Effective Transmission Conditions for Thin-Layer Transmission Problems in Elastodynamics

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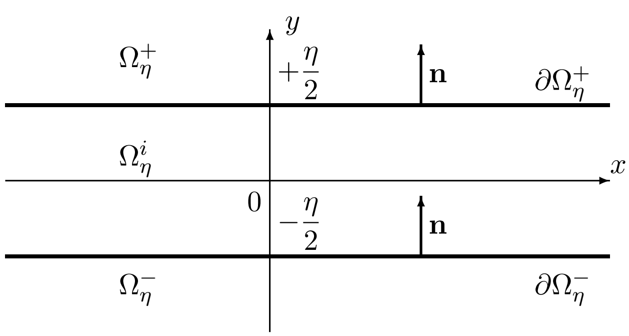
Introduction

This research is motivated by the numerical modeling of ultrasonic non-destructive testing experiments. Some tested media feature thin layers (made e.g. of resin), which are difficult to handle in numerical computations due to the very small element size required for meshing them. Similar issues arise in the treatment of thin coatings, see e.g. [2], [3]. To overcome these difficulties, one idea consists in using effective transmission conditions (ETCs) across the two interfaces bounding the layer. This work aims at establishing such ETCs by means of a formal asymptotic analysis with respect to the (small) layer thickness, in the spirit of [1] for Maxwell's equations.

1 Problem Setting

We consider the case of a thin layer of an isotropic elastic material occupying the strip $\Omega^i_\eta = \mathbb{R} \times [-\frac{\eta}{2}, \frac{\eta}{2}]$. In addition, let Ω^+_η (resp. Ω^-_η) denote the remaining portions of the propagation domain situated above (resp. below) the thin layer. The upper and lower boundaries of the layer are denoted $\partial\Omega^+_\eta$ and $\partial\Omega^-_\eta$; they are connected to Ω^+_η and Ω^-_η, respectively. We assume that the layer material is isotropic and homogeneous, with mass density ρ^i and Lamé coefficients λ^i, μ^i. We denote by ρ, λ, and μ the (possibly heterogeneous) coefficients of the material in $\Omega^\pm_\eta$.



The displacement field $\mathbf{u}^i_\eta$ in Ω^i_η, as well as the displacement fields $\mathbf{u}^\pm_\eta$ inside $\Omega^\pm_\eta$ satisfy the elastodynamics equations:

$$\rho^i \partial_t^2 \mathbf{u}^i_\eta - \text{ div } \sigma^i(\mathbf{u}^i_\eta) = 0, \qquad \text{in } \Omega^i_\eta, \tag{1}$$

$$\rho\, \partial_t^2 \mathbf{u}^\pm_\eta - \text{ div } \sigma(\mathbf{u}^\pm_\eta) = 0, \qquad \text{in } \Omega^\pm_\eta. \tag{2}$$

where $\sigma(\mathbf{u})$ and $\sigma^i(\mathbf{u})$ respectively denote the stress tensor in the surrounding and layer material, respectively, given by Hooke's law applied to a given displacement $\mathbf{u}$. Equations (1) and (2) are coupled with transmission conditions on the interfaces $\partial\Omega^\pm_\eta$ (we omit the time variable t for simplicity):

$$\begin{cases} \mathbf{u}^\pm_\eta\Big(x, \pm\dfrac{\eta}{2}\Big) = \mathbf{u}^i_\eta\Big(x, \pm\dfrac{\eta}{2}\Big) \\ \mathbf{t}(\mathbf{u}^\pm_\eta)\ \Big(x, \pm\dfrac{\eta}{2}\Big) = \mathbf{t}^i(\mathbf{u}^i_\eta)\ \Big(x, \pm\dfrac{\eta}{2}\Big), \end{cases} \tag{3}$$

where $\mathbf{t}(\mathbf{u}) := \sigma(\mathbf{u})\mathbf{n}$ and $\mathbf{t}^i(\mathbf{u}) := \sigma^i(\mathbf{u})\mathbf{n}$ are the traction vectors relative to $\Omega^\pm_\eta$ and Ω_η and $\mathbf{n}$ is the normal vector to $\partial\Omega^\pm_\eta$ (see the figure above).

