# Closing geodesics in C1 topology

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# Closing geodesics in $C^1$ topology

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**Abstract**

Given a closed Riemannian manifold, we show how to close an orbit of the geodesic flow by a small perturbation of the metric in the $C^1$ topology.

## 1   Introduction

Given a dynamical system and a recurrent point $x$, the Closing Problem is concerned with the existence of a nearby dynamical system with a closed orbit through $x$. The statement of the Closing Problem for vector fields in the $C^r$ topology is as follows.

$C^r$**-Closing Problem for vector fields.** Let $M$ be a smooth compact manifold, $r \geq 0$ an integer, $X$ be a vector field of class $C^{\max\{1,r\}}$ on $M$, and $x$ be a recurrent point of $X$. Does there exist a $C^r$ vector field $Y$ arbitrary close to $X$ in the $C^r$ topology so that $x$ is a periodic point of $Y$ ?

The answer to the Closing Problem in the $C^0$ topology is trivially affirmative (see [8, §1 p. 958]). The Closing Problem in the $C^1$ topology is much more difficult. In the 60's, Charles Pugh [8] solved by a tour de force the Closing Problem in the $C^1$ topology.

**Theorem 1** ($C^1$-Closing Lemma for vector fields)**.** *Let $M$ be a smooth compact manifold. Suppose that some vector field $X$ has a nontrivial recurrent trajectory through $x \in M$ and suppose that $\mathcal{U}$ is a neighborhood of $X$ in the $C^1$ topology. Then there exists $Y \in \mathcal{U}$ such that $Y$ has a closed orbit through $x$.*

Since then, the Pugh $C^1$-Closing Lemma has been developed in several directions. Pugh himself [9] extended it to the case of nonwandering points for vector fields, diffeomorphisms and flows. Then, in the 80's, Charles Pugh and Clark Robinson [10] studied

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the Closing Problem for conservative dynamical systems such as the Hamiltonian systems.

**Theorem 2** (Closing Lemma for Hamiltonian vector fields in the $C^2$ topology)**.** *Let $(N, \omega)$ be a symplectic manifold of dimension $2n \geq 2$ and $H : N \to \mathbb{R}$ be a given Hamiltonian of class $C^2$. Let $X$ be the Hamiltonian vector field associated with $H$ and $\phi^H$ the Hamiltonian flow. Suppose that $X$ has a nontrivial recurrent trajectory through $x \in N$ and suppose that $\mathcal{U}$ is a neighborhood of $X$ in the $C^1$ topology. Then there exists $Y \in \mathcal{U}$ such that $Y$ is a Hamiltonian vector field and $Y$ has a closed orbit through $x$.*

Note that a perturbation of the Hamiltonian in the $C^2$ topology induces a perturbation of the associated Hamiltonian vector field in the $C^1$ topology only. We refer the reader to the exhaustive memoir [1] of Marie-Claude Arnaud for a detailed presentation and proofs of various versions of the closing lemma as well as comments on the Closing Problem in the $C^2$ topology (almost nothing is known in that case). Knowing the Pugh-Robinson Closing Lemma for Hamiltonian vector fields (they prove actually Theorem 2 for nonwandering points), it is natural to ask what happens for geodesics flows.

$C^r$**-Closing Problem for geodesic flows.** Let $(M, g)$ be a smooth compact manifold, $r \geq 0$ an integer, and $(x, v)$ be fixed in the unit tangent bundle $U^g M$. If $(x, v)$ is recurrent with respect to the geodesic flow of $g$, do there exist smooth metrics arbitrary close to $g$ in the $C^r$ topology so that the unit speed geodesic starting at $x$ with initial velocity $v$ is periodic ?

For that problem, nothing is known. Even the $C^0$-Closing Lemma for geodesic flows is unproved (see [10, §10 p. 309]). Let us explain why in few words. A geodesic flow may indeed be viewed as an Hamiltonian flow on the cotangent bundle $N = T^*M$ equipped with the canonical symplectic form. Given a smooth Riemannian metric $g$, we may define a smooth Hamiltonian $H : T^*M \to \mathbb{R}$ by (in local coordinates)

$$H(x, p) = \frac{1}{2} \left( \|p\|_x^* \right)^2 \qquad \forall (x, p) \in T^*M,$$

where $\| \cdot \|^*$ denotes the dual metric on $T^*M$. In that way, the Closing Problem for geodesic flows becomes a Closing Problem for Hamiltonian vector fields with a specific type of perturbation. As a matter of fact, a perturbation of a given metric in a small neighborhood $\Omega$ of some $x \in M$ induces a perturbation of the associated Hamiltonian in all the fibers $T_y^*M$ with $y \in \Omega$. However, in Theorem 2, one allows perturbations of the Hamiltonian in both variables. In other words, in contrast to Theorem 2, the perturbations allowed in the Closing Problem for geodesic flows cannot be localized in the phase space $T^*M$ but only in $M$.

2

The aim of the present paper is to prove a closing lemma for geodesic flows in the $C^1$ topology on the metric, that is in the $C^0$ topology for the associated dynamics. To state the result, let us make clear the notations which will be used throughout the paper.

Let $M$ be a smooth compact manifold without boundary of dimension $n \geq 2$ (throughout the paper, smooth always means of class $C^\infty$). For every Riemannian metric $g$ on $M$ of class $C^k$ with $k \geq 2$, denote by $|v|_x^g$ the norm of a vector $v \in T_x M$, by $U^g M$ the unit tangent bundle, and by $\phi_t^g$ the geodesic flow on $U^g M$. Moreover, for every $(x, v) \in U^g M$, denote by $\gamma_{x,v}^g : \mathbb{R} \to M$ the unit speed geodesic starting at $x$ with initial velocity $v$. The aim of the present paper is to show how to close an orbit of the geodesic flow with a small conformal perturbation of the metric in the $C^1$ topology. Pick a Riemannian distance on $TM$ and denote by $d_{TM}(\cdot, \cdot)$ the geodesic distance associated to it on $TM$. Note that since all Riemannian distances are Lipschitz equivalent on compact subsets, the choice of the metric on $TM$ is not important. Our main result is the following:

**Theorem 3.** *Let $g$ be a Riemannian metric on $M$ of class $C^k$ with $k \geq 3$ (resp. $k = \infty$), $(x, v) \in U^g M$ and $\epsilon > 0$ be fixed. Then there exist a metric $\tilde{g} = e^f g$ with $f : M \to \mathbb{R}$ of class $C^{k-1}$ (resp. $C^\infty$) satisfying $\|f\|_{C^1} < \epsilon$, and $(\tilde{x}, \tilde{v}) \in U^{\tilde{g}} M$ with $d_{TM}\big((x,v), (\tilde{x}, \tilde{v})\big) < \epsilon$, such that the geodesic $\gamma_{(\tilde{x},\tilde{v})}^{\tilde{g}}$ is periodic.*

The idea of our proof is first to observe that thanks to the Poincaré recurrence theorem, the geodesic flow is nonwandering on $U^g M$. Then we perform the construction of a connecting metric which preserves the transverse pieces of the geodesics crossing the box. This is done thanks to Lemma 5.

There is a constant $C > 0$ such that if $(x, v), (\tilde{x}, \tilde{v}) \in TM$ satisfy $(x, v) \in U^g M$ and $d_{TM}\big((x,v), (\tilde{x}, \tilde{v})\big) < \epsilon$ with $\epsilon > 0$ small enough, then there is a smooth diffeomorphism $\Phi : M \to M$ such that

$$\Phi(x) = \Phi(\tilde{x}), \quad d\Phi(x, v) = (\tilde{x}, \tilde{v}), \quad \text{and} \quad \|\Phi - Id\|_{C^2} < C\epsilon.$$

Therefore, the following result is an easy consequence of Theorem 3:

**Corollary 4.** *Let $g$ be a Riemannian metric on $M$ of class $C^k$ with $k \geq 3$ (resp. $k = \infty$), $(x, v) \in U^g M$ and $\epsilon > 0$ be fixed. Then there exists a metric $\tilde{g}$ of class $C^{k-1}$ (resp. $C^\infty$) with $\|\tilde{g} - g\|_{C^1} < \epsilon$ such that the geodesic $\gamma_{(x,v)}^{\tilde{g}}$ is periodic.*

The Pugh $C^1$-Closing Lemma has strong consequences on the structure of the flow of generic vector fields (see [9, §1 p. 1010]). It is worth noticing that our result is not striking enough to infer relevant properties for generic geodesic flows (for instance, the existence of an hyperbolic periodic orbit is not stable under $C^0$ perturbations on

3

the dynamics). Such interesting properties would follow from the following conjecture which is tempting in view of Pugh's Closing Lemma. (We refer the reader to [2] and references therein for known generic properties of geodesic flows in the $C^2$ topology.)

**Conjecture.** Let $(M, g)$ be a smooth compact manifold and $(x, v)$ be fixed in the unit tangent bundle $U^g M$. There exist smooth metrics arbitrary close to $g$ in the $C^2$ topology so that the unit speed geodesic starting at $x$ with initial velocity $v$ is periodic.

In 1951, Lyusternik and Fet proved that at least one closed geodesic exists on every smooth compact Riemannian manifold (see [6, 7]). Our Corollary 4 shows that any pair $(x, v) \in U^g M$ may indeed be seen as a pair $\big(\gamma_k(0), \dot{\gamma}_k(0)\big)$ for some sequence of closed orbits $\{\gamma_k\}$ with respect to smooth Riemannian metrics $\{g_k\}$ converging to $g$ in the $C^1$ topology.

The paper is organized as follows: In Section 2, we state and prove a result which is crucial to the proof of Theorem 3. This result, Proposition 5, shows how to connect two close geodesics while preserving a finite set of transverse geodesics, by a conformal perturbation of the initial metric with control on the support of the conformal factor and on its $C^1$ norm. Then, the proof of Theorem 3 is given in Section 3 and the proofs of some technical results are postponed to the appendix.

