 + \lambda)[1 - r(M-1)] - \lambda^2}{2(r\bar{r}^{M-1} - \lambda)}, \quad r > 1/(M-1) \quad (42a)$$

$$E_S^{M-1} N = \frac{(\sigma^2 + \lambda)\bar{r} - \lambda^2}{2(r\bar{r} - \lambda)} \quad (42b)$$

It is easy to notice in the presence of (39) that $E_S^1 N$ is tight bound for $\lambda \rightarrow r\bar{r}^{M-1}$ while $E_S^{M-1} N$ is more appropriate for $\lambda \rightarrow 0$.

To derive upper bounds from (40) we argue as above but now we must assume that for $k = 1, 2, \dots, \alpha-1$ $G_1(1, 1^{M-k-1}, 0^k) > 0$ while for $k \geq \alpha+1$ $G_1(1, 1^{M-k-1}, 0^k) < EN$. Then

Corollary 7: For all $\alpha = 1, 2, \dots, M-1$

$$EN \leq \frac{(\sigma^2 + \lambda)[1 - r(M-1)/\alpha] - \lambda^2}{2(r\bar{r}^{M-1} - \sum_{k=\alpha+1}^{M-1} \frac{k-\alpha}{\alpha} r^{k+1} \bar{r}^{M-k-1} \binom{M-1}{k} - \lambda)} \stackrel{def}{=} E_S^\alpha \bar{N} \quad (43a)$$

iff

$$r\bar{r}^{M-1} - \sum_{k=\alpha+1}^{M-1} \frac{k-\alpha}{\alpha} r^{k+1} \bar{r}^{M-k-1} \binom{M-k}{k} > 0 \quad (43b)$$

□

As above two values of α are very attractive, namely $\alpha = 1$ and $\alpha = M-1$. Then

$$E_S^1 \bar{N} = \frac{(\sigma^2 + \lambda)[1 - r(M-1)] - \lambda^2}{2[r(1 - r(M-1)) - \lambda]} \quad r < 1/(M-1) \quad (44a)$$

$$E_S^{M-1} \bar{N} = \frac{(\sigma^2 + \lambda)\bar{r} - \lambda^2}{2(r\bar{r}^{M-1} \lambda)} \quad (44b)$$

We note also that $E_S^1 \bar{N}$ is better than $E_S^{M-1} \bar{N}$ for $\lambda \rightarrow 0$ while for $\lambda \rightarrow r\bar{r}^{M-1}$ the reverse is true.

The upper bound (43) might be further improved if one finds better bounds for $G_1(1, 1^{M-k-1}, 0^k)$. But noting that $G_1(1, 0^{M-1}) < G_1(1, 1^{M-k-1}, 0^k) < G_1(1, 1^{M-2}, 0)$ and substituting it into (36) a simple computation reveals that

$$2rG_1(1, 0^{M-1}) \leq \sigma^2 + \lambda \leq 2rG_1(1, 1^{M-2}, 0) \quad (45)$$

Using RHS of (45) and arguing as above we show that

$$EN \leq \frac{(\sigma^2 + \lambda)[1 - r(M-1)/\alpha + r^{M-1}(M-1-\alpha)/\alpha] - \lambda^2}{2(r\bar{r}^{M-1} - \sum_{k=\alpha+1}^{M-2} \frac{k-\alpha}{\alpha} r^{k+1}\bar{r}^{M-k-1} \binom{M-1}{k} - \lambda)} \quad (46)$$

for $\alpha = 1, 2, \dots, M-1$ under the following assumption

$$r\bar{r}^{M-1} - \sum_{k=\alpha+1}^{M-2} \frac{k-\alpha}{\alpha} r^{k+1}\bar{r}^{M-k-1} \binom{M-1}{k} > 0$$

Finally, applying (45) directly to (37) for $\alpha = M-1$ we enhance (44b) obtaining

$$EN \leq \frac{(\sigma^2 + \lambda)\bar{r}(1 - r\bar{r}^{M-3}) - \lambda^3}{2(r\bar{r}^{M-1} - \lambda)} \stackrel{def}{=} E^{M-1}\bar{N} \quad (47)$$

This bound is very attractive for small values of M .

