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Abstract: In this paper, the analysis of a schistosomiasis infection model that involves human and intermediate snail hosts as well as an additional mammalian host and a competitor snail species is studied by constructing Lyapunov functions and using a Krasnoselkii sublinearity trick.
We derive the basic reproduction number $\mathcal{R}_0$ for the deterministic model, and establish that the global dynamics are completely determined by the values of $\mathcal{R}_0$. We obtain the global stability of the disease-free equilibrium $\mathbb{E}_0$ when $\mathcal{R}_0 \leq 1$ and we prove the existence and local stability of the endemic equilibrium $\mathbb{E}^*$ when $\mathcal{R}_0 > 1$.

Key-words: Nonlinear dynamical systems, global stability, Lyapunov methods.

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Analyse de stabilité d'un modèle d'infection de la bliharziose avec controle biologique

Résumé : On considère un modèle d'infection de la bilharziose qui prend en compte les humains et les hôtes intermédiaires d'escargots aussi bien des hôtes mammifères supplémentaires et une espèce d'escargot résistant . L'analyse de stabilité est étudié en construisant des fonctions de Lyapunov et une manie de Krasnoselskii de sous-linéarité. Nous établirons le taux de reproduction de base $\mathcal{R}_0$ pour le modèle posé et nous montrerons que la dynamique globale est complétement determinée par $\mathcal{R}_0$. Nous obtenons la stabilté globale du point d'équilibre sans maladie $\mathbb{E}_0$ lorsque $\mathcal{R}_0 \leq 1$ et quand $\mathcal{R}_0 > 1$ nous prouvons l'éxistence et la stabilité du point d'équilibre endémique $\mathbb{E}^*$.

Mots-clés : Systèmes dynamiques non-linéaires, stabilité globale, méthodes de Lyapunov

1 Introduction

Schistosomiasis also known as bilharzia is a parasite-induced disease. The disease is generally associated with rural poverty. The parasites, schistosomes, have to go through an intermediate host (snails in most cases) to complete their life cycle: from eggs, to miracidia, to cercaria, finally to adult worms. Schistosomes have two stages or reproduction-sexual reproduction in humans and asexual amplification in snails, see F.A Milner and R. Zhao [1].

Control methods for schistosomiasis range from environmental modification to eliminate the host snails, chemical molluscicides, chemotherapy and more permanent methods such as the provision of safe water and sanitary facilities.

For many endemic situations, chemotherapy is a major component and is focused on school age children and other high-risk groups. However, it is acknowledged that the price of a chemical antischistosomial control is beyond the health budget of many countries.

On effective control which may require relatively little funding is biological control. In particular, trematode parasites or competitive snails of the intermediate snail hosts have proved to be effective in controlling schistosomiasis in the Caribbean area, Pointier and Jourdane [2].

Mathematical modelling and analysis of schistosomiasis has drawn many attentions since the first paper by Macdonald in [3]. Thereafter many others researchers built excellent models and developed a decent understanding of transmission mechanism of schistosomiasis (see [4, 5, 6]).

Recently, a schistosomiasis infection model described by E.J Allen and H.D Victory [7] are proposed. This model generalizes in some way, previous mathematical models such as those described by Anderson and May [9]; Kimbir [10]; Wu and Feng [11].

However, our model allows competition between the intermediate host snails and a resistant snail species to study the advantages of biological control. In this paper, taking these specific characteristics into consideration and based on Allen and Victory's model, see [7], we propose a mathematical analysis. A stability analysis is also provided to study the epidemiological consequences of control strategies. We show that the DFE is globally asymptotically stable by constructing a lyapunov functions. The existence of at least one positive solution is shown by a simple application of fixed point Theorem in cones due by Thieme [32] and its locally asymptotically stability using a Krasnoselkii sub linearity trick.

The paper is organized as follows. In Section 2 we present the model described by E.J Allen and H.D Victory [7]. Its well-posedness is established and a reduced model is proposed. In Section 3 the local and global stability of the disease-free equilibrium is studied with the Lyapounov method. In Section 4 the existence of an endemic equilibria is investigated as well as its local stability.

Finally, in Section 5, we present some discussions and conclusions.

2 Model frame work

We consider the model presented in [7]. In their work, four definite mammalian host sub-populations, three intermediate snail host sub-populations, and a population of resistant competitor snails are considered. Its shall be assumed that the total time interval considered is sufficiently small so that human births and deaths can be neglected.

Further, it is assumed that infected snails and infected mammals do not recover from schistosomiasis as their life span are short in comparison to that for humans. The dynamical quantities of the model are:

$u_1(t)$ = the susceptible (uninfected), see [8], human population size,
$u_2(t)$ = the infected human population size,
$u_3(t)$ = the susceptible snail host population size,
$u_4(t)$ = the population size of the infected snails which are not yet shedding cercariae,
$u_5(t)$ = the infected and shedding snail population size (shedding population size),
$u_6(t)$ = the competitor snail population size (resistant to infection),
$u_7(t)$ = the susceptible mammal population size,
$u_8(t)$ = the infected mammal population size.

In addition, the population of snails as well as mammals are assumed to be competitive. Birth and death rates for the various sub populations will be denoted by b_i et d_i, respectively, for $i = 1, 2, ...8$. For simplicity the birth rate of different sub-populations of mammals and intermediate host snails will be assumed to be equal, i.e $b_3 = b_4 = b_5$ and $b_8 = b_7$. The transmission parameters for the model are:

t_{15} = transmission rate from infected snails to uninfected humans,
t_{32} = transmission rate from infected humans to uninfected snails,
t_{38} = transmission rate from infected mammals to susceptible snail,
t_{75} = transmission rate from infected snails to susceptible mammals.

Competition parameters are defined for the populations:
c_{33} is the competition parameter between u_3 and u_3, u_4, u_5,
c_{44} and c_{55} are the competition parameters between u_4 and u_5, respectively, and u_3, u_4, and u_5,
c_{36} is the competition parameter for snails u_6 with snails u_3,
c_{46} and c_{56} are defined analogously,
c_{64} is the competition parameter for snails u_3, u_4 and u_5 with u_6,
c_{66} is the competition parameter for u_6 with u_6,
c_{77} and c_{88} are the competition parameter for the mammals populations.

Also, r_{12} is the rate that infected humans recover and r_{54} denotes the rate that the latent snail population u_4 becomes shedding u_5.

The following system of equations, for $0 \leq t \leq T$ where $T > 0$, relate the various populations:

$$
\left\{
\begin{aligned}
\frac{du_1}{dt} &= -t_{15}\, u_5\, u_1 + r_{12}\, u_2, \\
\frac{du_2}{dt} &= t_{15}\, u_5\, u_1 - r_{12}\, u_2, \\
\frac{du_3}{dt} &= b_3\,(u_3 + u_4 + u_5) - t_{32}\, u_2\, u_3 - d_3\, u_3 - c_{33}\, u_3(u_3 + u_4 + u_5) \\
&\quad -c_{36}\, u_3\, u_6 - t_{38}\, u_3\, u_8, \\
\frac{du_4}{dt} &= t_{32}\, u_2\, u_3 + t_{38}\, u_3\, u_8 - d_4\, u_4 - c_{44}\, u_4(u_3 + u_4 + u_5) \\
&\quad -c_{46}\, u_4\, u_6 - r_{54}\, u_4, \\
\frac{du_5}{dt} &= r_{54}\, u_4 - d_5\, u_5 - c_{55}\, u_5(u_3 + u_4 + u_5) - c_{56}\, u_5\, u_6, \\
\frac{du_6}{dt} &= b_6\, u_6 - c_{64}\, u_6(u_3 + u_4 + u_5) - c_{66}\, u_6\, u_6 - d_6\, u_6, \\
\frac{du_7}{dt} &= b_7(u_7 + u_8) - t_{75}\, u_5\, u_7 - c_{77}\, u_7(u_7 + u_8) - d_7\, u_7, \\
\frac{du_8}{dt} &= t_{75}\, u_5\, u_7 - d_8 u_8 - c_{88}\, u_8(u_7 + u_8).
\end{aligned}
\right.
\tag{1}
$$

