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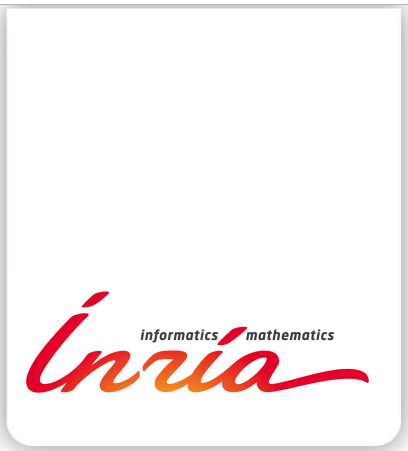

Global stability of a SI epidemic model with two infected stages and mass-action incidence

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Abstract: The work done in this paper consists in the establishment of the global stability of the model SI containing two classes of infected stages. The incidence used is non-linear and given by $(\beta_1 I_1 + \beta_2 I_2)\frac{S}{N}$.
Existence and uniqueness of the endemic equilibrium is established. A Lyapunov function is used to prove the stability of the disease free equilibrium, and the Poincarré-Bendixson theorem allows to prove the global asymptotic stability of the endemic equilibrium when it exists.

Key-words: Nonlinear dynamical systems, global asymptotic stability, Lyapunov methods, Poincarré-Bendixson theorem.

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Analyse de stabilité d'un modèle SI avec deux stades d'infection

Résumé : On considère un modèle epidémiologique du type SI avec deux stades d'infection et une surmortalité due à la maladie. La transmission est donnée par une loi d'action de masse. En utilisant les proportions, on prouve que l'état d'équilibre sans maladie est globalement asymptotiquement stable quand le taux de reproduction de base $\mathcal{R}_0 \leq 1$ et quand $\mathcal{R}_0 > 1$, on montre l'existence d'un unique état endemique qui est globalement asymptotiquement stable.

Mots-clés : Systèmes dynamiques non-linéaires, stabilité asymptotique globale, méthodes de Lyapunov, Théorème de Poincarré-Bendixson.

1 Introduction

Mathematical analysis became a major tool in the study of the evolution of epidemics. Indeed, more and more models were born for the study of some epidemics. In order to model an epidemic disease, the population is divided into various classes. In some cases the population is divided into two senior classes: the class of the susceptible, noted by S, and the class of the infected, noted I. Sometimes, the class of the infected can be split into several classes which allow to highlight the state of the disease. In our case, the infectious are divided into two categories, noted I_1 and I_2, with I_1 the first stage of the disease and I_2 the worsened case.
If β_1 and β_2 are the per capita contact rate of the infectious in respectively the compartments I_1 and I_2, there are $\beta_1 I_1 + \beta_2 I_2$ infective contacts. If any contact with a susceptible gives a new infected, then there is $(\beta_1 I_1 + \beta_2 I_2)P(S)$ new infected, where $P(S)$ is the probability for an infected to meet a susceptible. The quantity $(\beta_1 I_1 + \beta_2 I_2)P(S)$ is known in the literature as the mass action incidence rates. It is necessary to notice that most of the classical models of disease use a bilinear mass action incidence rates. For example, some of the most famous: the models of Kermack-Mckendrick (1927) and that of Lotka-Voltera (1926).
The goal of our study is to analyze the global stability of the SI_1I_2 model. The system considered represents the modeling of the HIV. For this model we suppose that an infected does $\frac{S}{N}$ contact of susceptible, then $P(S)$ is given by $\frac{S}{N}$. Also the incidence is given by $(\beta_1 I_1 + \beta_2 I_2)\frac{S}{N}$, where N represents the total population size: $N = S + I_1 + I_2$. The stability study of systems using this form of incidence is a very interesting subject to which some authors have already devoted some works. The work of C. Simon and J. Jacquez in [5] can be cited. Indeed, these authors deal with the subjects for n classes of infected, by a geometrical analysis, and they use a constant recruitment and also they suppose the duration in a class of infected and the rate of disease-induced are equal. However, in our study, the recruitment is variable and the duration in a class of infected is different of the rate of disease-induced. We can also cite the work of Melese and Gumel in [14], where for the proof of the endemic equilibrium stability, authors make a very stong assumption, which is very difficult to have. We cite also and specially the work of M. Li, J. Graef, L. Wang and J. Karsai in [4], which deals with a similar system, but the authors use one contact rate.
Also, to deal with the stability of the system, we derive a system of proportion. And, the determination of the endemic equilibrium is a serious question, its existence and uniqueness are proved in the section 3. After that, the global stability of the disease free equilibrium is given in section 4 by Lyapunov direct's method applicated to the initial system, and some considerations allow the passage in the stability of the proportion system, when its basic reproduction number is less than one. The local stability of the disease free equilibrium is given in the section 5. We prove in section 6 that the proportion system has no periodic orbit and the endemic equilibrium is globally asymptotically stable. For the stability of the endemic equilibrium, the Poincaré-Bendixson theory is used. To illustrate results of global asymptotically stability of the equilibrium points, we end this work by numerical simulations in section 7.