Notations: Throughout this paper, we denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product and by $|\cdot|$ the Euclidean norm in $\mathbb{R}^k$, and for any $x \in \mathbb{R}^k$ and any $r \geq 0$, we set $B^k(x, r) := \{y \in \mathbb{R}^k : |y - x| < r\}$.

## 2 Connecting geodesics with obstacles

### 2.1 Statement of the result

Let $n \geq 2$ be an integer, $\tau > 0$ be fixed, and let $\bar{g}$ be a complete Riemannian metric of class $C^k$ with $k \geq 3$ or $k = \infty$ on $\mathbb{R}^n$. Denote by $|v|_x^{\bar{g}}$ the norm with respect to $\bar{g}$ of a vector $(x, v) \in T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, denote by $\phi_t^{\bar{g}}$ the geodesic flow of $\bar{g}$ on $\mathbb{R}^n \times \mathbb{R}^n$ and for every $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$, denote by $\bar{\gamma}_{x,v}$ the geodesic with respect to $\bar{g}$ which starts at $x$ with velocity $v$. Assume that the curve $\bar{\gamma} : [0, \tau] \to \mathbb{R}^n$ is a geodesic with respect to $\bar{g}$ satisfying the following property ($e_1$ denotes the first vector in the canonical basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$):

4

(A) $|\dot{\bar{\gamma}}(t) - e_1| \leq 1/10$, for every $t \in [0, \tau]$.

Set
$$\begin{aligned}
\bar{x}^0 &= (\bar{x}_1^0, \ldots, \bar{x}_n^0) := \bar{\gamma}(0), &\quad \bar{v}^0 &= (\bar{v}_1^0, \ldots, \bar{v}_n^0) := \dot{\bar{\gamma}}(0), \\
\bar{x}^\tau &= (\bar{x}_1^\tau, \ldots, \bar{x}_n^\tau) := \bar{\gamma}(\tau), &\quad \bar{v}^\tau &= (\bar{v}_1^\tau, \ldots, \bar{v}_n^\tau) := \dot{\bar{\gamma}}(\tau).
\end{aligned}$$

Our aim is to show that, given $(x, v), (y, w) \in \mathbb{R}^n \times \mathbb{R}^n$ with $|v|_x^{\bar{g}} = |w|_y^{\bar{g}} = 1$ sufficiently close to $(\bar{x}^0, \bar{v}^0)$, there exists a Riemannian metric $\tilde{g}$ of class $C^{k-1}$ which is conformal to $\bar{g}$ and whose support and $C^1$-norm are controlled, which connects $(x, v)$ to $(\bar{\gamma}_{y,w}(\tau), \dot{\bar{\gamma}}_{y,w}(\tau)) = \phi_\tau^{\bar{g}}(y, w)$ and which preserves finitely many transverse geodesics. Set
$$\mathcal{R}(\rho) := \left\{ (t, z) \,|\, t \in \left[\bar{x}_1^0, \bar{x}_1^\tau\right], \, z \in B^{n-1}(0, \rho) \right\} \qquad \forall \rho > 0.$$

Let us state our result.

**Proposition 5.** *Let $\tau > 0$ and $\bar{\gamma} : [0, \tau] \to \mathbb{R}^n$ satisfying assumption (A) be fixed. Let $\rho > 0$ be such that $\bar{\gamma}([0, \tau]) \subset \mathcal{R}(\rho/2)$ be fixed. There are $\bar{\delta} = \bar{\delta}(\tau, \rho) \in (0, \tau/3)$ and $C = C(\tau, \rho) > 0$ such that the following property is satisfied: For every $(x, v), (y, w) \in U^{\bar{g}}\mathbb{R}^n$ satisfying*
$$\left| x - \bar{x}^0 \right|, \, \left| y - \bar{x}^0 \right|, \, \left| v - \bar{v}^0 \right|, \, \left| w - \bar{v}^0 \right| < \bar{\delta}, \tag{2.1}$$

*and for every finite set of unit speed geodesics*
$$\bar{c}_1 : I_1 = [a_1, b_1] \longrightarrow \mathbb{R}^n, \quad \cdots, \quad \bar{c}_L : I_L = [a_L, b_L] \longrightarrow \mathbb{R}^n$$

*satisfying*
$$\bar{c}_l(a_l), \bar{c}_l(b_l) \notin \mathcal{R}(\rho) \qquad \forall l \in \{1, \ldots, L\}, \tag{2.2}$$

$$(\bar{c}_l(s), \dot{\bar{c}}_l(s)) \neq \phi_t^{\bar{g}}(x, v), \, \phi_t^{\bar{g}}(y, w) \qquad \forall l \in \{1, \ldots, L\}, \, \forall s \in I_l, \, \forall t \in [0, \tau], \tag{2.3}$$

$$and \quad \left| \dot{\bar{c}}_l(s) - \dot{\bar{c}}_l(s') \right| < 1/8 \qquad \forall l \in \{1, \ldots, L\}, \, \forall s, s' \in I_l, \tag{2.4}$$

*there are $\tilde{\tau} > 0$ and a Riemannian metric $\tilde{g} = e^f \bar{g}$ on $\mathbb{R}^n$ with $f : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$ (or $f$ of class $C^\infty$ if $\bar{g}$ is itself $C^\infty$) satisfying the following properties:*

*(i) $\mathrm{Supp}\,(f) \subset \mathcal{R}(\rho)$;*

*(ii) $\|f\|_{C^1} < C \, |(x, v) - (y, w)|$;*

*(iii) $|\tilde{\tau} - \tau| < C \, |(x, v) - (y, w)|$;*

*(iv) $\phi_{\tilde{\tau}}^{\tilde{g}}(x, v) = \phi_\tau^{\bar{g}}(y, w)$;*

5

*(v) for every $l \in \{1, \ldots, L\}$ $\bar{c}_l$ is, up to reparametrization, a geodesic with respect to $\tilde{g}$.*

The proof of Proposition 5 occupies Sections 2.2 to 2.4. First, in Section 2.2, we restrict our attention to assertions (i)-(iv) by showing how to connect two unit speed geodesics in a constructive way (compare [4, Proposition 3.1] and [5, Proposition 2.1]). Then, in Section 2.3, we provide a lemma (Lemma 7) which explains how a conformal factor may preserve geodesic curves. Finally, in Section 2.4, we invoke transversality arguments together with Lemma 7 to conclude the proof of Proposition 5.

## 2.2 Connecting geodesics without obstacles

Let us first forget about assertion (v). For every $x \in \mathbb{R}^n$, denote by $\bar{G}(x)$ the $n \times n$ matrix whose coefficients are the $(\bar{g}_x)_{i,j}$, set $\bar{Q} := \bar{G}^{-1}$ and define the Hamiltonian $\bar{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of class $C^k$ by

$$\bar{H}(x,p) := \frac{1}{2} \langle p, \bar{Q}(x)p \rangle \qquad \forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}^n.$$

There is a one-to-one correspondence between the geodesics associated with $\bar{g}$ and the Hamiltonian trajectories of $\bar{H}$. For every $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$, the trajectory $(x(\cdot), p(\cdot)) : [0, \infty) \to \mathbb{R}^n \times \mathbb{R}^n$ defined by

$$\big(x(t), p(t)\big) := \Big(\bar{\gamma}_{x,v}(t), \bar{G}\big(\bar{\gamma}_{x,v}(t)\big) \dot{\bar{\gamma}}_{x,v}(t)\Big) \qquad \forall t \geq 0,$$

is the solution of the Hamiltonian system

$$\begin{cases} \dot{x}(t) &= \frac{\partial \bar{H}}{\partial p}\big(x(t), p(t)\big) \\ \dot{p}(t) &= -\frac{\partial \bar{H}}{\partial x}\big(x(t), p(t)\big) \end{cases} \tag{2.5}$$

such that $\big(x(0), p(0)\big) = \big(x, \bar{G}(x)\, v\big)$. Let $(x,v), (y,w) \in U^{\bar{g}}\mathbb{R}^n$ be fixed, set

$$x^0 := x, \quad p^0 := \bar{G}(x)\, v, \quad x^\tau := \bar{\gamma}_{y,w}(\tau), \quad v^\tau := \dot{\bar{\gamma}}_{y,w}(\tau), \quad p^\tau := \bar{G}(x^\tau)\, v^\tau. \tag{2.6}$$

Our aim is first to find a metric $\tilde{g}$ whose associated matrices $\tilde{G}, \tilde{Q}$ have the form

$$\tilde{G}(x)^{-1} = \tilde{Q}(x) = e^{-f(x)}\bar{Q}(x) \qquad \forall x \in \mathbb{R}^n,$$

in such a way that the trajectory $\big(x(\cdot), p(\cdot)\big) : [0, \infty) \to \mathbb{R}^n \times \mathbb{R}^n$ of the Hamiltonian system

$$\begin{cases} \dot{x}(t) &= \frac{\partial \tilde{H}}{\partial p}\big(x(t), p(t)\big) \\ \dot{p}(t) &= -\frac{\partial \tilde{H}}{\partial x}\big(x(t), p(t)\big) \end{cases} \tag{2.7}$$