With initial conditions

$$u_i(0) \geq 0, for\ all\ 1 \leq i \leq 8. \tag{2}$$

It is assumed for simplicity that $d_3 = d_4 = d_5$, $c_{33} = c_{44} = c_{55}$, $c_{77} = c_{88}$ and $c_{46} = c_{56} = c_{36}$. The total human population $N_H = u_1 + u_2$ is constant since

$$\frac{dN_H}{dt} = 0.$$

The given initial conditions make sure that $N_H(0) \geq 0$. Thus the total population $N_H(t)$ remains positive and bounded for all time $t > 0$. The dynamics of no resistant snails total population is

$$\frac{dN_{Si}}{dt} = (b_3 - d_3)\, N_{Si} - c_{33}\, N_{Si}^2 - c_{36}\, u_6\, N_{Si}.$$

It follows that

$$\frac{dN_{Si}}{dt} \leq (b_3 - d_3)\, N_{Si} - c_{33}\, N_{Si}^2.$$

So, using comparison principle, we get

$$N_{Si} \leq \frac{(b_3 - d_3)\, N_0}{c_{33}\, N_0 + (b_3 - d_3 - c_{33}\, N_0)\, \exp -(b_3 - d_3)\, t}.$$

Then

$$\lim_{t \to \infty} \sup N_{Si} \leq \frac{b_3 - d_3}{c_{33}}.$$

The dynamics of the resistant snails total population and the total mammals are respectively

$$\begin{aligned}\frac{du_6}{dt} &= (b_6 - d_6)\,u_6 - c_{66}\,u_6\,u_6 - c_{64}\,u_6(u_3 + u_4 + u_5),\\ \frac{dN_M}{dt} &= (b_7 - d_7)N_M - c_{77}\,N_M^2.\end{aligned}$$

It follows that

$$\lim_{t\to\infty} \sup u_6 \le \frac{b_6 - d_6}{c_{66}},$$
$$\lim_{t\to\infty} \sup N_M \le \frac{b_7 - d_7}{c_{77}}.$$

Thus the feasible region for the system (1) is

$$\mathbb{D} = \{(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8) \in \mathbb{R}^8_+ : N_{Si} \le \frac{b_3 - d_3}{c_{33}}, N_M \le \frac{b_7 - d_7}{c_{77}}, u_6 \le \frac{b_6 - d_6}{c_{66}}\}.$$

Proposition 2.1 *Let $(u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8)$ be a solution of the system (1) with initial conditions (2) and the closed set $\mathbb{D}$. Then $\mathbb{D}$ is positively invariant and attracting under the flow described by (1).*

Proof:
It is sufficient to consider the system on the faces of $\mathbb{D}$ and to show that for each face, the vector fields associated to the system points inside $\mathbb{D}$.

If $N_{Si} = 0$ then $\frac{dN_{Si}}{dt} = 0$. If $N_{Si} = \frac{b_3 - d_3}{c_{33}}$ then $\frac{dN_{Si}}{dt} \le 0$.
If $N_M = 0$ then $\frac{dN_M}{dt} = 0$. If $N_M = \frac{b_7 - d_7}{c_{77}}$ then $\frac{dN_M}{dt} \le 0$.
If $u_6 = 0$ then $\frac{du_6}{dt} = 0$.
If $u_6 \le \frac{b_6 - d_6}{c_{66}}$ then $\frac{du_6}{dt} \le -c_{64}\,\frac{b_6 - d_6}{c_{66}}(u_3 + u_4 + u_5) \le 0$.

Furthermore, the model (1) is well-posed epidemiologically. Hence, it is sufficient to study the dynamics of the basic model in $\mathbb{D}$.

2.1 Reduction of the system

We will reduce the stability analysis of (1), to the study of a smaller and simpler system. The following theorem(see [13]) will permit us to reduce the stability analysis to a smaller system.

Theorem 2.1 *Consider the following $\mathbb{C}^1$ system*

$$\begin{cases} \dot{x} = f(x); \qquad x \in \mathbb{R}^n \qquad y \in \mathbb{R}^m, \\ \dot{y} = g(x, y); \\ \textit{with a equilibrium point}, (x^*, y^*)\, i.e, \\ f(x^*) = 0 \textit{ and } g(x^*, y^*) = 0. \end{cases} \tag{3}$$

If x^* is globally asymptotically stable (GAS) in $\mathbb{R}^n$ for the system $\dot{x} = f(x)$, and if y^* is GAS in $\mathbb{R}^m$, for the system $\dot{y} = g(x^*, y)$, then (x^*, y^*) is (locally) asymptotically stable for (3).

Moreover, if all the trajectories of (3) are forward bounded, then (x^*, y^*) is GAS for (3).

We define $x_i = \dfrac{u_i}{N_{Si}}$ for $i = 3, 4, 5$. $x_i = \dfrac{u_i}{N_M}$ for $i = 7, 8$.

Using $\dot{x}_i = \dfrac{\dot{u}_i}{N_{Si}} - \dfrac{\dot{N}_{Si}}{N_{Si}} x_i = \dfrac{\dot{u}_i}{N_{Si}} - (a_3 - c_{33}N_{Si} - c_{36}u_6)\, x_i$, for $i = 3, 4, 5$.
$\dot{x}_i = \dfrac{\dot{u}_i}{N_M} - \dfrac{\dot{N}_M}{N_M} x_i = \dfrac{\dot{u}_i}{N_M} - (b_7 - d_7 - c_{77}N_M)\, x_i$, for $i = 7, 8$ and the fact that $\sum x_i = 1$.

These new variables satisfy:

$$\left\{\begin{aligned}
\frac{du_2}{dt} &= t_{15}(N_H - u_2)\, x_5 N_{Si} - r_{12}u_2,\\
\frac{dx_4}{dt} &= (t_{32}u_2 + t_{38}\, N_M\, x_8)(1 - x_4 - x_5) - (b_3 + r_{54})\, x_4,\\
\frac{dx_5}{dt} &= r_{54}x_4 - b_3x_5,\\
\frac{dx_8}{dt} &= t_{75}\, N_{Si}\, x_5(1 - x_8) - b_7x_8,\\
\frac{dN_{Si}}{dt} &= \overbrace{(b_3 - d_3)}^{a_3}\, N_{Si} - c_{33}N_{Si}^2 - c_{36}u_6N_{Si} = X_1(N_{Si}, u_6),\\
\frac{du_6}{dt} &= \overbrace{(b_6 - d_6)}^{a_6}\, u_6 - c_{64}u_6N_{Si} - c_{66}u_6u_6 = X_2(N_{Si}, u_6),\\
\frac{dN_M}{dt} &= (b_7 - d_7)N_M - c_{77}N_M^2.
\end{aligned}\right. \tag{4}$$

This system is triangular.

Let us consider the following subsystem:

$$\left\{\begin{aligned}
\frac{dN_H}{dt} &= 0,\\
\frac{dN_{Si}}{dt} &= \overbrace{(b_3 - d_3)}^{a_3}\, N_{Si} - c_{33}N_{Si}^2 - c_{36}u_6N_{Si} = X_1(N_{Si}, u_6),\\
\frac{du_6}{dt} &= \overbrace{(b_6 - d_6)}^{a_6}\, u_6 - c_{64}u_6N_{Si} - c_{66}u_6u_6 = X_2(N_{Si}, u_6),\\
\frac{dN_M}{dt} &= (b_7 - d_7)N_M - c_{77}N_M^2.
\end{aligned}\right. \tag{5}$$

The last equation has an equilibrium $N_M^* = \dfrac{b_7 - d_7}{c_{77}}$ which is GAS.