2 The model

The SI models are well known in the dynamic of population. In this section, we present the SI model used in this paper. We consider two stages of infected. The population of size N is divided into subclasses of individuals who are susceptible, infected into the first stage of the disease and infected into the second stage, with sizes denoted by S, I_1 and I_2.

The model study is given by the system

$$\begin{cases} \dot{S} = & bN - (\beta_1 I_1 + \beta_2 I_2)\dfrac{S}{N} - \mu S, \\ \dot{I_1} = & (\beta_1 I_1 + \beta_2 I_2)\dfrac{S}{N} - (\mu + \gamma) I_1, \\ \dot{I_2} = & \gamma I_1 - (\mu + d) I_2. \end{cases} \tag{1}$$

Where $N = S + I_1 + I_2$, b is the birth rate, β_1 and β_2 are respectively the per capita contact in the compartments I_1 and I_2. μ is the death rate, γ is the duration of an infected in the compartiment I_1 and d is the disease induced rate.
Doing the following notations: $s = \dfrac{S}{N}$, $i_1 = \dfrac{I_1}{N}$ and $i_2 = \dfrac{I_2}{N}$, we determinate the proportion model:

$$\dot{s} = \frac{\dot{S}}{N} - s\frac{\dot{N}}{N}.$$

We have:

$$\dot{N} = (b - \mu)N - dI_2 \text{ and}$$

$$\dot{s} = b(1 - s) - (\beta_1 i_1 + \beta_2 i_2)s + dsi_2.$$

$\dot{i_1}$ and $\dot{i_2}$ are also determined, and the proportion model is given by:

$$\begin{cases} \dot{s} = & b - bs - (\beta_1 i_1 + \beta_2 i_2)s + dsi_2, \\ \dot{i_1} = & (\beta_1 i_1 + \beta_2 i_2)s - (b + \gamma) i_1 + di_1 i_2, \\ \dot{i_2} = & \gamma i_1 - (b + d) i_2 + di_2^2. \end{cases} \tag{2}$$

We determine the basic reproduction number, which represents the number of secondary cases produced by one infective host in an entirely susceptible population.
We note by $\mathcal{F}_j(s, i_1, i_2)$ the rate of appearance of new infections in compartment j, and by $\mathcal{V}_j(s, i_1, i_2)$ the rate of transfer of individuals in and out the compartment j by all other means. The matrix $\mathcal{F}$ and $\mathcal{V}$ are given by:

$$\mathcal{F} = \begin{bmatrix} 0 \\ (\beta_1 i_1 + \beta_2 i_2)s \\ 0 \end{bmatrix}$$

and

$$\mathcal{V} = \begin{bmatrix} b - bs - (\beta_1 i_1 + \beta_2 i_2)s + dsi_2 \\ -(b + \gamma) i_1 + di_1 i_2 \\ \gamma i_1 - (b + d) i_2 + di_2^2 \end{bmatrix}.$$

The Jacobian matrices at the disease free equilibrium are:

$$\mathcal{DF} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_1 & \beta_2 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{DV} = \begin{pmatrix} -b & -\beta_1 & -\beta_2 + d \\ 0 & -(b + \gamma) & 0 \\ 0 & \gamma & -(b + d) \end{pmatrix}.$$

Let us put:

$$F = \begin{pmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{pmatrix} \text{ and } V = \begin{pmatrix} -(b+\gamma) & 0 \\ \gamma & -(b+d) \end{pmatrix}.$$

It is proved that the basic reproduction number is the spectral radius of the next generation matrix for the model, namely $-FV^{-1}$. The basic reproduction number of (2) is

$$\mathcal{R}_0 = \frac{\beta_1}{b+\gamma} + \frac{\beta_2\gamma}{(b+\gamma)(b+d)}.$$

3 The equilibrium points

The disease free equilibrium is given by DFE=(1,0,0). In the following, we show the existence and unicity of the endemic equilibrium for the system (2) assuming that $b \geq d$. Recall that b and d represent the birth and disease induced rate, respectively.

Lemma 3.1. *If $\mathcal{R}_0 > 1$, the endemic equilibrium exists and is unique.*

Proof. At the equilibrium, the last equation of the system (2) gives $i_1^* = \frac{b+d}{\gamma} i_2^* - \frac{d}{\gamma}(i_2^*)^2$. Replacing i_1^* by its expression in $(2)_2$, we have after simplification by γi_2^*:

$$(\beta_1(b+d) - \beta_1 d i_2^* + \beta_2\gamma)s^* - (b+\gamma)(b+d) + d(b+d)i_2^* + d(b+d)i_2^* - d^2(i_2^*)^2 = 0.$$