5. NUMERICAL RESULTS

In this section using numerical computations we compare the above obtained lower and upper bounds. Let us start with the symmetric case. Numerical results are contained in Table 1 and Table 2. Both tables show the best lower and the best upper bound, where the upper index at each position in the columns indicates what value of the splitting factor $\alpha (= 1, 2, \dots, M-1)$ was used to compute the best bounds. Table 1 for $M = 10$ compares the bounds for $(M-1)r < 1$ and $(M-1)r > 1$. In the first case the bounds are quite tight, while the latter case show quite a big gap between lower and upper bounds. This is true for all values of M and r as it is easy to notice comparing (41) and (43) (in particular, see (42) and (44)). In Table 2 the same bounds are shown for $M=50$ and $r=0.01$. To investigate the influence of M on the bounds we choose r in Table 1 (first two columns) and Table 2 such that $Mr = 0.5$. The results point out that greater M is worse the bounds are, however, for all $Mr < 1$ the bounds are still quite tight. Note also that from the practical point of view the probability r should be less then $1/M$ [3], so it is the most interesting case. Moreover, the maximum throughput $\lambda_{\max} = r\bar{r}^{M-1}$ is the greatest for $r = 1/M$, so restricting r and M to the case $Mr < 1$ is quite reasonable. Finally, note that for small values of λ bigger splitting factor α is better lower bound is, while for bigger values of $\lambda \rightarrow \lambda_{\max}$ lower splitting

factor α is better lower bound is. For upper bound the reverse is true.

Let us now discuss asymmetric system, and the bounds $E_l \tilde{N}_n, E_l N_n, l = 1, 2, 3, 4$ (Corollary 2-4). This case is much more difficult to study since a lot of factors must be taken into account.

We have decided to compare the following issues:

- the influence of M on the bounds - (Table 3),
- the influence of $\lambda_i, i = 1, 2, \dots, M$ on the bounds - (Table 4),
- $\sum_{i=1}^M r_i < 1$ versus $\sum_{i=1}^M r_i > 1$ - (Table 5).

All results in Tables 3-5 are computed for the first buffer, EN_1 versus λ_1 for different values of $r_i, \lambda_i, i = 2, \dots, M$ and M . To present the results in a compact form we assume that vector $R = (r_1, r_2, \dots, r_M)$ contains the values of probabilities $r_i, i = 1, 2, \dots, M$. Moreover, let $\lambda_k^{\max} = r_k \bar{r}^{-U-(k)}$ and in each row of the tables we increment λ_1 by $\lambda_1^{\max}/10$, what is denoted as $\lambda_1 = \lambda_1 + \lambda_1^{\max}/10$. However, since λ_1^{\max} strongly depends on the vector R and the number of users M in Table 3 and 5 we use different values of λ_1 for each case. Therefore, in the column λ_1 we show two values of λ_1 , namely, λ_1'/λ_1'' where λ_1' is valid for the first two columns of bounds, while λ_1'' for the next bounds. Moreover, the values of $\lambda_j, j = 2, \dots, M$ are chosen such that λ_2 and λ_3 are high fractions of λ_2^{\max} and λ_3^{\max} while $\lambda_k = \lambda_k^{\max}/10$ for $k = 4, 5, \dots, M$. It means that the input traffic to the second and third buffer is high, while for the remaining buffers we assume light traffic. Finally, with each value of the best lower and the best upper bound is associated an upper index (either 1 or 2 or 3 or 4) which indicates which bound was used to compute the best upper and the best lower bounds (Corollary 2-4).