6

associated with the new Hamiltonian $\tilde{H} = H_f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\tilde{H}(x,p) = H_f(x,p) := \frac{1}{2}\left\langle p, \tilde{Q}(x)p \right\rangle = \frac{e^{-f(x)}}{2}\left\langle p, \bar{Q}(x)p \right\rangle \quad \forall x \in \mathbb{R}^n, \forall p \in \mathbb{R}^n, \quad (2.8)$$

and starting at $\left(x^0, p^0\right)$ satisfies $(x(\tau), p(\tau)) = \left(x^\tau, p^\tau\right)$. Note that for any $x, p \in \mathbb{R}^n$,

$$\frac{\partial H_f}{\partial p}(x,p) = \tilde{Q}(x)p = e^{-f(x)}\bar{Q}(x)\,p \qquad (2.9)$$

and for every $i = 1, \ldots, n$,

$$\frac{\partial H_f}{\partial x_i}(x,p) = \frac{1}{2}\left\langle p, \frac{\partial \tilde{Q}}{\partial x_i}(x)\,p \right\rangle = \frac{e^{-f(x)}}{2}\left\langle p, \frac{\partial \bar{Q}}{\partial x_i}(x)\,p \right\rangle - \frac{1}{2}\left\langle p, \tilde{Q}(x)\,p \right\rangle \frac{\partial f}{\partial x_i}(x). \ (2.10)$$

Let us fix a smooth function $\psi : [0, \tau] \to [0, 1]$ satisfying

$$\psi(t) = 0 \quad \forall t \in [0, \tau/3] \quad \text{and} \quad \psi(t) = 1 \quad \forall t \in [2\tau/3, \tau].$$

Given $(x, v), (y, w) \in U^{\bar{g}}\mathbb{R}^n$, we define a trajectory

$$\mathcal{X}\bigl(\cdot; (x, v), (y, w)\bigr) : [0, \tau] \longrightarrow \mathbb{R}^n$$

of class $C^{k+1}$ by

$$\mathcal{X}\bigl(t; (x, v), (y, w)\bigr) := \bigl(1 - \psi(t)\bigr)\bar{\gamma}_{x,v}(t) + \psi(t)\,\bar{\gamma}_{y,w}(t) \qquad \forall\, t \in [0, \tau]. \qquad (2.11)$$

We note that the mapping $\bigl(t, (x, v), (y, w)\bigr) \mapsto \mathcal{X}\bigl(t; (x, v), (y, w)\bigr)$ is $C^{k+1}$ in the $t$ variable but only $C^{k-1}$ in the variables $x, v, y, w$. Let $\alpha\bigl(\cdot; (x, v), (y, w)\bigr) : [0, \tau] \to [0, +\infty)$ be the function defined as

$$\alpha\bigl(t; (x, v), (y, w)\bigr)$$
$$:= \int_0^t \sqrt{\left\langle \dot{\mathcal{X}}\bigl(s; (x, v), (y, w)\bigr), \bar{G}\left(\mathcal{X}\bigl(s; (x, v), (y, w)\bigr)\right) \dot{\mathcal{X}}\bigl(s; (x, v), (y, w)\bigr) \right\rangle}\,ds,$$

for every $t \in [0, \tau]$. We observe that $\alpha\bigl(\cdot; (x, v), (y, w)\bigr)$ is strictly increasing, of class $C^{k+1}$ in the $t$ variable, and of class $C^{k-1}$ in the variables $x, v, y, w$. Let

$$\theta\bigl(\cdot; (x, v), (y, w)\bigr) : \bigl[0, \tilde{\tau} = \tilde{\tau}\bigl((x, v), (y, w)\bigr) := \alpha\bigl(\tau; (x, v), (y, w)\bigr)\bigr] \longrightarrow [0, \tau]$$

denote its inverse, which is of class $C^{k+1}$ in $t$, $C^{k-1}$ in $x, v, y, w$, and satisfies (we set $\theta(\cdot) = \theta\left((\cdot; (x, v), (y, w)\right)$ and $\mathcal{X}(\cdot) = \mathcal{X}\left((\cdot; (x, v), (y, w)\right)$)

$$\dot{\theta}(s) = \frac{1}{\sqrt{\left\langle \dot{\mathcal{X}}\bigl(\theta(s)\bigr), \bar{G}\left(\mathcal{X}\bigl(\theta(s)\bigr)\right) \dot{\mathcal{X}}\bigl(\theta(s)\bigr) \right\rangle}} \qquad \forall s \in [0, \tilde{\tau}].$$

7

Then, we define a new trajectory

$$\tilde{x}(\cdot) = \tilde{x}\big(\cdot; (x, v), (y, w)\big) : \big[0, \tilde{\tau}\big((x, v), (y, w)\big)\big] \longrightarrow \mathbb{R}^n$$

of class $C^{k+1}$ by

$$\tilde{x}\big(t; (x, v), (y, w)\big) := \mathcal{X}\left(\theta(t)\right) \qquad \forall t \in [0, \tilde{\tau}].$$

By construction,

$$\begin{cases} \tilde{x}(t) = \mathcal{X}\big(t; (x, v), (y, w)\big) = \bar{\gamma}_{x,v}(t) & \forall t \in [0, \tau/3], \\ \tilde{x}(t) = \mathcal{X}\big(t; (x, v), (y, w)\big) = \bar{\gamma}_{y,w}(t) & \forall t \in [\tilde{\tau} - \tau/3, \tilde{\tau}], \end{cases} \qquad (2.12)$$

and

$$\big\langle \dot{\tilde{x}}(t), \bar{G}\left(\tilde{x}(t)\right) \dot{\tilde{x}}(t) \big\rangle = 1 \qquad \forall t \in [0, \tilde{\tau}].$$

This means that the adjoint trajectory

$$\tilde{p}(\cdot) = \tilde{p}\big(\cdot; (x, v), (y, w)\big) : \big[0, \tilde{\tau}\big((x, v), (y, w)\big)\big] \longrightarrow \mathbb{R}^n$$

defined by

$$\tilde{p}\big(t; (x, v), (y, w)\big) := \bar{G}\left(\tilde{x}(t)\right) \dot{\tilde{x}}(t) \qquad \forall t \in [0, \tilde{\tau}], \qquad (2.13)$$

satisfies

$$\dot{\tilde{x}}(t) = \frac{\partial \bar{H}}{\partial p}\left(\tilde{x}(t), \tilde{p}(t)\right) \qquad \forall t \in [0, \tilde{\tau}] \qquad (2.14)$$

and

$$\bar{H}\left(\tilde{x}(t), \tilde{p}(t)\right) = \frac{1}{2} \qquad \forall \in [0, \tilde{\tau}]. \qquad (2.15)$$

We now define the function

$$\tilde{u}(\cdot) = \Big( \tilde{u}_1\big(\cdot; (x, v), (y, w)\big), \ldots, \tilde{u}_n\big(\cdot; (x, v), (y, w)\big) \Big) : [0, \tilde{\tau}] \longrightarrow \mathbb{R}^n$$

by

$$\tilde{u}_i(t) := 2\dot{\tilde{p}}_i(t) + \Big\langle \tilde{p}(t), \frac{\partial \bar{Q}}{\partial x_i}\left(\tilde{x}(t)\right) \tilde{p}(t) \Big\rangle \qquad \forall i = 1, \ldots, n, \, \forall t \in [0, \tilde{\tau}]. \qquad (2.16)$$

By construction, the function $\tilde{p}$ is of class $C^k$ in the $t$ variable, $\tilde{u}$ is $C^{k-1}$ in the $t$ variable, and all the functions $\tilde{\tau}, \tilde{p}, \tilde{u}$ are $C^{k-1}$ in the $x, y, v, w$ variables. Furthermore, it follows that

$$\dot{\tilde{p}}(t) = -\frac{\partial \bar{H}}{\partial x}\left(\tilde{x}(t), \tilde{p}(t)\right) + \frac{1}{2}\tilde{u}(t) \qquad \forall t \in [0, \tilde{\tau}],$$

$$\begin{cases} \big(\tilde{x}(0), \tilde{p}(0)\big) = \big(x^0, p^0\big), \\ \big(\tilde{x}(\tilde{\tau}), \tilde{p}(\tilde{\tau})\big) = (x^\tau, p^\tau), \end{cases}$$

(using the notations (2.6) and remembering (2.12)), and

$$\tilde{u}\big(t; (x,v), (y,w)\big) = 0_n \qquad \forall\, t \in [0, \tau/3] \cup [\tilde{\tau} - \tau/3, \tilde{\tau}] \tag{2.17}$$

(by (2.12), (2.13), and (2.16)). Since $\bar{H}$ is of class $C^k$ with $k \geq 3$, the mapping

$$\mathcal{Q} : \big((x,v), (y,w), s\big) \in (\mathbb{R}^n \times \mathbb{R}^n) \times (\mathbb{R}^n \times \mathbb{R}^n) \times [0,1]$$
$$\longmapsto \big(\tilde{\tau}\big((x,v),(y,w)\big), \tilde{u}\big(s\tilde{\tau}\big((x,v),(y,w)\big); (x,v),(y,w)\big)\big)$$

is of class at least $C^1$. Therefore, since for all $(x,v) \in U^{\bar{g}}\mathbb{R}^n$ with $\big|x - \bar{x}^0\big| \leq 1$,

$$\mathcal{Q}\big((x,v),(x,v),s\big) = (\tau, 0) \qquad \forall\, s \in [0,1],$$

there exists a constant $K > 0$ such that, for every pair $(x,v), (y,w) \in U^{\bar{g}}\mathbb{R}^n$ with $\big|x - \bar{x}^0\big|, \big|y - \bar{x}^0\big| \leq 1$,

$$\begin{aligned} \big|\tilde{\tau}\big((x,v),(y,w)\big) - \tau\big| &\leq \big|\mathcal{Q}\big((x,v),(y,w),0\big) - \mathcal{Q}\big((x,v),(x,v),0\big)\big| \\ &\leq K\,\big|(x,v) - (y,w)\big|, \end{aligned} \tag{2.18}$$

and analogously

$$\big\|\tilde{u}\big(\cdot; (x,v),(y,w)\big)\big\|_{C^0} \leq K\,\big|(x,v) - (y,w)\big|. \tag{2.19}$$