Also, we replace s by its expression: $s^* = 1 - i_1^* - i_2^* = 1 - \frac{b+d}{\gamma} i_2^* - \frac{d}{\gamma}(i_2^*)^2 - i_2^*$, then i_2^* is solution of the polynom:

$$P(i_2^*) = a_3(i_2^*)^3 + a_2(i_2^*)^2 + a_1 i_2^* + a_0,$$

where

$$\begin{aligned} a_3 &= -\beta_1 \frac{d^2}{\gamma}, \\ a_2 &= 2\beta_1 d \frac{b+d}{\gamma} + \beta_1 d + \beta_2 d - d^2, \\ a_1 &= -\beta_1 \frac{(b+d)^2}{\gamma} - \beta_1(b+d) - \beta_1 d - \beta_2(b+d) - \beta_2\gamma + d(b+\gamma) + d(b+d), \text{ and} \\ a_0 &= \beta_1(b+d) + \beta_2\gamma - (b+d)(b+\gamma). \end{aligned}$$

The observation shows that: $a_3 < 0$, $a_2 > 0$, $a_1 < 0$ and $a_0 > 0$. We transform the problem of existence of convenable roots of P into the problem of intersection of a polynom Q to the line $y = \mathcal{R}_0$, where Q is given by:

$$Q(i_2^*) = -\frac{a_3}{k}(i_2^*)^3 - \frac{a_2}{k}(i_2^*)^2 - \frac{a_1}{k} i_2^* + 1, \text{and } k = (b+d)(b+\gamma).$$

We have:

$$Q(0) = 1, \text{and } Q(1) = (1/k)\{\beta_1(b^2 + b\gamma + d\gamma) + \gamma(b(b-d+\gamma) + \beta_2(b+\gamma))\}.$$

Also

$$Q(1) - \mathcal{R}_0 = (b/k\gamma)\{\beta_1 b + \beta_2\gamma + b\gamma + \gamma^2 - d\gamma\},$$

which is positive if and only if

$$\beta_1 b + \beta_2\gamma + b\gamma + \gamma^2 > d\gamma. \tag{3}$$

Due to the assumption $b \geq d$ (3) holds. In addition, we have: $i_1 + i_2 < 1$. By replacing i_1 by its expression, i_2 must verify this inequation:

$$R(i_2) = -di_2^2 + (b + d + \gamma)i_2 - \gamma < 0.$$

The discriminant of the polynom R is $\Delta_R = (b + d + \gamma)^2 - 4d\gamma = b^2 + 2b(d + \gamma) + (d - \gamma)^2 > 0$. The roots of R are $r_{1,2} = (b + d + \gamma \mp \sqrt{\Delta_R})/2d$. We have: $r_1 < \gamma/2d < r_2$, and with the supposition $b \geq d$ we have $r_2 > 1$. i_2 must belong to $I = (]-\infty, r_1[\cup]r_2, +\infty[)\cap]0, 1[$. We can reduce the intervall I to $I =]0, \min(r_1, 1)[\subset]0, \min(\gamma/2d, 1)[$. And

$$Q(r_1) - \mathcal{R}_0 = (1/(2k))[b(b + d + \gamma + \sqrt{\Delta_R})] > 0.$$

$Q(0) = 1 < \mathcal{R}_0$ and $Q(r_1) > \mathcal{R}_0$, also $Q(1) > \mathcal{R}_0$ then the graph of Q intersects the line $y = \mathcal{R}_0$ at least one time in I.
Now let us show that there is exactly one intersection in I.
The derivate of Q gives:

$$Q'(i_2) = -(1/k)(3a_3(i_2^*)^2 + 2a_2 i_2^* + a_1).$$

Note that by Descartes rules of signs there is no negative root. The discriminant is $\Delta = a_2^2 - 3a_3a_1$, we have two cases:
- If $\Delta \leq 0$, Q' is positive on $\mathbb{R}$,
- If $\Delta > 0$, we have two roots x_1 and x_2, and $x_1 + x_2 = -(2a_2/3a_3)$. However:

$$\begin{aligned}
-2a_2/3a_3 &= \frac{4}{3}\frac{b+d}{d} + \frac{2}{3}\{\frac{\gamma}{d} + \frac{\beta_2\gamma}{\beta_1 d} - \frac{\gamma}{\beta_1}\} \\
&= \frac{2}{3}\frac{b+d}{d} + \frac{2}{3}\{\frac{b+d}{d} + \frac{\gamma}{d} + \frac{\beta_2\gamma}{\beta_1 d} - \frac{\gamma}{\beta_1}\} \\
&= \frac{2}{3}\frac{b+d}{d} + \frac{2}{3\beta_1 d}\{\beta_1(b+d) + \beta_2\gamma + \beta_1\gamma - d\gamma\} \\
&= \frac{2}{3}\frac{b+d}{d} + \frac{2}{3\beta_1 d}\{(b+d)(b+\gamma)\mathcal{R}_0 + \beta_1\gamma - d\gamma\}.
\end{aligned}$$