The equilibria of (5) are:
$(0, 0)$ which is unstable: two positive eigenvalues a_3 and a_6.

$E_1 = \left(\dfrac{a_3}{c_{33}}, 0\right)$ with eigenvalues $-a_3$ and $a_6 - \dfrac{c_{64}a_3}{c_{33}} = \dfrac{c_{33}a_6 - c_{64}a_3}{c_{33}}$.

$E_2 = \left(0, \dfrac{a_6}{c_{66}}\right)$ with eigenvalues $-a_6$ and $a_3 - \dfrac{c_{36}a_6}{c_{66}} = \dfrac{c_{66}\, a_3 - c_{36}a_6}{c_{66}}$.

If $c_{33}a_6 - c_{64}a_3 < 0$ then E_1 is LAS. If $c_{66}\, a_3 - c_{36}a_6 < 0$ then E_2 is LAS.

We shall assume that E_1 and E_2 are unstable which implies that

$$c_{33}a_6 - c_{64}a_3 > 0 \textit{ and } c_{66}\, a_3 - c_{36}a_6 > 0. \tag{6}$$

This implies

$$c_{33}c_{66} - c_{36}c_{64} > 0.$$

The system (5) has a positive equilibrium E^*:

$$N_{Si}^* = \frac{c_{36}a_6 - c_{66}a_3}{c_{36}c_{64} - c_{33}c_{66}} = \frac{c_{66}a_3 - c_{36}a_6}{c_{33}c_{66} - c_{36}c_{64}},$$

$$u_6^* = \frac{c_{64}a_3 - c_{33}a_6}{c_{36}c_{64} - c_{33}c_{66}} = \frac{c_{33}a_6 - c_{64}a_3}{c_{33}c_{66} - c_{36}c_{64}}.$$

The equilibrium $E^* = (N_{Si}^*, u_6^*)$ exists if only if

1. 1st case: $c_{33}c_{66} - c_{36}c_{64} > 0$. In this case the existence of E^* implies that E^* is LAS and the other are unstable.

2. 2nd case: $c_{33}c_{66} - c_{36}c_{64} < 0$. In this case E^* exists if $c_{33}a_6 - c_{64}a_3 < 0$ and $c_{66}a_3 - c_{36}a_6 < 0$. In this case E_1 and E_2 are LAS but E^* is unstable.

Therefore we shall assume that

$$c_{33}c_{66} - c_{36}c_{64} > 0. \tag{7}$$

In this case E^* is LAS: eigenvalues with negative real part.

Let

$$V = (N_{Si} - N_{Si}^* \log N_{Si}) + d\,(u_6 - u_6^* \log u_6).$$

Then

$$\dot V = (N - N_{Si}^*)(a_3 - c_{33}N - c_{36}u_6) + d\,(u_6 - u_6^*)(a_6 - c_{64}N - c_{66}u_6).$$

Using equilibria relations, we obtain:

$$\begin{aligned}
\dot V &= (N - N_{Si}^*)(c_{33}N_{Si}^* + c_{36}u_6^* - c_{33}N - c_{36}u_6) + d\,(u_6 - u_6^*)(c_{64}N_{Si}^* \\
&\quad + c_{66}u_6^* - c_{64}N - c_{66}u_6) \\
&= -c_{33}(N - N_{Si}^*)^2 - d\,c_{66}(u_6 - u_6^*)^2 - c_{36}(N - N_{Si}^*)(u_6 - u_6^*) \\
&\quad - d\,c_{64}(N - N_{Si}^*)(u_6 - u_6^*) \\
&= -c_{33}(N - N_{Si}^*)^2 - d\,c_{66}(u_6 - u_6^*)^2 - (c_{36} + d\,c_{64})(N - N_{Si}^*)(u_6 - u_6^*).
\end{aligned}$$

We choose $d = \dfrac{c_{66}a_3^2}{c_{33}a_6^2}$. With this and using (7) we can show

$$(c_{36} + d\,c_{64})^2 - 4\,d\,c_{33}\,c_{66} < 0, \tag{8}$$

then $\dot V$ is definite negative and hence the equilibrium $(N_{Si}^*,\, u_6^*,\, N_M^*)$ is GAS.

Then, under the condition (7), $(N_{Si}^*,\, u_6^*,\, N_M^*)$ is GAS.

Remark: It is also possible to prove the GAS of $(N_{Si}^*,\, u_6^*)$ by using Dulac criterion with the function $\rho(N_{Si}, u_6) = \dfrac{1}{N_{Si}\, u_6}$ defined on $\mathcal{D} =]0, \dfrac{b_3 - d_3}{c_{33}}[\times]0, \dfrac{b_6 - d_6}{c_{66}}[$.

We have $\dfrac{\partial(\rho X_1)}{\partial N_{Si}} + \dfrac{\partial(\rho X_2)}{\partial u_6} = -\left(\dfrac{c_{33}}{u_6} + \dfrac{c_{66}}{N_{Si}}\right) < 0.$

Therefore, on the set $\mathcal{D}$ it is sufficient to consider the system:

$$\begin{cases} \dfrac{du_2}{dt} &= t_{15}(N_H - u_2)\, x_5 N_{Si}^* - r_{12}u_2, \\ \dfrac{dx_4}{dt} &= (t_{32}u_2 + t_{38}\, N_M^*\, x_8)(1 - x_4 - x_5) - (b_3 + r_{54})\, x_4, \\ \dfrac{dx_5}{dt} &= r_{54}x_4 - b_3x_5, \\ \dfrac{dx_8}{dt} &= t_{75}\, N_{Si}^*\, x_5(1 - x_8) - b_7x_8. \end{cases} \tag{9}$$

3 Disease-free Equilibrium and Stability Analysis

3.1 Main theorem

In this section, we will give an analytic expression for $\mathcal{R}_0$, for more details see [14, 15, 16] and completely answer the stability question of a disease-free equilibrium. As usual $\rho(M)$ is the spectral radius of a matrix M.

Proposition 3.1 *The origin is the DFE of (9) and*

$$R_0 = \sqrt[3]{\frac{r_{54}\, N_{Si}^* \;(b_7\, t_{15}\, t_{32}\, N_H \;+ r_{12}\, t_{38}\, t_{75}\, N_M^*)}{b_3\, b_7\, r_{12}\;(b_3 + r_{54})}} \;=\; T_0^{1/3}.$$

Moreover The DFE is LAS if $T_0 < 1$ and is unstable if $T_0 > 1$.

Proof:
It is clear that the DFE is $E_0 = (0, 0, 0, 0)$.
The Jacobian at E_0 is

$$J_0 = \begin{pmatrix} -r_{12} & 0 & t_{15}\, N_H\, N_{Si}^* & 0 \\ t_{32} & -(b_3 + r_{54}) & 0 & t_{38}\, N_M^* \\ 0 & r_{54} & -b_3 & 0 \\ 0 & 0 & t_{75}\, N_{Si}^* & -b_7 \end{pmatrix}.$$

J_0 is a Metzler matrix and $J_0 = F + V$ with $F = \begin{pmatrix} 0 & 0 & t_{15}\, N_H\, N_{Si}^* & 0 \\ t_{32} & 0 & 0 & t_{38}\, N_M^* \\ 0 & r_{54} & 0 & 0 \\ 0 & 0 & t_{75}\, N_{Si}^* & 0 \end{pmatrix}.$

We have $F > 0$ and V is Metzler stable, see [17, 18, 19, 20]. Thanks to Varga's Theorem in [22]: $s(J_0) \le 0$ if $\rho(-F\,V^{-1}) \le 1$.

