Sensitivity Analysis: A Variational Approach

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Overview

- General Sensitivity Analysis
- Sensitivity and Data Assimilation
- Second Order Analysis
- A 1-D Example
- An Application to a pollution problem

Sensitivity Analysis: Deterministic Approach

- Model: $\mathcal{F}$:

$$\mathcal{F}(\mathcal{X},\mathcal{U}) = 0 \tag{1}$$

- Scalar Response Function $\mathcal{G}$:

$$\mathcal{G}(\mathcal{X},\mathcal{U}) \tag{2}$$

- Sensitivity $\mathcal{S}$ is by definition the gradient of $\mathcal{G}$ with respect to $\mathcal{U}$:

$$\mathcal{S} = \nabla\mathcal{G}(\mathcal{X}(\mathcal{U}),\mathcal{U}) \tag{3}$$

Computing Sensitivity : Variational Method

- An adjoint variable $\mathcal{P}$ is introduced as the solution of :

$$\left[\frac{\partial \mathcal{F}}{\partial \mathcal{X}}\right]^t .\mathcal{P} = \left[\frac{\partial \mathcal{G}}{\partial \mathcal{X}}\right] \tag{4}$$

- Then we get :

$$\mathcal{S} = \left[\frac{\partial \mathcal{G}}{\partial \mathcal{U}}\right] - \left[\frac{\partial \mathcal{F}}{\partial \mathcal{U}}\right]^t .\mathcal{P} \tag{5}$$

- X is the state variable (velocity, surface elevation) governed by :

$$\begin{cases} \dfrac{dX}{dt} & = & F(X) \\ X(0) & = & U \end{cases} \tag{6}$$

- The concentration of pollutant C, produced by sources S verifies:

$$\begin{cases} \dfrac{dC}{dt} & = & G(X, C, S) \\ C(0) & = & V \end{cases} \tag{7}$$

- U and V are unknonw. The VDA problem is to evaluate them from observation X_{obs} and C_{obs}, in order to minimize the cost function J defined by:

$$J(U, V) = \frac{1}{2}\int_0^T \|EX - X_{obs}\|^2 dt + \frac{1}{2}\int_0^T \|DC - C_{obs}\|^2 dt \tag{8}$$

- For sake of simplicity regularization terms, of great practical importance, are not displayed

Data Assimilation for Pollution Modeling: Optimality System

- P and Q adjoint variables are introduced as the solution of the system :

$$\left\{ \begin{array}{rcl} \frac{dP}{dt} + \left[\frac{\partial F}{\partial X}\right]^t .P + \left[\frac{\partial G}{\partial X}\right]^t .Q & = & E^t(EX - X_{obs}) \\ P(T) & = & 0; \end{array} \right. \quad (9)$$

.;

$$\left\{ \begin{array}{rcl} \frac{dQ}{dt} + \left[\frac{\partial G}{\partial C}\right]^t .Q & = & D^t(DC - C_{obs}); \\ Q(T) & = & 0, \end{array} \right. \quad (10)$$

- Then the gradient of J with respect to U and V are given by :

$$\nabla J_U = -P(0) \quad (11)$$

$$\nabla J_V = -Q(0) \quad (12)$$

Sensitivity with respect to Observations and Sources

- If some response function $\mathcal{S}$ is introduced, how to evaluate the sensitivity with respect to observations? For instance how to evaluate the impact of an error of observation on a prediction?
- What should be the "model" $\mathcal{F}$ of the general sensitivity analysis?
- Because only the Optimality System contains the observation, the sensitivity analysis must be carried out on the O.S. considered as a Generalized Model
- Deriving the O.S. leads to carry out a **Second Order Analysis.**

Computing the sensitivity with respect to sources : second order adjoint.