Furthermore, we notice that differentiating (2.15) yields

$$\left\langle \frac{\partial \bar{H}}{\partial x}\big(\tilde{x}(t), \tilde{p}(t)\big), \dot{\tilde{x}}(t) \right\rangle + \left\langle \frac{\partial \bar{H}}{\partial p}\big(\tilde{x}(t), \tilde{p}(t)\big), \dot{\tilde{p}}(t) \right\rangle = 0 \qquad \forall\, t \in [0, \tilde{\tau}],$$

which together with (2.14) and (2.16) gives

$$\big\langle \tilde{u}(t), \dot{\tilde{x}}(t) \big\rangle = 0 \qquad \forall\, t \in [0, \tilde{\tau}]. \tag{2.20}$$

In conclusion, for every $(x,v), (y,w) \in U^{\bar{g}}\mathbb{R}^n$ satisfying $\big|x - \bar{x}^0\big|, \big|y - \bar{x}^0\big| \leq 1$, the function

$$t \in \big[0, \tilde{\tau}\big((x,v),(y,w)\big)\big] \longmapsto \big(\tilde{x}\big(t; (x,v),(y,w)\big), \tilde{p}\big(t; (x,v),(y,w)\big), \tilde{u}\big(t; (x,v),(y,w)\big)\big)$$

satisfies for every $t \in \big[0, \tilde{\tau}\big((x,v),(y,w)\big)\big]$ and every $i = 1, \ldots, n$,

$$\begin{cases} \dot{\tilde{x}}(t) &= \bar{Q}\big(\tilde{x}(t)\big)\tilde{p}(t) \\ \dot{\tilde{p}}_i(t) &= -\frac{1}{2}\left\langle \tilde{p}(t), \frac{\partial \bar{Q}}{\partial x_i}\big(\tilde{x}(t)\big)\tilde{p}(t) \right\rangle - \frac{1}{2}\left\langle \tilde{p}(t), \bar{Q}\big(\tilde{x}(t)\big)\tilde{p}(t) \right\rangle \tilde{u}_i(t), \end{cases} \tag{2.21}$$

9

and properties (2.18)-(2.20) hold. In particular, taking the constant $K > 0$ larger if necessary, (2.18)-(2.19) and (2.21) together with Gronwall's Lemma imply that

$$\left|\dot{\tilde{x}}(t) - e_1\right| \leq K\left|(x,v) - (y,w)\right|. \qquad \forall t \in [0, \tilde{\tau}]. \tag{2.22}$$

The proof of the following lemma (taken from [4]) is postponed to Section A.1.

**Lemma 6.** *Let* $T, \beta, \mu \in (0,1)$ *with* $3\mu \leq \beta < T$*, and let* $y(\cdot), w(\cdot) : [0,T] \to \mathbb{R}^n$ *be two functions of class respectively at least* $C^k$ *and* $C^{k-1}$ *satisfying*

$$|\dot{y}(t) - e_1| \leq 1/5 \qquad \forall t \in [0,T], \tag{2.23}$$

$$w(t) = 0_n \qquad \forall t \in [0, \beta] \cup [T - \beta, T], \tag{2.24}$$

$$\langle \dot{y}(t), w(t) \rangle = 0 \qquad \forall t \in [0, T]. \tag{2.25}$$

*Then, there exist a constant* $K$ *depending only on the dimension and* $T$*, and a function* $W : \mathbb{R}^n \to \mathbb{R}$ *of class* $C^k$ *such that the following properties hold:*

*(i)* $\mathrm{Supp}(W) \subset \left\{ y(t) + (0, z) \,|\, t \in [\beta/2, T - \beta/2], z \in B^{n-1}(0, \mu) \right\};$

*(ii)* $\|W\|_{C^1} \leq \frac{K}{\mu} \|w(\cdot)\|_{C^0};$

*(iii)* $\nabla W(y(t)) = w(t)$ *for every* $t \in [0, T];$

*(iv)* $W(y(t)) = 0$ *for every* $t \in [0, T].$

Therefore taking $\bar{\delta} \in (0, \tau/3)$ in (2.1) small enough, applying the above Lemma with $y(\cdot) = \tilde{x}(\cdot), w(\cdot) = \tilde{u}(\cdot), T = \tilde{\tau}, \beta = \tau/3$, and $\mu > 0$ small enough, and remembering assumption (A), that $\bar{\gamma}([0, \tau]) \subset \mathcal{R}(\rho/2)$, (2.17), (2.19)-(2.20), and (2.22) yields a universal constant $C = C(\tau, \rho) > 0$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ of class $C^k$ satisfying the following properties:

(a) $\mathrm{Supp}\,(f) \subset \mathcal{R}(\rho);$

(b) $\|f\|_{C^1} < C\,|(x,v) - (y,w)|;$

(c) for every $t \in [0, \tilde{\tau}]$, $\nabla f(\tilde{x}(t)) = \tilde{u}(t);$

(d) for every $t \in [0, \tilde{\tau}]$, $f(\tilde{x}(t)) = 0.$

Then, there is a one-to-one correspondence between the geodesics of $\tilde{g} := e^f \bar{g}$ and the solutions of the Hamiltonian system (2.7) associated with $\tilde{H} = H_f$ given by (2.8). For every $t \in [0, \tilde{\tau}]$, by construction of $f$, the function $\big(\tilde{x}(\cdot), \tilde{p}(\cdot)\big) : [0, \tilde{\tau}] \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ satisfies

$$\dot{\tilde{x}}(t) = e^{-f\big(\tilde{x}(t)\big)} \bar{Q}\big(\tilde{x}(t)\big) \tilde{p}(t)$$

and for every $i = 1, \ldots, n$,

$$\dot{\tilde{p}}_i(t) = -\frac{e^{-f\big(\tilde{x}(t)\big)}}{2} \left\langle \tilde{p}(t), \frac{\partial \bar{Q}}{\partial x_i}\big(\tilde{x}(t)\big) \tilde{p}(t) \right\rangle - \frac{e^{-f\big(\tilde{x}(t)\big)}}{2} \left\langle \tilde{p}(t), \bar{Q}\big(\tilde{x}(t)\big) \tilde{p}(t) \right\rangle \frac{\partial f}{\partial x_i}\big(\tilde{x}(t)\big).$$

This means that $\tilde{x}(\cdot)$ is a geodesic on $[0, \tilde{\tau}]$ with respect to $\tilde{g}$ starting from $\tilde{x}(0) = x^0 = x$ with initial velocity $v = \bar{G}(x^0)^{-1} p^0 = \tilde{G}(x^0)^{-1} \tilde{p}(0)$ and ending at $\tilde{x}(\tau) = x^\tau$ with final velocity $v^\tau = \bar{G}(x^\tau)^{-1} p^\tau = \tilde{G}(x^\tau)^{-1} \tilde{p}(\tau)$. This proves assertions (i)-(iv) of Proposition 5.

## 2.3 One remark about reparametrization

The following result will be useful to insure that the geodesic curves $\bar{c}_l(I_l)$ are preserved.

**Lemma 7.** *Let $\bar{c} : I = [a, b] \to \mathbb{R}^n$ be a unit speed geodesic with respect to $\bar{g}$, $\bar{f} : \mathbb{R}^n \to \mathbb{R}$ be a function of class at least $C^2$, and $\bar{\lambda} : \mathbb{R}^n \to \mathbb{R}$ be such that*

$$\nabla \bar{f}\,(\bar{c}(t)) = \bar{\lambda}(t)\bar{p}(t) := \bar{\lambda}(t)\,\bar{G}\,(\bar{c}(t))\,\dot{\bar{c}}(t) \qquad \forall t \in I, \tag{2.26}$$

*where $\nabla \bar{f}$ denotes the gradient of $\bar{f}$ with respect to the Euclidean metric. Then up to reparametrization, $c$ is a unit speed geodesic with respect to the metric $e^{\bar{f}}\bar{g}$.*

Of course, Lemma 7 is a consequence of the fact that the gradient of $\bar{f}$ with respect to $\bar{g}$ at $\bar{c}(t)$ is always colinear with the velocity $\dot{\bar{c}}(t)$. Such a result could be found in textbooks of Riemannian geometry. For sake of completeness, we prove Lemma 7 with the Hamiltonian point of view.

*Proof of Lemma 7.* Define the function $\beta : I \to \mathbb{R}$ by

$$\beta(t) := \int_0^t e^{\bar{f}(\bar{c}(s))/2}\, ds \qquad \forall t \in I. \tag{2.27}$$

It is a strictly increasing function of class at least $C^3$ from $I$ to $\tilde{I} = [0, \tilde{\tau}] := \beta(I)$. Denote by $\theta : \tilde{I} \to I$ its inverse. Note that $\theta$ is at least $C^3$ and satisfies

$$\dot{\theta}(s) = e^{-\bar{f}(\bar{c}(\theta(s)))/2} \qquad \forall s \in [0, \tilde{\tau}]. \tag{2.28}$$

11

Define $\tilde{c}, \tilde{p} : \tilde{I} \to \mathbb{R}^n$ by

$$\tilde{c}(s) := \bar{c}\big(\theta(s)\big) \quad \text{and} \quad \tilde{p}(s) := e^{\bar{f}(\tilde{c}(s))/2}\,\bar{p}(\theta(s)) \qquad \forall s \in \tilde{I}.$$

The metric $\hat{g} := e^{\bar{f}} \bar{g}$ is associated with matrices $\hat{G}, \hat{Q}$ given by

$$\hat{G}(x)^{-1} = \hat{Q}(x) = e^{-\bar{f}(x)} \bar{Q}(x) \qquad \forall x \in \mathbb{R}^n.$$