A simple computation gives:

$$\mathcal{R}_0 = \rho(-F\,V^{-1}) = \sqrt[3]{\frac{r_{54}\, N_{Si}^* \;(b_7\, t_{15}\, t_{32}\, N_H \;+ r_{12}\, t_{38}\, t_{75}\, N_M^*)}{b_3\, b_7\, r_{12}\;(b_3 + r_{54})}} \;=\; T_0^{1/3}.$$

Hence, E_0 is LAS if $T_0 < 1$ and is unstable if $T_0 > 1$.

3.2 A stability theorem

Theorem 3.1 *If $T_0 \le 1$ then the DFE is GAS.*

Proof:
Consider the candidate Lyapunov function:

$$V = \frac{t_{32}}{r_{12}}\, u_2 + x_4 + \frac{b_3 + r_{54}}{r_{54}}\, x_5 + \frac{t_{38}\, N_M^*}{b_7}\, x_8.$$

Its derivative along the solutions of (9) satisfies:

$$\begin{aligned}
\dot{V} = \quad & \frac{t_{32}}{r_{12}} t_{15}(N_H - u_2)\, x_5 N_{Si}^* - t_{32} u_2 \\
& + (t_{32}u_2 + t_{38}\, N_M^*\, x_8)(1 - x_4 - x_5) - (b_3 + r_{54})\, x_4 \\
& + \frac{b_3 + r_{54}}{r_{54}} (r_{54} x_4 - b_3 x_5) + \frac{t_{38}\, N_M^*}{b_7} \left(t_{75}\, N_{Si}^*\, x_5 (1 - x_8) - b_7 x_8\right) \\
= \quad & -t_{32} u_2 x_5 - t_{38}\, N_M^*\, x_8 x_5 \\
& + x_4 \left(\frac{b_3 + r_{54}}{r_{54}}\, r_{54} - (b_3 + r_{54}) - t_{32} u_2 - t_{38}\, N_M^*\, x_8 \right) \\
& + x_5 \left(\frac{t_{32}}{r_{12}} N_{Si}^*\, (N_H - u_2) - \frac{b_3 + r_{54}}{r_{54}}\, b_3 + \frac{t_{38}\, t_{75}\, N_M^*}{b_7}\, N_{Si}^*\, (1 - x_8) \right) \\
= \quad & -(x_4 + x_5)\, (t_{32} u_2 + t_{38}\, N_M^*\, x_8) \\
& + x_5 \left(\frac{t_{32}}{r_{12}} N_{Si}^*\, (N_H - u_2) - \frac{b_3 + r_{54}}{r_{54}}\, b_3 + \frac{t_{38}\, t_{75}\, N_M^*}{b_7}\, N_{Si}^*\, (1 - x_8) \right).
\end{aligned}$$

Then

$$\begin{aligned}
\dot{V} \leq \quad & -(x_4 + x_5)\, (t_{32} u_2 + t_{38}\, N_M^*\, x_8) \\
& + x_5 \left(\frac{t_{32}}{r_{12}} N_{Si}^*\, N_H - \frac{b_3 + r_{54}}{r_{54}}\, b_3 + \frac{t_{38}\, t_{75}\, N_M^*}{b_7}\, N_{Si}^* \right) \\
& = -(x_4 + x_5)\, (t_{32} u_2 + t_{38}\, N_M^*\, x_8) \\
& + x_5\, \frac{b_3 + r_{54}}{r_{54}}\, b_3 \left(\frac{r_{54}\, N_{Si}^*}{b_3\, (b_3 + r_{54})} \left(\frac{t_{32}}{r_{12}}\, N_H + \frac{t_{38}\, t_{75}\, N_M^*}{b_7} \right) - 1 \right) \\
& = -(x_4 + x_5)\, (t_{32} u_2 + t_{38}\, N_M^*\, x_8) \\
& + x_5\, \frac{b_3 + r_{54}}{r_{54}}\, b_3 \left(\frac{r_{54}\, N_{Si}^*\, (b_7\, t_{32}\, N_H + r_{12}\, t_{38}\, t_{75}\, N_M^*)}{b_3\, (b_3 + r_{54})\, b_7\, r_{12}} - 1 \right).
\end{aligned}$$

Hence $\dot{V} \leq -(x_4 + x_5)\, (t_{32} u_2 + t_{38}\, N_M^*\, x_8) + \dfrac{b_3 + r_{54}}{r_{54}}\, b_3\, (T_0 - 1)\, x_5 \leq 0$ if $T_0 \leq 1$.

If $T_0 < 1$, then $\dot{V} = 0$ implies $x_5 = x_4 = 0$, or $x_5 = x_8 = u_2 = 0$. Thanks to Lasalle [23, 24, 25] we conclude.

If $T_0 = 1$, then $\dot{V} = 0$ implies $x_5 = x_4 = 0$, or $x_8 = u_2 = 0$. Again thanks to Lasalle [23, 24, 25] we conclude.

4 Endemic Equilibrium and Stability Analysis

4.1 Existence of Endemic Equilibrium

Next, we will find the equilibrium points of system (9). To this end we express u_2, x_4, x_8 in terms of x_5.

Then we use a theorem for the existence and uniqueness of a positive fixed point of a multi-variable function. We labelled this theorem as follows

Theorem 4.1 *(Thieme [32], theorem 2.1) Let $F(x)$ be a continuous, monotone non-decreasing, strictly sub linear, bounded function which maps the non-negative orthant $\mathbb{R}^n_+ = [0, \infty)$ into itself. Let $F(0) = 0$ and $F'(0)$ exists and be irreducible. Then $F(x)$ does not have a non-trivial fixed point on the boundary of $\mathbb{R}^n_+$. Moreover, $F(x)$ has a positive fixed point iff $\rho(F'(0)) > 1$. If there is a positive fixed point, then it is unique.*

An equilibrium point is a solution of the simultaneous non-linear equations obtained by setting the right hand sides of the equation (9) to 0. Now we reformulate the non-linear equations as a fixed point equation. Solving the third equilibrium-point equation for x_5 and substituting into the first equation, we obtain

$$\begin{aligned} x_5 &= \frac{r_{54}}{b_3}\, x_4, \\ t_{15}(N_H - u_2)\, x_5\, N^*_{Si} - r_{12} u_2 &= 0, \end{aligned}$$

This implies

$$u_2 = \frac{t_{15}\, N^*_{Si}\, N^*_H \dfrac{r_{54}}{b_3}\, x_4}{r_{12} + t_{15}\, N^*_{Si} \dfrac{r_{54}}{b_3}\, x_4}.$$

Solving the equations in (9) at steady state gives

$$x^*_4 = \frac{b_3}{r_{54}}\, x^*_5, \qquad x^*_8 = \frac{t_{75} N^*_{Si}}{t_{75} N^*_{Si}\, x^*_5 + b_7}\, x^*_5, \qquad u^*_2 = \frac{t_{15} N_H N^*_{Si}\, x^*_5}{t_{15} N^*_{Si}\, x^*_5 + r_{12}}. \tag{10}$$

$$x_4 = \frac{(t_{32}\, u_2 + t_{38}\, N_M\, x_8)}{(b_3 + r_{54}) + (t_{32}\, u_2 + t_{38}\, N_M\, x_8)(1 + \dfrac{r_{54}}{b_3})}$$

the fourth equation gives

$$x_8 = \frac{N^*_{Si}\, t_{75} \dfrac{r_{54}}{b_3}\, x_4}{b_7 + N^*_{Si}\, t_{75} \dfrac{r_{54}}{b_3}}$$

We write this as

$$U = F(U),$$

where

$$U = \begin{pmatrix} u_2 \\ x_4 \\ x_8 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 := \dfrac{t_{15}\, N_{Si}^*\, N_H^* \dfrac{r_{54}}{b_3}\, x_4}{r_{12} + t_{15}\, N_{Si}^* \dfrac{r_{54}}{b_3}\, x_4} \le N_H \\ F_2 := \dfrac{(t_{32}\, u_2 + t_{38}\, N_M\, x_8)}{(b_3 + r_{54}) + (t_{32}\, u_2 + t_{38}\, N_M\, x_8)(1 + \dfrac{r_{54}}{b_3})} \le 1 \\ F_3 := \dfrac{N_{Si}^*\, t_{75} \dfrac{r_{54}}{b_3}\, x_4}{b_7 + N_{Si}^*\, t_{75} \dfrac{r_{54}}{b_3}\, x_4} \le 1 \end{pmatrix}$$

Thus the equilibrium points are fixed points of F given by $U = F(U)$ and this is the formulation that we use to prove existence and uniqueness of an endemic equilibrium point.