Eliminating formally $\mathbf{u}^i_\eta$, we can write a transmission problem for $\mathbf{u}_\eta := (\mathbf{u}^+_\eta, \mathbf{u}^-_\eta)$. For any function $f : \mathbb{R}^2 \to \mathbb{R}^d$, using the notation

$$\{f\}_\eta = \big([f]_\eta, \langle f\rangle_\eta\big) : \mathbb{R} \to \mathbb{R}^{2d},$$
$$[f]_\eta(x) := f(x, \eta/2) - f(x, -\eta/2),$$
$$\langle f\rangle_\eta(x) := \big(\, f(x, \eta/2) + f(x, -\eta/2)\,\big)/2,$$

this transmission condition can be written in the form

$$\{\mathbf{t}(\mathbf{u}_\eta)\}_\eta + \mathbf{T}_\eta\{\mathbf{u}_\eta\}_\eta = 0$$

where $\mathbf{T}_\eta$ is a (nonlocal) DtN transmission operator that can easily be defined implicitly from the solution of the interior Dirichlet problem in the strip Ω^i_η. The next idea is that, when η tends to 0, $\mathbf{T}_\eta$ becomes local and that one can get explicit analytical approximations of it.

2 Principle of construction of ETCs

This construction is based on an ansatz for the interior solution $\mathbf{u}^i_\eta$ of the form

$$\mathbf{u}^i_\eta(x,y) = \mathbf{U}^0\big(x, \frac{y}{\eta}\big) + \eta\, \mathbf{U}^1\big(x, \frac{y}{\eta}\big) + \eta^2\, \mathbf{U}^2\big(x, \frac{y}{\eta}\big) + ... \tag{4}$$

where $\mathbf{U}^k : \Omega^i_1 \to \mathbb{R}^2$. This implies in particular a similar expansion for the traces

$$\mathbf{u}^i_\eta(x, \pm\frac{\eta}{2}) = \mathbf{u}^0_\pm(x) + \eta\, \mathbf{u}^1_\pm(x) + \eta^2\, \mathbf{u}^2_\pm(x) + ... \tag{5}$$

Substituting (4) into (1) allows us to compute explicitly the $\mathbf{U}^k$ from the $\mathbf{u}^k_\pm$ by induction on k: these are polynomial functions in y/η. These expressions lead us to introduce a family of differential operators of order ℓ, $\mathcal{A}_\ell(\partial_x,\partial_t), \ell \geq 0$, such that

$$\mathcal{A}_0(\partial_x,\partial_t)\{\mathbf{u}^0\} = 0 \tag{6}$$

and

$$\mathbf{t}^i(\mathbf{u}^i_\eta)\left(x, \pm\frac{\eta}{2}\right) = \mathbf{t}^0_\pm(x) + \eta\, \mathbf{t}^1_\pm(x) + \eta^2\, \mathbf{t}^2_\pm(x) + ... \tag{7}$$

where $$\{\mathbf{t}^k\} = \sum_{j=0}^{k+1} \mathcal{A}_{k+1-j}(\partial_x,\partial_t)\{\mathbf{u}^j\} \tag{8}$$

and where we have defined

$$\{\mathbf{t}^k\} = \big([\mathbf{t}^k], \langle \mathbf{t}^k\rangle\big),\ [\mathbf{t}^k] = \mathbf{t}^k_+ - \mathbf{t}^k_-,\ \langle \mathbf{t}^k\rangle = \frac{\mathbf{t}^k_+ + \mathbf{t}^k_-}{2}$$

and the same for $\{\mathbf{u}^k\}$. Note that from (5) and (7)

$$\begin{cases} \{\mathbf{u}^i_\eta\}_\eta = \{\mathbf{u}^0\} + \eta\,\{\mathbf{u}^1\} + \eta^2\,\{\mathbf{u}^2\} + ... \\ \{\mathbf{t}^i(\mathbf{u}^i_\eta)\}_\eta = \{\mathbf{t}^0\} + \eta\,\{\mathbf{t}^1\} + \eta^2\,\{\mathbf{t}^2\} + ... \end{cases} \tag{9}$$