Then, for every $s \in \tilde{I}$, $\dot{\tilde{c}}(s)$ and $\tilde{p}(s)$ are given by

$$\dot{\tilde{c}}(s) = \dot{\theta}(s)\dot{\bar{c}}\big(\theta(s)\big) = \dot{\theta}(s)\bar{Q}\big(\bar{c}(\theta(s))\big)\,\bar{p}(\theta(s)) = \hat{Q}\big(\tilde{c}(s)\big)\,\tilde{p}(s)$$

and (using (2.28))

$$
\begin{aligned}
\big(\dot{\tilde{p}}\big)_i(s) &= \frac{d}{ds}\left(e^{\bar{f}(\tilde{c}(s))/2}\right)\big(\bar{p}\big)_i(\theta(s)) + e^{\bar{f}(\tilde{c}(s))/2}\,\dot{\theta}(s)\,\big(\dot{\bar{p}}\big)_i(\theta(s)) \\
&= \frac{d}{ds}\left(e^{\bar{f}(\tilde{c}(s))/2}\right)\big(\bar{p}\big)_i(\theta(s)) - \frac{1}{2}\left\langle \bar{p}(\theta(s)), \frac{\partial \bar{Q}_i}{\partial x_i}\big(\bar{c}(\theta(s))\,\bar{p}(\theta(s))\big) \right\rangle \\
&= \frac{d}{ds}\left(e^{\bar{f}(\tilde{c}(s))/2}\right)\big(\bar{p}\big)_i(\theta(s)) - \frac{e^{-\bar{f}(\tilde{c}(s))}}{2}\left\langle \tilde{p}(s), \frac{\partial \bar{Q}_i}{\partial x_i}\big(\tilde{c}(s)\,\tilde{p}(s)\big) \right\rangle,
\end{aligned}
$$

where the first term is equal to (using (2.26))

$$
\begin{aligned}
\frac{d}{ds}\left(e^{\bar{f}(\tilde{c}(s))/2}\right)\big(\bar{p}\big)_i(\theta(s)) &= \frac{e^{\bar{f}(\tilde{c}(s))/2}}{2}\left\langle \nabla\bar{f}\big(\tilde{c}(s)\big), \dot{\tilde{c}}(s) \right\rangle \big(\bar{p}\big)_i(\theta(s)) \\
&= \frac{1}{2}\,e^{\bar{f}(\tilde{c}(s))/2}\left\langle \bar{\lambda}(\theta(s))\,\bar{p}(\theta(s)), \hat{Q}\big(\tilde{c}(s)\,\tilde{p}(s)\big) \right\rangle \big(\bar{p}\big)_i(\theta(s)) \\
&= \frac{1}{2}\left\langle \tilde{p}(s), \hat{Q}\big(\tilde{c}(s)\big)\,\tilde{p}(s) \right\rangle \big(\bar{\lambda}(\theta(s))\,\big(\bar{p}\big)_i(\theta(s))\big) \\
&= \frac{1}{2}\left\langle \tilde{p}(s), \hat{Q}\big(\tilde{c}(s)\big)\,\tilde{p}(s) \right\rangle \frac{\partial \bar{f}}{\partial x_i}\big(\tilde{c}(s)\big).
\end{aligned}
$$

Remembering (2.9)-(2.10) with $f = \bar{f}$ and $\tilde{Q} = \hat{Q}$, this proves that $\big(\tilde{c}(\cdot), \tilde{p}(\cdot)\big) : \tilde{I} \to \mathbb{R}^n \times \mathbb{R}^n$ is a trajectory of the Hamiltonian system associated with $\tilde{H} = H_{\bar{f}}$ and in turn concludes the proof of the lemma. $\qquad\square$

## 2.4 Dealing with obstacles

We now proceed to explain how to modify our construction in order to get assertion (v) of Proposition 5. We fix $(x, v), (y, w) \in U^{\bar{g}}\mathbb{R}^n$ satisfying (2.1) and consider a finite set of unit speed geodesics

$$\bar{c}_1 : I_1 \longrightarrow \mathbb{R}^n, \quad \cdots, \quad \bar{c}_L : I_L \longrightarrow \mathbb{R}^n$$

satisfying assumptions (2.2)-(2.3). We set

$$\bar{\Gamma} := \bigcup_{l=1}^{L} \bar{c}_l\left(I_l\right).$$

The construction that we performed in the previous section together with transversality arguments yield the following result. (We recall that for any function $\tilde{u}(\cdot) : [0, \tilde{\tau}] \to \mathbb{R}^n$, $\mathrm{Supp}\big(\tilde{u}(\cdot)\big)$ denotes the closure of the set of $t \in [0, \tilde{\tau}]$ such that $\tilde{u}(t) = 0$.)

**Lemma 8.** *Taking $\bar{\delta} > 0$ in (2.1) small enough, there are a positive constant $C = C(\tau, \rho)$, $\tilde{\tau} = \tilde{\tau}\big((x, v), (y, w)\big) > 0$, a function*

$$\big(\tilde{x}(\cdot), \tilde{p}(\cdot)\big) = \big(\tilde{x}\big(\cdot; (x, v), (y, w)\big), \tilde{p}\big(\cdot; (x, v), (y, w)\big)\big) \; : \; [0, \tilde{\tau}] \longrightarrow \mathbb{R}^n$$

*of class $C^k$, and a function*

$$\tilde{u}(\cdot) = \tilde{u}\big(\cdot; (x, v), (y, w)\big) \; : \; [0, \tilde{\tau}] \longrightarrow \mathbb{R}^n$$

*of class $C^{k-1}$ satisfying (2.20), (2.21),*

$$|\tilde{\tau} - \tau| < C \; |(x, v) - (y, w)|, \tag{2.29}$$

$$\mathrm{Supp}\big(\tilde{u}(\cdot)\big) \subset [\tau/5, 4\tau/5], \tag{2.30}$$

$$\big\|\tilde{u}\big\|_{C^0} \leq C \; |(x, v) - (y, w)|, \tag{2.31}$$

$$\big(\tilde{x}(0), \tilde{p}(0)\big) = \big(x^0, p^0\big), \quad \big(\tilde{x}(\tilde{\tau}), \tilde{p}(\tilde{\tau})\big) = \big(x^\tau, p^\tau\big), \tag{2.32}$$

*such that the following properties are satisfied:*

   (i) *the curve $\tilde{x}\big(\mathrm{Supp}\big(\tilde{u}(\cdot)\big)\big)$ is transverse to $\bar{\Gamma}$;*

   (ii) *the set $\mathcal{T}_{\tilde{u}} \subset \mathrm{Supp}\big(\tilde{u}(\cdot)\big)$ defined by*

$$\mathcal{T}_{\tilde{u}} := \Big\{t \in \mathrm{Supp}\big(\tilde{u}(\cdot)\big) \,|\, \tilde{x}(t) \in \bar{\Gamma}\Big\}$$

   *is empty.*

13

*Proof of Lemma 8.* Let us consider the trajectory

$$\mathcal{X}(\cdot) = \mathcal{X}\big(\cdot\,; (x,v),(y,w)\big) : [0,\tau] \longrightarrow \mathbb{R}^n$$

of class $C^{k+1}$ defined by (2.11). Since $\mathcal{X}(\cdot)$ coincides respectively with $\bar{\gamma}_{x,v}$ and $\bar{\gamma}_{y,w}$ on the intervals $[0,\tau/3]$ and $[2\tau/3,\tau]$ and since the $\bar{c}_l$'s are unit speed geodesics satisfying (2.3), there are $t_1 \in (0,\tau/3)$, $t_2 \in (2\tau/3,\tau)$ and $\nu \in (0,\tau/100)$ such that

$$\mathcal{X}(t) \notin \bar{\Gamma} \qquad \forall t \in [t_1 - \nu, t_1 + \nu] \cup [t_2 - \nu, t_2 + \nu]. \tag{2.33}$$

Moreover, since $\mathcal{X}$ is a reparametrization of $\tilde{x}(\cdot)$ satisfying (2.22), we have

$$\left|\dot{\mathcal{X}}(t) - e_1\right| \le K' \left|(x,v) - (y,w)\right|. \qquad \forall t \in [0,\tau],$$

for some positive constant $K'$. Then taking $\bar{\delta} > 0$ in (2.1) small enough and remembering (2.4), to prove (i) it is sufficient to show that we can perturb the curve $\mathcal{X}([0,\tau])$ to make it transverse to all the geodesic curves $\bar{c}(I_l)$ verifying

$$|\dot{\bar{c}}_l(s) - e_1| < 1/2 \qquad \forall s \in I_l = [a_l, b_l].$$

Without loss of generality, we may assume that for each such curve (denote by $\mathcal{L}$ the set of such $l$), we have $\big(\bar{c}_l(a_l)\big)_1 \le \bar{x}^0$ and $\big(\bar{c}_l(b_l)\big)_1 \ge \bar{x}^\tau$ (remember (2.2)). Let us parametrize both curves $\mathcal{X}(\cdot)$ and $\bar{c}_l(\cdot)$ by their first coordinates (where $l \in \mathcal{L}$ is fixed). Namely, there are two diffeomorphisms $\theta_1 : J_1 = [\alpha,\beta] \to [0,\tau], \theta_2 : J_2 = [\alpha', \beta'] \to I_l$ of class $C^{k+1}$ such that

$$\big((\mathcal{X} \circ \theta_1)(s)\big)_1 = s \quad \forall s \in J_1 \quad \text{and} \quad \big((\bar{c}_l \circ \theta_2)(s)\big)_1 = s \quad \forall s \in J_2. \tag{2.34}$$

Extending $I_l$ if necessary, we may indeed assume that $J_1 \subset J_2$. Define the function $h_l : I \to \mathbb{R}^n$ of class $C^{k+1}$ by

$$h_l(s) := \big(\mathcal{X} \circ \theta_1\big)(s) - \big(\bar{c}_l \circ \theta_2\big)(s) \qquad \forall s \in J_1 = [\alpha,\beta].$$