In Table 3 we assume $R = (0.1; 0.15; 0.2; 0.05; 0.1; 0.05; 0.1; 0.05; 0.1; .05)$, and results for $M = 5$ and $M = 10$ are compared. We put $\lambda_2 = \frac{3}{4} \lambda_2^{\max}, \lambda_3 = \frac{1}{4} \lambda_3^{\max}$ and $\lambda_j = \lambda_j^{\max}/10$ ($j = 4, 5$ for $M = 5$ and $j = 4, 5, \dots, 10$ for $M = 10$). The table shows that:

Table 1. $M = 10$

λ	$r=0.05 < 1/(M-1)$		$r=0.15 > 1/(M-1)$	
	Best $E_S N$	Best $E_S \bar{N}$	Best $E_S N$	Best $E_S \bar{N}$
0.005	0.113 ²⁾	0.122 ¹⁾	0.034 ⁹⁾	0.112 ⁴⁾
0.01	0.266 ²⁾	0.311 ¹⁾	0.073 ³⁾	0.268 ⁴⁾
0.02	0.939 ¹⁾	1.441 ¹⁾	0.167 ³⁾	0.915 ⁴⁾
0.025	2.067 ¹⁾	3.478 ⁹⁾	0.213 ³⁾	1.985 ⁴⁾
0.03	10.714 ¹⁾	18.001 ⁹⁾	0.294 ³⁾	4.321 ⁴⁾
0.0315	∞	∞	0.326 ²⁾	8.275 ⁴⁾
0.034	∞	∞	∞	∞

Table 2. $M = 10$ $r = 0.01$

λ	Best $E_S N$	Best $E_S \bar{N}$
0.005	0.0529 ⁵⁾	0.0554 ¹⁾
0.001	0.1139 ⁵⁾	0.1242 ¹⁾
0.003	0.4905 ¹⁾	0.7264 ¹⁾
0.004	0.9630 ¹⁾	1.8010 ⁴⁹⁾
0.005	2.2860 ¹⁾	4.4213 ⁴⁹⁾
0.00611	∞	∞

- (i) in most cases the best lower bound is $E_1 N_1$
- (ii) for small values of λ_1 the best upper bound is $E_1 \bar{N}_1$, while for larger λ_1 $E_3 \bar{N}_1$ is better
- (iii) the smaller M is the tighter the bounds are.

In Table 4 we compare the best lower bound and the best upper bound for $M = 10$ and two sets of input rates to buffers $i = 3, 4, \dots, 10$, namely, either $\lambda_i = \lambda_i^{\max}/20$ or $\lambda_i = \lambda_i^{\max}/5$, $i = 3, 4, \dots, 10$. We assume the same values of probabilities r_i as in Table 3. It is easy to notice that conclusions (i) and (ii) from the previous results are valid here too, and in addition,

- (iv) the smaller input rates to other buffers are, the tighter bounds are.

Finally, in Table 5 we compare the bounds for two sets of probabilities r_i , namely $R_1 = (0.05; 0.1; 0.05; 0.02; 0.03; 0.05; 0.05; 0.02; 0.03)$ and $R_2 = (0.05; 0.1; 0.1; 0.2; 0.3; 0.1; 0.2;$

0.3; 0.1; 0.05). Note that in the first case $\sum_{i=1}^M r_i = 0.5 < 1$ while in the second case $\sum_{i=1}^M r_i = 1.5 > 1$

(we identify these cases as $\sum r_i < 1$ and $\sum r_i > 1$). Once again we confirm our conclusions (i) and (ii) from Tables 3 and 4, however, in addition we establish the next remark:

- (v) for $\sum_{i=1}^M r_i < 1$ the bounds are quite tight, while the results for $\sum_{i=1}^M r_i > 1$ cannot be accepted and more research is needed to find tighter bounds. However, from the practical point of view only the case $\sum_{i=1}^M r_i < 1$ is important by the same reasons as in the symmetric case.

Finally in Fig. 1 simulation results are compared with the bounds from Table 3 for $M = 5$.

As one may expect for small values of λ_1 the lower bound is quite close to the simulation results.

For bigger values of λ the simulation curve lies between lower and upper bound.