- We need to introduce four second order adjoint variables Γ, Λ, Φ and Ψ as the solution of :

$$
\left\{
\begin{array}{rcl}
\dfrac{d\Gamma}{dt} + \left[\dfrac{\partial F}{\partial X}\right]^t \cdot \Gamma + \left[\dfrac{\partial F}{\partial X}\right]^t \cdot \Lambda + \left[\dfrac{\partial^2 F}{\partial X^2} P\right]^t \cdot \Phi & & \\
+ \left[\dfrac{\partial^2 G}{\partial X^2} Q\right]^t \cdot \Phi + \left[\dfrac{\partial^2 G}{\partial C \partial X} Q\right]^t \cdot \Psi - E^t E \Phi & = & 0; \\
\Gamma(0) & = & 0; \\
\Gamma(T) & = & 0,
\end{array}
\right.
\tag{13}
$$

Computing the sensitivity with respect to sources 2

- $$\left\{\begin{array}{rcl} \dfrac{d\Lambda}{dt} + \left[\dfrac{\partial F}{\partial C}\right]^t \cdot \Lambda + \left[\dfrac{\partial^2 G}{\partial C \partial X} Q\right]^t \cdot \Phi & & \\ + \left[\dfrac{\partial^2 G}{\partial X^2} Q\right]^t \cdot \Psi - D^t D \Psi & = & \dfrac{\partial \varphi}{\partial C}; \\ \Lambda(0) & = & 0; \\ \Lambda(T) & = & 0, \end{array}\right. \quad (14)$$

$$\frac{d\Phi}{dt} + \left[\frac{\partial F}{\partial X}\right]^t \cdot \Phi = 0, \quad (15)$$

$$\frac{d\Psi}{dt} + \left[\frac{\partial G}{\partial C}\right]^t \cdot \Psi = 0, \quad (16)$$

- Then it comes :

$$\nabla \varphi = \left[\frac{\partial F}{\partial S}\right]^t \cdot \Lambda + \left[\frac{\partial^2 G}{\partial X^2} Q\right]^t \cdot \Phi + \left[\frac{\partial^2 G}{\partial C \partial S} Q\right]^t \cdot \Psi + \frac{\partial \varphi}{\partial S} \quad (17)$$

Remarks

- The sensitivity is obtained by solving the coupled system of four equations
- The System involves second order terms.
- We found a **non-standard problem : two equations have two conditions an initial condition and a final condition, the other two equations have no condition**

Solving the Non-Standard problem

- The Non-Standard problem can be symbolically written :

$$\left\{\begin{array}{rcl}\frac{dX}{dt} & = & K(X,Y),\ \ t\in[0,T];\\ \frac{dY}{dt} & = & L(X,Y),\ \ t\in[0,T]\end{array}\right. \tag{18}$$

- with :

$$\left\{\begin{array}{rcl}X(0) & = & 0;\\ X(T) & = & 0\end{array}\right. \tag{19}$$

and no condition on Y.
NSP is transformed into a problem of optimal control by introducing the control U and a cost-function $J_P(U)$ with :

$$\left\{\begin{array}{rcl}X(0) & = & 0;\\ Y(0) & = & U.\end{array}\right. \tag{20}$$

Solving the Non-Standard problem 2

A cost function $J_P(U)$ is defined by:

$$J_P(U) = \frac{1}{2}\|X(T,U)\|^2 + \frac{1}{2}\|U\|^2 \tag{21}$$

If Z and W are defined as the solution of:

$$\frac{dW}{dt} + \left[\frac{\partial K}{\partial X}\right]^t \cdot W + \left[\frac{\partial L}{\partial X}\right]^t \cdot Z = 0; \tag{22}$$

$$\frac{dZ}{dt} + \left[\frac{\partial K}{\partial Y}\right]^t \cdot W + \left[\frac{\partial L}{\partial Y}\right]^t \cdot Z = 0; \tag{23}$$

$$Z(T) = 0; W(T) = X(T), \tag{24}$$

then we get

$$\nabla J_P(U) = -Z(0) + U \tag{25}$$

Solving the Non-Standard problem 3

This problem involved third derivatives of the original model.
Recent developments on the NSP have been recently carrier out by V. Shutyaev and F.-X. Le Dimet
The existence of a solution is demonstrated
Another method to solve NSP is proposed.