We rewrite the transmission conditions (3) as

$$\{\mathbf{t}(\mathbf{u}_\eta)\}_\eta = \{\mathbf{t}^i(\mathbf{u}^i_\eta)\}_\eta, \quad \{\mathbf{u}_\eta\}_\eta = \{\mathbf{u}^i_\eta\}_\eta$$

so that, using (9) and (8), we get

$$\eta\,\{\mathbf{t}(\mathbf{u}_\eta)\}_\eta = \sum_{k\geq 0} \eta^{k+1}\,\Big(\sum_{j=0}^{k+1} \mathcal{A}_{k+1-j}(\partial_x,\partial_t)\{\mathbf{u}^j\}\Big)$$

which, thanks to (6), can be rearranged as

$$\begin{aligned}\eta\,\{\mathbf{t}(\mathbf{u}_\eta)\}_\eta &= \Big(\sum_\ell \eta^\ell \mathcal{A}_\ell(\partial_x,\partial_t)\Big)\Big(\sum_j \eta^j\{\mathbf{u}^j\}\Big) \\ &= \Big(\sum_{\ell=0}^{k} \eta^\ell \mathcal{A}_\ell(\partial_x,\partial_t)\Big)\{\mathbf{u}^\eta\}_\eta + O(\eta^{k+1})\end{aligned}$$

The transmission condition of order $k+1$ is then obtained formally by dropping the $O(\eta^{k+1})$ term.

3 A stable transmission condition

Applying the above method with $k = 2$ leads to the following transmission conditions

$$\begin{cases} A[\mathbf{u}^\eta]_\eta = \eta\,\langle \mathbf{t}(\mathbf{u}^\eta)\rangle_\eta - \eta\, BJ\,\partial_x\,\langle \mathbf{u}^\eta\rangle_\eta, \\ [\mathbf{t}(\mathbf{u}^\eta)]_\eta = \eta\,(\rho\,\partial_t^2 - JAJ\,\partial_x^2)\,\langle \mathbf{u}^\eta\rangle_\eta - JB\,\partial_x[\mathbf{u}^\eta]_\eta, \end{cases} \tag{10}$$

where A, B and J are the following 2×2 matrices:

$$A = \begin{pmatrix}\mu^i & 0 \\ 0 & \lambda^i + 2\mu^i\end{pmatrix},\ B = \begin{pmatrix}\mu^i & 0 \\ 0 & \lambda^i\end{pmatrix},\ J = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}. \tag{11}$$

A fundamental point is the well-posedness and uniform stability in η of the transmission problem (2), (10). This is in fact a consequence of an energy conservation result: any smooth enough solution of (2), (10) satisfies:

$$\frac{d}{dt}\Big(\mathcal{E}_\eta + \mathcal{E}^i_\eta\Big) = 0 \tag{12}$$

where

$$\mathcal{E}_\eta = \frac{\rho}{2}\int_{\Omega_\eta} |\partial_t \mathbf{u}^\eta|^2\ dx + \frac{1}{2}\int_{\Omega_\eta} \sigma(\mathbf{u}^\eta) : \varepsilon(\mathbf{u}^\eta)\ dx$$

$$\mathcal{E}^i_\eta = \frac{\eta\,\rho}{2}\int_{\mathbb{R}} |\langle \partial_t \mathbf{u}^\eta\rangle|^2\ d\gamma + \frac{\eta}{2}\int_{\mathbb{R}} Q\Big(\langle \partial_x \mathbf{u}^\eta\rangle, \frac{[\mathbf{u}^\eta]}{\eta}\Big)\ d\gamma$$

and $Q(\mathbf{x},\mathbf{y}) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is the symmetric quadratic form ($J^T = J$ and $(BJ)^T = JB$)

$$Q(\mathbf{x},\mathbf{y}) := \mathbf{x}^T JAJ\,\mathbf{x} + 2\,\mathbf{x}^T JB\,\mathbf{y} + \mathbf{y}^T A\,\mathbf{y} \tag{13}$$

The well-posedness and stability result is then a consequence of the

Theorem. *The quadratic form $Q(\mathbf{x},\mathbf{y})$ is positive.*

At the conference, we shall present various numerical simulations to illustrate the accuracy and the efficiency of our approximate model. Moreover, some insights about the error analysis will be given.

References

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