Table 3:

Probability vector $R = (0.1; 0.15; 0.2; 0.05; 0.1; 0.05; 0.1; 0.05; 0.1; 0.05);$
 $(\sum_{i=1}^5 r_i = 0.6; \sum_{i=1}^{10} r_i = 0.95)$

$\lambda_2 = \frac{3}{4} \lambda_2^{\max}, \lambda_3 = \lambda_2^{\max}/4, \lambda_i = \lambda_i^{\max}/10, i = 4, 5, \dots, 10$					
$M=5$	$M=10$	$M=5$		$M=10$	
$\lambda_1 = \lambda_1 + \lambda_1^{\max}/10$		BEST EN_1	BEST EN_1	BEST EN_1	BEST EN_1
0.012/0.049		0.131 ¹⁾	0.248 ¹⁾	0.087 ¹⁾	0.249 ¹⁾
0.023/0.016		0.300 ¹⁾	0.613 ²⁾	0.191 ¹⁾	0.651 ²⁾
0.035/0.024		0.526 ¹⁾	1.136 ³⁾	0.316 ¹⁾	1.060 ³⁾
0.047/0.032		0.850 ¹⁾	1.900 ³⁾	0.470 ¹⁾	1.519 ³⁾
0.058/0.040		1.346 ¹⁾	3.463 ²⁾	0.663 ¹⁾	2.231 ³⁾
0.069/0.048		5.654 ⁴⁾	8.507 ³⁾	0.917 ¹⁾	3.488 ³⁾
0.081/0.0565				1.263 ¹⁾	6.305 ³⁾

Table 4:
Probability vector R - as in Table 3. $M = 10$

λ_1	$\lambda_2 = \lambda_2^{\max}/2$		$\lambda_3 = \lambda_3^{\max}/2$	
	$\lambda_i = \lambda_i^{\max}/20, i = 3, \dots, 10$	BEST EN_1	$\lambda_i = \lambda_i^{\max}/5, i = 3, \dots, 10$	BEST EN_1
0.008	0.087 ¹⁾	0.249 ¹⁾	0.087 ¹⁾	0.249 ¹⁾
0.016	0.191 ¹⁾	0.603 ²⁾	0.191 ¹⁾	0.661 ¹⁾
0.024	0.316 ¹⁾	0.914 ³⁾	0.316 ¹⁾	1.240 ³⁾
0.032	0.469 ¹⁾	1.334 ³⁾	0.469 ¹⁾	1.756 ³⁾
0.0404	0.663 ¹⁾	1.983 ³⁾	0.663 ¹⁾	2.572 ³⁾
0.0485	1.409 ⁴⁾	3.117 ³⁾	0.917 ¹⁾	4.045 ³⁾
0.0565	1.710 ⁴⁾	5.622 ³⁾	1.263 ¹⁾	7.527 ³⁾

Table 5:

Probability vectors:

a) $R_1 = (0.05; 0.1; 0.1; 0.05; 0.02; 0.03; 0.05; 0.05; 0.02; 0.03); \sum_{i=1}^M r_i = 0.5 < 1$

b) $R_2 = (0.05; 0.1; 0.1; 0.2; 0.3; 0.1; 0.2; 0.3; 0.1; 0.05); \sum_{i=1}^M r_i = 1.5 > 1$

$M = 10$

$\lambda_2 = \lambda_2^{\max}/2, \lambda_3 = \lambda_3^{\max}/2, \lambda_i = \lambda_i^{\max}/10, i = 4, 5, \dots, 10$					
$\sum r_i < 1$	$\sum r_i > 1$	$\sum r_i = 0.5$		$\sum r_i = 1.5$	
$\lambda_1 = \lambda_1 + \lambda_1^{\max}/10$		BEST EN_1	BEST EN_1	BEST EN_1	BEST EN_1
0.006/0.002		0.143 ¹⁾	0.249 ¹⁾	0.041 ¹⁾	0.231 ¹⁾
0.013/0.004		0.333 ¹⁾	0.663 ¹⁾	0.085 ¹⁾	0.481 ²⁾
0.019/0.006		0.598 ¹⁾	1.223 ³⁾	0.133 ¹⁾	0.729 ³⁾
0.025/0.008		0.996 ¹⁾	2.056 ³⁾	0.185 ¹⁾	0.924 ³⁾
0.0314/0.009		1.658 ¹⁾	3.850 ³⁾	0.242 ¹⁾	1.169 ³⁾
0.038/0.012		1.338 ⁴⁾	10.562 ³⁾	0.305 ¹⁾	1.485 ³⁾
0.044/0.014		16.130	∞	1.046 ⁴⁾	1.907 ³⁾
/0.016		∞	∞	1.194 ⁴⁾	2.500 ³⁾

6. CONCLUSIONS

The finite number of buffered users in a packet contention broadcast environment was considered. We assumed throughout the paper random access protocol, however, it is not essential

for the analysis. Some upper and some lower bounds for the average queue lengths were presented. We found also stability conditions for the system, however, they are only sufficient but not necessary.