Thus

$$-2a_2/3a_3 = \frac{2}{3}\frac{b+d}{d} + \frac{2}{3\beta_1 d}\{b(b+\gamma)\mathcal{R}_0 + bd\mathcal{R}_0 + \beta_1\gamma + d\gamma(\mathcal{R}_0 - 1)\}.$$

We know that $b(b + \gamma)\mathcal{R}_0 = \beta_1 b + \dfrac{\beta_2 b\gamma}{b+d}$. Due to $b \geq d$, we have $-2a_2/3a_3 > 2$, thus there is at least one root of Q' larger than one.
All this observations show that Q intersects the line $y = \mathcal{R}_0$ only once. i_1^* is deduced by $i_1^* = \dfrac{b+d}{\gamma}i_2^* - \dfrac{d}{\gamma}(i_2^*)^2$, and $s^* = 1 - \dfrac{b+d+\gamma}{\gamma}i_2^* + \dfrac{d}{\gamma}(i_2^*)^2$. Then, the endemic equilibrium exists and is unique. □

4 Global Stability of the DFE

Theorem 4.1. *If $\mathcal{R}_0 < 1$, the DFE is globally asymptotically stable.*

Proof. We do the proof in two steps, in the first we suppose $\beta_2 \geq d$ and in the second $\beta_2 < d$. Into both steps, we use Lyapunov functions.

- If $\beta_2 \geq d$

We consider the following Lyapunov function:

$$V = i_1 + \frac{\beta_2}{b+d} i_2.$$

$$\dot{V} = (\beta_1 i_1 + \beta_2 i_2)s - (b+\gamma)i_1 + di_1 i_2 + \frac{\beta_2 \gamma}{b+d} i_1 - \beta_2 i_2 + \beta_2 \frac{d}{b+d} i_2^2.$$

Due to $\beta_1 i_1 s \leq \beta_1 i_1$ then

$$\begin{aligned} \dot{V} &\leq \beta_1 i_1 + \beta_2 i_2 s - (b+\gamma)i_1 + di_1 i_2 + \frac{\beta_2 \gamma}{b+d} i_1 - \beta_2 i_2 + \beta_2 \frac{d}{b+d} i_2^2 \\ &\leq (b+\gamma)[\frac{\beta_1}{b+\gamma} + \frac{\beta_2 \gamma}{(b+\gamma)(b+d)} - 1]i_1 + \beta_2 i_2(s-1) + di_1 i_2 + \beta_2 \frac{d}{b+d} i_2^2. \end{aligned}$$

We know that $\beta_2 i_2(s-1) = -\beta_2 i_2(i_1 + i_2)$ then

$$\begin{aligned} \dot{V} &\leq (b+\gamma)[\frac{\beta_1}{b+\gamma} + \frac{\beta_2 \gamma}{(b+\gamma)(b+d)} - 1]i_1 - \beta_2 i_2(i_1+i_2) + di_1 i_2 + \beta_2 \frac{d}{b+d} i_2^2 \\ &= (b+\gamma)(\mathcal{R}_0 - 1)i_1 + (d-\beta_2)i_1 i_2 + (\frac{d}{b+d} - 1)\beta_2 i_2^2. \end{aligned}$$

Thus

$$\dot{V} \leq (b+\gamma)(\mathcal{R}_0 - 1)i_1 + (d-\beta_2)i_1 i_2 - \frac{\beta_2 b}{b+d}\beta_2 i_2^2 < 0.$$

- If $\beta_2 < d$

We consider the following Lyapunov function:

$$V = s - \ln s + i_1 + (\frac{b+\gamma}{\gamma} - \frac{\beta_1}{\gamma})i_2.$$

We obtain

$$\begin{aligned} \dot{V} &= \dot{s}(1 - \frac{1}{s}) + \dot{i_1} + (\frac{b+\gamma}{\gamma} - \frac{\beta_1}{\gamma})\dot{i_2} \\ &= (b - bs)(1 - \frac{1}{s}) - (\beta_1 i_1 + \beta_2 i_2)s + (\beta_1 i_1 + \beta_2 i_2) + dsi_2 - di_2 \\ &\quad + (\beta_1 i_1 + \beta_2 i_2)s - (b+\gamma)i_1 + di_1 i_2 + (b+\gamma)i_1 - \frac{(b+\gamma)(b+d)}{\gamma} i_2 \\ &\quad + \frac{d(b+\gamma)}{\gamma} i_2^2 - \beta_1 i_1 + \beta_1 \frac{b+d}{\gamma} i_2 - \beta_1 \frac{d}{\gamma} i_2^2, \end{aligned}$$

we get :

$$\dot{V} = -\frac{b}{s}(1-s)^2 + \beta_2 i_2 + di_2(s + i_1 - 1) - \frac{(b+\gamma)(b+d)}{\gamma} i_2 + \frac{bd}{\gamma} i_2^2 + di_2^2 + \beta_1 \frac{b+d}{\gamma} i_2 - \beta_1 \frac{d}{\gamma} i_2^2.$$