In this case $F(U)$ is continuous, bounded function which maps $\Omega = \{(u_2, x_4, x_8) : 0 \le u_2 \le N_H, 0 \le x_4 \le 1, 0 \le x_8 \le 1\}$ into itself and infinitely differentiable with Jacobian as follows

$$J_e = \begin{pmatrix} 0 & \dfrac{b_3 N_H N_{Si} r_{12} r_{54} t_{15}}{(b_3 r_{12} + N_{Si} r_{54} t_{15} x_4)^2} & 0 \\ \dfrac{b_3^2 t_{32}}{(b_3 + r_{54})(b_3 + t_{32} u_2 + N_M t_{38} x_8)^2} & 0 & \dfrac{b_3^2 N_M t_{38}}{(b_3 + r_{54})(b_3 + t_{32} u_2 + N_M t_{38} x_8)^2} \\ 0 & \dfrac{b_3 b_7 N_{Si} r_{54} t_{75}}{(b_3 b_7 + N_{Si} r_{54} t_{75} x_4)^2} & 0 \end{pmatrix}$$

where the off-diagonal elements are non-negative. Thus the function $F(U)$ is monotone non-decreasing and $F(0) = 0$. Note that $\rho(F'(0)) = \mathcal{R}_0 > 1$. Thanks to the graph theory, we claim that $F'(0)$ is irreducible because the associated graph of the matrix is strongly connected.

Let us now prove that F is strictly sub linear in Ω, i.e., $F(r\,U) > r\,F(U)$, for any $U \in \Omega$ with $U > 0$, and $r \in (0,1)$. Some calculations give

$$\frac{r_1\, F_1(U)}{F_1(r_1\, U)} = r_1\, \frac{t_{15}\, N_{Si}^*\, N_H^* \dfrac{r_{54}}{b_3}\, x_4}{r_{12} + t_{15}\, N_{Si}^* \dfrac{r_{54}}{b_3}\, x_4} * \frac{r_{12} + t_{15}\, N_{Si}^* \dfrac{r_{54}}{b_3}\, r_1\, x_4}{t_{15}\, N_{Si}^*\, N_H^* \dfrac{r_{54}}{b_3}\, r_1\, x_4} \tag{11}$$

$$= \frac{r_{12} + t_{15}\, N_{Si}^* \dfrac{r_{54}}{b_3}\, r\, x_4}{r_{12} + t_{15}\, N_{Si}^* \dfrac{r_{54}}{b_3}\, x_4} < 1$$

$$\frac{r_2\, F_2(U)}{F_2(r_2\, U)} = \frac{r_2\,(t_{32}\, u_2 + t_{38}\, N_M\, x_8)}{(b_3 + r_{54}) + (t_{32}\, u_2 + t_{38}\, N_M\, x_8)(1 + \dfrac{r_{54}}{b_3})} * \frac{(b_3 + r_{54}) + r_2\,(t_{32}\, u_2 + t_{38}\, N_M\, x_8)(1 + \dfrac{r_{54}}{b_3})}{r_2\,(t_{32}\, u_2 + t_{38}\, N_M\, x_8)} \tag{12}$$

$$= \frac{(b_3 + r_{54}) + (t_{32}\, u_2 + t_{38}\, N_M\, x_8)(1 + \dfrac{r_{54}}{b_3})}{(b_3 + r_{54}) + r\,(t_{32}\, u_2 + t_{38}\, N_M\, x_8)(1 + \dfrac{r_{54}}{b_3})} < 1$$

$$\frac{r_3\,F_3(U)}{F_3(r_3\,U)} = \frac{r_3\,N_{Si}^*\,t_{75}\,\dfrac{r_{54}}{b_3}\,x_4}{b_7+N_{Si}^*\,t_{75}\,\dfrac{r_{54}}{b_3}\,x_4} * \frac{b_7+N_{Si}^*\,t_{75}\,\dfrac{r_{54}}{b_3}\,r_3\,x_4}{N_{Si}^*\,t_{75}\,\dfrac{r_{54}}{b_3}\,r_3\,x_4} \tag{13}$$

$$= \frac{b_7+N_{Si}^*\,t_{75}\,\dfrac{r_{54}}{b_3}\,r_3\,x_4}{b_7+N_{Si}^*\,t_{75}\,\dfrac{r_{54}}{b_3}\,x_4} < 1$$

So the function $F(U)$ is strictly sub linear with $r = \min(r_1, r_2, r_3)$. In this way we have proved the following theorem

Theorem 4.2 *If $\mathcal{R} \leq 1$, the only equilibrium point of the system is the disease-free equilibrium E_0. If $\mathcal{R} > 1$, there also exists a unique endemic equilibrium E^* in $int(\Omega)$ whose coordinates are given by (10).*

4.2 Local Stability of the Endemic Equilibrium

In this section, we shall prove the local stability of the endemic equilibrium when $\mathcal{R}_0 > 1$. For this we shall follow the method given by Hethcote and Thieme, which is based on a Krasnoselkii technique. A usual way to prove the local asymptotic stability of an equilibrium point $\bar{x}_0$ of the system of differential equations

$$\bar{x}' = f(\bar{x}) \tag{14}$$

is proving that the linearised equation

$$\bar{Z}' = Df(\bar{x}_0)\bar{Z} \tag{15}$$

has no solutions of the form

$$\bar{Z}(t) = \bar{Z}_0 \exp(wt) \tag{16}$$

with $\bar{Z}_0 \in \mathbb{C}^n - \{0\}$, $w \in \mathbb{C}$ and $\mathcal{R}e\,w \geq 0$, where $\mathbb{C}$ denotes the complex numbers i.e., $w\,\bar{Z} = Df(\bar{x}_0)\,\bar{Z}$ with $\bar{Z} \in \mathbb{C}^n - \{0\}$, $w \in \mathcal{C}^n$ implies $\mathcal{R}e\,w < 0$.