Numerical results showed that the accuracy of the method depends on the sign of $1 - \sum_{i=1}^M r_i$,

that is, for $\sum_{i=1}^M r_i < 1$ the upper and lower bounds are quite tight, but for $\sum_{i=1}^M r_i > 1$ more research is needed to improve (mainly) the lower bounds. For symmetric systems the bounds for $Mr < 1$ are very attractive, however, even for $1 < Mr < 3$ the bounds are acceptable. We also pointed out that from the practical point of view the case $Mr < 1$ ($\sum r_i < 1$ for asymmetric case) is the most important.

Further research should go into two directions. To recognize the behavior of the system we should establish sufficient and necessary conditions for stability of the system. Although the conditions are known for the symmetric case, an asymmetric system is very difficult to handle. It seems that establishing such conditions for this type of systems needs some general consideration for a wide class of multidimensional Markov chains. Such an effort was undertaken in [9].

The second problem is to find tighter bounds for queue length, waiting time and so forth. It is reasonable to assume that in the near future an exact analysis for two users will be available (Riemman-Hilbert approach), however, for more dimensional systems only bounds and approximations seems to be in these days achievable. For symmetric case the bounds might be quite well improved if one notices that by (34a) and (36) evaluation of the unknown denominator in (34a) is reduced to a solution of the following problem. Let X be a Bernoulli distributed random variable and $h(X)$ an unknown function (in our case $h(k) = G_1(1, 1^{M-k-1}, 0^k)$). Then the problem is to find $Eh(X)$ assuming that $2Eh(X)X = (M-1)(\sigma^2 + \lambda)$. On the other hand, in the asymmetric case stochastic dominance approach seems to be promising, e.g. by creating two systems with well known solutions we might upper and lower bound the ALOHA-type system (see for

example [3]).

APPENDIX A: Generating function $G(z)$

In this appendix we shall derive the formula (4) for the generating function of the M -dimensional Markov chain (N_1^k, \dots, N_M^k) defined in Eqs.(1). Let us denote by N_i' the queue length in the i -th buffer if $Z_i=1$. Then, in the steady state the generating function $G(z_1, z_2, \dots, z_M) = G(z)$ may be obtained as:

$$G(z) = \prod_{i=1}^M H_i(z_i) \left\{ \prod_{i=1}^M \bar{r}_i G(z) + \sum_{k=1}^M \sum_{I \in C_k} r_{i_1} \dots r_{i_k} \bar{r}^{U-(i_1, \dots, i_k)} E z_{i_1}^{N_{i_1}'} z_{i_2}^{N_{i_2}'} \dots z_{i_k}^{N_{i_k}'} z_{j_1}^{N_{j_1}} \dots z_{j_{M-k}}^{N_{j_{M-k}}} \right\} \quad (A1)$$

where $(i_1, \dots, i_k) = I \in C_k$ and $(j_1, \dots, j_{M-k}) = U - I$. Hence, to evaluate $G(z)$ we must compute the pseudo-generating function under the sums of Eq.(A1). Note that for $k=1$ we have $I = \{i\}$ and $N_i' = (N_i - 1)^+$. Then

$$E z_1^{N_1'} \dots z_i^{(N_i-1)'} \dots z_M^{N_M} = z_i^{-1} \{G(z) + (z_i - 1)G(z_1, \dots, z_{i-1}, 0, z_{i+1}, \dots, z_M)\} \\ = z_i^{-1} \{G(z) + (z_i - 1)G(0^i, z^{U-i})\} \quad (A2)$$