We have the followings equalities:
$\frac{b}{s}(1-s)^2 = \frac{b}{s}(i_1+i_2)^2$ and $di_2(s+i_1-1) = -di_2^2$. Then $\dot{V}$ becomes:

$$\begin{aligned}\dot{V} &= -\frac{b}{s}(i_1+i_2)^2 + \beta_2 i_2 - \frac{(b+\gamma)(b+d)}{\gamma} i_2 + \frac{bd}{\gamma} i_2^2 + \beta_1 \frac{b+d}{\gamma} i_2 - \beta_1 \frac{d}{\gamma} i_2^2 \\ &= -\frac{b}{s}(i_1+i_2)^2 + \frac{(b+\gamma)(b+d)}{\gamma}(\mathcal{R}_0 - 1) i_2 + \frac{bd}{\gamma} i_2^2 - \beta_1 \frac{d}{\gamma} i_2^2 \\ &= -\frac{b}{s}(i_1+i_2)^2 - \frac{(b+\gamma)(b+d)}{\gamma}(1-\mathcal{R}_0) i_2 - \beta_1 \frac{d}{\gamma} i_2^2 + \frac{bd}{\gamma} i_2^2.\end{aligned}$$

As $1/s > 1$ and $i_2 > i_2^2$ then

$$\begin{aligned}\dot{V} &\leq -b(i_1+i_2)^2 - \frac{(b+\gamma)(b+d)}{\gamma}(1-\mathcal{R}_0) i_2^2 - \beta_1 \frac{d}{\gamma} i_2^2 + \frac{bd}{\gamma} i_2^2 \\ &= -bi_1^2 - 2bi_1 i_2 - bi_2^2 - \frac{(b+\gamma)(b+d)}{\gamma}(1-\mathcal{R}_0) i_2^2 - \beta_1 \frac{d}{\gamma} i_2^2 + \frac{bd}{\gamma} i_2^2 \\ &= -bi_1^2 - 2bi_1 i_2 - \frac{i_2^2}{\gamma}[b\gamma + (b+\gamma)(b+d)(1-\mathcal{R}_0) + \beta_1 d - bd].\end{aligned}$$

$\dot{V} < 0$ if $D = b\gamma + (b+\gamma)(b+d)(1-\mathcal{R}_0) + \beta_1 d - bd > 0$.
If we have $bd < b\gamma + \beta_1 d$ then $D > 0$.
If not

$$\begin{aligned}D &= b\gamma + (b+\gamma)(b+d) - \beta_1(b+d) - \beta_2\gamma + \beta_1 d - bd \\ &= b\gamma + b^2 + bd + b\gamma + d\gamma - \beta_1(b+d) - \beta_2\gamma + \beta_1 d - bd \\ &= b^2 + 2b\gamma + d\gamma - \beta_1 b - \beta_2\gamma \\ &= b(b-\beta 1) + \gamma(2b + d - \beta_2).\end{aligned}$$

The equality $bd \geq b\gamma + \beta_1 d$ gives $b > \beta_1$, then $D > 0$.
In the two cases $\dot{V} < 0$, then the disease free equilibrium is globally asymptotically stable. □

5 Local Stability of the endemic equilibrium

With the assumption $b \geq d$ we have the following result:

Theorem 5.1. *The endemic equilibrium is asymptotically stable.*

Proof. Due to $s + i_1 + i_2 = 1$, we eliminate s in the system (2). We get the following system:

$$\begin{cases} \dot{i_1} = & (\beta_1 i_1 + \beta_2 i_2)(1 - i_1 - i_2) - (b+\gamma) i_1 + d i_1 i_2, \\ \dot{i_2} = & \gamma i_1 - (b+d) i_2 + d i_2^2. \end{cases} \tag{4}$$

Let us consider the jacobian of (4) at the endemic equilibrium ($EE = (i_1^*, i_2^*)$), we have:

$$J(EE) = \begin{pmatrix} \beta_1 - 2\beta_1 i_1^* - \beta_1 i_2^* - \beta_2 i_2^* - (b+\gamma) + d i_2^* & \beta_2 - 2\beta_2 i_2^* - \beta_2 i_1^* - \beta_1 i_1^* + d i_1^* \\ \gamma & -b - d + 2d i_2^* \end{pmatrix}.$$

At the endemic equilibrium we have:

$$\beta_1 - 2\beta_1 i_1^* - \beta_1 i_2^* - \beta_2 i_2^* - (b+\gamma) + d i_2^* = -\beta_2 i_2^* \frac{1 - i_2^*}{i_1^*} - \beta_1 i_1^*.$$