Fix a smooth function $\psi : [0,\tau] \to [0,1]$ satisfying

$$\psi(t) = 0 \quad \forall t \in [0, t_1 - \nu] \cup [t_2 + \nu, \tau] \quad \text{and} \quad \psi(t) = 1 \quad \forall t \in [t_1 + \nu, t_2 - \nu]. \tag{2.35}$$

For every $\omega \in \mathbb{R}^n$ with $\omega_1 = 0$, define the curve $\mathcal{X}_\omega : [0,\tau] \to \mathbb{R}^n$ by

$$\mathcal{X}_\omega(t) := \mathcal{X}(t) + \psi(t)\,\omega \qquad \forall t \in [0,\tau].$$

If $\mathcal{X}_\omega\left([0,\tau]\right)$ intersects $\bar{c}_l(I_l)$, then

$$\begin{aligned} 0_n &= \mathcal{X}_\omega(t) - \bar{c}_l(s) \\ &= \mathcal{X}(t) - \bar{c}_l(s) + \psi(t)\,\omega \\ &= (\mathcal{X} \circ \theta_1)\big(\theta_1^{-1}(t)\big) - (\bar{c}_l \circ \theta_2)\big(\theta_2^{-1}(s)\big) + \psi(t)\,\omega, \end{aligned}$$

14

for some $t \in [0, \tau]$ and $s \in J_1$. Since $\omega_1 = 0$ and (2.34) is satisfied, we must have $\theta_1^{-1}(t) = \theta_2^{-1}(s)$, then we obtain

$$0_n = (\mathcal{X} \circ \theta_1)\left(\theta_1^{-1}(t)\right) - (\bar{c}_l \circ \theta_2)\left(\theta_1^{-1}(t)\right) + \psi(t)\,\omega = h_l\left(\theta_1^{-1}(t)\right) + \psi(t)\,\omega.$$

Furthermore, by (2.33), if $\omega$ is small enough, the restriction of $\mathcal{X}_\omega(\cdot)$ to the two intervals $[t_1 - \nu, t_1 + \nu]$ and $[t_2 - \nu, t_2 + \nu]$ cannot intersect $\bar{\Gamma}$. By (2.35), we infer that

$$h_l\left(\theta_1^{-1}(t)\right) + \omega = 0_n \quad \text{for some } t \in [t_1 + \nu, t_2 - \nu].$$

By Sard's Theorem (see for instance [3]), almost every value of $h_l$ is regular. In addition, if $-\omega$ is a regular value of $h_l$, then $\dot{h}_l(s) \neq 0_n$ for all $s$ such that $h_l(s) = -\omega$. This shows that if $-\omega$ is a small enough regular value of $h_l$, then $\mathcal{X}_\omega\left([t_1 - \nu, t_2 + \nu]\right)$ is transverse to $\bar{c}_l(I_l)$. Finally, we observe that

$$\begin{cases} \dot{\mathcal{X}}_\omega(t) = \dot{\mathcal{X}}(t) + \dot{\psi}(t)\,\omega \\ \ddot{\mathcal{X}}_\omega(t) = \ddot{\mathcal{X}}(t) + \ddot{\psi}(t)\,\omega \end{cases} \qquad \forall t \in [0, \tau]. \tag{2.36}$$

Then taking a small enough $\omega \in \mathbb{R}^n$ with $\omega_1 = 0$ such that $-\omega$ is a regular value for all the $h_l$'s and proceeding as in Section 2.2 provides $\tilde{\tau} = \tilde{\tau}\big((x, v), (y, w)\big) > 0$ and a triple

$$(\tilde{x}(\cdot), \tilde{p}(\cdot), \tilde{u}(\cdot)) = \big(\tilde{x}\big(\cdot; (x, v), (y, w)\big), \tilde{p}\big(\cdot; (x, v), (y, w)\big), \tilde{u}\big(\cdot; (x, v), (y, w)\big)\big)$$
$$: [0, \tilde{\tau}] \longrightarrow \mathbb{R}^n$$

satisfying (2.20), (2.21), and (2.32). Moreover, $\tilde{\tau}$ is given by

$$\tilde{\tau} := \int_0^\tau \sqrt{\left\langle \dot{\mathcal{X}}_\omega(s), \bar{G}\left(\mathcal{X}_\omega(s)\right) \dot{\mathcal{X}}_\omega(s) \right\rangle}\, ds$$

and for every $t \in [0, \tilde{\tau}]$,

$$\begin{aligned} \tilde{u}(t) &= 2\dot{\tilde{p}}(t) + 2\frac{\partial \bar{H}}{\partial x}(\tilde{x}(t), \tilde{p}(t)) \\ &= 2\frac{d}{dt}\left\{\bar{G}(\tilde{x}(t))\,\dot{\tilde{x}}(t)\right\} + 2\frac{\partial \bar{H}}{\partial x}(\tilde{x}(t), \tilde{p}(t)). \end{aligned}$$

From (2.36) and (2.18)-(2.19), we deduce that taking $\omega$ small enough yields (2.29) and (2.31) for some universal constant $C = C(\tau, \rho) > 0$. All in all, this shows assertion (i).

To show assertion (ii), replace the curve $\tilde{x}(\cdot)$ (which is a reparametrization of $\mathcal{X}_\omega$) by a piece of unit speed geodesic (with respect to $\bar{g}$) in a neighborhood of each $t \in [0, \tilde{\tau}]$ such that $\tilde{x}(t) \in \bar{\Gamma}$ and reparametrize it as in Section 2.2. Let us explain briefly how

15

to proceed. Given $\bar{t} \in (0, \tilde{\tau})$ such that $\tilde{x}(\bar{t}) \in \bar{\Gamma}$ and $\lambda > 0$, define $\tilde{x}_\lambda(\cdot) : [0, \tilde{\tau}] \to \mathbb{R}^n$ a small perturbation of $\tilde{x}(\cdot)$ by

$$\tilde{x}_\lambda(t) := \varphi\left(\frac{t - \bar{t}}{\lambda}\right) \tilde{x}(t) + \left[1 - \varphi\left(\frac{t - \bar{t}}{\lambda}\right)\right] \bar{\gamma}_{\tilde{x}(\bar{t}), \dot{\tilde{x}}(\bar{t})}(t - \bar{t}) \qquad \forall t \in [0, \tilde{\tau}],$$

where $\varphi : \mathbb{R} \to [0, 1]$ is a smooth function satisfying

$$\varphi(t) = 1 \quad \forall t \in (-\infty, -1] \cup [1, +\infty) \quad \text{and} \quad \varphi(t) = 0 \quad \forall t \in [-1/2, 1/2].$$

We leave the reader to check that taking $\lambda > 0$ small enough yields the desired result.

$\square$

Proposition 5 follows easily from the following result whose technical proof is postponed to Appendix A.2.

**Lemma 9.** *There are $C = C(\tau, \rho) > 0$ and a function $f : \mathbb{R}^n \to \mathbb{R}$ of class $C^{k-1}$ such that the following properties are satisfied:*

(i) *$\text{Supp}(f) \subset \mathcal{R}(\rho)$;*

(ii) *$\|f\|_{C^1} < C \, |(x, v) - (y, w)|$;*

(iii) *for every $t \in [0, \tilde{\tau}]$, $\nabla f(\tilde{x}(t)) = \tilde{u}(t)$;*

(iv) *for every $l \in \{1, \ldots, L\}$ and every $s \in I_l$, , there is $\lambda_l(s)$ such that*

$$\nabla f(\bar{c}_l(s)) = \lambda_l(s) \bar{p}_l(s) := \lambda_l(s) \, \bar{G}(\bar{c}_l(s)) \, \dot{\bar{c}}_l(s).$$

# 3 Proof of Theorem 3

Let $\gamma = \gamma_{x,v} : \mathbb{R} \to M$ be the geodesic starting from $x$ with velocity $v \in U_x^g M$ and $\epsilon > 0$ be fixed. Let $\tau \in (0, 1/20)$ be a small enough time such that the curve $\gamma_{x,v}([-10\tau, 10\tau])$ has no self-intersection. There exist an open neighborhood $\mathcal{U}_x$ of $x$ and a smooth diffeomorphism

$$\theta_x : \mathcal{U}_x \longrightarrow B^n(0, 1) \quad \text{with} \quad \theta_x(x) = 0_n \quad \text{and} \quad \frac{d}{dt}\left(\theta_x \circ \gamma_{x,v}\right)(0) = e_1.$$

Set

$$\bar{\gamma}(t) := \theta_x\left(\gamma_{x,v}(t)\right) \quad \forall t \in [-10\tau, 10\tau]$$

and

$$\bar{x}^0 := \bar{\gamma}(0) = 0_n, \quad \bar{v}^0 := \dot{\bar{\gamma}}(0) = e_1, \quad \bar{x}^\tau := \bar{\gamma}(\tau), \quad \bar{v}^\tau := \dot{\bar{\gamma}}(\tau).$$

16

The metric $g$ is sent, via the smooth diffeomorphism $\theta_x$, onto a Riemannian metric $\bar{g}$ of class $C^k$ on $B^n(0,1)$. Without loss of generality, we may assume that $\bar{g}$ is the restriction to $B^n(0,1)$ of a complete Riemannian metric of class $C^k$ defined on $\mathbb{R}^n$. Denote by $\phi_t^{\bar{g}}$ the geodesic flow on $\mathbb{R}^n \times \mathbb{R}^n$. Set

$$\mathcal{H}_0 := \left\{ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_1 = 0 \right\}.$$

Since $\bar{\gamma}(0) = 0_n$ and $\dot{\bar{\gamma}}(0) = e_1$, taking $\tau$ smaller if necessary we may assume that

$$\text{and} \quad \left| \frac{d}{dt} \left( \theta_x \circ \gamma_{x,v} \right)(t) - e_1 \right| \leq 1/10 \qquad \forall t \in [0, \tau]. \tag{3.1}$$