For $k > 1$ the computations are much complicated. For example for $k=2$ ($I = \{1, 2\}$) we find

$$E z_1^{N_1'} z_2^{N_2'} z_3^{N_3} \dots z_M^{N_M} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_M=0}^{\infty} z_1^{k_1} z_2^{k_2} z_3^{k_3} \dots z_M^{k_M} Pr\{N_1=k_1, N_2=k_2, \dots, N_M=k_M\} = \\ \sum_{k_2=0}^{\infty} \dots \sum_{k_M=0}^{\infty} z_3^{k_3} \dots z_M^{k_M} Pr\{N_1=0, N_2=0, N_3=k_3, \dots, N_M=k_M\} + \\ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=0}^{\infty} \dots \sum_{k_M=0}^{\infty} z_1^{k_1} z_2^{k_2} z_3^{k_3} \dots z_M^{k_M} Pr\{N_1=k_1, N_2=k_2, N_3=k_3, \dots, N_M=k_M\} + \\ \sum_{k_2=1}^{\infty} \sum_{k_3=0}^{\infty} \dots \sum_{k_M=0}^{\infty} z_2^{k_2-1} z_3^{k_3} \dots z_M^{k_M} Pr\{N_1=0, N_2=k_2, \dots, N_M=k_M\} + \\ \sum_{k_1=1}^{\infty} \sum_{k_3=0}^{\infty} \dots \sum_{k_M=0}^{\infty} z_1^{k_1-1} z_3^{k_3} \dots z_M^{k_M} Pr\{N_1=k_1, N_2=0, N_3=k_3, \dots, N_M=k_M\} = \\ G(z) + \prod_{j=1}^M z_j^{-1} \{G(0, 0, z_3, \dots, z_M) [(z_1-1) \prod_{j=2}^M z_j + (z_2-1) \prod_{j=2}^M z_j] - G(0, z_2, z_3, \dots, z_M) (z_1-1) \prod_{j=2}^M z_j - \\ G(z_1, 0, z_3, \dots, z_M) (z_2-1) \prod_{j=2}^M z_j\}$$

Generalizing the above we prove that for $I = (i_1, \dots, i_k)$

$$E z_{i_1}^{N_{i_1}'} \dots z_{i_k}^{N_{i_k}'} z_{j_1}^{N_{j_1}} \dots z_{j_{M-k}}^{N_{j_{M-k}}} = G(z) + \\ \prod_{j=1}^M z_j^{-1} \{G(0^I, z^{U-I}) \sum_{j=1}^k (z_{i_j} - 1) \prod_{\substack{l=1 \\ l \neq i_j}}^M z_l - \sum_{j=1}^M G(0^{I-(i_j)}, z^{U-I \cup \{i_j\}}) (z_{i_j} - 1) \prod_{\substack{l=1 \\ l \neq i_j}}^M z_l\} \quad (A3)$$

Taking into account (A1) - (A3) we finally obtain Eq.(4).

Appendix B: Proof of Theorem 1.

We prove Theorem 1 using so called *comparison test* established in [9 , Lemma 2]. Let us define a two-dimensional Markov chain $(\bar{N}_n^k, \bar{N}_m^k)$ which represents the queue lengths in the n -th and m -th buffer under the condition that all other buffers are never empty, that is $N_j^k > 0$ for all $k \geq 0$ and $j \in U - \{n, m\}$. It is easy to notice that [9]

$$(N_n^k, N_m^k) \leq_{st} (\bar{N}_n^k, \bar{N}_m^k) \quad (B1)$$

i.e. the two-dimensional process (N_n^k, N_m^k) is stochastically smaller than the two-dimensional Markov chain $(\bar{N}_n^k, \bar{N}_m^k)$. In [9] it is shown that M -dimensional Markov chain is ergodic if one finds such ergodic two-dimensional Markov chains $(\bar{N}_n^k, \bar{N}_m^k)$ that (B1) is satisfied for all $n, m \in U$. Therefore, to prove the theorem we must establish ergodic condition for $(\bar{N}_n^k, \bar{N}_m^k)$. But Malyshev conditions [2] [9] state that the two-dimensional Markov chain is ergodic if the following conditions are satisfied:

- (i) $a_n > 0, a_m > 0$ implies $q_{mn} d_n(m) > 0$ and $q_{mn} d_m(n) > 0$
- (ii) $a_n \leq 0, a_m > 0$ implies $q_{mn} d_n(m) > 0$
- (iii) $a_n > 0, a_m \leq 0$ implies $q_{mn} d_m(n) > 0$