The determinant is given by:

$$\begin{aligned} det(J(EE)) &= \beta_2(b+d)i_2^*\frac{1-i_2^*}{i_1^*} + \beta_1(b+d)i_1^* - 2\beta_2 d(i_2^*)^2\frac{1-i_2^*}{i_1^*} - 2\beta_1 d i_1^* i_2^* \\ &\quad -\beta_2\gamma + 2\beta_2\gamma i_2^* + \beta_2\gamma i_1^* + \beta_1\gamma i_1^* - d\gamma i_1^* \\ &= \beta_2(b+d)i_2^*\frac{1-i_2^*}{i_1^*} + [\beta_1(b+d)+\beta_2\gamma]i_1^* + 2\beta_2 i_2^*[\gamma - d i_2^*\frac{1-i_2^*}{i_1^*}] \\ &\quad -2\beta_1 d i_1^* i_2^* - \beta_2\gamma + \beta_1\gamma i_1^* - d\gamma i_1^*. \end{aligned}$$

In the first term of the determinant, we replace $(b+d)i_2^*$ by $\gamma i_1^* + d(i_2^*)^2$ and we get:

$$\begin{aligned} det(J(EE)) &= \beta_2[\gamma i_1^* + d(i_2^*)^2]\frac{1-i_2^*}{i_1^*} + (b+d)(b+\gamma)\mathcal{R}_0 i_1^* + 2\beta_2\frac{i_2^*}{i_1^*}[\gamma i_1^* - d i_2^* + d(i_2^*)^2] \\ &\quad -2\beta_1 d i_1^* i_2^* - \beta_2\gamma + \beta_1\gamma i_1^* - d\gamma i_1^*. \end{aligned}$$

We replace again $\gamma i_1^* - d i_2^* + d(i_2^*)^2$ by $b i_2^*$ and by developping the first term of the determinant, we get:

$$\begin{aligned} det(J(EE)) &= \beta_2\gamma - \beta_2\gamma i_2^* + \beta_2 d(i_2^*)^2\frac{1-i_2^*}{i_1^*} + (b+d)(b+\gamma)\mathcal{R}_0 i_1^* + 2\beta_2 b\frac{(i_2^*)^2}{i_1^*} \\ &\quad -2\beta_1 d i_1^* i_2^* - \beta_2\gamma + \beta_1\gamma i_1^* - d\gamma i_1^* \\ &= \beta_2 i_2^*[-\gamma + d i_2^*\frac{1-i_2^*}{i_1^*} + b\frac{i_2^*}{i_1^*}] + (b+d)(b+\gamma)\mathcal{R}_0 i_1^* + \beta_2 b\frac{(i_2^*)^2}{i_1^*} \\ &\quad -2\beta_1 d i_1^* i_2^* + \beta_1\gamma i_1^* - d\gamma i_1^* \\ &= \beta_2\frac{i_2^*}{i_1^*}[-\gamma i_1^* + (b+d)i_2^* - d(i_2^*)^2] + (b+d)(b+\gamma)\mathcal{R}_0 i_1^* + \beta_2 b\frac{(i_2^*)^2}{i_1^*} \\ &\quad -2\beta_1 d i_1^* i_2^* + \beta_1\gamma i_1^* - d\gamma i_1^*. \end{aligned}$$

Thus

$$det(J(EE)) = b(b+d+\gamma)\mathcal{R}_0 i_1^* + \beta_2 b\frac{i_2^*}{i_1^*} + d\gamma i_1^*(\mathcal{R}_0 - 1) + \beta_1 i_1^*(\gamma - 2d i_2^*).$$

The deteriminant is positive because $i_2^* \in]0, \frac{\gamma}{2d}[$. Futhermore the trace is negative, because it is given by:

$$trJ(EE) = -\beta_2 i_2^*\frac{1-i_2^*}{i_1^*} - \beta_1 i_1^* - b - d + 2d i_2^*.$$

Then the endemic equilibrium is asymptotically stable. □

6 Globlal stability of the endemic equilibrium

Because of $s+i_1+i_2 = 1$, we can reduce (2) to a planar system. We use the Poincarre-Bendixson theorem to determine the stablity for (2). To do it let us consider the following system:

$$\begin{cases} \dot{s} = & b(1-s) - (\beta_1 i_1 + \beta_2(1-i_1-s))s + ds(1-i_1-s), \\ \dot{i_1} = & (\beta_1 i_1 + \beta_2(1-i_1-s))s - (b+\gamma)i_1 + di_1(1-i_1-s). \end{cases} \tag{5}$$

One establishes by the Dulac-Bendixson criteria that there is no periodic orbit for (5).