Keeping the notations of Section 2.1, we may also assume that there is $\rho > 0$ such that the following properties are satisfied:

(i) $\bar{\gamma}(t) \in \mathcal{R}(\rho/2) = \left\{ (t, z) \mid t \in [0, \bar{x}_1^{\tau}], \ z \in B^{n-1}(0, \rho/2) \right\} \subset B^n(0,1)$,

(ii) for every unit speed geodesic $\bar{c} : I = [a_1, b_1] \longrightarrow \mathbb{R}^n$ with $\bar{c}(I) \subset \mathcal{R}(2\rho) \subset B^n(0,1)$, there holds

$$\left| \dot{\bar{c}}_l(s) - \dot{\bar{c}}_l(s') \right| < 1/8 \qquad \forall s, s' \in I.$$

Then, we can apply Proposition 5 to the curve $\bar{\gamma} : [0, \tau] \to \mathbb{R}^n$. Consequently, there are $\bar{\delta} = \bar{\delta}(\tau, \rho) \in (0, \tau/3)$ and $C = C(\tau, \rho) > 0$ such that the property stated in Proposition 5 is satisfied. Define the section $\mathcal{S} \subset TM$ by

$$\mathcal{S} := d\theta_x^{-1} \left( \mathcal{H}_0 \times \mathbb{R}^n \right).$$

Since $M$ is assumed to be compact and the geodesic flow preserves the Liouville measure, the Poincaré recurrence theorem implies that the geodesic flow is nonwandering on $U^g M$. Thus, for every neighborhood $\mathcal{V}$ of $(x, v)$ in $U^g M$, there exist $t \geq 1$ and $(x', v') \in \mathcal{V}$ such that $\phi_t^g(x', v') \in \mathcal{V}$. Then, since $\gamma_{x,v}$ is transverse to $\mathcal{S}$ at time zero, for every $r > 0$ small, there exist $(x^r, v^r), (x_*^r, v_*^r) \in \mathcal{S} \cap U^g M$, $T^r > 0$ and $y^r, y_*^r, w^r, w_*^r \in B^n(0,1)$ such that

(a) $(x_*^r, v_*^r) = \phi_{T^r}^g(x^r, v^r)$.

(b) $(y^r, w^r) = d\theta_x(x^r, v^r), (y_*^r, w_*^r) = d\theta_x(x_*^r, v_*^r)$;

(c) $(y^r, w^r), (y_*^r, w_*^r) \in U^{\bar{g}} \mathbb{R}^n$;

(d) $y^r, y_*^r \in \mathcal{H}_0$;

(e) $\left| x - \bar{x}^0 \right|, \left| y - \bar{x}^0 \right|, \left| v - \bar{v}^0 \right|, \left| w - \bar{v}^0 \right| < \bar{\delta}$;

17

(f) $|(y^r, w^r) - (y^r_*, w^r_*)| < r$.

Recall that the cylinder $\mathcal{R}(\rho/2)$ is defined by

$$\mathcal{R}(\rho/2) := \left\{ (t, z) \mid t \in [0, \bar{x}^\tau_1], \ z \in B^{n-1}(0, \rho/2) \right\} \subset B^n(0, 1).$$

The intersection of the curve $\gamma_{x^r, v^r} ([5\tau, T^r - 5\tau])$ with the open set $\theta_x^{-1} (\mathcal{R}(\rho/2))$ can be covered by a finite number of connected curves. More precisely, there are a finite number of unit speed geodesic arcs

$$\bar{c}_1 \ : \ I_1 = [a_1, b_1] \longrightarrow B^n(0, 1), \quad \cdots, \quad \bar{c}_L \ : \ I_L = [a_L, b_L] \longrightarrow B^n(0, 1)$$

such that the following properties are satisfied:

(g) For every $l \in \{1, \ldots, L\}$, $\bar{c}_l(a_l), \bar{c}_l(b_l) \in \mathcal{R}(2\rho) \setminus \mathcal{R}(\rho/2)$;

(h) there are disjoint closed intervals $\mathcal{J}_1, \ldots, \mathcal{J}_L \subset [-5\tau, T^r - 5\tau]$ such that

$$\gamma_{x^r, v^r}(\mathcal{J}_l) \subset \mathcal{U}_x, \quad \bar{c}_l(I_l) = \theta_x \left( \gamma_{x^r, v^r}(\mathcal{J}_l) \right) \qquad \forall l = 1, \ldots, L,$$

$$\text{and} \quad \left( \theta_x \left( \gamma_{x^r, v^r} ([5\tau, T_r - 5\tau]) \cap \mathcal{U}_x \right) \cap \mathcal{R}(\rho/2) \right) \subset \bigcup_{l=1}^{L} \bar{c}_l(I_l).$$

From the above properties and (ii), we can connect $(y^r_*, w^r_*)$ to $\phi^{\bar{g}}_\tau(y^r, w^r)$ by preserving the curves $\bar{c}_1(I_1), \ldots, \bar{c}_L(I_L)$. We define the metric $\tilde{g}$ on $M$ by

$$\tilde{g} = \left\{ \begin{array}{l} \bar{g} \text{ on } M \setminus \mathcal{U}_x \\ \theta_x^* \left( e^f \bar{g} \right) \text{ on } \mathcal{U}_x. \end{array} \right.$$

We leave the reader to check that by construction the geodesic starting from $x^r_*$ with initial velocity $v^r_*$ is periodic. Taking $r > 0$ small enough yields $d_{TM}\left( (x, v), (x^r_*, v^r_*) \right) < \epsilon$ and $\|f\|_{C^1} < \epsilon$.

## A    Proof of Lemmas 6 and 9

### A.1    Proof of Lemma 6

Define the function $\Phi : [0, T] \times \mathbb{R}^{n-1} \to \mathbb{R}^n$ by

$$\Phi(t, z) := y(t) + (0, z) \qquad \forall (t, z) \in [0, T] \times \mathbb{R}^{n-1}.$$

18

We can easily check that, thanks to (2.23), $\Phi$ is a diffeomorphism of class $C^k$ from $[0, T] \times \mathbb{R}^{n-1}$ into $[y_1(0), y_1(\tau)] \times \mathbb{R}^{n-1}$ which sends the cylinder $[\beta/2, T - \beta/2] \times B^{n-1}(0, \mu)$ into the "cylinder"

$$\mathcal{C}_y(\mu) := \Big\{ y(t) + (0, z) \,|\, t \in [\beta/2, T - \beta/2], z \in B^{n-1}(0, \mu) \Big\},$$

and which satisfies

$$\|\Phi\|_{C^1}, \big\|\Phi^{-1}\big\|_{C^1} \leq K_0,$$

for some positive constant $K_0$ depending on $T$ only. Define the function $\tilde{w}(\cdot) : [0, T] \to \mathbb{R}^n$ by

$$\tilde{w}(t) := \big( d\Phi\big(t, 0_{n-1}\big)\big)^* \big(w(t)\big) \qquad \forall t \in [0, T].$$

The function $\tilde{w}$ is $C^{k-1}$; in addition, by (2.24) and (2.25), it follows that

$$\tilde{w}(t) = 0_n \qquad \forall t \in [0, \beta] \cup [T - \beta, T] \quad \text{and} \quad \tilde{w}_1(t) = 0 \qquad \forall t \in [0, T].$$

Let $\psi : \mathbb{R} \to [0, 1]$ be an even function of class $C^\infty$ satisfying the following properties:

- $\psi(s) = 1$ for $s \in [0, 1/3]$;

- $\psi(s) = 0$ for $s \geq 2/3$;

- $|\psi(s)|, |\psi'(s)| \leq 10$ for any $s \in [0, +\infty)$.

Extend the function $\tilde{w}(\cdot)$ on $\mathbb{R}$ by $\tilde{w}(t) := 0$ for $t \leq 0$ and $t \geq T$, and define the function $\tilde{W} : [0, T] \times \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\tilde{W}(t, z) = \psi\left(\frac{|z|}{\mu}\right) \left[ \sum_{i=2}^n \int_0^{z_i} \tilde{w}_i(t + s) ds \right] \qquad \forall (t, z) \in [0, T] \times \mathbb{R}^{n-1}.$$

Since $\tilde{w}$ is $C^{k-1}$, $\psi$ is $C^k$, and $\tilde{W}(t, z)$ can be written as

$$\tilde{W}(t, z) = \psi\left(\frac{|z|}{\mu}\right) \left[ \sum_{i=2}^n \int_t^{t+z_i} \tilde{w}_i(t + s) ds \right],$$

it is easy to check that $\tilde{W}$ is of class $C^k$. Moreover, (using that $3\mu \leq \beta < T$) we check easily that

$$\text{Supp}\,\big(\tilde{W}\big) \subset [\beta/2, T - \beta/2] \times B^{n-1}\big(0, 2\mu/3\big),$$

19

$$\nabla \tilde{W}(t,0) = \tilde{w}(t), \quad \tilde{W}(t,0) = 0 \qquad \forall t \in [0,T],$$

and that (see the proof of [4, Lemma 3.3])

$$\left\| \tilde{W} \right\|_{C^1} \leq \frac{K_1}{\mu} \left\| \tilde{w}(\cdot) \right\|_{C^0},$$

for some constant $K_1 > 0$. Finally, define the function $W : \mathbb{R}^n \to \mathbb{R}$ by

$$W(x) := \begin{cases} \tilde{W}(\Phi^{-1}(x)) & \text{if } x \in \mathcal{C}_y(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $W$ satisfies (i)-(iv).

## A.2   Proof of Lemma 9

We proceed in several steps.