A 1-D Example 1

Let us assume that the one dimensional velocity field $u = u(x, t)$ evolves according to the Burgers equation given by :

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} & = & f, x \in \Omega =]-1, 1[, t \in [0, T]; \\ u & = & u_0, t = 0, \\ u & = & u_1, x \in \{-1, 1\}, \end{cases} \tag{26}$$

Evolution of the pollutant's concentration:

$$\begin{cases} \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} & = & \eta\frac{\partial^2 c}{\partial x^2} + s, x \in]-1, 1[, t \in [0, T] \\ c & = & c_0, t = 0; \\ c & = & c_1, x \in \{-1, 1\} \end{cases} \tag{27}$$

A 1-D Example: cost function

The cost function takes the form (with continuous observation in space and time):

$$J(u_0, c_0) = \frac{1}{2}\int_0^T \|u - u_{obs}\|_\Omega^2 dt + \frac{1}{2}\int_0^T \|c - c_{obs}\|_\Omega^2 dt. \qquad (28)$$

where $\|f\|_\Omega^2 = \int_\Omega f(x)f(x)dx = \int_0^1 f(x)f(x)dx.$

A 1-D Example: adjoint model

The adjoint variables p and q are introduced as the solutions of :

$$\left\{ \begin{array}{rcl} \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \nu\frac{\partial^2 p}{\partial x^2} + q\frac{\partial c}{\partial x} & = & u - u_{obs} \\ p(t = T) & = & 0 \\ p & = & 0, x \in \{-1, 1\} \end{array} \right. \tag{29}$$

$$\left\{ \begin{array}{rcl} \frac{\partial q}{\partial t} + \frac{\partial uq}{\partial x} + \eta\frac{\partial^2 q}{\partial x^2} & = & c - c_{obs} \\ q(t = T) & = & 0 \\ q & = & 0, x \in \{-1, 1\} \end{array} \right. \tag{30}$$

And the gradient of the cost function is given by:

$$\begin{array}{rcl} \nabla_{u_0} J & = & -p(0) \\ \nabla_{c_0} J & = & -q(0) \end{array}$$

Sensitivity of a response function

Let φ be a function a the concentration and the source functions, the response function is given by:

$$\Phi_A(t, s) = \int_{\Omega_A} \varphi(c, s) dx \tag{31}$$

where $\Omega_A \subset \Omega$ is the response region. Following the guidelines of the derivation of the gradient, we introduces the adjoint variables Γ ϕ,ψ and Λ as the solution of:

Sensitivity of a response function

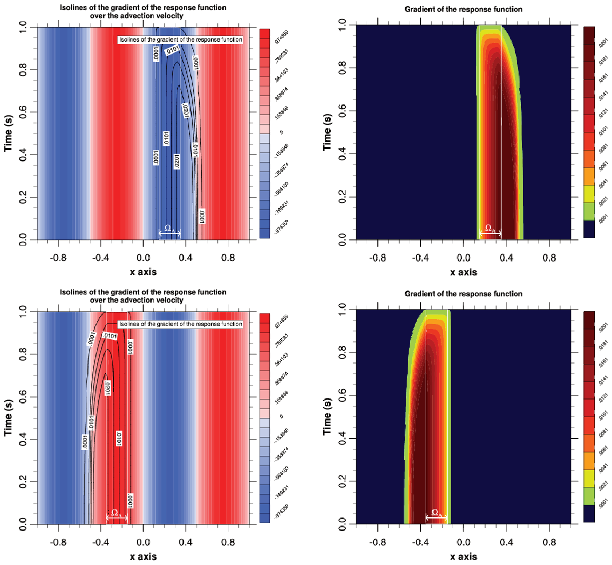
$$
\begin{cases}
\frac{\partial \Gamma}{\partial t} + u\frac{\partial \Gamma}{\partial x} + \nu\frac{\partial^2 \Gamma}{\partial x^2} - \Lambda\frac{\partial c}{\partial x} \\
\qquad -\phi\frac{\partial p}{\partial x} + q\frac{\partial \psi}{\partial x} - \phi = 0 \\
\Gamma = 0, t \in \{0, T\} \\
\Gamma = 0, x \in \{-1, 1\}
\end{cases}
\tag{32}
$$

$$
\begin{cases}
\frac{\partial \Lambda}{\partial t} + \frac{\partial u\Lambda}{\partial x} + \eta\frac{\partial^2 \Lambda}{\partial x^2} + \frac{\partial q\phi}{\partial x} - \psi = -\frac{\partial \varphi}{\partial c} \\
\Lambda = 0, t \in \{0, T\} \\
\Lambda = 0, x \in \{-1, 1\}
\end{cases}
\tag{33}
$$