(B2)

where

$$q_{nm} = \prod_{\substack{j=1 \\ j \neq n, m}}^M (1 - r_j) \quad (B3)$$

Note now that (see (11))

$$d_n(m) = r_n \bar{r}_n q_{nm} - \bar{r}_n \lambda_n - r_n \lambda_n, \quad d_m(n) = r_m \bar{r}_m q_{nm} - r_m \lambda_n - \bar{r}_m \lambda_m \quad (B4)$$

that is, $d_n(m)$ and $d_m(n)$ are linear functions of λ_n and λ_m , (see Fig.B1). Let us assume now that $q_{nm} \neq 0$ for all $n, m \in U$, i.e. $0 < r_n < 1, n \in U$ (if any of $r_i, i=1, \dots, M$ is equal to one then ergodicity conditions might be established using Lyapunov function method [9]). Then, because of (11) the condition (i) is satisfied whenever $a_n > 0$ and $a_m > 0$. Moreover, $d_n(m) = d_m(n) = 0$

for $a_n = a_m = 0$. Therefore, the conditions (i) - (iii) hold if

$d_n(m) > 0$ and $d_m(n) > 0$ for $r_n \leq \bar{r}_m$ (Fig.B 1a)
or if

$d_n(m) > 0$ or $d_m(n) > 0$ for $r_n > \bar{r}_m$ (Fig.B 1b)

what proves (19).

□

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NOTATIONS

Definitions of sets:

$$U = \{1, 2, \dots, M\}$$

$$C_k = \{(i_1, i_2, \dots, i_k) : i_j \in U; 1 \leq j \leq k, i_l \neq i_j \text{ iff } l \neq j\}$$

$$C_k - \{n_1, \dots, n_h\} = \{(i_1, i_2, \dots, i_k) \in C_k : i_j \in U - \{n_1, \dots, n_h\}\}$$

E - ergodicity subsets defined in (21).

Random variables:

N_n^k - queue length in the n -th buffer at the beginning of the k -th slot

EN_n - the average queue length in the n -th buffer

$E_k N_n$ - the k -th lower bound for the queue length EN_n

$E_k \bar{N}_n$ - the k -th upper bound for the queue length EN_n

$G(z)$ - the generating function of the queue lengths (N_1, \dots, N_M)

X_n - arrival process to the n -th buffer; $\lambda_n = EX_n$, $\sigma_n^2 = \text{var } X_n$

$H_n(z)$ - generating function of X_n

Z_n - control variable; $\Pr\{Z_n = 1\} = r_n$, $\Pr\{Z_n = 0\} = 1 - r_n = \bar{r}_n$

Boundary functions:

For $I = (i_1, \dots, i_k) \in C_k$, $(j_1, \dots, j_{M-k}) \in U-I$:

$$r^I \bar{r}^{U-I} = r_{i_1} r_{i_2} \dots r_{i_k} \bar{r}_{j_1} \dots \bar{r}_{j_{M-k}}$$

$$G(0^I, z^{U-I}) = G(z) | z_{i_1} = \dots = z_{i_k} = 0$$

$$G_n(1^n, 0^I, 1^{U-I-n}) = \frac{d}{dz_n} G(z_n, 0^I, 1^{U-I-n}) | z_n = 1, \quad I \in C_k - [n]$$

Miscellaneous

M - number of users

$$a_n = r_n \bar{r}^{U-[n]} - \lambda_n; \quad d_n(m) = a_n \bar{r}_n + a_m r_n \quad \text{for } n, m \in U$$

$$\Omega_n = \sigma_n^2 + \lambda_n - \lambda_n^2 \quad ; \quad q_{nm} = \prod_{j=1, j \neq n, m}^M (1 - r_j);$$

$$a_{\Pi} = \prod_{k=1}^M a_k; \quad a_{\Pi}(n) = \prod_{k=1, k \neq n}^M a_k; \quad a_{\Sigma} = \sum_{k=1}^M a_k; \quad a_{\Sigma}(n) = \sum_{k=1, k \neq n}^M a_k$$