Substituting a solution of the form (16) in the linearised equation of the endemic equilibrium, we obtain the following linear equations.

$$\begin{cases} w\,Z_1 = & -(t_{15}\,x_5^*\,N_{Si}^* + r_{12})\;Z_1 + t_{15}\;(N_H - u_2^*)\;N_{Si}^*\,Z_3 \\ w\,Z_2 = & (1 - x_4^* - x_5^*\,)\,t_{32}\,Z_1 - (t_{32}\,u_2^* + t_{38}\,N_M\,x_8^*)\;Z_2 \\ & -(b_3 + r_{54})\;Z_2 - (t_{32}\,u_2^* + t_{38}\,N_M\,x_8^*)\;Z_3 + N_M t_{38}\,(1 - x_4 - x_5)\;Z_4 \\ w\,Z_3 = & r_{54}\,Z_2 - b_3\,Z_3 \\ w\,Z_4 = & t_{75}\,N_{Si}\;(1 - x_8^*)\;Z_3 - (t_{75}\,N_{Si}\,x_5^* + b_7)\;Z_4 \end{cases} \tag{17}$$

Solving for Z_3 from the third equation of (17), and substituting the result into the second equation (and simplifying), give the equivalent system

$$\begin{aligned}
\left(1+\frac{w+t_{15}\,x_5^*\,N_{Si}^*}{r_{12}}\right)Z_1 &= \frac{t_{15}\,(N_H-u_2^*)\,N_{Si}^*}{r_{12}}\,Z_3\\
(1+G_2\,(w))\;Z_2 &= \frac{(t_{32})\,(1-x_4^*-x_5^*)}{b_3+r_{54}}\,Z_1+\frac{(t_{38}\,N_M\,x_8^*)\,(1-x_4^*-x_5^*)}{b_3+r_{54}}\,Z_4\\
\left(1+\frac{w}{b_3}\right)Z_3 &= \frac{r_{54}}{b_3}\,Z_2\\
\left(1+\frac{t_{75}\,N_{Si}\,x_5^*}{b_7}\right)Z_4 &= \frac{t_{75}\,N_{Si}\,(1-x_8^*)}{b_7}\,Z_3
\end{aligned} \tag{18}$$

where

$$G_2(w)=\frac{w}{b_3+r_{54}}+\frac{(t_{32}\,u_2+N_M\,t_{38}\,x_8)}{b_3+r_{54}}\left(1+\frac{r_{54}}{w+b_3}\right)$$

Denoting in the same way

$$\begin{aligned}
G_1(w) &= \frac{w+t_{15}\,x_5^*\,N_{Si}^*}{r_{12}}\\
G_3(w) &= \frac{w}{b_3}\\
G_4(w) &= \frac{w+t_{75}\,N_{Si}\,x_5^*}{b_7}
\end{aligned}$$

we obtain the system

$$\begin{aligned}
[1+G_1(w)]\;Z_1 &= \left(H\,\bar{Z}\right)_3\\
[1+G_2(w)]\;Z_2 &= \left(H\,\bar{Z}\right)_1+\left(H\,\bar{Z}\right)_4\\
[1+G_3(w)]\;Z_3 &= \left(H\,\bar{Z}\right)_2\\
[1+G_4(w)]\;Z_4 &= \left(H\,\bar{Z}\right)_3
\end{aligned} \tag{19}$$

with

$$H=\begin{pmatrix}
0 & 0 & \dfrac{t_{15}\,(N_H-u_2^*)\,N_{Si}^*}{r_{12}} & 0\\
\dfrac{t_{32}\,(1-x_4^*-x_5^*)}{b_3+r_{54}} & 0 & 0 & \dfrac{t_{38}\,N_M\,(1-x_4^*-x_5^*)}{b_3+r_{54}}\\
0 & \dfrac{r_{54}}{b_3} & 0 & 0\\
0 & 0 & \dfrac{t_{75}\,N_{Si}\,(1-x_8^*)}{b_7} & 0
\end{pmatrix}$$

Note that the notation $H\,(\bar{Z})_i$ (with $i=1,...,4$)) denotes the ith coordinate of the vector $H(\bar{Z})$. It should further be noted that the matrix H has non-negative entries, and the equilibrium $E^*=(u_{2^*},x_{4^*},x_{5^*},x_{8^*})$ satisfies $E^*=H\,E^*$. Furthermore, since the coordinates of E^* are all positive, its follows then that if $\bar{Z}$ is a solution of (19), then it is possible to find a minimal positive real numbers s, depending on $\bar{Z}$, such that

$$\|\bar{Z}\| \leq s\, E^* \tag{20}$$

where $\|\bar{Z}\| = \left(\bar{Z}_1, \bar{Z}_2, \bar{Z}_3, \bar{Z}_4\right)$ with the lexicographic order, and $\|\|$ is a norm in $\mathbb{C}$. now we want to show that $\mathcal{R}e\, w < 0$. Deny it, we distinguish two cases : $w = 0$ and $w \neq 0$. In the first case, the determinant of the homogeneous linear system (17) in the variable Z_i $(i = 1, ..., 4)$ corresponds to that of the Jacobian of the matrix

$$\begin{pmatrix} -1 - G_1(0) & 0 & \dfrac{N_{Si} t_{15} \left(N_H - u_2^*\right)}{r_{12}} & 0 \\ \dfrac{\left(1 - x_4^* - x_5^*\right) t_{32}}{b_3 + r_{54}} & -1 - G_2(0) & 0 & \dfrac{\left(1 - x_4^* - x_5^*\right) N_M t_{38}}{b_3 + r_{54}} \\ 0 & \dfrac{r_{54}}{b_3} & -1 - G_3(0) & 0 \\ 0 & 0 & \dfrac{N_{Si} t_{75} \left(1 - x_8^*\right)}{b_7} & -1 - G_4 \end{pmatrix}$$

which is given by

$$\begin{aligned} = \quad & \left(-1 - G_4(0)\right) \Big(\left(1 + G_1(0) + G_2(0) + G_1(0) G_2(0)\right) \left(-1 - G_3(0)\right) \\ + \quad & \frac{N_{Si}\, r_{54}\, t_{15}\, t_{32}\, \left(N_H - u_2\right) \left(1 - x_4 - x_5\right)}{b_3\, r_{12}\, \left(b_3 + r_{54}\right)} \Big) \\ + \quad & \frac{\left(-1 - G_1(0)\right)\, N_M\, N_{Si}\, r_{54}\, t_{38}\, t_{75}\, \left(1 - x_4 - x_5\right) \left(1 - x_8\right)}{b_3\, b_7\, \left(b_3 + r_{54}\right)} \end{aligned}$$

Since $G_3(0) = 0$

$$\begin{aligned} = \quad & \left(-1 - G_4(0)\right) \left(-1 - G_1(0) - G_2(0) - G_1(0) G_2(0) + \frac{t_{32}\, \left(1 - x_4 - x_5\right)\, u_{2^*}}{x_{4^*}\, \left(b_3 + r_{54}\right)} \right) \\ + \quad & \frac{\left(-1 - G_1(0)\right) N_M\, t_{38}\, \left(1 - x_4 - x_5\right)\, x_{8^*}}{x_{4^*}\, \left(b_3 + r_{54}\right)} \\ = \quad & \left(-1 - G_4(0)\right) \left(-1 - G_1(0) - G_2(0) - G_1(0) G_2(0)\right) + \left(-1 - G_4(0)\right) \left(\frac{t_{32}\, \left(1 - x_4 - x_5\right)\, u_{2^*}}{x_{4^*}\, \left(b_3 + r_{54}\right)} \right) \\ + \quad & \frac{\left(-1 - G_1(0)\right) N_M\, t_{38}\, \left(1 - x_4 - x_5\right)\, x_{8^*}}{x_{4^*}\, \left(b_3 + r_{54}\right)} \\ = \quad & \left(-1 - G_4(0)\right) \left(-1 - G_1(0) - G_2(0) - G_1(0) G_2(0)\right) \\ - \quad & \frac{1}{x_{4^*}} \left(\frac{t_{32}\, \left(1 - x_4 - x_5\right)\, u_{2^*}}{\left(b_3 + r_{54}\right)} \left(1 + G_4(0)\right) + \frac{N_M\, t_{38}\, \left(1 - x_4 - x_5\right)\, x_{8^*}}{\left(b_3 + r_{54}\right)} \left(1 + G_1(0)\right) \right) \end{aligned}$$

Denoting $\alpha = \max\{1 + G_1(0), 1 + G_4(0)\}$, we have

$$\begin{aligned} \Delta \quad > \quad & 1 + G_1(0) + G_2(0) + G_1(0)\, G_2(0) + G_4(0) + G_1(0)\, G_4(0) + G_2(0)\, G_4(0) \\ + \quad & G_1(0)\, G_2(0)\, G_4(0) - \alpha \end{aligned}$$