Theorem 6.1. *System (5) has not periodic orbit.*

Proof. Consider the function $B(x,y) = \dfrac{1}{xy}$. We have:

$$B\dot{s}(s,i_1) = \frac{b}{si_1} - \frac{b}{i_1} - \beta_1 - \frac{\beta_2}{i_1} + \beta_2 + \frac{\beta_2 s}{i_1} + \frac{d}{i_1} - d - \frac{ds}{i_1},$$

thus

$$\frac{\partial B\dot{s}(s,i_1)}{\partial s} = -\frac{b}{s^2 i_1} + \frac{\beta_2}{i_1} - \frac{d}{i_1}.$$

And

$$B\dot{i_1}(s,i_1) = \beta_1 + \frac{\beta_2}{i_1} - \beta_2 - \frac{\beta_2 s}{i_1} - \frac{b+\gamma}{s} + \frac{d}{s} - \frac{di_1}{s} - d,$$

also

$$\frac{\partial B\dot{i_1}(s,i_1)}{\partial i_1} = -\frac{\beta_2}{i_1^2} + \frac{\beta_2 s}{i_1^2} - \frac{d}{s}.$$

It leads to

$$\frac{\partial B\dot{s}(s,i_1)}{\partial s} + \frac{\partial B\dot{i_1}(s,i_1)}{\partial i_1} = -\frac{b}{s^2 i_1} + \frac{\beta_2}{i_1^2}(s + i_1 - 1) - \frac{d}{i_1} - \frac{d}{s}$$

$$\frac{\partial B\dot{s}(s,i_1)}{\partial s} + \frac{\partial B\dot{i_1}(s,i_1)}{\partial i_1} < 0 \ \forall s, i_1 \in]0,1].$$

By Dulac-Bendixson criteria, we conclude that there is no closed orbitre for the system (5). □

Due to the Theorem 6.1 and the Poincaré-Bendixson theorem we have the following result:

Theorem 6.2. *If $\mathcal{R}_0 > 1$ the endemic equilibrium exists and is globally asymptotically stable in $\Omega - \Gamma$, where Γ is the stable manifold of the disease endemic equilibrium.*

Proof. If $\mathcal{R}_0 > 1$, by Taylor we linearize (5) at the disease free equilibrium and show that the DFE is unstable. Indeed, the linearized system at the point (1,0) has a negative determinant. Then the DFE is unstable, but the eigenvalues of the system (5) at the DFE are equal to:

$$\lambda_{1,2} = \beta_1 - (b+\gamma) - (b+d) \pm \sqrt{[\beta_1 - (b+\gamma) - (b+d)]^2 - 4(b+\gamma)(b+d)(1-\mathcal{R}_0)}.$$

One of the two eigenvalues is negative, which gives that the disease free equilibrium has one dimensional stable manifold. The $\omega - limit$ set of the system (5) is reduced to the endemic equilibrium point. Because of the local stability of the endemic equilibrium for $\mathcal{R}_0 > 1$, the endemic equilibrium is globally asymptotically stable. □

7 Numerical Simulations

In this part we aim to illustrate the theoretical results by numerical simulations. Thus, we draw first the curves of the infected for parameters verifying $\mathcal{R}_0$ less than 1, and we shall do the same for parameters verifying $\mathcal{R}_0$ upper to 1.
We take the value of the parameters as:

$$\beta_1 = 0.0001,\ \beta_2 = 0.0015,\ \gamma = 0.02,\ b = 0.4,\ d = 0.015,$$

which corresponds to $\mathcal{R}_0 = 0.0004102$. We have theoretically proved that, in this case, $\mathcal{R}_0 > 1$, the disease free equilibrium is globally asymptotically stable. The figure 1 gives the simulations of the infected i_1 and i_2.

In the figure 1, we see that the curves aim towards zero, which illustrates that the disease free equilibrium is globally asymptotically stable.
In the second case, we take the value of the parameters as:

$$\beta_1 = 0.3,\ \beta_2 = 0.8,\ \gamma = 0.5,\ b = 0.4,\ d = 0.1,$$

which corresponds to $\mathcal{R}_0 = 1.2222222$. In this case, We have proved that, the endemic equilibrium is globally asymptotically stable. The figure 2 gives the simulations of the infected i_1 and i_2.
For $\mathcal{R}_0 > 1$, the simulations show (see figure 2) that $i_1(t)$ and $i_2(t)$ converge to positive and finite limits, which are i_1^* and i_2^*, respectively.

8 Conclusion

The model SI is one of the most importing epidemiological models. This paper gives a qualitative analysis of the stability of the model by using a non-linear incidence. For this incidence, the system is returned to a proportional system, and the theory of Poincarré-Bendixson is used.
It would be interesting to generalize the work by doing the study of the sytem for n-stages of infected stages. It is also a very important thing to prove the stability of the system of proportion by using the Lyapunov function.