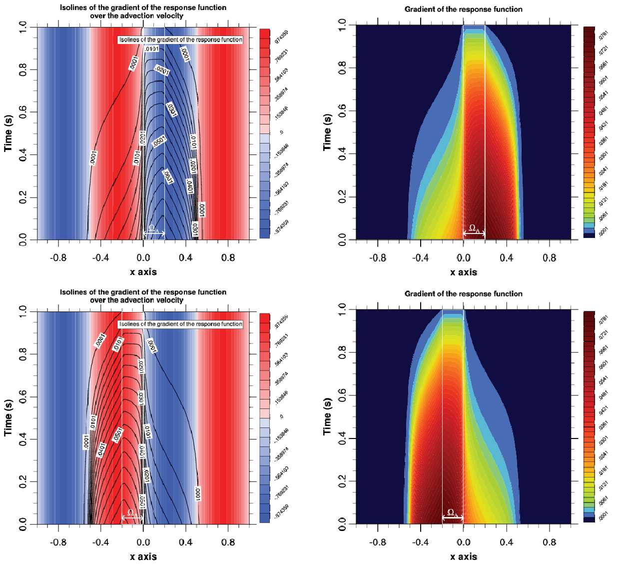
Where the function 1_{Ω_A} is:

$$
1_{\Omega_A}(x) = \begin{cases} 1, & \text{if } x \in \Omega_A \\ 0, & \text{if not.} \end{cases}
$$

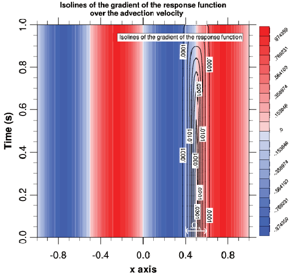
Numerical results

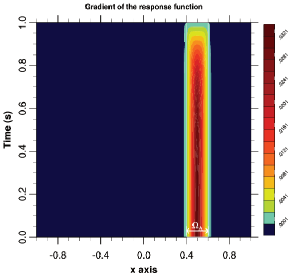


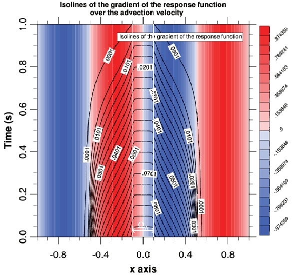
Numerical results

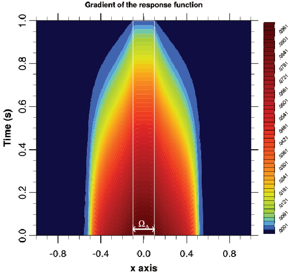


Numerical results









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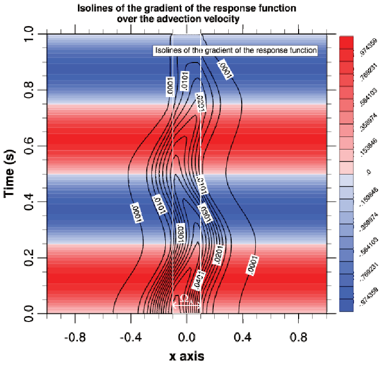

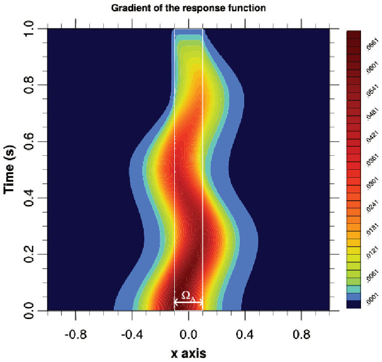

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