Step 1: Applying Lemma 6, we get a universal constant $C_1 = C_1(\tau, \rho) > 0$ and a function $f_1 : \mathbb{R}^n \to \mathbb{R}$ of class $C^k$ such that the following properties are satisfied:

$(i)_1$ Supp $(f_1) \subset \mathcal{R}(2\rho/3)$;

$(ii)_1$ $\left\| f_1 \right\|_{C^1} < C_1 \, |(x,v) - (y,w)|$;

$(iii)_1$ $\nabla f_1(\tilde{x}(t)) = \tilde{u}(t)$, for every $t \in [0, \tilde{\tau}]$;

$(iv)_1$ $f_1(\tilde{x}(t)) = 0$, for every $t \in [0, \tau]$.

Step 2: Let $x_1, \ldots, x_N$ be a set of points in $\mathcal{R}(2\rho/3)$ such that

$$\left( \bigcup_{k,l=1, k \neq l}^{L} \left( \bar{c}_k(I_k) \cap \bar{c}_l(I_l) \right) \right) \cap \mathcal{R}(2\rho/3) = \left\{ x_1, \ldots, x_N \right\}.$$

Note that by Lemma 8 (ii), the set $\{x_1, \ldots, x_N\}$ does not intersect the curve $\tilde{x}$ (Supp $(\tilde{u}(\cdot))$). Let $\mu > 0$ be such that the $N$ balls $B^n(x_1, 2\mu), \ldots, B^n(x_N, 2\mu)$ are disjoint and do not intersect neither the curve $\tilde{x}$ (Supp $(\tilde{u}(\cdot))$ nor the boundary of $\mathcal{R}(2\rho/3)$. Define the $C^k$ function $f_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_2(x) := f_1 \left( \sum_{k=1}^{N} \left[ \psi \left( \frac{|x - x_k|}{3\mu} \right) x_k + \left( 1 - \psi \left( \frac{|x - x_k|}{3\mu} \right) \right) x \right] \right) \qquad \forall x \in \mathbb{R}^n.$$

By construction, there is a universal constant $C_2 = C_2(\tau, \rho) > 0$ such that $f_2$ satisfies the following properties:

$(i)_2$ Supp $(f_2) \subset \mathcal{R}(2\rho/3)$;

$(ii)_2$ $\left\| f_2 \right\|_{C^1} < C_2 \left| (x,v) - (y,w) \right|$;

$(iii)_2$ $\nabla f_2\big(\tilde{x}(t)\big) = \tilde{u}(t)$, for every $t \in [0,\tilde{\tau}]$;

$(iv)_2$ $f_2\big(\tilde{x}(t)\big) = 0$, for every $t \in [0,\tilde{\tau}]$;

$(v)_2$ $f_2(x) = f_1(x)$ for every $x \in \mathbb{R}^n \setminus \left( \bigcup_{k=1}^{N} B^n(x_k, 2\mu) \right)$;

$(vi)_2$ $\nabla f_2(x) = 0$ for every $x \in \bigcup_{k=1}^{N} B^n(x_k, \mu)$.


Step 3: Let $t_1, \ldots, t_K \in [0,\tau]$ be the set of times such that

$$\tilde{x}\big(\mathrm{Supp}\ (\tilde{u}(\cdot))\big) \cap \left( \bigcup_{l=1}^{L} \bar{c}_l(I_l) \right) = \left\{ \tilde{x}(t_k) \mid k = 1, \ldots K \right\}.$$

Taking $\mu > 0$ smaller if necessary, we may assume that the balls $B^n\big(\tilde{x}(t_1), 5\mu\big)$, $\ldots, B^n\big(\tilde{x}(t_K), 5\mu\big)$ are disjoint, do not intersect the boundary of $\mathcal{R}(\rho/2)$, and such that $\tilde{u}(t) = 0$ for every $t \in [0,\tilde{\tau}]$ with $\tilde{x}(t) \in \bigcup_{k=1}^{Q} B^n\big(\tilde{x}(t_k), 5\mu\big)$ (remember Lemma 8 (ii)). Set

$$\Omega := \bigcup_{k=1}^{Q} B^n\big(\tilde{x}(t_k), 2\mu\big).$$

Taking $\mu > 0$ smaller if necessary again, the projection (with respect to the Euclidean metric) $\mathcal{P}_0 : \Omega \to \mathbb{R}^n$ to the set

$$S := \bigcup_{k=1}^{K} \left( B^n\big(\tilde{x}(t_k), 2\mu\big) \cap \tilde{x}\big([0,\tilde{\tau}]\big) \right),$$

is of class $C^{k-1}$, has a $C^1$ norm $\left\| \mathcal{P}_0 \right\|_{C^1}$ which is bounded by a universal constant, and satisfies

$$\mathcal{P}_0(x) = x \qquad \forall x \in S,$$
$$\mathcal{P}_0(x) \in S \qquad \forall x \in \Omega,$$
$$\left| x - \mathcal{P}_0(x) \right| < \frac{\mu}{2} \qquad \forall x \in \bigcup_{k=1}^{K} \left( B^n\big(\tilde{x}(t_k), \mu/2\big) \right).$$

Define the $C^{k-1}$ function $f_3 : \mathbb{R}^n \to \mathbb{R}$ by

$$f_3(x) := \begin{cases} f_2\Big(h(x)\mathcal{P}_0(x) + \big(1 - h(x)\big)x\Big) & \text{if } x \in \Omega \\ f_2(x) \text{ otherwise,} \end{cases}$$

where $h : \Omega \to \mathbb{R}$ is defined by

$$h(x) := \psi\left(\sum_{q=1}^{Q} \frac{2|x - \tilde{x}(t_q)|}{3\mu}\right) \qquad \forall x \in \Omega.$$

We note that $h(x) = 1$ for every $x \in \bigcup_{k=1}^{K}\Big(B^n\big(\tilde{x}(t_k), \mu/2\big)\Big)$ and $h(x) = 0$ for every $x \in \Omega$ which does not belong to the set $\bigcup_{k=1}^{K}\Big(B^n\big(\tilde{x}(t_k), \mu\big)\Big)$. Consequently, by construction, there is a universal constant $C_3 = C_3(\tau, \rho) > 0$ such that $f_3$ satisfies the following properties:

$(i)_3$ Supp $(f_3) \subset \mathcal{R}(2\rho/3)$;

$(ii)_3$ $\left\|f_3\right\|_{C^1} \le C_3 \, |(x, v) - (y, w)|$;

$(iii)_3$ $\nabla f_3\big(\tilde{x}(t)\big) = \tilde{u}(t)$, for every $t \in [0, \tilde{\tau}]$;

$(iv)_3$ $f_3\big(\tilde{x}(t)\big) = 0$, for every $t \in [0, \tilde{\tau}]$;

$(v)_3$ $f_3(x) = f_2(x)$ for every $x \in \mathbb{R}^n \setminus \Omega$;

$(vi)_3$ $\nabla f_3(x) = 0$ for every $x \in \bigcup_{k=1}^{K} B^n\big(\tilde{x}(t_k), \mu/2\big)$.

Step 4: Denote by $d_{\bar{g}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Riemannian distance with respect to the Riemannian metric $\bar{g}$. Denote by $\mathrm{dist}_{\bar{g}}^{\bar{\Gamma}}(\cdot)$ the distance function (with respect to $\bar{g}$) to the set $\bar{\Gamma}$. For every $\delta > 0$, let $\mathcal{S}_\delta \subset \mathcal{R}(2\rho/3 + \delta)$ be the subset of $\bar{\Gamma}$ defined by

$$\mathcal{S}_\delta := \Big(\bar{\Gamma} \cap \mathcal{R}(\tau, 2\rho/3 + \delta)\Big) \setminus \left(\bigcup_{k=1}^{N} B^n\big(x_k, \mu/2\big) \cup \bigcup_{k=1}^{K} B^n\big(\tilde{x}(t_q), \mu/4\big)\right).$$

For every $\delta, \mu > 0$, we denote by $\mathcal{S}_\delta^\mu$ the open set of points whose distance (with respect to $\bar{g}$) to $\mathcal{S}_\delta$ is strictly less than $\mu$. There are $\delta, \mu > 0$ such that the function $\mathrm{dist}_{\bar{g}}^{\bar{\Gamma}}(\cdot)$ is of class $C^k$ on $\mathcal{S}_\delta^\mu$, the projection $\mathcal{P}_{\bar{g}}^{\bar{\Gamma}}$ to $\bar{\Gamma}$ with respect to $\bar{g}$ is $C^{k-1}$ on $\mathcal{S}_\delta^\mu$, and both $\big\|\mathrm{dist}_{\bar{g}}^{\bar{\Gamma}}(\cdot)\big\|_{C^1(\mathcal{S}_\delta^\mu)}, \big\|\mathcal{P}_{\bar{g}}^{\bar{\Gamma}}(\cdot)\big\|_{C^1(\mathcal{S}_\delta^\mu)}$ are bounded by a universal constant. Define the function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) := \begin{cases} f_3\left(P(x)\right) & \text{if } x \in \mathcal{S}_\delta^\mu \\ f_3(x) \text{ otherwise,} \end{cases}$$

where the mapping $P : \mathcal{S}_\delta^\mu \to \mathbb{R}^n$ is defined by

$$P(x) := \psi\left(\frac{2\mathrm{dist}_{\bar{g}}^{\bar{\Gamma}}(x)}{3\mu}\right) \mathcal{P}_{\bar{g}}^{\bar{\Gamma}}(x) + \left(1 - \psi\left(\frac{2\mathrm{dist}_{\bar{g}}^{\bar{\Gamma}}(x)}{3\mu}\right)\right) x \qquad \forall x \in \mathcal{S}_\delta^\mu.$$

We leave the reader to check that if $\mu > 0$ is small enough, the function $f$ is of class $C^{k-1}$ and satisfies assertions (i)-(iv) of Lemma 9 for some universal constant $C = C(\tau, \rho) > 0$.

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