References

[1] R.M.Anderson and R.M.May. *Infection diseases of Humans, Oxford University Press, London 1991.*

[2] (MR0766910) H.W.Hethcote and J.A.Yorke. *Gonorrhea Transmission Dynamics and control, Springer Verlag, 1984.*

[3] S. A. Levin. *Descartes' Rule of Sign-How hard can it be?.*

[4] (MR1706068) M. Y. Li,J. R. Graef, L Wang and J Karsai. *Global dynamics of a SEIR with varying total population size, Math. Biosciences 160 (1999) 191-213.*

[5] (MR1154787) C. P. Simon and J. A. Jacquez. *Reproduction Numbers and the stability of equilibria of SI models for heteregeneous population,SIAM Vol 52,No. 2, pp 541-576,April 1992.*

[6] A. Korobeinikov and P. K. Maini. *Non-linear incidence and stability of infectious disease models,Math.Med.Biology (2005) 22,pp. 113-128.*

[7] (MR2130156) A. Korobeinikov. *Lyapunov functions and global properties for SEIR and SEIS epidemic models, Math.Med.Biology (2004) 21,pp.75-83.*

[8] (MR1814049) H. W. Hethcote. *The Mathematics of infectious diseases, SIAM Re.42 (2000), pp.599-653.*

[9] (MR0810991) H.W.Hethcote and and H.R.Thieme. *Stability of the endemic equilibrium in epidemic model with subpopulations, Math.Biosciences 75 (1985),pp.205-227.*

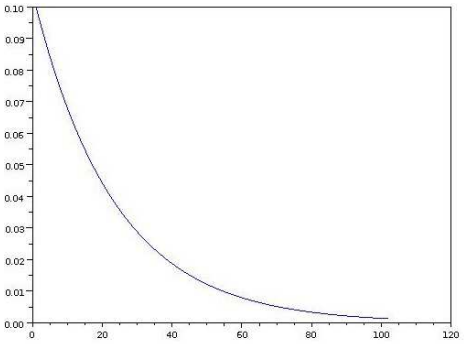
[10] (MR1943226) C.C.McCluskey. *A model of HIV/AIDS with staged progression and amelioration, Math.Biosciences 181 (2003), pp.1-16.*

[11] (MR0481301) J.P.LaSalle. *The stability of dynamics systems, CBMS-NSF Regional Conf. Ser. in Appl.Math. 25, SIAM, Philadelphia, 1976.*

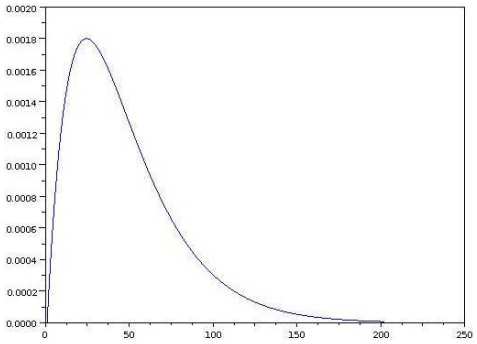
[12] (MR2773185) H.R.Thieme. *Global stability of the endemic equilibrium in infinite dimension:Lyapunov functions and positive operator,Journal of differentiel equation 2011,250(9),pp.3772-3801.*

[13] (MR2537442) Li-Ming Cai, Xue-Zhi Li and Ghosh. *Global satbility of staged-structured model with a non linear incidence,Applied Math. Comput. 214(2009), pp.73-82.*

[14] (MR2593646) D. Y. Melesse and A. B. Gumel. *Global asymptotic properties of an SEIRS model with multiple infectious stages, Math. Anal. and Appl. 366 (2010), pp.202-217.*

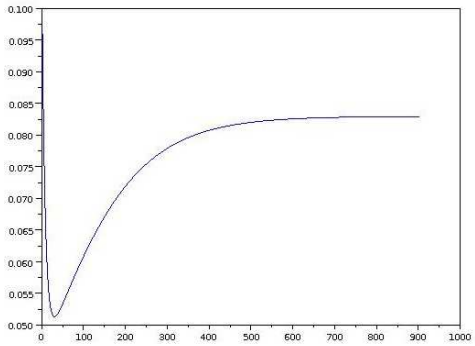


(a) curve of i_1

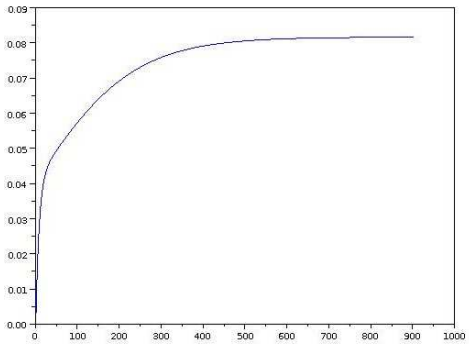


(b) curve of i_2

Figure 1: $\mathcal{R}_0 < 1$

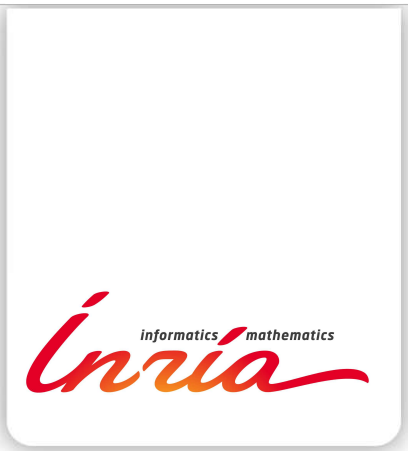


(a) curve of i_1



(b) curve of i_2

Figure 2: $\mathcal{R}_0 > 1$



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