Then if $\alpha = 1 + G_1$, we obtain

$$\Delta \quad > \quad G_2 + G_1\,G_2 + G_4 + G_1\,G_4 + G_2\,G_4 + G_1\,G_2\,G_4 > 0$$

Else $\alpha = 1 + G_2$, we obtain

$$\Delta \quad > \quad G_1 + G_2 + G_1\,G_2 + G_1\,G_4 + G_2\,G_4 + G_1\,G_2\,G_4 > 0$$

since $G_1(0)$, $G_2(0)$, $G_4(0)$ are positive. Then, for $w = 0$, the only solution of the system (19) is the trivial one which implies that $w \neq 0$. Assume now that $w \neq 0$, and $\mathcal{R}e\,w \geq 0$. Let $G(w) = \min\{|1 + G_i(w)|\,, i = 1, ..., 4\}$. It is easy to prove that in this case $|1 + G_i(w)| > 1$ for all i, and therefore $G(w) > 1$. Taking norms on both sides of (19), and using the fact that H is non-negative, we obtain the following inequality:

$$G(w)\,\|\bar{Z}\| \leq H\,\|\bar{Z}\|. \tag{21}$$

Using (20) and (21), we get

$$G(w)\,\|\bar{Z}\| \leq s\,H\,E^* = s\,E^*.$$

which implies

$$\|\bar{Z}\| \leq \frac{s}{G(w)}\,E^* < s\,E^*.$$

but this contradicts the minimality of s. Therefore $\mathcal{R}_e\,w < 0$. In this way we proved the following theorem.

Theorem 4.3 *If $\mathcal{R}_0 > 1$, then the positive endemic equilibrium stated E^* of the system (9) is locally asymptotically stable on the set $\mathcal{D}$.*

Table 1: Parameter values corresponding to $\mathcal{R}_0 < 1$

Parameters	Values
c_{36}	$5.11 * 10^{-8}$ (case $1-2$)
c_{44}	$3.11 * 10^{-7}$(case $1-2$)
c_{55}	$5.11 * 10^{-7}$ (case $1-2$)
c_{46},	$5.11 * 10^{-7}$(case $1-2$)
c_{88}	$7.00 * 10^{-8}$ (case $1-2$)
c_{77}	$7.00 * 10^{-8}$ (case $1-2$)
c_{64}	$25.11 * 10^{-9}$ (case $1-2$)
c_{56}	$5.11 * 10^{-7}$(case $1-2$)
c_{33}	$5.11 * 10^{-7}$(case $1-2$)
c_{66}	$1.50 * 10^{-8}$(case $1-2$)
r_{12}	$4.47 * 10^{-3}$(case $1-2$)
r_{54}	$2.50 * 10^{-6}$(case $1-2$)
d_3	$8.86 * 10^{-3}$ (case $1-2$)
d_4	$8.86 * 10^{-3}$ (case $1-2$)
d_5	$1.79 * 10^{-3}$ (case $1-2$)
d_6	$1.00 * 10^{-2}$ (case $1-2$)
d_7	$5.00 * 10^{-3}$ (case $1-2$)
d_8	$5.00 * 10^{-3}$ (case $1-2$)
b_6	$6.60 * 10^{-2}$ (case $1-2$)
b_3	$6.00 * 10^{-2}$ (case 1), $6.00 * 10^{-1}$ (case 2)
b_7	$1.20 * 10^{-2}$ (case1), $1.20 * 10^{-5}$ (case 2)
t_{15}	$2.23 * 10^{-7}$ (case 1), $2.23 * 10^{-9}$ (case 2)
t_{38}	$2.0 * 10^{-7}$ (case 1), $2.0 * 10^{-9}$ (case 2)
t_{32}	$1.05 * 10^{-7}$ (case 1), $1.05 * 10^{-}9$ (case 2)
t_{75}	$1.02 * 10^{-7}$ (case 1), $1.02 * 10^{-9}$ (case 2)

Table 2: Parameter values for which $\mathcal{R}_0 > 1$

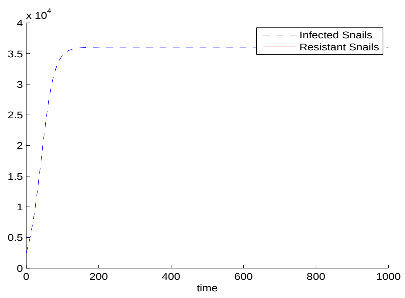
Parameters	Values
c_{36}	$5.11 * 10^{-8}$
c_{44}	$3.11 * 10^{-7}$
c_{55}	$5.11 * 10^{-7}$
c_{46}	$5.11 * 10^{-7}$
c_{88}	$7.00 * 10^{-8}$
c_{77}	$7.00 * 10^{-8}$
c_{64}	$5.11 * 10^{-7}$
c_{56}	$5.11 * 10^{-7}$
c_{33}	$5.11 * 10^{-7}$
c_{66}	$2.50 * 10^{-7}$
r_{12}	$4.47 * 10^{-4}$
r_{54}	$2.50 * 10^{-2}$
d_3	$8.86 * 10^{-3}$
d_4	$8.86 * 10^{-3}$
d_5	$1.79 * 10^{-3}$
d_6	$8.00 * 10^{-3}$
d_7	$5.00 * 10^{-3}$
d_8	$5.00 * 10^{-3}$
b_6	$6.60 * 10^{-2}$
b_3	$6.00 * 10^{-2}$
b_7	$1.20 * 10^{-2}$
t_{15}	$2.23 * 10^{-7}$
t_{38}	$1.05 * 10^{-7}$
t_{32}	$1.04 * 10^{-5}$
t_{75}	$2 * 10^{-6}$

5 Numerical studies

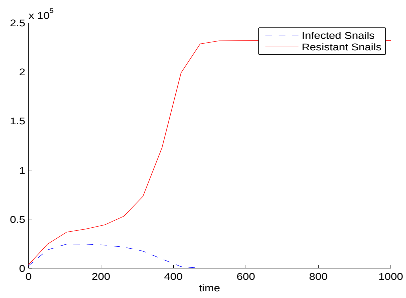
To illustrate the various theoretical results contained in the paper, the whole system with the eight equations is simulated and parameter values using data of Allen and al. and summarize in the following table.

In the present investigation, simulations were performed to study the effect of applying biological treatments strategy. Infected humans and snails and Latent snails were compared by introducing or not competitor snails. Notice, introduction of resistant that out compete the intermediate host snails can eradicate the infection more rapidly.

In addition, the tables presents with a minor modification the sets of values of the parameters discussed in Allen and al., which are used in the numerical simulations.



(a)



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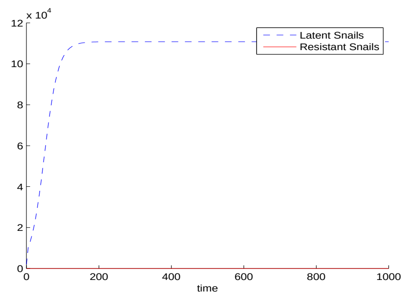

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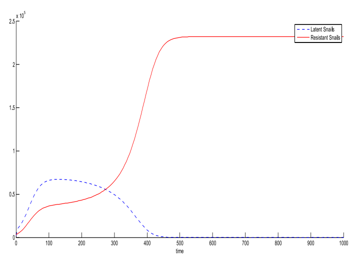
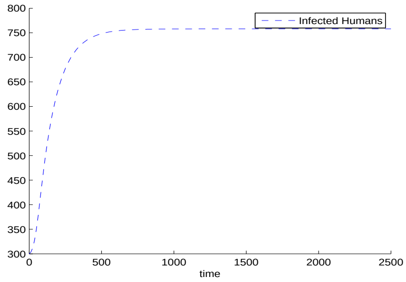


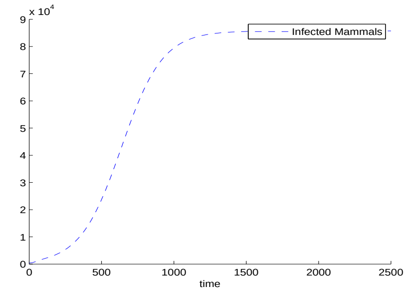
(b)

Figure 1: Trajectories of the model without resistant snails and parameters values in Table 2 when $\mathcal{R}_0 > 1$

Figure 2: Trajectories of the model with resistant snails conditions and parameters values in Table 2 when $\mathcal{R}_0 > 1$

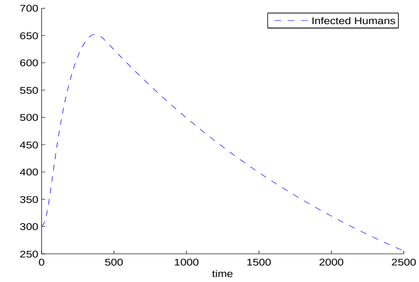


(a)

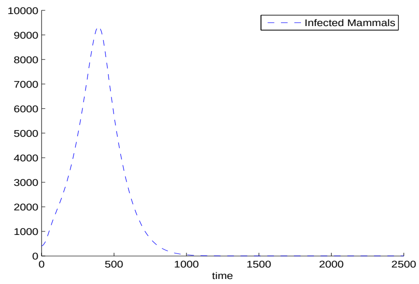


(b)

Figure 3: Trajectories of the model without resistant snails and parameters values in Table 2 when $\mathcal{R}_0 > 1$



(a)



(b)

Figure 4: Trajectories of the model with resistant snails conditions and parameters values in Table 2 when $\mathcal{R}_0 > 1$

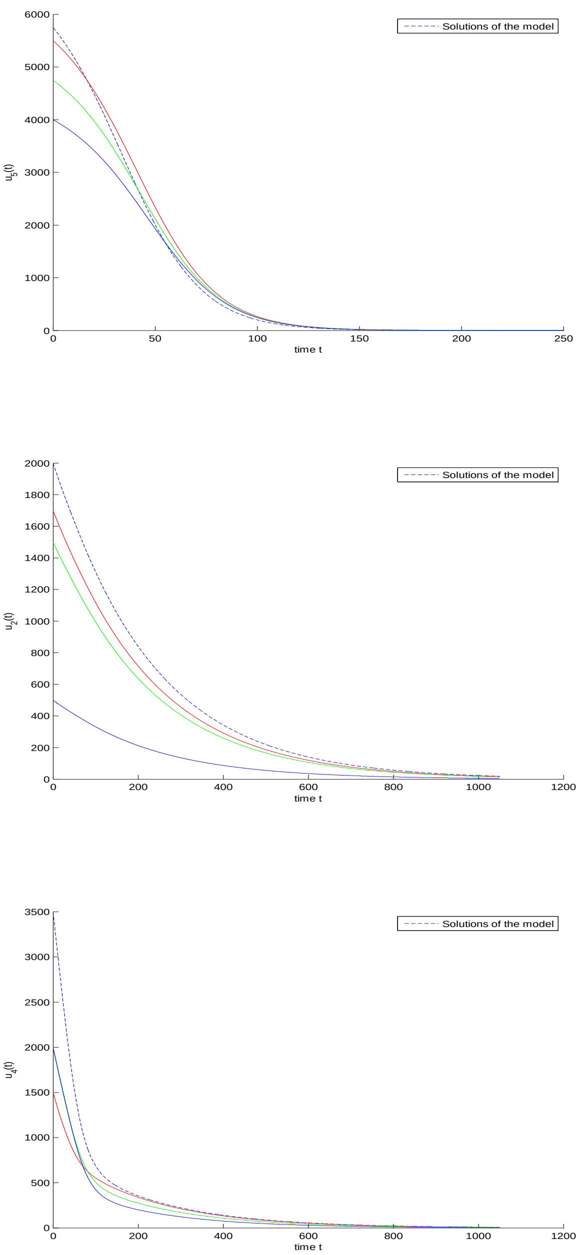


Figure 5: Trajectories of the infected populations with different initial conditions when $\mathcal{R}_0 < 1$

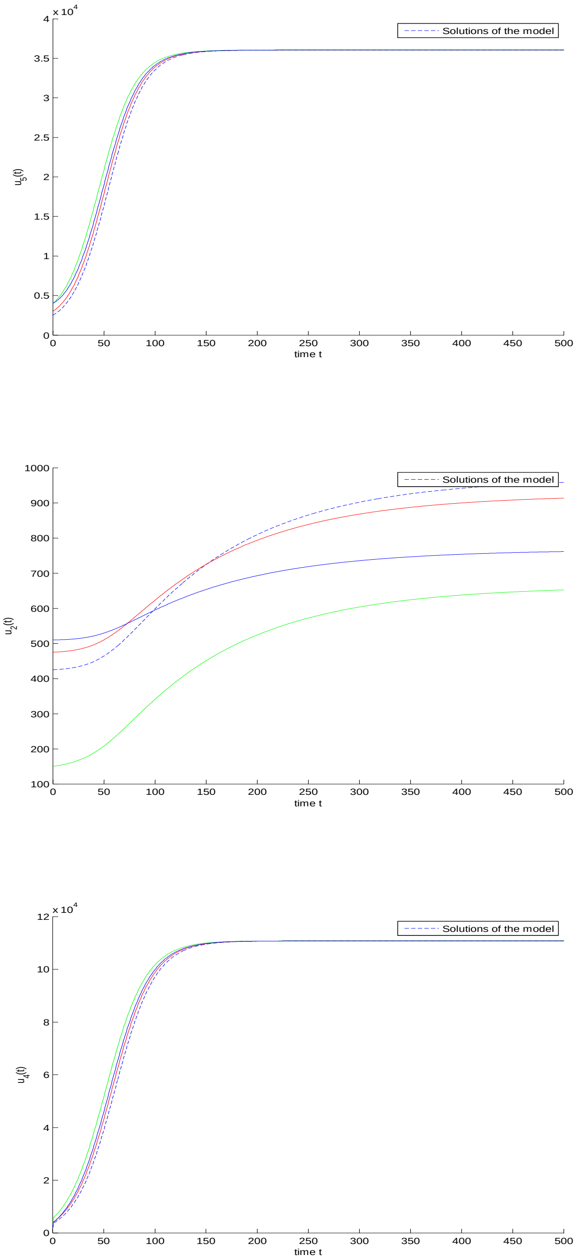


Figure 6: Trajectories of the infected populations with different initial conditions when $\mathcal{R}_0 > 1$

6 Summary and conclusions

In this paper, we have presented a stability analysis of a deterministic model for the transmission dynamics of a schistosomiasis infection.

Eight subpopulation sizes were modeled: human host susceptible and infected, snail intermediate host susceptible, latent, and shedding, resistant competitor snail, mammal host susceptible and infected. The snails competition is needed to study control to the infection by biological control.

Mathematical properties of the model are analyzed in terms of the stability of the possible steady states. The reproductive number $\mathcal{R}_0$ is calculated. We proved that the disease-free steady state $\mathbb{E}_0$ is globally asymptotically stable if $\mathcal{R}_0 < 1$ and it is unstable if $\mathcal{R}_0 > 1$. We proved also the existence and uniqueness of the endemic equilibrium $\mathbb{E}^*$ in the case where $\mathcal{R}_0 > 1$ as well as its local asymptotic stability.

In a more realistic situation, the speed of a river should affect the transmission dynamics of schistosomiasis by assuming flush-away of only free-swimming miracidia and cercariae. We are working on a model with spatial structure (modeling with a ODEs equations coupled with a shallow water system) that characterizes the density change of parasites following the flush-away of larvae, see [33].

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